Estimation of long memory in the presence of a smooth nonparametric trend

Clifford Hurvich, Gabriel Lang, and Philippe Soulier

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Abstract. We consider semiparametric estimation of the long-memory parameter of a stationary process in the presence of an additive nonparametric mean function. We use a semiparametric Whittle-type estimator, applied to the tapered, differenced series. Since the mean function is not necessarily a polynomial of finite order, no amount of differencing will completely remove the mean. We establish a central limit theorem for the estimator of the memory parameter, assuming that a slowly increasing number of low frequencies are trimmed from the estimator's objective function. We find in simulations that tapering and trimming are essential for the good performance of the estimator in practice.

Keywords: Nonparametric regression, long-range dependence, tapering, periodogram.

1 Introduction

The semiparametric estimation of long memory for weakly stationary univariate series has been extensively studied. See, for example, Robinson (1994, 1995a,b), Hurvich, Deo and Brodsky (1998), Moulines and Soulier (1999). Generalizations to the case where additive polynomial trends may be present were considered by Velasco (1999a,b), Hurvich and Chen (2000), Hurvich, Moulines and Soulier (2002). All four of these papers employ tapering schemes, and the latter two employ differencing prior to tapering.

The idea of differencing to detrend the data followed by tapering to handle difficulties induced by possible non-invertibility was suggested by Hart (1989), in a nonparametric context. In the presence of polynomial trends, adequate differencing will completely annihilate the trends, but if the trend is an arbitrary smooth function, differencing serves as only an approximate detrending device. The focus of Hart (1989) was on estimation of the autocovariances of the noise process (stochastic component), which was assumed to have short memory, in the presence of a smooth additive nonparametric signal (trend). Here, we will explore the use of differencing and tapering for estimation of the memory parameter of a long-memory noise process in the presence of a smooth additive nonparametric signal.

There is an existing literature on long memory in the presence of non-polynomial trends. Künsch (1986) discussed the difficulty of distinguishing certain monotonic trends from long memory. Hall and Hart (1990), Csörgo and Mielnikzuk (1995a,b) and Deo (1997) discussed the properties of kernel estimators of of the mean function in the presence of a long-memory noise. Robinson (1997) discussed this same topic, and also provided a method for estimating the memory parameter in the presence of a nonparametric signal. Our focus here is only on estimation of the memory parameter of the noise, not on estimation of the signal. Nevertheless, though we do not pursue it here, our estimator of the memory parameter of the noise, which does not require any preliminary estimator of the signal, might be useful for estimating standard errors for the signal, or in constructing optimal estimators of the signal.

Robinson (1997) showed that the memory parameter of the noise may be estimated consistently from the raw data, even in the presence of an unknown nonparametric signal. However, he only established that the estimator of the memory parameter is $\log^{1/2}(n)$–consistent, and furthermore required conditions on the bandwidth that become extremely stringent as the short-memory case is approached. Our procedure, which applies the Gaussian semiparametric estimator (see Robinson 1995b) to the tapered, differenced data, yields $n^{2/5-\delta}$–consistency and asymptotic normality for any sufficiently small positive value of δ, assuming that the process is stationary with a spectral pole at zero frequency.

Section 2 presents the main theoretical results on consistency and asymptotic normality of the estimator. It is assumed in these theorems that an increasing number of low frequencies is trimmed from the objective function of the estimator. Stronger tapers lead to a decrease in the amount of trimming required. Simulation results, presented in Section 3, indicate that tapering and trimming are essential for the good performance of the estimator in practice. Furthermore, a finite-sample correction to the asymptotic variance agrees well with the variances of the estimators as found in our simulations. The remaining sections present the theory needed to establish the main theorems. The central limit theorem presented in Section 4 is notably general in terms of the taper allowed, and the weights used in a linear combination of the tapered periodogram of a Martingale difference sequence.

2 Assumptions and main results

We consider the following model:

$$X_t = r(t/n) + \epsilon_t \ \ , \ \ t = 0, \dots n \tag{2.1}$$

where r is a sufficiently smooth function and ϵ is a linear process with long range dependence. Denote

$$Y_t = X_t - X_{t-1} \ \ , \ \ \Delta r(t/n) = r(t/n) - r((t-1)/n), \ \ \text{and } \eta_t = \epsilon_t - \epsilon_{t-1}.$$

This yields

$$Y_t = \Delta r(t/n) + \eta_t \ \ , \ \ t = 1, \dots, n.$$

We assume that the process η is linear with respect to a zero mean unit variance white noise $Z = (Z_t)_{t \in \mathbb{Z}}$:

$$\eta_t = \sum_{j \in \mathbb{Z}} a_j Z_{t-j}, \ \ \sum_{j \in \mathbb{Z}} a_j^2 < \infty. \tag{2.2}$$

We further assume that the spectral density of η, denoted by f, can be expressed as

$$f(x) = |x|^{-2d_0} f^*(x) \tag{2.3}$$

where d_0 is referred to as the memory parameter of the differenced series and $f^*(x)$ satisfies some smoothness condition in the neighborhood of zero frequency (see the assumptions in the statement of Theorem

1). It will be assumed in the sequel that the original series is stationary with long-memory: thus, $d_0 \in I \subset (-1, -1/2)$.

Let h be a complex-valued function defined on $[0,1]$. For any positive integer n, define $H_n = \sum_{t=1}^n |h(t/n)|^2$. The tapered Discrete Fourier Transform (DFT) and the tapered periodogram ordinates of a process ξ are defined as:

$$d_{\xi,k} = \frac{1}{\sqrt{2\pi H_n}} \sum_{t=1}^{n} h(t/n)\xi_t \mathrm{e}^{\mathrm{i}x_k t} \quad \text{and} \quad I_{\xi,k} = |d_{\xi,k}|^2, \tag{2.4}$$

where $x_k := 2\pi k/n$ ($k = 1, \ldots, [(n-1)/2]$) are the Fourier frequencies.

The following assumptions on the mean function r and on the taper h will be used. Throughout the paper, p is a fixed integer, $p \geq 1$.

(A1) r is a p+1 times continuously differentiable function on $[0,1]$.

(A2) $h(x) = \sum_{u=0}^{v} b_u \mathrm{e}^{2\mathrm{i}\pi x u}$ for a non negative integer $v \geq p$ and real coefficients b_u, $0 \leq u \leq v$ such that

$$\sum_{u=0}^{v} b_u u^k = 0, \quad \text{for } k = 0, \ldots, p-1.$$

A function h that satisfies **(A2)** has the following important properties.
(i) There exists a constant C such that for all $x \in [-\pi, \pi]$,

$$\left| \sum_{t=1}^{n} h(t/n) \mathrm{e}^{\mathrm{i}tx} \right| \leq C \frac{n}{(1+n|x|)^{p+1}}. \tag{2.5}$$

(ii) h has at least $p-1$ vanishing derivatives at 0 and 1:

$$h^{(j)}(0) = h^{(j)}(1) = 0, \; j = 0, \ldots, p-1. \tag{2.6}$$

An example of a function h that satisfies **(A2)** is $h(x) = (1 - e^{2\mathrm{i}\pi x})^p$. This yields the family of tapers introduced by Hurvich and Chen (2000). For this taper, which has $v = p$, (2.6) clearly holds and (2.5) was proved by Hurvich and Chen (2000). (2.5) is proved under **(A2)** in Lemma A.1. Chen (2001) has constructed certain tapers that satisfy **(A2)**. These new tapers may have improved efficiency properties compared to the Hurvich-Chen tapers.

The following bound for the tapered DFT of the differenced mean function $d_{\Delta r,k}$ is crucial for the derivation of the properties of our estimator. It should be noted that the Lemma does not require the specific form for the taper as given in assumption **(A2)**.

Lemma 2.1. *Assume* **(A1)** *and let h be a complex-valued p times continuously differentiable function that satisfies (2.6). Then, there exists a constant C such that, for $1 \leq k \leq n/2$,*

$$|d_{\Delta r,k}| \leq C(k^{-p} n^{-1/2} + n^{-3/2}). \tag{2.7}$$

Proof. By a Taylor expansion, and since $\sum_{t=1}^{n} |h(t/n)| \le H_n^{1/2} n^{1/2}$, we have,

$$\left| d_{\Delta r,k} - n^{-1}(2\pi H_n)^{-1/2} \sum_{t=1}^{n} h(t/n) r'(t/n) \exp(\mathrm{i} x_k t) \right| \le C \|r''\|_\infty n^{-3/2}$$

Under assumption **(A1)**, hr' is continuously differentiable. Hence its Fourier series, defined as $c_j := \int_0^1 h(s) r'(s) \mathrm{e}^{2\mathrm{i}\pi js} \,\mathrm{d}s$, is absolutely summable and $h(s)r'(s) = \sum_{j\in\mathbb{Z}} c_j \mathrm{e}^{-2\mathrm{i}\pi js}$. Moreover, by assumption, the derivatives up to the order $p-1$ of hr' vanish at 0 and 1. This implies that $\sum_{j\in\mathbb{Z}} j^{2p}|c_j|^2 < \infty$. Hence, we have

$$n^{-1}\sum_{t=1}^{n} h(t/n) r'(t/n) \exp(2\mathrm{i}\pi k t/n) = \sum_{j\in\mathbb{Z}} c_j n^{-1} \sum_{t=1}^{n} \exp(2\mathrm{i}\pi (k-j) t/n) = \sum_{\lambda\in\mathbb{Z}} c_{k+\lambda n}.$$

By applying Hölder's inequality and noting that $k \le n/2$, we bound this last series:

$$\sum_{\lambda\in\mathbb{Z}} |c_{k+\lambda n}| \le \left\{ \sum_{\lambda\in\mathbb{Z}} (k+\lambda n)^{2p} |c_{k+\lambda n}|^2 \right\}^{1/2} \left\{ \sum_{\lambda\in\mathbb{Z}} (k+\lambda n)^{-2p} \right\}^{1/2} \le C k^{-p},$$

for some constant C depending only on h, r' and their derivatives up to the order p. □

The local Whittle contrast is defined as

$$W_m(C,d) = \sum_{k=\ell}^{m} \left\{ \log(C x_k^{-2d}) + C^{-1} x_k^{2d} I_{Y,k} \right\} \tag{2.8}$$

where $m < n/2$ is a bandwidth parameter and $\ell < m$ is a lower trimming number. Concentrating C out of W_m yields the profile likelihood:

$$\hat{J}_{\ell,m}(d) = \log\left(\frac{1}{m-\ell+1} \sum_{k=\ell}^{m} k^{2d} I_{Y,k} \right) - 2d\gamma_{\ell,m} \tag{2.9}$$

where $\gamma_{\ell,m} = \frac{1}{m-\ell+1} \sum_{k=\ell}^{m} \log(k)$. We define the local Whittle (or Gaussian semiparametric in the terminology of Robinson 1995b) estimate of d_0 as:

$$\hat{d}_n = \arg \min_{d\in(-1,-1/2)} \hat{J}_{\ell,m}(d).$$

We now introduce some additional assumptions.

(A3) (Z_l) is a fourth-order homoscedastic martingale difference sequence *i.e.* almost surely,

$$E[Z_k|\mathcal{F}_{k-1}] = 0, \quad \mathbb{E}[Z_k^2|\mathcal{F}_{k-1}] = 1 \text{ and } \mathbb{E}[Z_k^4|\mathcal{F}_{k-1}] = \mu_4,$$

where $\mathcal{F}_k = \sigma(Z_l, l \le k)$.

(A4) $a(x) := \sum_{j\in\mathbb{Z}} a_j \mathrm{e}^{\mathrm{i}jx}$ can be expressed as $a(x) = x^{-d_0} a^*(x)$ $(x > 0)$ where $d_0 \in (-1,-1/2)$ and a^* is twice continuously differentiable in a neighbourhood of zero and absolutely integrable on $[-\pi,\pi]$.

Theorem 1. *Assume that* **(A1)**, **(A2)**, **(A3)** *and* **(A4)** *hold. If ℓ and m are non increasing sequences of integers such that*

$$m/n + n/m\ell^{1+2p} = O(n^{-\zeta}), \tag{2.10}$$

for some $\zeta > 0$, then $\hat{d}_n - d_0 = O_P(n^{\eta})$ for some $\eta > 0$.

Remark 1. A suitable choice of the sequences ℓ and m is $\ell = [n^{\delta/2p}]$ and $m = [n^{1-\delta}]$ for some arbitrarily small $\delta > 0$.

Proof. Define

$$J_{\ell,m}(d) = \log\left(\frac{1}{m-\ell+1}\sum_{k=\ell}^{m} k^{2d-2d_0}\right) - 2(d-d_0)\gamma_{\ell,m},$$

$$E_n(d) = \left(\sum_{j=\ell}^{m} j^{2d-2d_0}\right)^{-1} \sum_{k=\ell}^{m} k^{2d-2d_0}\left(x_k^{2d_0} f^*(0)^{-1} I_{Y,k} - 1\right).$$

With these notations, we obtain

$$\hat{J}_{\ell,m}(d) = \log(1+E_n(d)) + J_{\ell,m}(d) + \log(f^*(0)) - 2d_0\log(2\pi/n) - 2d_0\gamma_{\ell,m}. \tag{2.11}$$

The strict concavity of the function log implies that $J_{\ell,m}$ is minimized by d_0. Moreover, there exists a constant $C_0 > 0$ such that for all $d \in (-1,-1/2)$ and all $m \geq 1$:

$$J_{\ell,m}(d) - J_{\ell,m}(d_0) \geq C_0(d-d_0)^2. \tag{2.12}$$

By Proposition 1 in section 6, we know that there exists an $\eta > 0$ such that

$$\sup_{d\in(-1,-1/2)} |E_n(d)| = O_P(n^{-\eta}).$$

This implies that

$$\sup_{d\in(-1,-1/2)} |\log(1+E_n(d))| = O_P(n^{-\eta}). \tag{2.13}$$

For any positive real A and positive integer n, define $\mathcal{D}_{A,n} = \{d \in (-1,-1/2);\ n^{\eta/2}|d-d_0| > A\}$. Applying (2.12), (2.11), and the fact that $\hat{d}_n$ minimizes $\hat{J}_{\ell,m}$, we obtain:

$$\begin{aligned}\mathbb{P}(\hat{d}_n \in \mathcal{D}_{A,n})) &= \mathbb{P}(n^{\eta}(\hat{d}_n - d_0)^2 \geq A^2) \leq \mathbb{P}(J_{\ell,m}(\hat{d}_n) - J_{\ell,m}(d_0) \geq C_0A^2n^{-\eta}) \\ &= \mathbb{P}(\hat{J}_{\ell,m}(\hat{d}_n) - \hat{J}_{\ell,m}(d_0) + \log(1+E_n(d_0)) - \log(1+E_n(\hat{d}_n)) \geq C_0A^2n^{-\eta}) \\ &\leq \mathbb{P}(2\sup_{d\in(-1,-1/2)} |\log(1+E_n(d))| \geq C_0A^2n^{-\eta}).\end{aligned}$$

We conclude by applying (2.13) which implies that $\lim_{A\to\infty}\sup_{n\to\infty}\mathbb{P}(\hat{d}_n \in \mathcal{D}_{A,n}) = 0$, which exactly means that $\hat{d}_n - d_0 = O_P(n^{-\eta/2})$. □

Theorem 2. *Assume* **(A1)**-**(A4)**. *Let ℓ and m be non decreasing sequences of integers such that (2.10) holds and*

$$\lim_{n\to\infty}\left(n\log(m)/(m^{1/2}\ell^{1+2p}) + m^5/n^4\right) = 0 \tag{2.14}$$

Then $m^{1/2}(\hat{d}_n - d_0)$ converges weakly to the Gaussian distribution with zero mean and variance $T_v/4$ with

$$T_v = \frac{\sum_{z=-v}^{v}(\sum_{u=0}^{v-|z|} b_u b_{u+|z|})^2}{\left(\sum_{u=0}^{v} b_u^2\right)^2}.$$

Remark 2. A suitable choice of the sequences ℓ and m is $\ell = [n^{\frac{3}{5(1+2p)}+\frac{\delta}{4p}}]$ and $m = [n^{4/5-\delta}]$ for some arbitrarily small $\delta \in (0, 4/5)$. Hence the number of trimmed lower frequencies is not too high.

Remark 3. In the case of the Hurvich-Chen taper of order p, $v = p$ and $T_p = (4p)!(p!)^4/\{4(2p)!^4\}$.

Proof. Since $\hat{d}_n$ is consistent, for large enough n, it satisfies

$$0 = \frac{\partial \hat{J}_{\ell,m}(\hat{d}_n)}{\partial d} = \frac{2\sum_{k=\ell}^m k^{2\hat{d}_n}\log(k) I_{Y,k}}{\sum_{k=\ell}^m k^{2\hat{d}_n} I_{Y,k}} \; - \; 2\gamma_{\ell,m}.$$

This implies, by a Taylor expansion

$$\begin{aligned} 0 &= \sum_{k=\ell}^m k^{2\hat{d}_n}(\log(k) - \gamma_{\ell,m}) I_{Y,k} \\ &= \sum_{k=\ell}^m k^{2d_0}(\log(k) - \gamma_{\ell,m}) I_{Y,k} + 2(\hat{d}_n - d_0)\sum_{k=\ell}^m k^{2\tilde{d}_n}\log(k)(\log(k) - \gamma_{\ell,m}) I_{Y,k}, \end{aligned}$$

where $\tilde{d}_n$ lies between $\hat{d}_n$ and d_0. Define $\nu_k = \log(k) - \gamma_{\ell,m}$, $s_m^2 = \sum_{k=\ell}^m \nu_k^2$ and

$$T_m = s_m^{-2}\sum_{k=\ell}^m k^{2\tilde{d}_n - 2d_0}\log(k)\nu_k \frac{I_{Y,k}}{x_k^{-2d_0} f^*(0)}.$$

With these notations, we obtain

$$s_m(\hat{d}_n - d_0) = -\frac{1}{2} T_m^{-1} s_m^{-1} \sum_{k=\ell}^m \nu_k \frac{I_{Y,k}}{x_k^{-2d_0} f^*(0)}.$$

We will prove that T_m converges in probability to 1 (Proposition 2). Hence the asymptotic distribution of $s_m(\hat{d}_n - d_0)$ is the same as that of $U_n := -\frac{1}{2}s_m^{-1}\sum_{k=\ell}^m \nu_k \frac{I_{Y,k}}{x_k^{-2d_0} f^*(0)}$. Write now $U_n = V_n + R_n$, with

$$V_n = -\frac{1}{2}s_m^{-1}\sum_{k=\ell}^m \nu_k 2\pi I_{Z,k} \;\text{ and }\; R_n = -\frac{1}{2}s_m^{-1}\sum_{k=\ell}^m \nu_k \left\{ \frac{I_{Y,k}}{x_k^{-2d_0} f^*(0)} \; - \; 2\pi I_{Z,k} \right\}.$$

Theorem 3 applied with $\beta_{n,k} = \nu_k / s_m$ implies that V_n converges weakly to $\mathcal{N}(0, T_v)$ and Proposition 3 implies R_n converges in probability to 0. □

3 Simulations

We investigated the properties of the estimator $\hat{d}_n$ based on first-differences of $\{X_t\}$, where $\{X_t\}$ is generated by model (2.1). We took the noise process $\{\epsilon_t\}$ to be a Gaussian $ARFIMA(1, d_0 + 1, 0)$ process with memory parameter $d_0 + 1 = 0.4$, and $AR(1)$ parameter (lag-one autocorrelation of short memory component) equal to 0.2. We considered two values of the mean function: $r(x) = 10x^4$ and $r(x) \equiv 0$. The bandwidth for the estimators was $m = n^{0.7}$. We applied p'th order Hurvich-Chen tapers $h(x) = (1 - e^{i2\pi x})^p$ with $p = 0, 1, 2$ to the first-differences, which have a memory parameter of $d_0 = -0.6$.

Since the differences are non-invertible, the standard theory of Robinson (1995b) for the non-tapered case ($p = 0$) does not apply, but we present this non-tapered case for the sake of comparison. We considered three sample sizes, $n = 100, 1000, 5000$. For each of these sample sizes, and each choice of p and the mean function r, we simulated two hundred realizations of the process, using the method of Davies and Harte (1987).

For computation of $\hat{d}_n$, we used a slightly modified version the profile likelihood (2.9):

$$\hat{J}^*_{\ell,m}(d) = \log\left(\frac{1}{m-\ell+1}\sum_{k=\ell}^{m}(k+p/2)^{2d}I_{Y,k}\right) - \frac{2d}{m-\ell+1}\sum_{k=\ell}^{m}\log(k+p/2).$$

We use $k + p/2$ here in place of k in (2.9) because it is advisable to consider $I_{Y,k}$ as an estimator of $f(2\pi(k + p/2)/n)$ rather than as an estimator of $f(2\pi k/n)$. More discussion on this slight frequency shifting due to the taper is given in Hurvich and Chen (2000) and Hurvich, Moulines and Soulier (2002).

The profile likelihood $\hat{J}^*_{\ell,m}$ was minimized for d on a grid of mesh size 0.01, in the range $[-1.49, 0.49]$. This range is wider than allowed by our theoretical results.

The degree of trimming was set according to the value of p. For $p = 0$, no trimming was used ($\ell = 1$). For $p = 1$, we used $\ell = n^{0.25}$. For $p = 2$ we used $\ell = n^{0.15}$. The values of ℓ used for $p = 1$ and $p = 2$ satisfy the condition given in Remark 2.

Figure 1 gives boxplots for the values of $\hat{d}_n$. The first and second panels correspond to the mean functions $r(x) = 10x^4$ and $r(x) \equiv 0$, respectively. Tables 1 and 2 give the bias, $m^{1/2}\cdot$ Bias, and variance for $\hat{d}_n$, for the cases of the quartic mean function and the constant mean function, respectively.

For the quartic mean function, the failure to use a taper ($p = 0$) leads to substantially biased estimators, while the bias is considerably reduced by using the first-order taper ($p = 1$) and is somewhat further reduced by using the second-order taper ($p = 2$). The reduced bias for $p = 2$ compared to $p = 1$ is consistent with Lemma 2.1. The bias in the non-tapered case is so strong that $m^{1/2}\cdot$ Bias remains nearly constant with n, whereas it decreases with n in the cases $p = 1$ and $p = 2$. For the case of the constant mean function, the bias in all of the estimators is attributable to the short-memory component of the process, and $m^{1/2}\cdot$ Bias decreases with n for all values of p.

We next consider the variance of the estimators. For comparison, Table 3 presents some theoretical variances, for the same values of n and m used in the simulations. In Table 3, the asymptotic variance, based on Theorem 2, is $\Phi_p/(4m)$, where $\Phi_p = (4p)!(p!)^4((2p)!)^{-4}$, and the finite-sample corrected variance is given by

$$\frac{\Phi_p}{4\sum_{k=\ell}^{m}[\log(k+p/2) - (m-\ell+1)^{-1}\sum_{j=\ell}^{m}\log(j+p/2)]^2}.$$

The corrected expression, which is asymptotically equivalent to the asymptotic expression, is justified heuristically in Hurvich and Chen (2000).

For $p = 0$, the variances of $\hat{d}_n$ obtained in the simulations (Tables 1 and 2) differ quite substantially for the two different mean functions, and almost none of the simulation variances agree well with either of the theoretical variance expressions from Table 3. Perhaps these difficulties are due to the non-invertibility of the differenced noise ($d_0 < -0.5$), combined with the failure to use a taper. There is no contradiction with Theorem 2, since the theorem assumes that $p \geq 1$. For $p = 1$ and $p = 2$, the simulation variances agree well with the corresponding finite-sample corrected variances, but differ substantially from the asymptotic variances. It should also be noted that the inflation in the observed and corrected asymptotic variance for $p = 2$ compared to the case $p = 1$ is minimal, and in some cases nonexistent, since fewer frequencies are trimmed for $p = 2$ than for $p = 1$.

The above results show that if no tapering or trimming is used, the estimator is severely biased, whereas if both tapering and trimming are used, the bias is substantially reduced. Further simulations, not shown here, indicate that while tapering alone reduces bias, if the true value of d_0 is small then trimming is required in addition to tapering in order to yield a nearly unbiased estimator. Thus, in general, it seems that both tapering and trimming are essential, both in theory and in practice. The practical justification for trimming may be based on examination of log-log periodogram plots (not shown here), indicating that the behavior of $I_{Y,k}$ for very small values of k is substantially affected by the presence of a nonzero signal, as suggested by Lemma 2.1.

The reader may still ask whether it might be possible to avoid this trimming in the theory for the estimator. The question seems relevant since we are not aware of any other situation where the Gaussian semiparametric estimator requires trimming. Furthermore, for stationary Gaussian processes (which therefore have a constant mean function) it is known that if the log-periodogram regression (GPH) estimator of Geweke and Porter-Hudak (1983) is used, then no trimming is required in order to achieve $\sqrt{m}$-consistency and asymptotic normality of the estimator (see Hurvich, Deo and Brodsky 1998), even though trimming was used in the earlier paper of Robinson (1995a) on this estimator. Finally, even a small amount of trimming can noticeably inflate the variability of the estimator in finite samples. We believe that unfortunately, the answer to the question, for the Gaussian semiparametric estimator used in the presence of a nonparametric additive mean function, is no: If trimming is not used, then for certain mean functions the Gaussian semiparametric estimator will be inconsistent for d_0. An interesting open question is whether the GPH estimator would also require trimming in order to ensure consistency. We hope to explore this in future work.

4 A central limit theorem for general linear combinations of tapered periodogram ordinates of a martingale difference sequence

Theorem 3. *Assume* **(A3)**. *Let* $(\beta_{n,k})_{1\leq k\leq K(n)}$ *be a triangular array of real numbers such that*

$$\sum_{k=1}^{K} \beta_{n,k} = 0, \tag{4.1}$$

$$\sum_{k=1}^{K} \beta_{n,k}^2 = 1, \tag{4.2}$$

$$\lim_{n\to\infty} \left(|\beta_{n,K}| + \sum_{k=1}^{K-1} |\beta_{n,k+1} - \beta_{n,k}| \right)^2 \log(n) = 0. \tag{4.3}$$

Let v *be a non negative integer and* $b_0, \ldots, b_v$ *be real numbers. Define* $h_{n,t} = \sum_{u=0}^{v} b_u \mathrm{e}^{\mathrm{i}tx_u}$, *and* $J_{Z,k} = \left(n \sum_{u=0}^{v} b_u^2\right)^{-1} \left|\sum_{t=1}^{n} h_{n,t} Z_t \mathrm{e}^{\mathrm{i}tx_k}\right|^2$. *Then* $\sum_{k=1}^{K} \beta_{n,k} \left(J_{Z,k} - 1\right)$ *converges to the Gaussian distribution* *with zero mean and variance* $T_v = \left(\sum_{u=0}^{v} b_u^2\right)^{-1} \sum_{z=-v}^{v} (\sum_{u=0}^{v-|z|} b_u b_{u+|z|})^2$.

Proof. Write first

$$\sum_{k=1}^{K} \beta_{n,k} \left(J_{n,k}^Z - 1\right) = \frac{\sum_{k=1}^{K} \beta_{n,k}}{n B_v} \sum_{t=1}^{n} |h_{n,t}|^2 (Z_t^2 - 1) + \sum_{t=2}^{n} \sum_{s=1}^{t-1} c_{t,s} Z_s Z_t, \tag{4.4}$$

where $c_{t,s} := \frac{2}{nB_v} \sum_{k=1}^{K} \beta_{n,k} \Re\left(h_{n,t} \bar{h}_{n,s} e^{\mathrm{i}(t-s)x_k}\right)$. Under assumption (4.1), the first term on the rhs of (4.4) vanishes. By the Martingale central limit theorem, cf. for instance Hall and Heyde (1980) or Dacunha-Castelle and Duflo (1991), the last term in (4.4) is asymptotically $\mathcal{N}(0, T_v)$ if we prove that:

$$\sum_{t=1}^{n} \left(\sum_{s=1}^{t-1} c_{t,s} Z_s\right)^2 \xrightarrow{P} T_v, \tag{4.5}$$

$$\sum_{t=1}^{n} \mathbb{E}\left[\left(\sum_{s=1}^{t-1} c_{t,s} Z_s\right)^4\right] \to 0. \tag{4.6}$$

We start by proving (4.5). Write $\sum_{t=2}^{n} \left(\sum_{s=1}^{t-1} Z_s c_{t,s}\right)^2 =: A_n + B_n + 2C_n$, with

$$A_n := \sum_{t=2}^{n} \sum_{s=1}^{t-1} c_{t,s}^2, \quad B_n := \sum_{t=2}^{n} \sum_{s=1}^{t-1} (Z_s^2 - 1) c_{t,s}^2, \quad C_n := \sum_{t=2}^{n} \sum_{1 \le r < s < t} c_{t,r} c_{t,s} Z_r Z_s.$$

We will prove in Lemma B.1 below that $\lim_{n\to\infty} A_n = T_v$ and that $\max_{1 \le s \le n} \sum_{t=1}^{n} c_{t,s}^2 = O(n^{-1})$.

Consider now B_n. Assumption **(A3)** implies that $\mathbb{E}[Z_t^2 - 1] = 0$ and $\mathbb{E}[(Z_s^2 - 1)(Z_t^2 - 1)] = \delta_{s,t}$. Hence B_n has zero mean and variance

$$\mathbb{E}(B_n^2) = \sum_{s=1}^{n-1} \sum_{s<t,u\le n} c_{t,s}^2 c_{u,s}^2 \le A_n \max_{1\le s\le n} \sum_{t=1}^{n} c_{t,s}^2 = O(n^{-1}).$$

To bound the last term C_n, note first that assumption **(A3)** implies that if $s < t$ and $u < v$, then $\mathbb{E}[Z_s Z_t Z_u Z_v] = 0$ if $s \ne u$ or $t \ne v$. Define $\beta_n = \left(|\beta_{n,K}| + \sum_{k=1}^{K-1} |\beta_{n,k+1} - \beta_{n,k}|\right)$. Applying the Cauchy-Schwarz inequality, (B.2) and (B.3), we obtain:

$$\begin{aligned} \mathbb{E}[C_n^2] &= \sum_{1\le r<s<n} \left(\sum_{s<t\le n} c_{t,r} c_{t,s}\right)^2 \le \sum_{1\le r<s<n} \sum_{s<t\le n} c_{t,r}^2 \sum_{s<t\le n} c_{t,s}^2 \\ &\le C n^{-1} \beta_n^2 \sum_{1\le r<s<n} \sum_{s<t\le n} (t-r)^{-2} \le C \beta_n^2 \log(n) = o(1), \end{aligned}$$

under assumption (4.3). We now prove (4.6). Under **(A3)**, by applying Rosenthal's inequality for martingales, and Lemma B.1, we obtain:

$$\sum_{t=1}^{n} \mathbb{E}\left[\left(\sum_{s=1}^{t-1} c_{t,s} Z_s\right)^4\right] \le C \sum_{t=1}^{n} \sum_{s=1}^{t-1} c_{t,s}^4 + C \sum_{t=1}^{n} \left(\sum_{s=1}^{t-1} c_{t,s}^2\right)^2 = O(n^{-1}).$$

□

5 Bounds for the Bartlett Approximation

The primary goal of this section is to establish Corollary 1 stated at the end of the section. Corollary 1 gives a so-called Bartlett approximation, whereby the periodogram of a tapered series, normalized by a proxy for the spectral density, is approximated by the periodogram of the noise appearing in a linear representation of the series. Corollary 1 is needed for proving that $R_n \xrightarrow{P} 0$ in Theorem 2. To establish Corollary 1, we first give, in Lemma 5.1, general conditions guaranteeing the L^1 convergence to zero of a weighted sum of the difference between the periodogram of a tapered series, normalized by an arbitrary sequence, and the periodogram of the driving noise. The conditions for Lemma 5.1 are "unprimitive": they bear on some integrals involving the DFT of the taper, the transfer function and a normalising factor. The reason we use these assumptions is to separate the purely analytical properties of the DFT of the taper (which yield bounds for these integrals) and the probabilistic properties of the tapered DFT and periodogram ordinates.

Lemma 5.1 as stated here can be used in deriving the properties of a tapered Gaussian semiparametric estimator in the absence of a signal, and without trimming, since Lemma 5.1 does not require that ℓ tends to ∞. Furthermore, Lemma 5.1 allows for a general choice of the coefficients $\tilde{a}_k$. For proving our main results, we use $\tilde{a}_k = x_k^{-d_0} a^*(0)$. However, other choices may be used depending on the problem at hand. See, for instance, Hurvich, Moulines and Soulier (2002), Andrews and Sun (2001). An alternative choice for the present context could be $\tilde{a}_k = (1 - \mathrm{e}^{-\mathrm{i}x_k})^{-2d} a^*(0)$. In order for the conditions of the lemma to apply, the coefficients $\tilde{a}_k$ should approximate the behavior of the underlying transfer function $a(x)$ in a neighborhood of zero frequency.

Once Lemma 5.1 is proved, it is necessary to check that its conditions hold under the assumptions of Theorem 2. This checking is carried out, under weaker assumptions than needed for Theorem 2, in Lemma A.2, stated and proved in the Appendix.

Lemma 5.1. *Let η be a covariance stationary process which admits a linear representation with respect to a white noise Z which satisfies assumption* **(A3)** *and with transfer function $a(x) = \sum_{j \in \mathbb{Z}} a_j \mathrm{e}^{\mathrm{i}jx}$. Let $(\tilde{a}_k)_{\ell \le k \le m}$ be complex numbers. Let h be any function on $[0,1]$, $D_n(x) = (2\pi H_n)^{-1/2} \sum_{t=1}^n h(t/n) \mathrm{e}^{\mathrm{i}tx}$ and*

$$p_{k,j} = \left| \int_{-\pi}^{\pi} \left(\frac{a(x)}{\tilde{a}_k} - 1 \right) D_n(x_k - x) \overline{D_n(x_j - x)} \, \mathrm{d}x \right|, \tag{5.1}$$

$$q_k = \left| \int_{-\pi}^{\pi} \left(\frac{|a(x)|^2}{|\tilde{a}_k|^2} - 1 \right) |D_n(x_k - x)|^2 \, \mathrm{d}x \right|. \tag{5.2}$$

Let m be a non decreasing sequence of integers and let $(c_{n,k})_{\ell \le k \le m}$ be a triangular array of real numbers such that

$$\sum_{k=\ell}^{m} c_{n,k}^2 \le 1, \tag{5.3}$$

$$\lim_{n \to \infty} \sum_{k=\ell}^{m} |c_{n,k}| (q_k + 2p_{k,k}) = 0, \tag{5.4}$$

$$\lim_{n \to \infty} \sum_{\ell \le k \le j \le m} |c_{n,k} c_{n,j}| p_{k,j} p_{j,k} = 0. \tag{5.5}$$

Define $\tilde{f}_{\eta,k} = |\tilde{a}_k|^2/(2\pi)$ and let $I_{\eta,k}$ and $I_{Z,k}$ be the tapered DFT of the processes η and Z with tapering coefficients $h(t/n)$, as defined in (2.4). Then

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\sum_{k=\ell}^{m} c_{n,k}\left(\frac{I_{\eta,k}}{\tilde{f}_{\eta,k}} - 2\pi I_{Z,k}\right)\right|\right] = 0. \tag{5.6}$$

Proof. Define, $u_j = \tilde{a}_j^{-1}\sqrt{2\pi}d_{\eta,j}$ and $v_j = \sqrt{2\pi}d_{Z,j}$. Using these notations, we obtain the following decomposition:

$$\tilde{f}_{\eta,k}^{-1} I_{\eta,k} - 2\pi I_{Z,k} = |u_k - v_k|^2 + 2\mathrm{Re}(\bar{v}_k(u_k - v_k)).$$

By straightforward algebra, we obtain:

$$\mathbb{E}[|u_j - v_j|^2] = \int_{-\pi}^{\pi}\left(\frac{|a(x)|^2}{|\tilde{a}_j|^2} - 1\right)|D_n(x - x_j)|^2 dx \\ - 2\mathrm{Re}\left(\int_{-\pi}^{\pi}\left(\frac{a(x)}{\tilde{a}_j} - 1\right)|D_n(x - x_j)|^2 \,\mathrm{d}x\right) \leq q_j + 2p_{j,j}. \tag{5.7}$$

Applying assumption (5.4), we obtain

$$\mathbb{E}\left[\sum_{k=\ell}^{m}|c_{n,k}||u_k - v_k|^2\right] \leq \sum_{k=\ell}^{m}|c_{n,k}|(q_k + 2p_{k,k}) = o(1). \tag{5.8}$$

Define $W_n := \sum_{k=\ell}^{m} c_{n,k} v_k(\bar{u}_k - \bar{v}_k)$. By some well known formula, we have $\mathbb{E}[W_n^2] = S_{1,n} + S_{2,n} + S_{3,n}$, with

$$S_{1,n} = \sum_{k=\ell}^{m} c_{n,k}^2 \mathbb{E}[|u_k - v_k|^2],$$

$$S_{2,n} = \sum_{\ell\leq k\leq j\leq m} c_{n,k}c_{n,j}\Big\{\mathbb{E}[\bar{v}_k(u_k - v_k)]\;\mathbb{E}[\bar{v}_j(u_j - v_j)] + \mathbb{E}[v_k(\bar{u}_k - \bar{v}_k)]\;\mathbb{E}[\bar{v}_j(u_j - v_j)] + \\ \mathbb{E}[\bar{v}_k(u_j - v_j)]\;\mathbb{E}[\bar{v}_j(u_k - v_k)] + \mathbb{E}[v_k(\bar{u}_j - \bar{v}_j)]\;\mathbb{E}[\bar{v}_j(u_k - v_k)]\Big\},$$

$$S_{3,n} = 2\sum_{\ell\leq k\leq j\leq m} c_{n,k}c_{n,j}\mathrm{cum}(v_k, \bar{u}_k - \bar{v}_k, \bar{v}_j, u_j - v_j).$$

Applying (5.14), we obtain:

$$S_{1,n} \leq \sum_{k=\ell}^{m} c_{n,k}^2(q_k + 2p_{k,k}) \leq \sum_{k=\ell}^{m}|c_{n,k}|(q_k + 2p_{k,k}) = o(1)$$

by assumption (5.4). To bound $S_{2,n}$, note that

$$|\mathbb{E}[\bar{v}_j(u_k - v_k)]| = \left|\int_{-\pi}^{\pi}\left(\frac{a(x)}{\tilde{a}_k} - 1\right)D_n(x_k - x)\overline{D_n(x_j - x)}\,\mathrm{d}x\right| = p_{k,j}.$$

Hence, by assumptions (5.4) and (5.5), we obtain, for some numerical constant c:

$$|S_{2,n}| \leq c\sum_{\ell\leq k\leq j\leq m}|c_{n,k}c_{n,j}|(p_{k,k}p_{j,j} + p_{k,j}p_{j,k}) = o(1).$$

In order to compute and bound the cumulants that appear in the term $S_{3,n}$, we need to introduce the following kernel:

$$\tilde{D}_{p,n}(x) = \int_{-\pi}^{\pi} D_{p,n}(z) D_{p,n}(x-z)\,\mathrm{d}z = H_n^{-1} \sum_{t=1}^{n} h(t/n)^2 \mathrm{e}^{\mathrm{i}tx}. \tag{5.9}$$

It is easily seen that $\tilde{D}_{p,n}$ satisfies

$$\int_{-\pi}^{\pi} |\tilde{D}_{p,n}(x)|^2 dx = O(n^{-1}), \tag{5.10}$$

Some more algebra and (5.9) yields:

$$\begin{aligned}
\mathrm{cum}(\bar{v}_k, u_k - v_k, \bar{v}_j, u_j - v_j) &= \frac{\kappa_4}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(\frac{a(x)}{\tilde{a}_k} - 1 \right) \left(\frac{a(y)}{\tilde{a}_j} - 1 \right) \\
&\times\ D_n(x_k - x)\ D_n(x_j - y) \overline{D_n(x_k - x - y + z)}\ \overline{D_n(x_j - z)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \frac{\kappa_4}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(\frac{a(x)}{\tilde{a}_k} - 1 \right) \left(\frac{a(y)}{\tilde{a}_j} - 1 \right) \\
&\qquad \times\ D_n(x_k - x)\ D_n(x_j - y) \overline{\tilde{D}_n(x_k + x_j - x - y)}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, (5.10), and (5.7), we obtain

$$\begin{aligned}
|\mathrm{cum}(\bar{v}_k, u_k - v_k, \bar{v}_j, u_j - v_j)| &\le c n^{-1/2} \left(\int_{-\pi}^{\pi} \left| \frac{a(x)}{\tilde{a}_k} - 1 \right|^2 |D_n(x_k - x)|^2 dx \right)^{1/2} \\
&\times \left(\int_{-\pi}^{\pi} \left| \frac{a(x)}{\tilde{a}_j} - 1 \right|^2 |D_n(x_j - x)|^2 dx \right)^{1/2} \le c\kappa_4 n^{-1/2} \sqrt{(q_k + 2p_{k,k})(q_j + 2p_{j,j})}.
\end{aligned}$$

Hence, applying the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}
|S_{n,3}| &\le c n^{-1/2} \sum_{\ell \le k \le j \le m} |c_{n,k} c_{n,j}| \sqrt{(q_k + 2p_{k,k})(q_j + 2p_{j,j})} \le c n^{-1/2} \left(\sum_{k=\ell}^{m} |c_{n,k}| \sqrt{(q_k + 2p_{k,k})} \right)^2 \\
&\le c\kappa n^{-1/2} \sum_{k=\ell}^{m} |c_{n,k}| \sum_{k=\ell}^{m} |c_{n,k}| (q_k + 2p_{k,k}).
\end{aligned}$$

Assumption (5.3) implies that $n^{-1/2} \sum_{k=\ell}^{m} |c_k| \le \sqrt{m/n} \le 1$, hence, by assumption (5.4), we obtain:

$$|S_{n,3}| \le c(m/n)^{1/2} \sum_{k=\ell}^{m} |c_{n,k}| (q_k + 2p_{k,k}) = o(1).$$

□

Assumptions **(A2)**, **(A3)**, **(A4)**, Lemma A.1 and Lemma A.2 applied with $q = p$ imply that if we choose

$$\tilde{a}_k = x_k^{-d_0} a^*(0) \tag{5.11}$$

then the coefficients defined in (5.1) and (5.2) satisfy

$$q_k + 2p_{k,k} \le C(k^{-1} + (k/n)^2) \quad \text{and} \quad p_{k,j}p_{j,k} \le C(k^{-1}j^{-1} + ((k \vee j)/n)^4). \tag{5.12}$$

Hence (5.4) and (5.5) hold for any sequence of weights $(c_{n,k})_{\ell \le k \le m}$ such that (5.3) holds and

$$\lim_{n\to\infty} \max_{\ell \le k \le m} |c_{n,k}| = 0, \tag{5.13}$$

which is the so-called Lindeberg condition, and for any sequence m such that $\lim_{n\to\infty}(m^5/n^4) = 0$. No restriction is imposed on ℓ.

If in (5.12) the terms $(k/n)^2$ and $(k \vee j)/n)^4$ are replaced by $(k/n)^\beta$ and $(k \vee j)/n)^{2\beta}$, respectively, then (5.4) and (5.5) hold for any sequence of weights $(c_{n,k})_{\ell \le k \le m}$ such that (5.3), (5.13) hold and any sequence m such that $\lim_{n\to\infty}(m^{2\beta+1}/n^{2\beta}) = 0$.

We give a brief proof of this. Since we make no assumption on ℓ, we assume without loss of generality that $\ell = 1$. Let m' be a sequence of integers such that $1 \le m' \le m$ and $\lim_{n\to\infty} m' = +\infty$. Define $c_n^* = \max_{1\le k\le m} |c_{n,k}|$. If (5.13) holds, m' can be chosen such that $\lim_{n\to\infty} c_n^* \log(m) = 0$. Then, splitting the sum at m' and applying the Cauchy Schwarz inequality to the sum above m' and (5.3), we have:

$$\sum_{k=1}^{m} |c_{n,k}| k^{-1} \le c_n^* \sum_{k=1}^{m'} k^{-1} + \left\{ \sum_{k=m'+1} k^{-2} \right\}^{1/2} \le c_n^* \log(m') + {m'}^{-1},$$

$$\sum_{k=1}^{m} |c_{n,k}|(k/n)^\beta \le m^{\beta+1/2} n^{-\beta},$$

$$\sum_{1\le k\le j\le m} |c_{n,k}||c_{n,j}|(j/n)^{2\beta} \le m^{2\beta+1} n^{-2\beta}.$$

For further reference, we gather these remarks in a corollary.

Corollary 1. *Assume* **(A2)**, **(A3) (A4)**. *Let m be a non decreasing sequence of integers such that $\lim_{n\to\infty}(m^5/n^4) = 0$, let $\tilde{f}_{\eta,k} = |\tilde{a}_k|^2/(2\pi)$ where $\tilde{a}_k$ is given by (5.11), and let $(c_{n,k})_{\ell\le k\le m}$ be a triangular array of real numbers such that (5.3) and (5.13) hold. Then*

$$\lim_{n\to\infty} \mathbb{E}\left[\left| \sum_{k=\ell}^{m} c_{n,k} \left(\frac{I_{\eta,k}}{\tilde{f}_{\eta,k}} - 2\pi I_{Z,k} \right) \right| \right] = 0. \tag{5.14}$$

6 Propositions

Proposition 1. *Assume* **(A1)- (A4)**. *Let ℓ and m be non decreasing sequences such that*

$$\lim_{n\to\infty} \left(m/n + n/(\ell^{1+2p} m) \right) = 0.$$

Then $\sup_{d\in(-1,-1/2)} |E_n(d)| = o_P(1)$. If moreover m and ℓ are such that

$$m/n + n/(\ell^{1+2p} m) \le n^{-\zeta},$$

for some $\zeta > 0$, then there exists a real $\eta > 0$ such that

$$\mathbb{E}[\sup_{d\in(-1,-1/2)} |E_n(d)|] = O(n^{-\eta}).$$

Proof. Denote, for $k = \ell, \dots, m$,

$$\gamma_k(d) = \frac{k^{2(d-d_0)}}{\sum_{j=\ell}^m j^{2(d-d_0)}}, \; w_k = f^*(0)^{-1/2} x_k^{d_0} d_{\eta,k} - \sqrt{2\pi} d_{Z,k}, \; \zeta_k = \frac{I_{\eta,k}}{x_k^{-2d_0} f^*(0)} - 2\pi I_{Z,k}.$$

Then,

$$\begin{aligned}
\frac{I_{Y,k}}{x_k^{-2d_0} f^*(0)} - 1 &= \frac{I_{\Delta r,k}}{x_k^{-2d_0} f^*(0)} + 2\mathrm{Re}\left(\frac{d_{\Delta r,k} d_{\eta,k}}{x_k^{-2d_0} f^*(0)}\right) + \zeta_k + 2\pi I_{Z,k} - 1, \\
E_n(d) &= \sum_{k=\ell}^m \gamma_k(d)\zeta_k + \sum_{k=\ell}^m \gamma_k(d)(2\pi I_{Z,k} - 1) \\
&\quad + \sum_{k=\ell}^m \gamma_k(d)\left(\frac{I_{\Delta r,k}}{x_k^{-2d_0} f^*(0)} + 2\mathrm{Re}\left(\frac{d_{\Delta r,k} d_{\eta,k}}{x_k^{-2d_0} f^*(0)}\right)\right) =: R_1(d) + R_2(d) + R_3(d).
\end{aligned}$$

The terms R_1 and R_2 can be dealt with by using the techniques introduced by Robinson (1995b) in the non tapered case and used by Hurvich and Chen (2000) in the case of the Hurvich and Chen taper. We nevertheless give a streamlined proof since our assumptions are slightly different. The main tools are summation by parts, Corollary 1 and the following bounds: for all $d \in (-1, -1/2)$,

$$0 \le \gamma_k(d) \le C k^{-1} \left(\frac{k}{m}\right)^{-2d_0-1} \quad \text{and} \quad |\gamma_k(d) - \gamma_{k+1}(d)| \le C k^{-2} \left(\frac{k}{m}\right)^{-2d_0-1}. \tag{6.1}$$

By summation by parts, we obtain:

$$R_1(d) = \sum_{k=\ell}^{m-1} (\gamma_k(d) - \gamma_{k+1}(d)) \sum_{j=1}^k \zeta_j + \gamma_m(d) \sum_{j=1}^{m-1} \zeta_j.$$

By Corollary 1, we obtain the bound $\mathbb{E}\left[\left|\sum_{j=1}^k \zeta_j\right|\right] = O(k^{1/2})$. This and (6.1) yield:

$$\begin{aligned}
\mathbb{E}\left[\sup_{d \in (-1,-1/2)} |R_1(d)|\right] &\le C \sum_{k=\ell}^m k^{1/2} k^{-2} \left(\frac{k}{m}\right)^{-2d_0-1} + C m^{-1/2} \\
&\le C m^{2d_0+1} \sum_{k=\ell}^m k^{-5/2-2d_0} + C m^{-1/2} \le C(m^{2d_0+1} + m^{-1/2}).
\end{aligned}$$

To deal with the term R_2, we gather here some useful facts on the tapered DFT and periodogram ordinates of a white noise satisfying assumption **(A3)**. The main effect of the data taper is that even though Z is a zero mean unit variance white noise, its DFT ordinates are not uncorrelated. Nevertheless, by assumption **(A2)**, they satisfy $\mathbb{E}[d_{Z,k} \bar{d}_{Z,k+s}] = 0$ if $s > v$ and $|\mathbb{E}[d_{Z,k} \bar{d}_{Z,k+s}]| \le 1$ if $s \le v$. Hence for any sequence of complex numbers $(a_j)_{\ell \le j \le m}$, we have the bound:

$$\mathbb{E}\left[\left|\sum_{j=\ell}^m a_j d_{Z,j}\right|^2\right] \le \sum_{j=\ell}^m \sum_{s=0}^v |a_j a_{j+s}| \le v \sum_{j=\ell}^m |a_j|^2. \tag{6.2}$$

If $0 \le j \le v$, $\mathrm{cov}(I_{Z,k}, I_{Z,k+j})$ is uniformly bounded with respect to n, k and j by some constant which depends only on v and the distribution of Z. If $j > v$, then $\mathrm{cov}(I_{Z,k}, I_{Z,k+j}) = c(v, Z) n^{-1}$, for some constant $c(v,Z)$ which depends only on v and the distribution of Z. For instance, for $v=0$, it is well known that $\mathrm{cov}(I_{Z,k}, I_{Z,k+j}) = \kappa_4 n^{-1}$, where $\kappa_4 := \mathbb{E}[Z_0^4]-3$ is the fourth cumulant of Z_0 (cf. for instance Fay and Soulier (2001)). Hence we obtain, for some constant C which depends only on v and the distribution of Z:

$$\mathbb{E}\left[\left(\sum_{k=\ell}^m c_k(2\pi I_{Z,k}-1)\right)^2\right] \le C\sum_{k=\ell}^m c_k^2 + Cn^{-1}\left(\sum_{k=\ell}^m c_k\right)^2 \le C(v,Z)\sum_{k=\ell}^m c_k^2. \tag{6.3}$$

We now obtain a uniform bound for R_2, applying summation by parts, (6.1) and (6.3).

$$R_2(d) = \sum_{k=\ell}^{m-1}(\gamma_k(d)-\gamma_{k+1}(d))\sum_{j=1}^k(2\pi I_{Z,j}-1) + \gamma_m(d)\sum_{j=1}^{m-1}(2\pi I_{Z,j}-1),$$

$$\mathbb{E}\left[\sup_{d\in(-1,-1/2)}|R_2(d)|\right] \le C\sum_{k=\ell}^m k^{1/2}k^{-2}\left(\frac{k}{m}\right)^{-2d_0-1} + Cm^{-1/2} \le C(m^{2d_0+1}+m^{-1/2}).$$

We now bound $\mathbb{E}[\sup_{d\in(-1,-1/2)}|R_3(d)|]$. Write $R_3(d)=R_4(d)+R_5(d)+R_6(d)$ with

$$R_4(d) = \sum_{k=\ell}^m \gamma_k(d)\frac{I_{\Delta r,k}}{x_k^{-2d_0}f^*(0)},$$
$$R_5(d) = 2\sum_{k=\ell}^m \gamma_k(d)\mathrm{Re}\left(\frac{d_{\Delta r,k}}{x_k^{-d_0}\sqrt{f^*(0)}}w_k\right),$$
$$R_6(d) = 2\sum_{k=\ell}^m \gamma_k(d)\mathrm{Re}\left(\frac{\sqrt{2\pi}d_{\Delta r,k}d_{Z,k}}{x_k^{-d_0}\sqrt{f^*(0)}}\right).$$

By applying the first bound in (6.1) and Lemma 2.1, we obtain, for all $d, d_0\in(-1,-1/2)$,

$$|R_4(d)| \le C\sum_{k=\ell}^m k^{-1}(k/m)^{-2d_0-1}(k/n)^{2d_0}n^{-1}(k^{-2p}+n^{-2}) = C(m/n)^{2d_0+1}\sum_{k=\ell}^m(k^{-2-2p}+k^{-2}n^{-2})$$
$$\le Cnm^{-1}(\ell^{-1-2p}+\ell^{-1}n^{-2}) \le C(\ell^{-1-2p}m^{-1}n+n^{-1}). \tag{6.4}$$

Equations (5.7) and (5.12) imply that

$$\mathbb{E}[|w_k|^2] \le C(k^{-1}+(k/n)^2). \tag{6.5}$$

By Lemma 2.1, (6.5) and the Cauchy-Schwarz inequality, we obtain

$$\mathbb{E}[|d_{\Delta r,k}w_k|] \le Cn^{-1/2}(k^{-p}+n^{-1})(k^{-1/2}+kn^{-1}) \le Cn^{-1/2}(k^{-p-1/2}+n^{-1}).$$

Applying this bound, equation (6.1) and summation by parts, we obtain:

$$\mathbb{E}\left[\sup_{d\in(-1,-1/2)}|R_5(d)|\right] \le C\sum_{k=\ell}^m k^{-2}(k/m)^{-2d_0-1}\sum_{j=\ell}^k(j/n)^{d_0}n^{-1/2}(j^{-p-1/2}+n^{-1}) \tag{6.6}$$

$$+ Cm^{-1}\sum_{k=\ell}^m(k/n)^{d_0}n^{-1/2}(k^{-p-1/2}+n^{-1}). \tag{6.7}$$

We first obtain the following bound:

$$\sum_{j=\ell}^{k}(j/n)^{d_0}n^{-1/2}(j^{-p-1/2}+n^{-1}) \le C(n^{-d_0-1/2}\ell^{d_0-p+1/2}+n^{-d_0-3/2}k^{d_0+1}). \tag{6.8}$$

Thus the right hand side of (6.6) is bounded by a constant times

$$m^{2d_0+1}\left(\ell^{-d_0-p-3/2}n^{-d_0-1/2}+\ell^{-1-d_0}n^{-d_0-3/2}\right) \le \ell^{-1-p}m^{-1/2}n^{1/2}+n^{-1/2}. \tag{6.9}$$

The term (6.7) is bounded by a constant times

$$\ell^{d_0-p+1/2}m^{-1}n^{-d_0-1/2}+m^{d_0+1}n^{-d_0-3/2} \le \ell^{-p-1/2}m^{-1}n^{1/2}+n^{-1/2} \tag{6.10}$$

Applying (6.2) and Lemma 2.1, we obtain:

$$\mathbb{E}\left[\left|\sum_{j=\ell}^{k}x_k^{d_0}d_{\Delta r,k}d_{Z,j}\right|^2\right] \le v\sum_{j=\ell}^{k}x_k^{2d_0}I_{\Delta r,k} \le C(\ell^{2d_0-2p+1}n^{-2d_0-1}+n^{-2d_0-3}k^{2d_0+1}). \tag{6.11}$$

Applying (6.1), (6.11), the Cauchy-Schwarz inequality and summation by parts, we obtain:

$$\begin{aligned}\mathbb{E}\left[\sup_{d\in(-1,-1/2)}|R_6(d)|\right] &\le Cm^{2d_0+1}\sum_{k=\ell}^{m}k^{-2d_0-3}\mathbb{E}^{1/2}\left[\left|\sum_{j=\ell}^{k}x_k^{d_0}d_{\Delta r,k}d_{Z,j}\right|^2\right] \\ &+ Cm^{-1}\mathbb{E}^{1/2}\left[\left|\sum_{j=\ell}^{m}x_k^{d_0}d_{\Delta r,k}d_{Z,j}\right|^2\right] \\ &\le Cm^{2d_0+1}(\ell^{-d_0-p-3/2}n^{-d_0-1/2}+\ell^{-d_0-3/2}n^{-d_0-1/2}) \\ &+ Cm^{-1}(\ell^{d_0-p+1/2}n^{-d_0-1/2}+n^{-d_0-3/2}m^{d_0+1/2}) \\ &\le C(\ell^{-p-1}m^{-1/2}n^{1/2}+n^{-1/2}). \end{aligned} \tag{6.12}$$

Gathering (6.4), (6.9), (6.10) and (6.12) yields:

$$\mathbb{E}\left[\sup_{d\in(-1,-1/2)}|R_3(d)|\right] \le C\left(\ell^{-2p-1}m^{-1}n+n^{-1}\right)^{1/2},$$

which concludes the proof of Proposition 1. □

Proposition 2. *Assume* **(A1)**-**(A4)**. *Let ℓ and m be non decreasing sequences of integers such that (2.10) and (2.14) hold for some $\zeta>0$. Then T_m converges in probability to 1.*

Proof. Define $\xi_k = f^*(0)^{-1}x_k^{2d_0}I_{Y,k}-1$. Then T_m can then be expressed as

$$T_m = 1+s_m^{-2}\sum_{k=\ell}^{m}\log(k)\nu_k\xi_k + s_m^{-2}\sum_{k=\ell}^{m}(k^{2(\tilde{d}_n-d_0)}-1)\log(k)\nu_k\frac{I_{Y,k}}{x_k^{-2d_0}f^*(0)} =: 1+T_{1,m}+T_{2,m}.$$

We will prove that $T_{1,m} = o_P(1)$ and $T_{2,m}\mathbf{1}_{\{|\hat{d}_n - d_0| \leq n^{-\eta}\}} = o_P(1)$. This is sufficient to prove Proposition 2 since it follows from Proposition 1 that $\hat{d}_n$ is $n^{-\eta}$-consistent for some $\eta > 0$.

For brevity, denote $V_k = f^*(0)^{-1/2} x_k^{d_0} d_{\Delta r,k}$. Using the notations of Proposition 1, we then obtain:

$$\xi_k = 2\pi I_{Z,k} - 1 + \zeta_k + |V_k|^2 + 2\mathrm{Re}\,(V_k w_k) + 2\sqrt{2\pi}\mathrm{Re}\,(V_k d_{Z,k})\,.$$

Denote $c_k = s_m^{-2}\log(k)\nu_k$. Then $\sum_{k=\ell}^m c_k = 1$ and $|c_k| = O(\log^2(m)m^{-1})$. Applying Lemma 2.1, we obtain:

$$\begin{aligned}\sum_{k=\ell}^m |c_k| \frac{I_{\Delta r,k}}{x_k^{-2d_0} f^*(0)} &\leq C\frac{\log^2(m)}{m}\sum_{k=\ell}^m n^{-1}(k^{-2p} + n^{-2})(k/n)^{2d_0} \\ &\leq C\log^2(m)(\ell^{1-2p+2d_0} + \ell^{1+2d_0}n^{-2})m^{-1}n^{-1-2d_0} \leq C(\log^2(m)\ell^{-1-2p}m^{-1}n + n^{-1}). \quad (6.13)\end{aligned}$$

Applying Lemma 2.1, (6.5) and the bound (6.8) with $k = m$ we obtain:

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{k=\ell}^m c_k \frac{d_{\Delta r,k}}{x_k^{-d_0}\sqrt{f^*(0)}}\, w_k\right|\right] &\leq C\log^2(m)m^{-1}(n^{-d_0}l^{d_0-p+1/2} + n^{-d_0-3/2}m^{d_0+1}) \\ &\leq C\log^2(m)(l^{-p-1/2}m^{-1}n^{1/2} + n^{-1/2}). \quad (6.14)\end{aligned}$$

Applying (6.2) and (6.11), we now obtain:

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{k=\ell}^m c_k V_k d_{Z,k}\right|^2\right] &\leq p\sum_{k=\ell}^m c_k^2|V_k|^2 \leq C\log^4(m)m^{-2}(\ell^{2d_0-2p+1}n^{-2d_0-1} + n^{-2d_0-3}m^{2d_0+1}) \\ &\leq C\log^4(m)(l^{-1-2p}m^{-2}n + m^{-1}n^{-1}). \quad (6.15)\end{aligned}$$

Since the coefficients c_k satisfy the conditions of Corollary 1, we obtain that $\lim_{n\to\infty}\mathbb{E}[|\sum_{k=\ell}^m c_k\zeta_k|] = 0$.

To bound the last remaining term in the decomposition of $T_{1,m}$, we use (6.3):

$$\mathbb{E}\left[\left(\sum_{k=\ell}^m c_k(2\pi I_{Z,k} - 1)\right)^2\right] \leq C\sum_{k=\ell}^m c_k^2 = O(\log^2(m)m^{-1}).$$

We have proved that $T_{1,m} = o_P(1)$.

We now consider $T_{2,m}$. Denote $b_k = c_k(k^{2\tilde{d}_n - 2d_0} - 1)\mathbf{1}_{\{|\hat{d}_n - d_0| \leq n^{-\eta}\}}$. Then $|b_k| \leq \log^3(m)m^{-1}n^{-\eta}$ and

$$T_{2,m} = \sum_{k=\ell}^m b_k \frac{I_{\eta,k}}{x_k^{-2d_0} f^*(0)} + \sum_{k=\ell}^m b_k|V_k|^2 + 2\mathrm{Re}\left(\sum_{k=\ell}^m b_k \frac{d_{\eta,k}}{x_k^{-d_0}\sqrt{f^*(0)}}V_k\right).$$

By Lemmas 5.1 and A.2, $\mathbb{E}[x_k^{2d_0} I_{\eta,k}]$ is uniformly bounded. Hence

$$\mathbb{E}\left[\left|\sum_{k=\ell}^m b_k \frac{I_{\eta,k}}{x_k^{-2d_0} f^*(0)}\right|\right] \leq C\log^3(m)n^{-\eta}.$$

By Lemma 2.1 and (6.11), we obtain:

$$\mathbb{E}\left[\sum_{k=\ell}^{m}|b_k|V_k^2\right] \le C\log^3(m)m^{-1}n^{-1-\eta}\sum_{k=\ell}^{m}(k/n)^{2d_0}(k^{-2p}+k^2n^{-2}) \le \log^3(m)n^{-\eta}(\ell^{-1-2p}m^{-1}n+n^{-1-\eta}).$$

The last term is dealt with by applying the Cauchy-Schwarz inequality. □

Proposition 3. *Assume* **(A1)**-**(A4)**. *Let ℓ and m be non decreasing sequences of integers such that (2.14) holds. Then R_n converges in probability to 0.*

Proof. Write $R_n = \sum_{i=1}^4 R_{1,n}$, with

$$R_{1,n} = s_m^{-1}\sum_{k=\ell}^{m}\nu_k\frac{I_{\Delta r,k}}{x_k^{-2d_0}f^*(0)},\ R_{2,n} = s_m^{-1}\sum_{k=\ell}^{m}\nu_k\frac{\mathrm{Re}(\bar{d}_{\Delta r,k}w_k)}{x_k^{-d_0}\sqrt{2\pi}a^*(0)},$$
$$R_{3,n} = s_m^{-1}\sum_{k=\ell}^{m}\nu_k\frac{\mathrm{Re}(\bar{d}_{\Delta r,k}d_{Z,k})}{x_k^{-d_0}a^*(0)},\ R_{4,n} = s_m^{-1}\sum_{k=\ell}^{m}\nu_k\zeta_k,$$

where $\zeta_k = x_k^{2d_0}f^*(0)^{-1}I_{\eta,k} - 2\pi I_{Z,k}$.

By the same computations as in (6.13), (6.14) and (6.15) with $s_m^{-1}\nu_k$ instead of c_k, we obtain:

$$\begin{aligned}R_{n,1} &\le C\log(m)(\ell^{-1-2p}m^{-1/2}n + m^{1/2}n^{-1}),\\ \mathbb{E}[|R_{n,2}|] &\le C\log(m)(\ell^{-p-1/2}m^{-1/2}n^{1/2}+m^{1/2}n^{-1/2}),\\ &\le C\log^2(m)(\ell^{-1-2p}m^{-1}n+n^{-1}).\end{aligned}$$

Finally, Corollary 1 implies that $\mathbb{E}[|R_{4,n}|] = o(1)$. □

A Appendix

Lemma A.1. *Let h be a taper satisfying assumption* **(A2)**. *Then the corresponding kernel $D_n(x) = (2\pi H_n)^{-1/2}\sum_{t=1}^n h(t/n)\mathrm{e}^{\mathrm{i}tx}$ satisfies*

$$|D_n(x)| \le C\frac{n^{1/2}}{(1+n|x|)^{p+1}}.$$

Proof. The proof is exactly the same as in Hurvich and Chen (2000). First, we note that as $H_n = n\sum b_u^2$, $|D_n(x)| \le Cn^{1/2}$. We prove now that, for $x \neq 0$, $|D_n(x)| \le Cn^{-p-1/2}|x|^{-p-1}$. We have

$$D_n(x) = (2\pi H_n)^{-1/2}\sum_{t=1}^{n}\sum_{u=0}^{v}b_u\mathrm{e}^{\mathrm{i}(x+x_u)} = (2\pi H_n)^{-1/2}\sum_{u=0}^{v}b_u\mathrm{e}^{\mathrm{i}(n+1)x/2}\frac{\sin(nx/2)}{\sin((x+x_u)/2)}.$$

We bound the upper sine and the exponential by 1. Defining $g(x) = 1/\sin(x)$, we get

$$|D_n(x)| \le (2\pi H_n)^{-1/2}\left|\sum_{u=0}^{v}b_u g((x+x_u)/2)\right|.$$

Now we use a Taylor expansion for $g((x+x_u)/2)$:

$$g((x+x_u)/2) = \sum_{k=0}^{p-1} (x_u/2)^k \frac{g^{(k)}(x/2)}{k!} + (x_u/2)^p \frac{g^{(p)}(\xi_u)}{p!},$$

where $x/2 \le \xi_u \le (x+x_u)/2$. Summing over u, all the terms vanish except the order p term. Now (see Hurvich and Chen 2000) $|g^{(p)}(\xi_u)| < |C_p / \sin^{p+1}(x/2)| < C_p 2^p |x|^{-p-1}$ so that

$$\left| \sum_{u=0}^{v} b_u g((x+x_u)/2) \right| \le \frac{C_p \pi^p v^{p+1}}{p!} |x|^{-p-1} n^{-p}$$

□

Lemma A.2. *Let $q \in \mathbb{N}^*$ and let $\nu \in (-1, 2q+1)$. Let ϕ and ψ be integrable functions on $[-\pi,\pi]$, such that $\phi(-x) = \overline{\phi(x)}$, ψ is a symmetric real function, differentiable on $[-\vartheta,\vartheta] \setminus \{0\}$ and for some $\beta \in (0,2]$, $K > 0$ and all $x \in (0,\vartheta] \setminus \{0\}$,*

$$|x\psi'(x)| \le K|\psi(x)|, \tag{A.1}$$

$$K_1|x|^\nu \le |\psi(x)| \le K_2|x|^\nu, \tag{A.2}$$

$$|\phi(x) - \psi(x)| \le K\psi(x)x^\beta. \tag{A.3}$$

Let D_n be such that for $x \in [-\pi,\pi]$,

$$|D_n(x)| \le C \frac{n^{1/2}}{(1+n|x|)^{q+1}}. \tag{A.4}$$

Then, there exists a constant C such that, for all $n \ge 1$ and all k such that $0 < x_k \le \vartheta/2$,

$$\left| \int_{-\pi}^{\pi} \left(\frac{\phi(x)}{\psi(x_k)} - 1 \right) |D_n(x_k - x)|^2 \, \mathrm{d}x \right| \le C \left(k^{-1} + \delta(\phi,\psi) \left(\frac{k}{n} \right)^\beta \right), \tag{A.5}$$

with $\delta(\phi,\psi) = 1$ if $\phi \ne \psi$ and 0 otherwise.

If $\nu \in (-1/2, q+1/2)$, then for all k, j such that $0 < x_k \ne x_j \le \vartheta/2$,

$$\left| \int_{-\pi}^{\pi} \left(\frac{\phi(x)}{\psi(x_k)} - 1 \right) D_n(x_k - x)\overline{D_n(x_j - x)} \, \mathrm{d}x \right| \le C\alpha(k,j) + \delta(\phi,\psi)\beta(k,j)(k/n)^\beta, \tag{A.6}$$

with

$$\alpha(k,j) = k^{-q}j^{-q}(j \wedge k)^{-1} + (1 + (j/k)^\nu)(j \wedge k)^{-1}|j-k|^{-q},$$
$$\beta(k,j) = (j \wedge k)^{-2q-1} + (1 + (j/k)^\nu)|j-k|^{-q}.$$

Proof. Define $\psi_{n,k}(x) := \left(\psi(x_k)^{-1}\phi(x) - 1\right) |D_n(x_k - x)|^2$. Then, if $|x| \in [\vartheta,\pi]$, $|x - x_k| \ge \vartheta/2$ and applying (A.4), $|D_n(x_k - x)|^2 \le Cn^{-2q-1}$. We obtain:

$$\int_{\vartheta \le |x| \le \pi} \psi_{n,k}(x) \, \mathrm{d}x \le \frac{C}{n^{2q+1}} \left(1 + \frac{\int_{-\pi}^{\pi} |\phi(x)| \, \mathrm{d}x}{|\psi(x_k)|} \right) \le Ck^{-2q-1}.$$

because (A.2) implies that $|\psi(x_k)| \geq C x_k^{2q+1}$.
Consider now the integral over $[-\vartheta, \vartheta]$. Write $\psi_{n,k}(x) = \psi_{n,k}^{(1)}(x) + \psi_{n,k}^{(2)}(x)$ with

$$\psi_{n,k}^{(1)}(x) := \frac{\psi(x) - \psi(x_k)}{\psi(x_k)} |D_n(x_k - x)|^2,$$
$$\psi_{n,k}^{(2)}(x) := \frac{\phi(x) - \psi(x)}{\psi(x_k)} |D_n(x_k - x)|^2.$$

If $x \in [2x_k, \vartheta]$, (A.1) and (A.2) imply

$$|\psi(x) - \psi(x_k)| \leq C(x^{\nu-1} + x_k^{\nu-1})|x - x_k|.$$

Since $x \geq 2x_k$ implies that $x - x_k \geq x/2$, we obtain:

$$\int_{2x_k}^{\vartheta} |\psi_{n,k}^{(1)}(x)| \, \mathrm{d}x \leq \int_{2x_k}^{\vartheta} \frac{C(x^{\nu-1} + x_k^{\nu-1}) n |x - x_k| \, \mathrm{d}x}{|\psi(x_k)|(1 + n|x - x_k|)^{2q+2}}$$
$$\leq C x_k^{-\nu} n^{-2q-1} \int_{2x_k}^{\vartheta} (x^{\nu-1} + x_k^{\nu-1}) x^{-2q-1} \, \mathrm{d}x \leq C k^{-2q-1}.$$

If $x \in [-\vartheta, -x_k/2]$, then $|x - x_k| \geq |x|$. Hence

$$\int_{-\vartheta}^{-x_k/2} |\psi_{n,k}^{(1)}(x)| \, \mathrm{d}x \leq \int_{-\vartheta}^{-x_k/2} \frac{C(|x|^{\nu} + |x_k|^{\nu}) n \, \mathrm{d}x}{|x_k|^{\nu}(1 + n|x - x_k|^{2q+2})}$$
$$\leq C n^{-2q-1} |x_k|^{-\nu} \int_{-\vartheta}^{-x_k/2} (|x|^{\nu} + |x_k|^{\nu}) |x|^{-2q-2} \, \mathrm{d}x \leq C k^{-2q-1}.$$

Applying (A.2), $\int_{-x_k}^{x_k} |\psi(x)|/|\psi(x_k)| \, \mathrm{d}x \leq x_k$. Applying (A.4) with $|x - x_k| > x_k/2$, we obtain:

$$\int_{-x_k/2}^{x_k/2} |\psi_{n,k}^{(1)}(x)| \, \mathrm{d}x \leq \frac{C\, n}{(1 + n x_k/2)^{2q+2}} \left\{ x_k + \int_{-x_k}^{x_k} \frac{|\psi(x)|}{|\psi(x_k)|} \, \mathrm{d}x \right\} \leq C k^{-2q-1}.$$

If $x \in [x_k/2, 2x_k]$, (A.1) and (A.2) imply $|\psi(x) - \psi(x_k)| \leq C|x - x_k||x_k|^{\nu-1}$. Applying the bound (A.4), we obtain:

$$\int_{x_k/2}^{2x_k} |\psi_{n,k}^{(1)}(x)| \, \mathrm{d}x \leq \frac{C|x_k|^{\nu-1}}{|\psi(x_k)|} \int_{x_k/2}^{2x_k} |x - x_k||D_n(x - x_k)|^2 \, \mathrm{d}x \leq C k^{-1} \int_0^{2\pi k} \frac{t}{(1+t)^{2q+2}} \, \mathrm{d}t \leq C k^{-1}.$$

We now consider $\psi_{n,k}^{(2)}$. If $x \geq 2x_k$, then $|x - x_k| \geq x/2$. Applying (A.3) and (A.4), we obtain:

$$\int_{2x_k}^{\vartheta} |\psi_{n,k}^{(2)}(x)| \, \mathrm{d}x \leq \frac{C n^{-2q-1}}{|\psi(x_k)|} \int_{2x_k}^{\vartheta} x^{\nu+\beta-2q-2} \, \mathrm{d}x \leq C k^{-2q-1}.$$

Similarly, if $x \leq -x_k/2$, then $|x - x_k| \geq |x|/2$, hence, applying (A.3) and (A.4), we also obtain:

$$\int_{-\vartheta}^{-x_k/2} |\psi_{n,k}^{(2)}(x)| \, \mathrm{d}x \leq C k^{-2q-1}.$$

Applying (A.3) and (A.4) with $|x - x_k| > x_k/2$, we obtain:

$$\int_{-x_k/2}^{x_k/2} |\psi_{n,k}^{(2)}(x)| \, \mathrm{d}x \leq \frac{C\, n x_k^{\beta}}{(1 + n x_k/2)^{2q+2}} \int_{-x_k}^{x_k} \frac{|\psi(x)|}{|\psi(x_k)|} \, \mathrm{d}x \leq C k^{-2q-1}.$$

Applying (A.2) for$-x_k/2 \le x \le 2x_k$, we get $|\psi(x)||x|^\beta \le C|x_k|^{\nu+\beta}$ and:

$$\int_{x_k/2}^{2x_k} |\psi_{n,k}^{(2)}(x)|\,\mathrm{d}x \le Cx_k^\beta \int_{-x_k/2}^{x_k} |D_n(x)|^2\,\mathrm{d}x \le C(k/n)^\beta.$$

This concludes the proof of (A.5).

To prove (A.6), denote $E_{n,j,k}(x) = D_n(x-x_k)\overline{D_n(x-x_j)}$ and $\psi_{n,j,k}(x) = (\psi(x_k)^{-1}\phi(x)-1)E_{n,k,j}(x)$. If $|x| \in (\vartheta,\pi]$, $|x_k| \le \vartheta/2$ and $|x_j| \le \vartheta/2$, then (A.4) implies that $|E_{n,k,j}(x)| \le Cn^{-2q-1}$. Hence, applying (A.2) and (A.4), we obtain:

$$\int_{\vartheta\le|x|\le\pi} |\psi_{n,k,j}(x)|\,\mathrm{d}x \le \frac{C}{n^{2q+1}}\left(1+\frac{\int_{-\pi}^{\pi}|\phi(x)|\,\mathrm{d}x}{|\psi(x_k)|}\right) \le Cn^{-q-1/2}k^{-q-1/2} \le Cj^{-q}k^{-q-1}.$$

Denote $\psi_{n,j,k}^{(1)}(x) = (\psi(x_k)^{-1}\psi(x)-1)E_{n,j,k}(x)$, $\psi_{n,j,k}^{(2)}(x) = \psi(x_k)^{-1}\{\phi(x)-\psi(x)\}E_{n,j,k}(x)$ and $y = \min(x_k,x_j)$ and $z = \max(x_k,x_j)$.

$$\begin{aligned}\int_{2z}^{\vartheta} |\psi_{n,j,k}^{(1)}(x)|\,\mathrm{d}x &\le \int_{2z}^{\vartheta} \frac{C(x^{\nu-1}+x_k^{\nu-1})n|x-x_k|\,\mathrm{d}x}{|\psi(x_k)|(1+n|x-x_k|)^{q+1}(1+n|x-x_j|)^{q+1}} \\ &\le C(n(2z-x_j))^{-q}|x_k|^{-\nu}\int_{2z}^{\vartheta}(x^{\nu-1}+x_k^{\nu-1})(nx)^{-q-1}\,\mathrm{d}x \le Cj^{-q}k^{-q-1}.\end{aligned}$$

If $x \in [-\vartheta,-y/2]$, then $|x-x_k| \ge |x|$, $|x-x_j| \ge |x|$ and $|x-x_j| \ge x_j$. Hence

$$\begin{aligned}\int_{-\vartheta}^{-y/2} |\psi_{n,k,j}^{(1)}(x)|\,\mathrm{d}x &\le \int_{-\vartheta}^{-y/2} \frac{C(|x|^\nu+|x_k|^\nu)n\,\mathrm{d}x}{|x_k|^\nu(1+n|x-x_k|^{q+1})(1+n|x-x_j|^{q+1})} \\ &\le Cn^{-2q-1}x_j^{-q}x_k^{-\nu}\int_{-\vartheta}^{-y/2}(|x|^\nu+|x_k|^\nu)|x|^{-q-2}\,\mathrm{d}x \\ &\le Cn^{-2q-1}x_j^{-q}x_k^{-q-1} \le Cj^{-q}k^{-q-1}.\end{aligned}$$

If $x \in [-y/2,y/2]$, then $|x-x_k| \ge x_k/2$ and $|x-x_j| \ge x_j/2$. Applying (A.2) and (A.4), we obtain:

$$\int_{-y/2}^{y/2} |\psi_{n,j,k}^{(1)}(x)|\,\mathrm{d}x \le \frac{Cn\left\{y+\int_{-y}^{y}\frac{|\psi(x)|}{|\psi(x_k)|}\,\mathrm{d}x\right\}}{(1+n(x_k/2))^{q+1}(1+n(x_j/2))^{q+1}} \le Ck^{-q}j^{-q-1}.$$

If $x \in [y/2,2z]$, then applying (A.1), (A.2)

$$\begin{aligned}\int_{y/2}^{2z} |\psi_{n,j,k}^{(1)}(x)|\,\mathrm{d}x &\le \int_{y/2}^{2z} \frac{C(x_k^{\nu-1}+x_j^{\nu-1})n|x-x_k|\,\mathrm{d}x}{|x_k|^\nu(1+n|x-x_k|)^{q+1}(1+n|x-x_j|)^{q+1}} \\ &\le \frac{C(x_k^{\nu-1}+x_j^{\nu-1})}{x_k^\nu(n(z-y))^q}\left(\int_{y/2}^{(y+z)/2}\frac{\mathrm{d}x}{(1+n|x-y|)^{q+1}}+\int_{(y+z)/2}^{2z}\frac{\mathrm{d}x}{(1+n|x-z|)^{q+1}}\right) \\ &\le \frac{C(1+(j/k)^\nu)}{ny(1+n|z-y|)^q} = \frac{C(1+(j/k)^\nu)}{(j\wedge k)|j-k|^q}.\end{aligned}$$

Consider now $\psi^{(2)}_{n,j,k}$.

$$\int_{2z}^{\vartheta} |\psi^{(2)}_{n,j,k}(x)|\,\mathrm{d}x \le \int_{2z}^{\vartheta} \frac{C|x|^{\nu+\beta} n\,\mathrm{d}x}{|\psi(x_k)|(1+n|x-x_k|)^{q+1}(1+n|x-x_j|)^{q+1}}$$
$$\le Cn^{-2q-1}|x_k|^{-\nu}\int_{2z}^{\vartheta}(x^{\nu+\beta}+x_k^{\nu+\beta})x^{-2q-2}\,\mathrm{d}x \le Cn^{-2q-1}x_k^{2q+1+\beta} \le Ck^{-2q-1}(k/n)^{\beta}.$$

If $x \in [-\vartheta, -y/2]$, then $|x-x_k| \ge |x|$, $|x-x_j| \ge |x|$ and $|x-x_j| \ge x_j$. Hence

$$\int_{-\vartheta}^{-y/2} |\psi^{(2)}_{n,k,j}(x)|\,\mathrm{d}x \le \int_{-\vartheta}^{-y/2} \frac{C(|x|^{\nu+\beta}+|x_k|^{\nu+\beta})n\,\mathrm{d}x}{|x_k|^{\nu}(1+n|x-x_k|^{q+1})(1+n|x-x_j|^{q+1})}$$
$$\le Cn^{-2q-1}x_k^{-\nu}\int_{-\vartheta}^{-y/2}(|x|^{\nu+\beta}+|x_k|^{\nu+\beta})|x|^{-2q-2}\,\mathrm{d}x \le C|j\wedge k|^{-2q-1}(k/n)^{\beta}.$$

If $x \in [-y/2, y/2]$, then $|x-x_k| \ge x_k/2$ and $|x-x_j| \ge x_j/2$ Applying (A.2) and (A.4), we obtain:

$$\int_{-y/2}^{y/2} |\psi^{(2)}_{n,j,k}(x)|\,\mathrm{d}x \le \frac{Cn\left\{\int_{-y}^{y}|x|^{\nu+\beta}|x_k|^{-\nu}\,\mathrm{d}x\right\}}{(1+n(x_k/2))^{q+1}(1+n(x_j/2))^{q+1}} \le Ck^{-q}j^{-q-1}(k/n)^{\beta}.$$

If $\in [y/2, 2z]$, then applying (A.1), (A.2)

$$\int_{y/2}^{2z} |\psi^{(2)}_{n,j,k}(x)|\,\mathrm{d}x \le \int_{y/2}^{2z} \frac{Cn(x_k^{\nu+\beta}+x_j^{\nu+\beta})\,\mathrm{d}x}{|x_k|^{\nu}(1+n|x-x_k|)^{q+1}(1+n|x-x_j|)^{q+1}}$$
$$\le \frac{Cn^{-q}(x_k^{\nu+\beta}+x_j^{\nu+\beta})}{x_k^{\nu}|z-y|^{q+1}}\left(\int_{y/2}^{(y+z)/2}\frac{\mathrm{d}x}{(1+n|x-y|)^{q+1}}+\int_{(y+z)/2}^{2z}\frac{\mathrm{d}x}{(1+|x-z|)^{q+1}}\right)$$
$$\le C\left(\frac{(j\vee k)}{n}\right)^{\beta}\frac{1+(j/k)^{\nu}}{|j-k|^{q}}.$$

Altogether, we obtain:

$$\int_{-\pi}^{\pi} |\psi^{(1)}_{n,j,k}(x)|\,\mathrm{d}x \le C(j\wedge k)^{-1}\left(k^{-q}j^{-q}+(1+(j/k)^{\nu})|j-k|^{-q}\right),$$
$$\int_{-\pi}^{\pi} |\psi^{(2)}_{n,j,k}(x)|\,\mathrm{d}x \le C\left((j\wedge k)^{-2q-1}+(1+(j/k)^{\nu})|j-k|^{-q}\right)((j\vee k))/n)^{\beta}.$$

□

B Apendix

Lemma B.1. *Under the assumptions and notations of Theorem 3, it holds that:*

$$\lim_{n\to\infty}\sum_{t=2}^{n}\sum_{s=1}^{t-1}c_{t,s}^2 = T_v, \tag{B.1}$$

$$|c_{t,s}| \le C\beta_n|t-s|^{-1}, \tag{B.2}$$

$$\max_{1\le s\le n}\sum_{t=1}^{n}c_{t,s}^2 = O(1/n). \tag{B.3}$$

Proof. We will make repeated use of the following identities.

$$\sum_{t=2}^{n}\sum_{s=1}^{t-1}\mathrm{e}^{\mathrm{i}tx_u}\mathrm{e}^{-\mathrm{i}sx_v} = 0 \text{ if } u\ne v \text{ mod. n}, \tag{B.4}$$

$$\sum_{t=2}^{n}\sum_{s=1}^{t-1}\mathrm{e}^{\mathrm{i}tx_u}\mathrm{e}^{-\mathrm{i}sx_v} = -n/2 \text{ if } u=v\ne 0 \text{ mod. n}, \tag{B.5}$$

$$\sum_{t=2}^{n}\sum_{s=1}^{t-1}\mathrm{e}^{\mathrm{i}tx_u}\mathrm{e}^{-\mathrm{i}sx_v} = n(n-1)/2 \text{ if } u=v=0 \text{ mod. n}. \tag{B.6}$$

We start by proving (B.1). Write

$$A_n = \frac{4}{n^2B_p^2}\sum_{k,l}\beta_{n,k}\beta_{n,l}\sum_{t=1}^{n}\sum_{s=1}^{t-1}\Re\left(h_t\bar{h}_s\mathrm{e}^{\mathrm{i}(t-s)x_k}\right)\Re\left(h_t\bar{h}_s\mathrm{e}^{\mathrm{i}(t-s)x_l}\right).$$

Recall that $h_t = \sum_{u=0}^{v} b_u\mathrm{e}^{\mathrm{i}tx_u}$.

Thus,

$$\frac{4}{n^2B_p^2}\sum_{k,l}\beta_{n,k}\beta_{n,l}\sum_{t=1}^{n}\sum_{s=1}^{t-1}\Re\left(h_t\bar{h}_s\mathrm{e}^{\mathrm{i}(t-s)x_k}\right)\Re\left(h_t\bar{h}_s\mathrm{e}^{\mathrm{i}(t-s)x_l}\right) \tag{B.7}$$

$$= \frac{4}{n^2B_p^2}\sum_{k\ne l}\beta_{n,k}\beta_{n,l}\sum_{u,w,y,z}b_ub_wb_yb_z\sum_{t=1}^{n}\sum_{s=1}^{t-1}\mathrm{e}^{\mathrm{i}tx_{k+u+l+y}}\mathrm{e}^{-\mathrm{i}sx_{k+w+l+z}} + \mathrm{e}^{-\mathrm{i}tx_{k+u+l+y}}\mathrm{e}^{\mathrm{i}sx_{k+w+l+z}} \tag{B.8}$$

$$+\mathrm{e}^{\mathrm{i}tx_{k+u-l-y}}\mathrm{e}^{-\mathrm{i}sx_{k+w-l-z}} + \mathrm{e}^{-\mathrm{i}tx_{k+u-l-y}}\mathrm{e}^{\mathrm{i}sx_{k+w-l-z}}. \tag{B.9}$$

Denote $\Sigma_1(b) = \sum_{u,w,y,z,u+y=w+z} b_ub_wb_yb_z$ and $\Sigma_2(b) = \sum_{u,w,y,z,u-y=w-z} b_ub_wb_yb_z$. Using (B.4) and (B.5), we get

$$\sum_{u,w,y,z}b_ub_wb_yb_z\sum_{t=1}^{n}\sum_{s=1}^{t-1}\mathrm{e}^{\mathrm{i}tx_{k+u+l+y}}\mathrm{e}^{-\mathrm{i}sx_{k+w+l+z}} + \mathrm{e}^{-\mathrm{i}tx_{k+u+l+y}}\mathrm{e}^{\mathrm{i}sx_{k+w+l+z}} = -n\Sigma_1(b).$$

Similarly, using (B.4) (B.5) and (B.6), we get:

$$\sum_{u,w,y,z} b_u b_w b_y b_z \sum_{t=1}^{n} \sum_{s=1}^{t-1} \mathrm{e}^{\mathrm{i}tx_{k+u-l-y}} \mathrm{e}^{-\mathrm{i}sx_{k+w-l-z}} + \mathrm{e}^{-\mathrm{i}tx_{k+u-l-y}} \mathrm{e}^{\mathrm{i}sx_{k+w-l-z}}$$

$$= -n\Sigma_2(b) + n^2 \left(\sum_u b_u b_{u+k-l} \right)^2 .$$

Now, note that $\Sigma_1(b)$ and $\Sigma_2(b)$ do not depend on k, l and that the sum of the $\beta_{n,k}$ equals zero.

$$A_n = \frac{4}{B_v^2} \sum_{k,l} \beta_{n,k} \beta_{n,l} \left(\sum_u b_u b_{u+k-l} \right)^2$$

By definition, the terms $b_u b_{u+k-l}$ vanish if $|k-l| > v$. Assumptions (4.2) and (4.3) implies that for all fixed $u \in \mathbb{N}$, $\lim_{n\to\infty} \sum_{k=1}^{K-u} \beta_{n,k}\beta_{n,k+u} = 1$. Hence,

$$\lim_{n\to\infty} A_n = \frac{4}{(\sum_u b_u^2)^2} \sum_{1\le|z|\le v} \left(\sum_u b_u b_{u+z} \right)^2 .$$

We now prove (B.3). The computation is very similar, but we sum over t, not over s :

$$\sum_{s<t\le n} c_{t,s}^2 \le \sum_{1\le t\le n} c_{t,s}^2 = \sum_{k,l} \beta_{n,k}\beta_{n,l} \sum_{t=1}^{n} \Re\left(h_t \bar{h}_s \mathrm{e}^{\mathrm{i}(t-s)x_k} \right) \Re\left(h_t \bar{h}_s \mathrm{e}^{\mathrm{i}(t-s)x_l} \right)$$

$$= \frac{4}{n^2 B_p^2} \sum_{k\ne l} \beta_{n,k}\beta_{n,l} \sum_{u,w,y,z} b_u b_w b_y b_z \sum_{t=1}^{n} \mathrm{e}^{\mathrm{i}tx_{k+u+l+y}} \mathrm{e}^{-\mathrm{i}sx_{k+w+l+z}} + \mathrm{e}^{-\mathrm{i}tx_{k+u+l+y}} \mathrm{e}^{\mathrm{i}sx_{k+w+l+z}}$$

$$+ \mathrm{e}^{\mathrm{i}tx_{k+u-l-y}} \mathrm{e}^{-\mathrm{i}sx_{k+w-l-z}} + \mathrm{e}^{-\mathrm{i}tx_{k+u-l-y}} \mathrm{e}^{\mathrm{i}sx_{k+w-l-z}}$$

The first line gives no contribution. The second gives a contribution n when $k+u-l-y=0$. Hence

$$\sum_{s<t\le n} c_{t,s}^2 = \frac{4n}{n^2 B_p^2} \sum_{|z|<p} \sum_{k-l=z} \beta_{n,k}\beta_{n,l} \left(\sum_u b_u b_{u+z} \right)^2 = O(n^{-1}).$$

We now prove (B.2). Denote $\tilde{c}_{t,s} = \sum_{k=1}^{K} \beta_{n,k} \mathrm{e}^{\mathrm{i}(t-s)x_k}$. Since $|c_{t,s}| \le Cn^{-1}|\tilde{c}_{t,s}|$, it suffices to bound $\tilde{c}_{t,s}$. By summation by parts, we get

$$\tilde{c}_{t,s} = \sum_{k=1}^{K} \Big(\sum_{j=1}^{k} \mathrm{e}^{\mathrm{i}(t-s)x_j} \Big) (\beta_{n,k} - \beta_{n,k-1}) + \sum_{j=1}^{K} \mathrm{e}^{\mathrm{i}(t-s)x_k}.$$

We conclude by using the uniform bound $\sum_{j=1}^{k} \mathrm{e}^{\mathrm{i}(t-s)x_j} \le Cn|t-s|^{-1}$. □

Table 1: Simulation Results for Quartic Mean Function, $r(x) = 10x^4$

		$p=0$	$p=1$	$p=2$
$n=100$				
	Bias	0.336	0.186	0.151
	$m^{1/2}\cdot$ Bias	1.68	0.930	0.757
	Variance	0.00759	0.0571	0.0448
$n=1000$				
	Bias	0.130	0.0221	0.0167
	$m^{1/2}\cdot$ Bias	1.46	0.247	0.187
	Variance	0.00178	0.00599	0.00553
$n=5000$				
	Bias	0.0773	0.0112	0.00910
	$m^{1/2}\cdot$ Bias	1.52	0.220	0.179
	Variance	0.000629	0.00157	0.00161

Table 2: Simulation Results for Constant Mean Function, $r(x) \equiv 0$

		$p=0$	$p=1$	$p=2$
$n=100$				
	Bias	0.117	0.159	0.114
	$m^{1/2}\cdot$ Bias	0.585	0.795	0.573
	Variance	0.0178	0.0542	0.0460
$n=1000$				
	Bias	0.0384	0.0222	0.0169
	$m^{1/2}\cdot$ Bias	0.430	0.249	0.190
	Variance	0.00300	0.00602	0.00551
$n=5000$				
	Bias	0.0223	0.0112	0.00895
	$m^{1/2}\cdot$ Bias	0.438	0.221	0.176
	Variance	0.00104	0.00157	0.00162

Table 3: Asymptotic and Finite-Sample Corrected Variance Expressions for $\hat{d}_n$

		$p=0$	$p=1$	$p=2$
$n=100$				
	Asymptotic: $\Phi_p/(4m)$	0.0100	0.0150	0.0194
	Corrected	0.0150	0.0525	0.0413
$n=1000$				
	Asymptotic: $\Phi_p/(4m)$	0.00200	0.00300	0.00389
	Corrected	0.00229	0.00562	0.00574
$n=5000$				
	Asymptotic: $\Phi_p/(4m)$	0.000644	0.000966	0.00125
	Corrected	0.000685	0.00145	0.00158

Figure 1A: Quartic Mean Function, d=-0.6

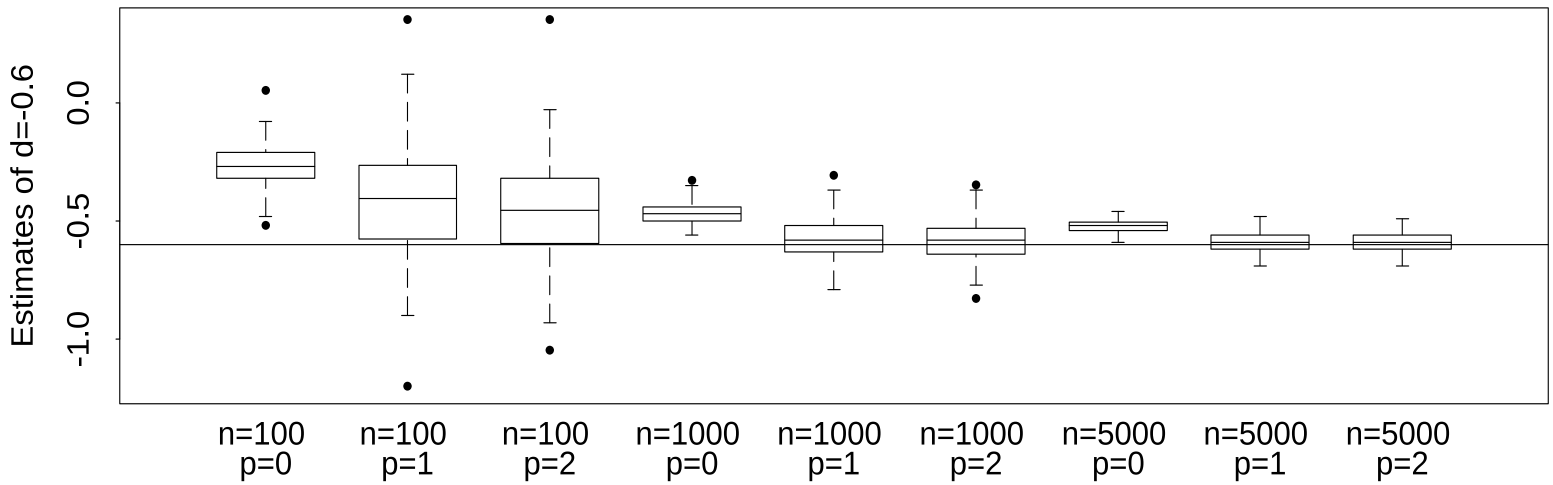
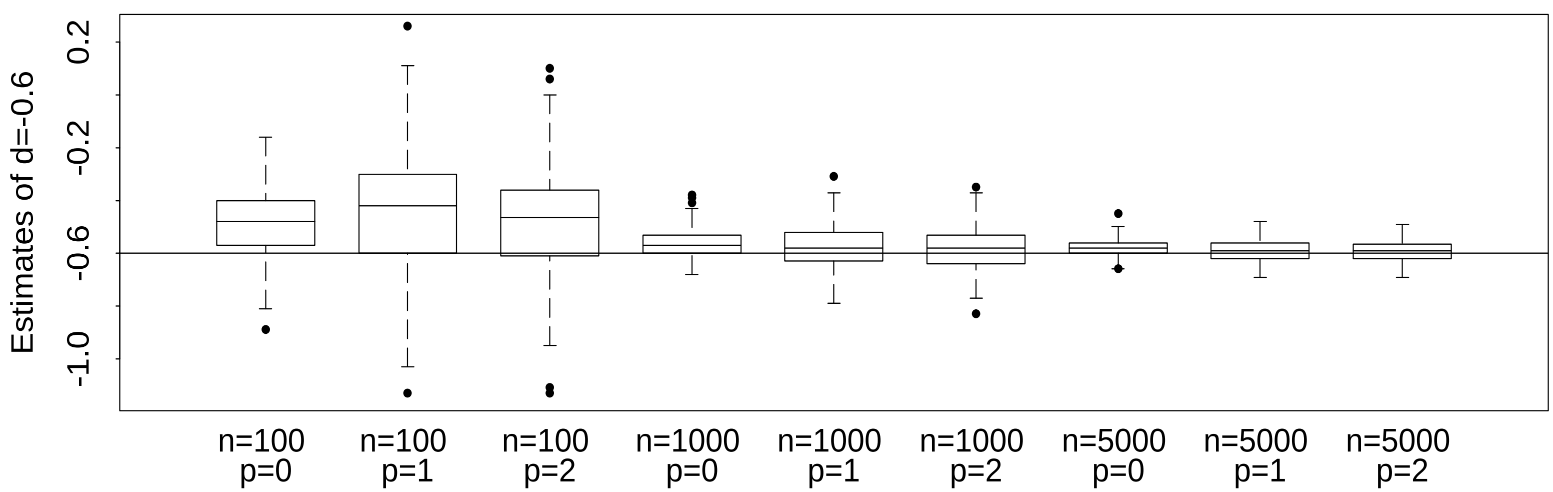


Figure 1B: Constant Mean Function, d=-0.6



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