NEW YORK UNIVERSITY
STERN SCHOOL OF BUSINESS
FINANCE DEPARTMENT

Working Paper Series, 1995

The Central Tendency: A Second Factor in Bond Yields.

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FIN-95-7

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by

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May 1994. This revision, September 1995.

Abstract: We assume that the instantaneous riskless rate reverts towards a central tendency which, in turn, is changing stochastically over time, and we derive a model of the term structure of interest rates. Our term-structure model implies that a linear combination of any two rates can be used as a proxy of the central tendency. Based on the central-tendency proxy, we estimate a model of the one-month rate which performs better than models which assume the central tendency to be constant.

Keywords: term structure. **JEL#** G12.

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We thank for helpful comments Ned Elton, Peter Jorgensen, Anthony Lynch, Eric Reiner, Frank Skinner, Bruce Tuckman, and participants at the 1995 Western Finance Association Meetings, and the 1995 European Finance Association Meetings.

1 Introduction

This paper develops a two-factor model of the term structure. We follow a consolidated tradition in the finance literature [see, *e.g.,* Brennan and Schwartz (1979), Cox, Ingersoll, and Ross (1981, 1985), and Longstaff and Schwartz (1992)], and identify the first factor with the *level* of the short-term rate. The novelty of our analysis is that we identify the second factor with the *central tendency* of the short-term rate. The approach advocated here complements that of Longstaff and Schwartz (1992), who propose a model in which the second factor is identified with the conditional volatility of the short-term rate. Also, in a spirit similar to ours, Naik and Lee (1994) postulate a process for the short-term rate where both the conditional volatility and the central tendency may experience discrete shifts, which they relate to changes in the interest-rate regime.

We provide preliminary evidence on the explanatory power of a central-tendency factor. Bond yields are regressed on the one-month T-bill rate, a proxy for the conditional *mean* of the one-month rate, and a proxy for the conditional *volatility* of the one-month rate. The conditional-mean proxy always has significant explanatory power, especially for shorter-maturity yields.

This motivates a two-factor term-structure model, which is based on the assumption that the instantaneous riskless rate reverts towards a central tendency, which, in turn, changes stochastically over time. We impose minimal restrictions on the dynamics of the central tendency, other than that its conditional mean should *not* be affected by the instantaneous riskless rate. The theoretical analysis suggests a proxy for the central-tendency factor which depends on a linear combination of two yields. The central-tendency proxy is then used to estimate the stochastic process for the instantaneous riskless rate, proxied by the one-month T-bill rate. This is also the main idea of our paper: we show how a term-structure model can be used to learn about the dynamics of the short-term rate.

In explaining the variation of the one-month rate, models which allow for a time-varying central tendency perform better than the standard Vasicek (1977) and Cox, Ingersoll, and Ross (1985) models, in which the central tendency is assumed constant. We replicate the analysis for different subperiods. Time variation in the central tendency is especially important for the 1952-71 and 1982-93 periods, when the Federal Reserve's operating policy took into account both the volatility of interest rates and the behavior of monetary aggregates.

The paper is organized as follows. Section 1 puts forward the motivating evidence, while Section 2 develops the theoretical model. In Section 3 we estimate various models of the one-month rate, and we examine the properties of the central-tendency proxy. Section 4 concludes.

2 Preliminary evidence

In a spirit similar to Longstaff and Schwartz (1993), we gather some preliminary evidence on the role of the central tendency as a factor affecting bond yields. We estimate the following model

$$Y_{\tau t} - Y_{\tau,t-1} = \alpha + \beta(r_t - r_{t-1}) + \gamma(r_{t+1} - r_t) + \delta(|r_{t+1} - r_t| - |r_t - r_{t-1}|) + \text{error}_t, \qquad (1)$$

where Y_τ is the yield on a discount bond of maturity $\tau = 1, 2, 3, 4, 5$ years, and r is the one-month interest rate. The realized rate r_{t+1} is used as a proxy for the conditional expectation, while the absolute value of the realized one-month rate change, $|r_{t+1} - r_t|$, proxies for the conditional volatility. This simple model summarizes the extent to which changes in yields are driven by changes in the *level* of the short-term rate (with coefficient β), changes in the *conditional mean* of the short-term rate (coefficient γ), and changes in the *conditional volatility* of the short-term rate (coefficient δ).

Note that if we assume a model of the short-term rate of the kind

$$E_t(r_{t+1}) = (1 - (k/12))r_t + (k/12)\theta_t,$$

where k is the "mean-reversion" parameter, and θ_t is a time varying central tendency, then $r_{t+1} - r_t$ proxies for $(k/12)(\theta_t - r_t)$. In this perspective, we interpret a significant estimate of γ in (1) as suggesting that θ has a significant impact on yields of different maturities.

The model is estimated using monthly data for the sample period 1952:7-93:11, and then for the four separate subperiods, 1952:7-71:8, 1971:9-79:9, 1979:10-82:9, and 1982:10-93:11. Monthly data on the one- and three-month Treasury-bill rates are from the CRSP RISKFREE file, while discount-bond prices are from the FAMABLIS file also on the CRSP tape. All interest rates are quoted on a *continuously compounded* basis.

The subperiods were selected to control for different operating procedures followed by the Federal Reserve: During the 1950s and 1960s the Fed took the view that monetary policy should be based on intuitive judgment of money-market conditions. This period also roughly coincides with the time during which the Bretton Woods exchange-rate agreement was in place, which ended in August, 1971. During the 1970s the Fed made a formal commitment to targeting the growth rate of key monetary aggregates, such as M1 and M2. In practice, this translated into a tight targeting of the federal funds rate at a level perceived to be consistent with the stated quantity targets. The October 1979–October 1982 period witnessed the Fed's "experiment." The operating target of monetary policy became the amount of non-borrowed reserves with the banking system, and both the level and the volatility of interest rates reached levels never experienced before. The post-October 1982 period is one of de-emphasis of monetary aggregates, and of renewed concern for reducing the volatility of interest rates.

Figure 1 summarizes the estimation results for the entire period (Panel a) and the subperiods (Panels b and c), displaying the estimated coefficients of the regression, with two-standard error bands, against the maturity of the yield used as dependent variable.[1]

The evidence from Figure 1 can be summarized as follows: When we look at the entire sample, the estimates of *all* three parameters appear to be significant. In subperiods other than the 1979:10-82:9, however, changes in the conditional volatility are not significant, while changes in the conditional mean are. Consistent with the notion that long-term rates are less sensitive to risk factors, the coefficients of all regressors tend to be smaller the longer the maturity of the yield used as regressand.

Overall, this evidence suggests that changes in the one-month rate and in its conditional expectation are always significant in explaining changes in yields, while changes in the conditional volatility are significant only at short maturities and during the two most recent subperiods.

3 Yields with a time-varying central tendency

The behavior of the instantaneous riskless rate r is described by the stochastic differential equation[2]

$$dr = k(\theta - r)dt + \sqrt{\sigma_0^2 + \sigma_1^2 r}\, dZ, \tag{2}$$

where k, σ_0, and σ_1 are constants, and Z is a standard Brownian motion; θ is the central tendency towards which the instantaneous rate reverts. As in Pearson and Sun (1994), the diffusion term of (2) generalizes the square-root process of Cox, Ingersoll, and Ross (1985) by allowing the lower bound for the instantaneous interest rate to be different from zero.

In turn, θ evolves over time according to the stochastic differential equation

$$d\theta = m(\theta)dt + s(\theta)dW, \tag{3}$$

where, like Z, W is a standard Brownian motion. We assume that the conditional drift $m(\theta)$ and the conditional volatility $s(\theta)^2$ are "smooth" (continuous with continuous derivatives) functions of θ alone. Specifically, θ may evolve according to

$$d\theta = (m_0 + m_1\theta)dt + \sqrt{s_0^2 + s_1^2\theta}\, dW. \tag{3'}$$

The covariance between the two factors, $\sigma_{r\theta}$, is assumed constant.

[1]Standard errors are adjusted for heteroskedasticity in the residuals, see White (1980).

[2]We use the same notation to indicate the instantaneous riskless rate and the one-month rate. In fact, in the empirical applications of Section 4 the instantaneous riskless rate is proxied by the one-month rate.

We assume the risk premium associated with r to be linear in r, $\lambda_0 + \lambda_1 r$, with λ_0, λ_1 constant, while the risk premium associated with θ, $l(\theta)$, is a smooth function of θ alone. Under the assumptions above, the price of a risk-free discount bond of maturity τ, $P = P(r, \theta; \tau)$ satisfies the partial differential equation [see Cox, Ingersoll, and Ross (1985)]

$$E(\mathcal{D}P) - rP - P_r(\lambda_0 + \lambda_1 r) - P_\theta l(\theta) = 0, \tag{4}$$

where $\mathcal{D}$ denotes the Dynkin differential operator.

We assume that there is a solution for the fundamental valuation equation (4); the solution then is of the form

$$P(r, \theta; \tau) = e^{-A(\tau) - B(\tau)r - C(\theta;\tau)}, \tag{5}$$

where $A(\tau)$ and $B(\tau)$ are functions of τ alone, while $C(\theta; \tau)$ is a smooth function of both θ and τ. In fact one can write equation (4) as

$$\begin{aligned} &B(\tau)[k\theta - \lambda_0 - (k + \lambda_1)r] + B^2(\tau)[\sigma_0^2 + \sigma_1^2 r + \sigma_2^2\theta]/2 + C_\theta(\theta; \tau)[m(\theta) - l(\theta)] \\ &+[C_{\theta\theta}(\theta; \tau) + C_\theta^2(\theta; \tau)]s(\theta)^2/2 + B(\tau)C_\theta(\theta; \tau)\sigma_{r\theta} - A_\tau(\tau) - B_\tau(\tau)r - C_\tau(\theta; \tau) - r = 0, \end{aligned}$$

and verify that for the previous equation to be uniformly satisfied over the domain of r

$$-B(\tau)(k + \lambda_1) + B^2(\tau)\sigma_1^2/2 - B_\tau(\tau) - 1 = 0.$$

The solution for $B(\tau)$, subject to the initial condition that $B(0) = 0$, is well-known [see Cox, Ingersoll, and Ross (1985)]:

$$B(\tau) = \frac{2(e^{\delta\tau} - 1)}{(\lambda_1 + \delta + k)(e^{\delta\tau} - 1) + 2\delta}, \quad \delta = \sqrt{(\lambda_1 + k)^2 + 2\sigma_1^2}, \tag{6}$$

which reduces to $B(\tau) = (1 - e^{-k\tau})/k$ when $\lambda_1 = \sigma_1^2 = 0$ [see Vasicek (1977)]. If $m(\theta)$, $s(\theta)^2$ and $l(\theta)$ were linear functions of θ as postulated in equation (3′), yields would be linear in r and θ, and bond prices would have the familiar form

$$P(r, \theta; \tau) = e^{-A(\tau) - B(\tau)r - D(\tau)\theta}. \tag{7}$$

Conditions for yields to be linear in the underlying state variables are given in Cox, Ingersoll, and Ross (1981), and Duffie and Kan (1993).

A useful implication of the analysis above is that $B(\tau)$ is constant for any given maturity τ. Since $B(\tau)$ represents the sensitivity of the maturity-τ bond yield to changes in the instantaneous rate r, choosing appropriate weights one can build a linear combination of two bond yields which is independent of r. The variation of such quantity mimics variation in the second factor. This implication that yields of different maturities can be used as instruments for unobservable factors is well known [Cox, Ingersoll, and Ross (1985), pp.398-401], and is the basis of a number of empirical studies, *e.g.,* Stambaugh (1988) and Sun (1992), before ours.

Formally, consider two bonds of maturity τ_1 and τ_2, respectively. From equation (5), the corresponding yields are

$$Y(r,\theta;\tau_i) = -\ln P(r,\theta;\tau_i)/\tau_i = [A(\tau_i) + B(\tau_i)r + C(\theta_i;\tau_i)]/\tau_i, \qquad \text{for } i = 1, 2. \tag{8}$$

Solving for r from the first yield, substituting into the second, and rearranging we obtain:

$$\tau_1 B(\tau_2)Y(r,\theta;\tau_1) - \tau_2 B(\tau_1)Y(r,\theta;\tau_2) = B(\tau_2)[A(\tau_1) + C(\theta;\tau_1)] - B(\tau_1)[A(\tau_2) + C(\theta;\tau_2)].$$

Note that this quantity does not depend on r.

If the drift and diffusion of the process for θ were also linear in θ, then prices would be of the form in equation (7), and the second factor θ could be written as

$$\theta = \frac{B(\tau_2)[\tau_1 Y(r,\theta;\tau_1) - A(\tau_1)] - B(\tau_1)[\tau_2 Y(r,\theta;\tau_2) - A(\tau_2)]}{B(\tau_2)D(\tau_1) - B(\tau_1)D(\tau_2)}. \tag{9}$$

Equation (9) justifies a *proxy* for θ which is used in the empirical analysis of the next section. We denote this proxy with $\hat{\theta}$:

$$\hat{\theta} = a_0 + a_1[B(\tau_2)\tau_1 Y(r,\theta;\tau_1) - B(\tau_1)\tau_2 Y(r,\theta;\tau_2)]. \tag{10}$$

Using the proxy (10) rather than the variable (9) is convenient in that even if m and s are not linear in θ, still we would expect (10) to be a reasonable approximation of the true, unknown, functional form relating θ to any two bond yields.

At this point, we may also note that if the conditional mean and volatility of the process for θ were in fact linear in θ, all proxies estimated from any bond-maturity pair would be *the same.* Thus, we shall interpret evidence that the statistical properties of $\hat{\theta}$ vary using different bond-maturity pairs as an indication of general misspecification of the model (3′). The misspecification may depend, for example, on θ following a process where m and s^2 are *nonlinear* functions of the level of θ, as it is allowed in equation (3).

4 Empirical analysis

An important issue in estimation is that of identifying empirical counterparts to the factors r and θ, which are in principle non observable. We follow a standard practice (see, for example, Chan, Karolyi, Longstaff, and Sanders (1992), and Longstaff and Schwartz (1992))] and treat the one-month T-bill rate as a proxy for the instantaneous rate r. While this practice is convenient, the one-month rate is only an approximation of the instantaneous rate of interest, because yields on any finite-maturity bond depend on *both* factors, r and θ, as well as on their risk premia, $\lambda_0 + \lambda_1 r$ and $l(\theta)$ (see, *e.g.*, Pearson and Sun (1994)).

Second, we proxy θ with the quantity proposed in the previous section in equation (10). This second approximation depends on the fact that *only* when m and s are linear, θ is a linear function of the yields in (10).

A second issue is that we need to discretize the stochastic differential equation describing dr to match it to discretely sampled data. When there is only one factor as, for example, when θ is constant, we could estimate an *exact* stochastic difference equation for the instantaneous rate implied by the differential equation for dr. However, since we allow θ to vary, we shall use the Euler discretization often used in the literature [see, for example, Chan, Karolyi, Longstaff, and Sanders (1992)],

$$r_{t+h} - r_t \approx kh(\theta_t - r_t) + \sqrt{\sigma_0^2 + \sigma_1^2 r}\, \epsilon_{t+h}. \tag{11}$$

Since we use monthly observations, we set $h = 1/12$ and estimate our model by maximizing the log-likelihood function

$$-.5 \sum \left[\ln \sigma_{t+1}^2 + (r_{t+1} - \bar{r}_{t+1})^2 / \sigma_{t+1}^2\right], \tag{12}$$

where

$$\begin{aligned} \bar{r}_{t+1} &= (1 - (k/12))r_t + (k/12)\hat{\theta}_t && (13) \\ \sigma_{t+1}^2 &= (\sigma_0^2/12) + (\sigma_1^2/12)r_t, && (14) \end{aligned}$$

and $\hat{\theta}_t$ is given by

$$\hat{\theta}_t = a_0 + a_1[B(\tau_2)\tau_1 Y(r_t, \theta_t; \tau_1) - B(\tau_1)\tau_2 Y(r_t, \theta_t; \tau_2)] \tag{10}$$

where

$$B(\tau) = \frac{2(e^{\delta\tau} - 1)}{(\lambda_1 + \delta + k)(e^{\delta\tau} - 1) + 2\delta}, \quad \delta = \sqrt{(\lambda_1 + k)^2 + 2\sigma_1^2}. \tag{6}$$

We estimate the model (6), (10), (12), (13), and (14) using monthly observations. The estimates presented here are obtained using one- and two-year bond yields to construct the proxy in equation (10).[3] The estimation period goes from 1952:7 to 1993:12. Results are presented in Table 1.

The Table shows parameter estimates for five different models:

[3]We experimented with different maturity pairs as well, with very similar, if slightly worse, results. The robustness of our results to the choice of maturity pairs is further discussed in Section 5.

1. Vasicek: $a_1 = \sigma_1 = 0$;
2. CIR: $a_1 = \sigma_0 = 0$;
3. Vasicek*: $\sigma_1 = 0$;
4. CIR*: $\sigma_0 = 0$;

In "CIR" and "Vasicek" we restrict $\hat{\theta}$ to be a constant. The first model corresponds to the Vasicek (1977) Ornstein-Uhlenbeck process, while the second model corresponds to the Cox, Ingersoll, and Ross (1985) square-root process. In "Vasicek*" and "CIR*" the mean of the square-root process is also affected by θ.

By equations (10) and (6) the coefficients relating the central-tendency proxy to bond yields depend on the parameter λ_1 which, in turn, describes the sensitivity of the market price of r-risk to r. In the empirical implementation, though, we found the log-likelihood to be largely insensitive to different values of λ_1: the Vasicek*, and CIR* models were estimated imposing $\lambda_1 = 0.0, -.5, -1.0, -2.0$, with slight decreases in likelihood value, the larger λ_1 in absolute value. Parameter estimates and the extracted θ series were also essentially the same. Table 1 reports the estimation results for the different models for $\lambda_1 = 0$.

The full-sample estimation results indicate that in the Vasicek* and CIR* models the estimate of a_1 is negative and significant: the central-tendency proxy is indeed time-varying. It is also apparent that the level of the one-month rate affects its conditional volatility: in the CIR* model the estimate of σ_1 is positive and significant. Also, it is worth noting that the assumption of a constant central tendency (Vasicek and CIR models) leads to lower estimates of the mean-reversion parameter k. This is consistent with the intuition that interest rates should converge more quickly towards a central tendency which is time varying than towards a constant one.

Turning now to the subperiod evidence, we find that during the two periods 1971:9-79:9 and 1979:10-82:9, the measure of the central tendency captures little if any time-variation in the interest rate. This happens in the two periods for different reasons. During the 1971:9-79:9 period, monthly interest rates were changing very smoothly, with a behavior closely resembling a random walk, making the very notion of a central tendency of little use, at least at the monthly frequency. In contrast, the Fed's aggressive management of non-borrowed reserves during the 1979:10-82:9 period resulted in such a high volatility of interest rates that the notion of a time-varying central tendency is again useless. During this period only the conditional volatility parameters seem to be estimated with reasonable precision, and the data show a slight preference for a model where volatility is tied to the level of the interest rate (CIR and CIR*) relative to a model with constant variance (Vasicek and Vasicek*).

In contrast, for the first and the last subperiod the central-tendency proxy seems to play an important role in explaining the conditional mean of the one-month rate. These two periods have in common a somewhat "eclectic" approach of the Federal Reserve to monetary policy, where both interest-rate volatility and the growth of monetary aggregates were taken into account when formulating operating targets.

Given the parameter estimates for the CIR* model of Table 1, for the entire sample, we obtain a time series of $\hat{\theta}$. Figure 2 plots $\hat{\theta}$ and the one-month rate. The visual evidence of Figure 2 confirms that $\hat{\theta}$ indeed behaves as a central tendency, and it is less volatile than the one-month rate of interest.[4]

We now examine the properties of the central-tendency proxy to gather information as to the appropriate specification of the model. The first test concerns the correlation between changes in $\hat{\theta}_t$ and the level of r_{t-1} which, according to (3), should be zero. Figure 3 plots changes in $\hat{\theta}$ extracted from different maturity pairs against lagged r. In plotting each variable we symmetrically trim 2.5 percent of the data at both ends of the distribution, to eliminate the visual effect of a few extreme outliers. To help visualizing the relation between central-tendency proxies estimated using different bond-maturity pairs, we add a locally linear scatterplot smooth estimated by Cleveland's (1979) *loess* method, with span covering 40 percent of the data. Figure 3 shows that, at least on the bulk of the data, there is very little correlation between changes in $\hat{\theta}$ and lagged r. This is consistent with the maintained assumption that the system of equations (2)-(3), describing the joint process for r and θ, is recursive, with θ affecting the conditional mean and volatility of r but not viceversa.

A second test concerns the properties of central-tendency proxies extracted using different yield-maturity pairs. The top three panels of Figure 3 present bivariate scatterplots of $\hat{\theta}$ series estimated from the one- and two-year maturity yields against the series estimated using different maturity pairs. Below each scatterplot, we show the corresponding quantile-quantile plot. Under the null that $m(\theta)$ and $s(\theta)$ are linear, the distribution of different central-tendency proxies should be identical, and their quantiles should plot on the 45 degree line displayed in the picture.

Evidence that the statistical properties of $\hat{\theta}$ vary using different bond-maturity pairs is an indication of general misspecification of the model (3′). The Figure is obtained using a similar technique to Figure 3. Figure 4 suggests that changes in the proxy estimated using different bond-maturity pairs are a linear function of one another, but not in a one-for-one fashion. If we maintain that (2) is correctly specified, this result does not support the notion that θ affects *linearly* the conditional mean and volatility of θ, and it is instead consistent with θ following a process where its conditional mean and/or volatility are nonlinear functions of its level, as in equation (3).

[4]The standard deviation of r (percentage points per month) for the 1952:7-1993:12 period is 2.89, as opposed to a standard deviation of 2.45 for $\hat{\theta}$ over the same period.

5 Conclusions

In one-factor models, such as Cox, Ingersoll, and Ross (1985) or Vasicek (1977), the conditional mean of the instantaneous rate changes with its current level. This papers gathers evidence that the conditional mean of the one-month rate explains variations in bond yields of different maturities, even after controlling for the effect of the current level of the one-month rate. This suggests the presence of a second factor driving the conditional mean, other the level of the one-month rate: we refer to this second factor as the central tendency. The above idea is captured in a two-factor model of the term structure where the instantaneous rate fluctuates around a stochastic central tendency. We then build a proxy for the central-tendency factor based on the information contained in the term structure of interest rates. We use the proxy to estimate the process for the one-month rate, and find the central-tendency proxy to be significant in explaining the conditional mean of the one-month rate.

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Figure 1.a: Yield sensitivities to level, conditional mean, and conditional volatility of the one-month rate (entire sample)

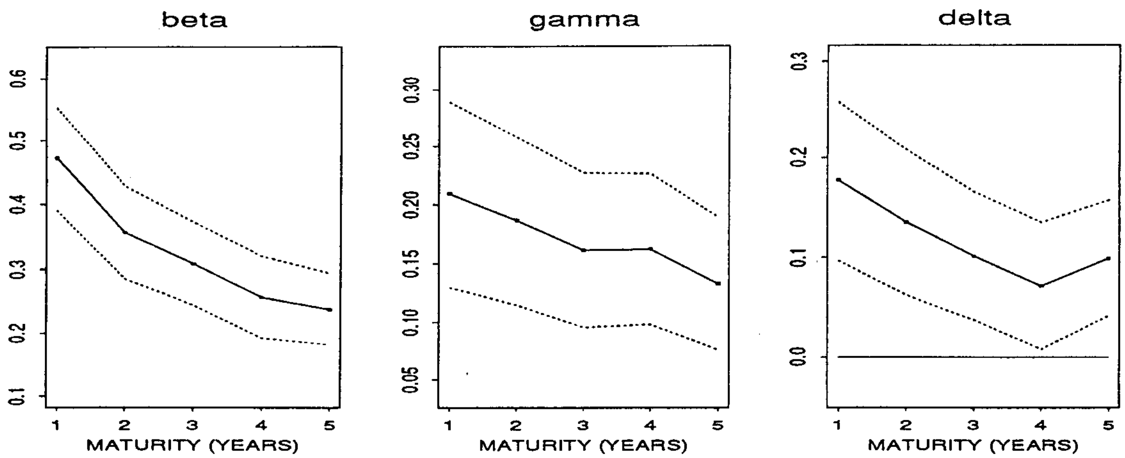

Figure 1.b: Yield sensitivities to level, conditional mean, and conditional volatility of the short-term rate (subperiods)

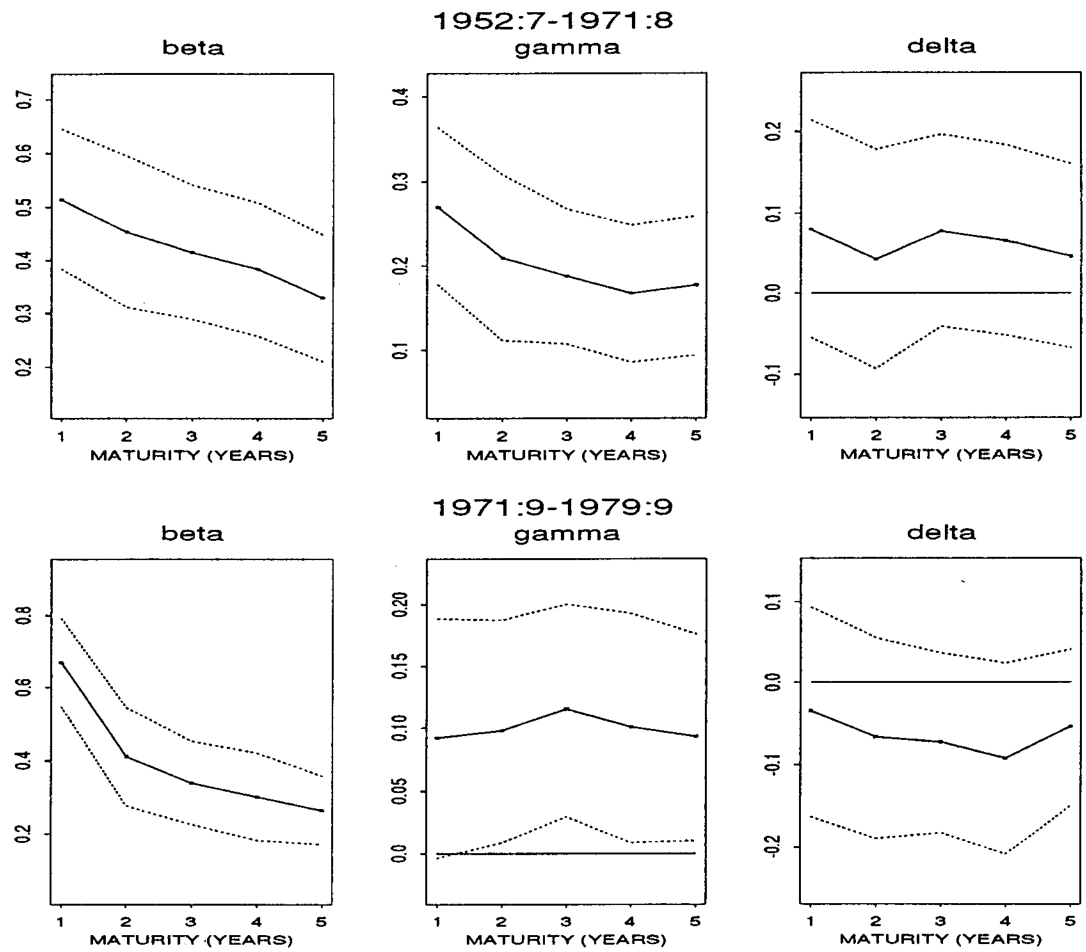

Figure 1.c: Yield sensitivities to level, conditional mean, and conditional volatility of the short-term rate (subperiods, contd.)

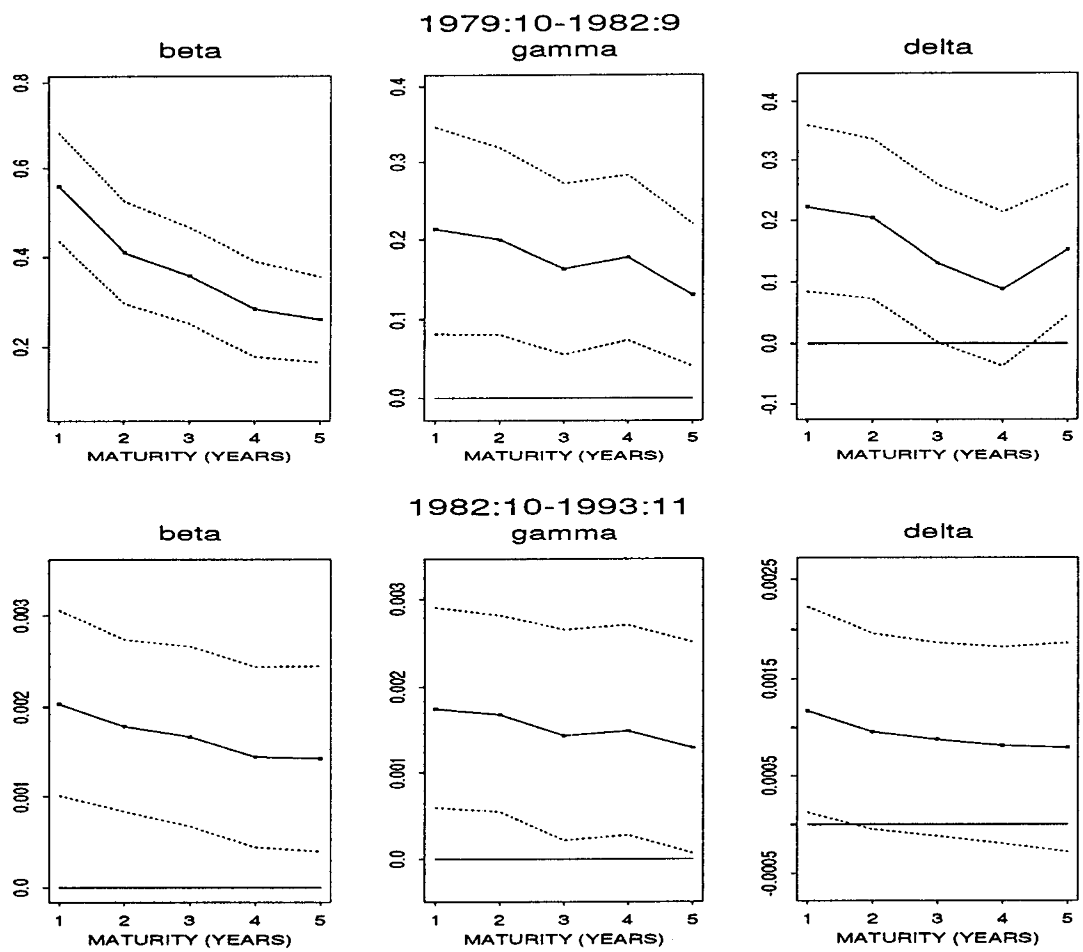

Figure 2: One-month rate and $\hat{\theta}$

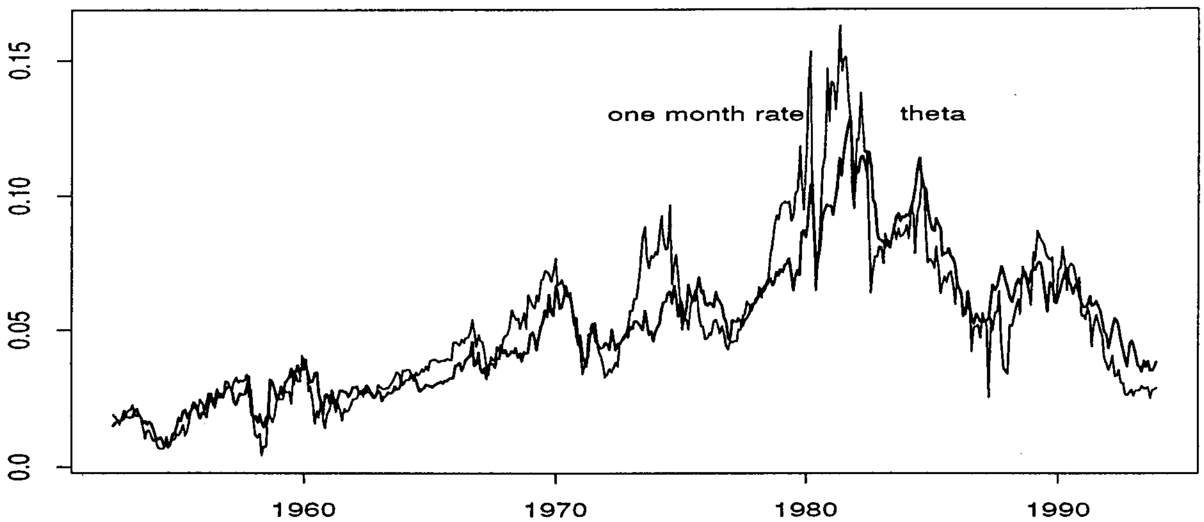

Figure 3: Scatterplot of $\hat{\theta}_t - \hat{\theta}_{t-1}$ and $r_{t-1} - r_{t-2}$

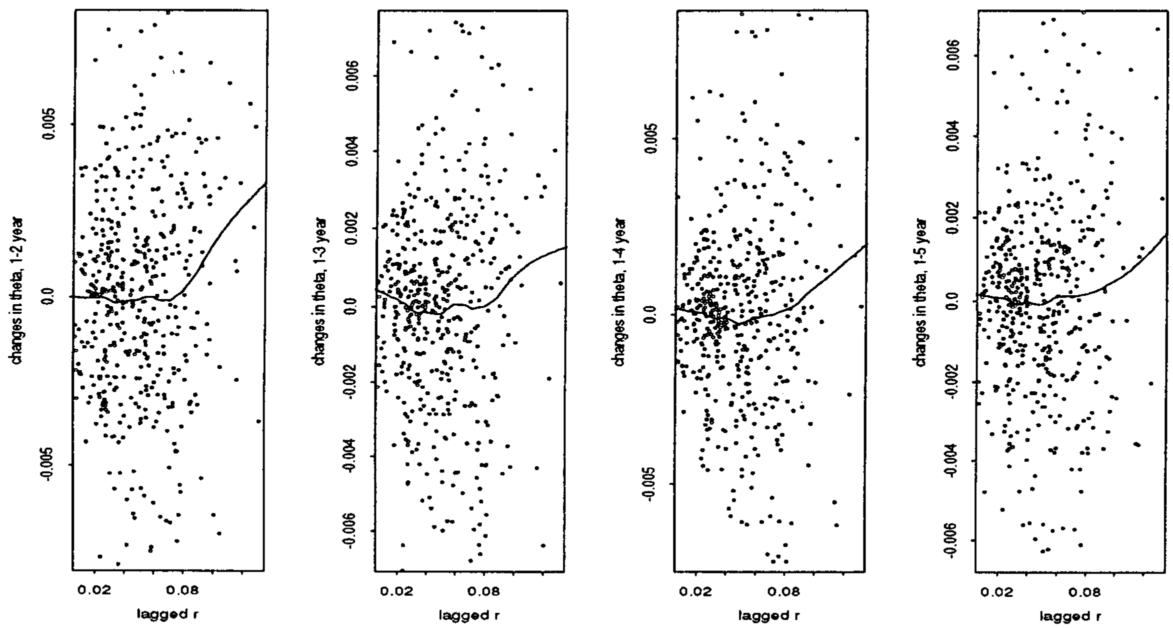

Figure 4: $\hat{\theta}$ generated from pairs of bonds of different maturities

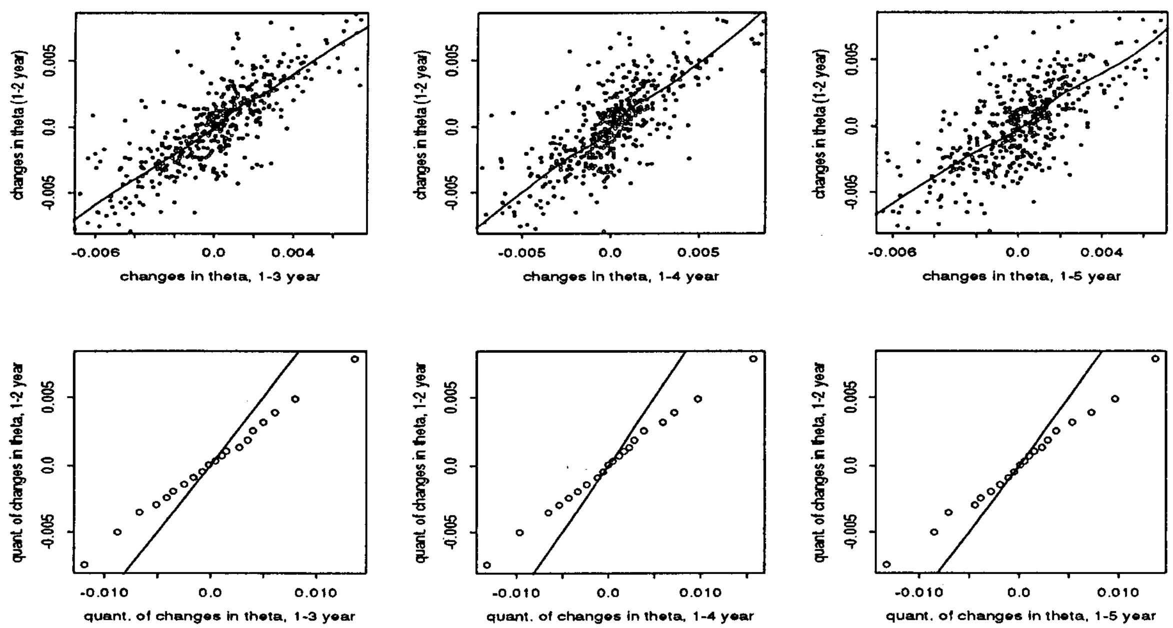

Table 1: Maximum likelihood estimation

Observations are monthly, 1952:7-93:12, using one- and two-year bond yields to construct the proxy in equation (10). We report the value of the parameter estimates, the maximized log likelihood, and standard errors (in parenthesis).

1952:7-93:12 (entire sample)

Model	k	a_0	a_1	σ_0	σ_1	log lik.
Vasicek	0.38368	0.05178		0.02523		2202
	(0.10715)	(0.01376)		(0.00037)		
CIR	0.29641	0.05198			0.10217	2291
	(0.09639)	(0.01151)			(0.00182)	
Vasicek*	1.77839	-0.00009	-2.06846	0.02486		2210
	(0.22894)	(0.00705)	(0.21658)	(0.00038)		
CIR*	2.12232	-0.00181	-2.26905		0.09947	2305
	(0.24666)	(0.00253)	(0.15907)		(0.00209)	

1952:7-71:8

Model	k	a_0	a_1	σ_0	σ_1	log lik.
Vasicek	0.35462	0.03526		0.01302		1169
	(0.18061)	(0.00840)		(0.00044)		
CIR	0.48909	0.03419			0.08287	1158
	(0.21139)	(0.00897)			(0.00236)	
Vasicek*	2.06495	-0.00403	-2.47920	0.01279		1173
	(0.52103)	(0.00423)	(0.35947)	(0.00045)		
CIR*	2.61681	-0.00396	-2.76994		0.07993	1166
	(0.52438)	(0.00254)	(0.38710)		(0.00246)	

1971:9-79:9

Model	k	a_0	a_1	σ_0	σ_1	log lik.
Vasicek	0.37097	0.07918		0.01992		452
	(0.49714)	(0.02442)		(0.00092)		
CIR	0.23899	0.08958			0.07512	461
	(0.41252)	(0.05703)			(0.00391)	
Vasicek*	0.38912	0.04438	-1.64259	0.01990		452
	(0.93049)	(0.08492)	(7.10060)	(0.00096)		
CIR*	0.24305	0.09517	0.36916		0.07511	461
	(0.51001)	(0.07253)	(5.23431)		(0.00405)	

Table 1: Maximum likelihood estimation (subperiods, contd.)

1979:10-82:9

Model	k	a_0	a_1	σ_0	σ_1	log lik.
Vasicek	2.79028	0.11263		0.06110		127
	(1.53551)	(0.01351)		(0.00792)		
CIR	2.46332	0.11214			0.18064	128
	(1.47522)	(0.01474)			(0.02449)	
Vasicek*	2.38119	0.16057	1.13098	0.06099		127
	(1.66958)	(0.15988)	(3.49793)	(0.00789)		
CIR*	1.92442	0.18962	1.67001		0.18008	128
	(1.49678)	(0.18155)	(3.57585)		(0.02423)	

1982:10-93:12

Model	k	a_0	a_1	σ_0	σ_1	log lik.
Vasicek	0.73703	0.05573		0.02601		593
	(0.41065)	(0.01099)		(0.00114)		
CIR	0.77285	0.05596			0.11365	587
	(0.45875)	(0.01100)			(0.00492)	
Vasicek*	3.98616	-0.01122	-3.58094	0.02397		604
	(0.62020)	(0.00862)	(0.59769)	(0.00115)		
CIR*	4.46751	-0.01381	-4.10881		0.10202	602
	(0.51897)	(0.00829)	(0.61231)		(0.00469)	