NEW YORK UNIVERSITY
STERN SCHOOL OF BUSINESS
FINANCE DEPARTMENT

Working Paper Series, 1994

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FD-94-9

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May 1994. This revision, January 1995.

Abstract: We assume the short-term rate to revert towards a central tendency which, in turn, is stochastically changing over time. We impose minimal restrictions on the joint behavior of the short-term rate and the central-tendency factor, and derive implications for the term structure of interest rates. The analysis suggests a proxy for the central tendency which is then used to estimate the short-term rate process. Our model captures variations in the short-term rate better than the Vasicek (1977) and Cox, Ingersoll and Ross (1985) models, where the central tendency is assumed to be constant. Also, the central-tendency proxy explains the conditional volatility of the short-term rate better than the short-term rate itself.

Keywords: term structure. **JEL#** G12.

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1 Introduction

There is consensus that one-factor models of the term structure cannot successfully account for a number of the salient features of bond yields. One reason for such widespread opinion is easily understood: in a one-factor model of the term structure, yields and returns of all bonds should be perfectly correlated. Stambaugh (1988), among others, argues that yields are driven by at least *two* risk factors.

In this paper we develop a two-factor model of the term structure. We follow a consolidated tradition in the finance literature [see, *e.g.,* Brennan and Schwartz (1979), Cox, Ingersoll, and Ross (1981, 1985), and Longstaff and Schwartz (1992)], and identify the first factor with the *level* of the short-term rate. The novelty of our analysis is that we identify the second factor with the *central tendency* of the short-term rate. The approach advocated here complements that of Longstaff and Schwartz (1992), who propose a model in which the second factor is identified with the conditional volatility of the short-term rate. Also, in a spirit similar to ours, Naik and Lee (1994) postulate a process for the short-term rate where both conditional volatility and central tendency may experience discrete shifts, which they relate to changes in the interest-rate regime.

We assume the short-term rate to revert towards a central tendency which, in turn, is stochastically changing over time. We also assume the conditional volatility of the short-term rate to be *linear* in the short-term rate itself and in the central-tendency factor. We impose minimal restrictions on the central-tendency factor's dynamics other than it should *not* be affected by the behavior of the short-term rate. Based on these assumptions, we derive implications for the term structure of interest rates (Section 2). The analysis suggests a proxy for the central tendency which is a *linear combination* of any two longer-maturity yields. This proxy is then used in Section 3 to estimate the short-term rate process.

The empirical analysis yields two main findings: First, our model captures variations in the short-term rate better than the Vasicek (1977) and Cox, Ingersoll and Ross (1985) models, in which the central tendency is assumed to be constant. Second, the central-tendency proxy explains the conditional volatility of the short-term rate better than the short-term rate itself.

2 The model

Bond prices

The behavior of the short-term rate r is described by the stochastic differential equation

$$dr = k(\theta - r)dt + \sqrt{\sigma_0^2 + \sigma_1^2 r + \sigma_2^2 \theta}\, dZ, \tag{1}$$

where k, σ_0, σ_1, and σ_2 are constants, and Z is a standard Brownian motion; θ is the central tendency towards which the short-term rate reverts. In turn, θ evolves over time according to the stochastic differential equation

$$d\theta = m(\theta)dt + s(\theta)dW, \tag{2}$$

where the conditional drift $m(\theta)$ and the conditional volatility $s(\theta)^2$ are "smooth" functions of θ alone. (A function is smooth if it is continuous with continuous derivatives of any order.) Like Z, W is a standard Brownian motion, and we assume $E(dZdW) = 0$.

The price of a risk-free discount bond of maturity τ, $P = P(r, \theta; \tau)$ satisfies the partial differential equation [see Cox, Ingersoll, and Ross (1985)]

$$E(\mathcal{D}P) - rP - P_r \lambda r - P_\theta l(\theta) = 0, \tag{3}$$

where $\mathcal{D}$ denotes the Dynkin differential operator. We assume the risk premium associated with r to be proportional to r, as in Cox, Ingersoll, and Ross (1985), with λ a constant; while the risk premium associated with θ, $l(\theta)$, is a smooth function of θ.

We guess a solution for the fundamental valuation equation (3) of the form

$$P(r, \theta; \tau) = e^{-A(\tau) - B(\tau)r - C(\theta;\tau)}, \tag{4}$$

where $A(\tau)$ and $B(\tau)$ are functions of τ alone, while $C(\theta; \tau)$ is a smooth function of both θ and τ. Equation (3) can then be rewritten as

$$\begin{aligned} &B(\tau)[k\theta - (k+\lambda)r] + B^2(\tau)[\sigma_0^2 + \sigma_1^2 r + \sigma_2^2 \theta]/2 + C_\theta(\theta;\tau)[m(\theta) - l(\theta)] \\ &\quad + [C_{\theta\theta}(\theta;\tau) + C_\theta^2(\theta;\tau)]s(\theta)^2/2 - A_\tau(\tau) - B_\tau(\tau)r - C_\tau(\theta;\tau) - r = 0, \end{aligned}$$

where we imposed the covariance between the two factors to be zero. A sufficient condition for the previous equation to be uniformly satisfied over the domain of r is that all the terms in r are equal to zero. Equating the terms in r to zero yields an ordinary differential equation which does not involve A and C:

$$-B(k+\lambda) + B^2\sigma_1^2/2 - B_\tau - 1 = 0.$$

The solution for B, subject to the initial condition that $B(0) = 0$, is well-known [see Cox, Ingersoll, and Ross (1985)]:

$$B(\tau) = \frac{2(e^{\delta\tau} - 1)}{(\lambda + \delta + k)(e^{\delta\tau} - 1) + 2\delta}, \quad \delta = \sqrt{(\lambda + k)^2 + 2\sigma_1^2}, \tag{5}$$

which reduces to $B(\tau) = (1 - e^{-k\tau})/k$ when $\lambda = \sigma_1^2 = 0$ [see Vasicek (1977)]. If $m(\theta)$, $s(\theta)^2$ and $l(\theta)$ were linear functions of θ, yields would be linear in r and θ, and bond prices would have the familiar form

$$P(r, \theta; \tau) = e^{-A(\tau) - B(\tau)r - D(\tau)\theta}. \tag{6}$$

Exact conditions for yields to be linear in the relevant state variables are given in Cox, Ingersoll, and Ross (1981), and Duffie and Kan (1993).

A proxy for the second factor

A useful implication of the analysis above is that $B(\tau)$ is constant for a given maturity, where $B(\tau)$ represents the *sensitivity* of the maturity-τ bond yield to changes in the short-term rate r. Choosing appropriate weights, one can build a linear combination of two bond yields which is independent of r. The variation of such quantity mimics variation in the second factor. This implication is well known [Cox, Ingersoll, and Ross (1985), pp.398-401]: yields of different maturities can be used as instruments for unobservable factors. Stambaugh (1988) and Sun (1992) base their empirical analysis on the same observation.

Formally, consider two bonds of maturity τ_1 and τ_2, respectively. From equation (4), the corresponding yields are

$$Y(r, \theta; \tau_i) = -\ln P(r, \theta; \tau_i)/\tau_i = [A(\tau_i) + B(\tau_i)r + C(\theta_i; \tau_i)]/\tau_i, \qquad \text{for } i = 1, 2. \tag{7}$$

Solving for r from the first yield, substituting into the second, and rearranging we obtain:

$$\tau_1 B(\tau_2) Y(r, \theta; \tau_1) - \tau_2 B(\tau_1) Y(r, \theta; \tau_2) = B(\tau_2)[A(\tau_1) + C(\theta; \tau_1)] - B(\tau_1)[A(\tau_2) + C(\theta; \tau_2)].$$

Note that this quantity does not depend on r.

If the drift and diffusion of the process for θ were also linear in θ, then prices would be of the form in equation (6), and the second factor θ could be estimated as

$$\theta = \frac{B(\tau_2)[\tau_1 Y(r, \theta; \tau_1) - A(\tau_1)] - B(\tau_1)[\tau_2 Y(r, \theta; \tau_2) - A(\tau_2)]}{B(\tau_2)D(\tau_1) - B(\tau_1)D(\tau_2)}. \tag{8}$$

Equation (8) justifies a proxy, or first-order approximation, for θ which is used in the empirical analysis of the next section. We denote this proxy with $\hat{\theta}$:

$$\hat{\theta} = a_0 + a_1[B(\tau_2)\tau_1 Y(r, \theta; \tau_1) - B(\tau_1)\tau_2 Y(r, \theta; \tau_2)]. \tag{9}$$

If the conditional mean and volatility of the process for θ were linear in θ, yields would be linear in both r and θ [see equation (6)], and all proxies estimated from any bond-maturity pair would be *the same.* Evidence that the statistical properties of $\hat{\theta}$ vary using different bond-maturity pairs is instead consistent with θ following a process where its conditional mean and/or volatility are nonlinear functions of the level of θ itself.

3 Empirical analysis

The model

In order to estimate the parameters of the process in (1) we implement an estimation procedure based on the following considerations.

If θ were constant and $\sigma_1 = \sigma_2 = 0$, we could estimate the following *exact* stochastic difference equation for the short-term rate:

$$r_{t+h} - r_t = (1 - e^{-kh})(\theta - r_t) + \sqrt{\sigma_0^2(1 - e^{-2kh})/(2k)}\ \epsilon_{t+h},$$

where ϵ_{t+h} is distributed standard normal.

Here, for simplicity, we shall use an (Euler) approximation to the exact stochastic difference equation above [see, for example, Chan, Karolyi, Longstaff, and Sanders (1992)],

$$r_{t+h} - r_t \approx kh(\theta_t - r_t) + \sigma_0 \epsilon_{t+h}. \tag{10}$$

Moreover, *both* factors underlying our term-structure model are unobservable. Hence, we shall need *observable* empirical counterparts.

First, we follow a standard practice [see, for example, Chan, Karolyi, Longstaff, and Sanders (1992), and Longstaff and Schwartz (1992)] and treat the one-month Treasury-bill rate as a proxy for the short-term rate r. While this practice is convenient, we are aware that the one-month rate may indeed deviate from the unobserved factor r, as Pearson and Sun (1994) point out. In fact, our model implies that the yield on any finite-maturity bond depends on *both* factors, r and θ, as well as on their risk premia λr and $l(\theta)$.

Second, we proxy θ with the quantity proposed in the previous section in equation (9). This second approximation depends on the fact that we have not imposed m and s to be linear, and θ may relate nonlinearly to the yields in (7).

Using the same approximation as in equation (10) above, we discretize the stochastic differential equation describing dr, for monthly observations ($h = 1/12$). We then estimate our interest-rate model by maximizing the log-likelihood function

$$-.5\sum_{t=1}^{T}\left[\ln(E_t(r_{t+1}-E_tr_{t+1})^2)+(r_{t+1}-E_tr_{t+1})^2/(E_t(r_{t+1}-E_tr_{t+1})^2)\right], \tag{11}$$

where

$$E_tr_{t+1} = (1-(k/12))r_t+(k/12)\hat{\theta}_t \tag{12}$$

$$E_t(r_{t+1}-E_tr_{t+1})^2 = (\sigma_0^2/12)+(\sigma_1^2/12)r_t+(\sigma_2^2/12)\hat{\theta}_t, \tag{13}$$

and $\hat{\theta}$ is given by

$$\hat{\theta}=a_0+a_1[B(\tau_2)\tau_1Y(r,\theta;\tau_1)-B(\tau_1)\tau_2Y(r,\theta;\tau_2)] \tag{9}$$

with

$$B(\tau)=\frac{2(e^{\delta\tau}-1)}{(\lambda+\delta+k)(e^{\delta\tau}-1)+2\delta},\quad \delta=\sqrt{(\lambda+k)^2+2\sigma_1^2}. \tag{5}$$

Data

Monthly data on the one-month Treasury-bill rate are from the CRSP risk-free rates file, while discount-bond prices are from the Fama-Bliss files also from the CRSP tape. All interest rates are on a continuously compounded basis and sampled at the end of the month.

Estimation results

We estimate the model (11), (12), (13), (9), and (5) using monthly observations. The estimates presented here are obtained using one- and two-year bond yields to construct the proxy in equation (9). The estimation period goes from 1960:2 to 1990:12. Results are presented in the following Table.

The Table shows parameter estimates for the restricted and unrestricted versions of the model. We estimate the following versions of the one-month-rate process: 1) CIR: $a_1=\sigma_0=\sigma_2=0$; 2) Vasicek: $a_1=\sigma_1=\sigma_2=0$; 3) CIR*: $\sigma_0=\sigma_2=0$; 4) Vasicek*: $\sigma_1=\sigma_2=0$; 5) Unrestricted. In "CIR" and "Vasicek" we restrict $\hat{\theta}$ to be a constant. The first model corresponds to the Cox, Ingersoll, and Ross (1985) square-root process, while the second model corresponds to the Vasicek (1977) Ornstein-Uhlenbeck process. In "CIR*" we allow the square-root process to be coupled with a time-varying central tendency which affects only the mean of the one-month rate. In "Vasicek*" we do the same for the Ornstein-Uhlenbeck process. Finally, the unrestricted model allows both factors to affect both mean and variance of the one-month rate.

Table: Maximum likelihood estimation

Observations are monthly for the 1960:2–1990:12 period. T-statistics are in parenthesis. We also report the value of the maximized log likelihood, and the χ^2 statistic of the likelihood-ratio test of each model against the unrestricted one, with the corresponding degrees of freedom in parenthesis.

Model	k	σ_0	σ_1	σ_2	a_0	a_1	log lik.	χ^2
CIR	0.4291		0.1047		0.0603		1707	33.26(3)
	(3.03)		(47.42)		(5.87)			
Vasicek	0.5353	0.0280			0.0602		1654	139.7(3)
	(3.77)	(54.91)			(5.01)			
CIR*	2.0951		0.1020		0.0005	-2.2200	1717	13.36(2)
	(7.86)		(41.84)		(0.08)	(-9.33)		
Vasicek*	1.8515	0.0276			0.0046	-1.9828	1659	128.76(2)
	(6.66)	(53.83)			(0.47)	(-6.74)		
Unrestricted	1.5260	-0.0000	0.0160	0.1011	-0.0195	-2.7044	1724	
	(4.62)	(0.00)	(0.29)	(10.29)	(-2.62)	(-11.76)		

Attempts at estimating the risk-premium parameter λ have been rather discouraging, leading either to numerical problems, or to very imprecise estimates. Hence, in our empirical implementation we restricted λ to equal zero.

The estimation results indicate that both the Cox, Ingersoll, and Ross and the Vasicek model are rejected. In "CIR*" and "Vasicek*" the coefficient of the central-tendency proxy, a_1, is significantly different from zero. It is also apparent that the level of the one-month rate affects its conditional volatility: in "CIR*," σ_1 is significantly different from zero. In other words, the data indicate that *both* the central tendency and the conditional volatility of the one-month rate are time varying. Also, it is interesting to note that the (rejected) assumption of a constant central tendency translates into a substantial underestimation of the mean-reversion parameter k.

Hence, we proceed to estimate the unrestricted model. We have two main findings. First, the constant term σ_0^2 is redundant. Second, the central-tendency proxy dominates the one-month rate in explaining volatility.

The first finding is specific to the sample period under scrutiny. For the sample period 1960:2–1990:12, the average level of $\hat{\theta}$ carries approximately the same "information" as σ_0^2. In fact, we have $\sigma_2^2 \times \text{average}\,\hat{\theta} = 0.000614$ (from the unrestricted model) and $\sigma_0^2 = 0.000761$ (from the Vasicek* model).

The second finding is consistent with the analysis of Longstaff and Schwartz (1992), who argue that the conditional volatility of the one-month rate should affect yields of longer maturity. Our proxy for the central tendency is essentially a linear combination of yields: longer-term yields contain information for the conditional volatility of the one-month rate over and above that contained in the one-month rate itself.

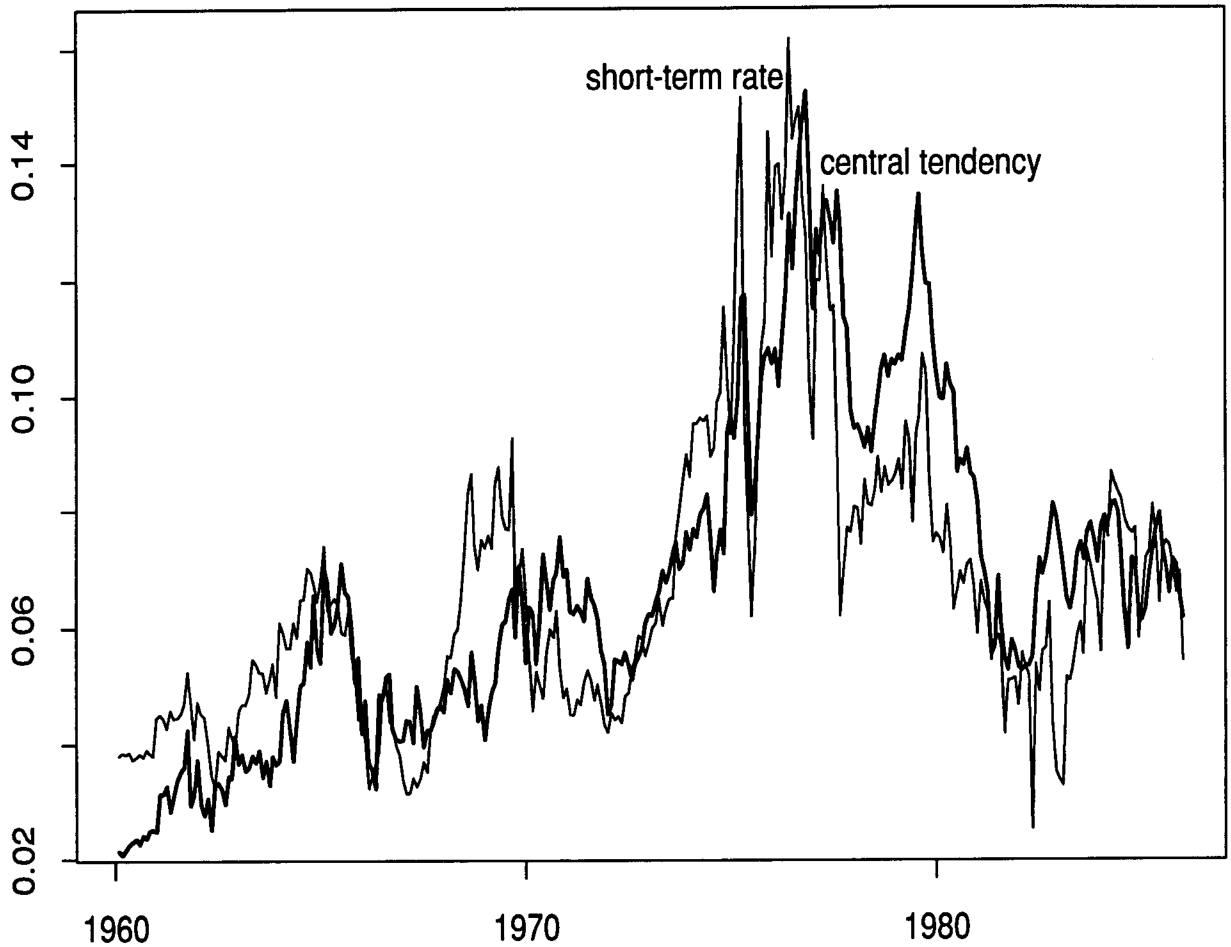

Figure 1: **One-month rate and $\hat{\theta}$**

Statistical properties of $\hat{\theta}$

Based on the empirical results above, we proceed to estimate a series for $\hat{\theta}$ where we impose $\sigma_0 = \sigma_1 = 0$. Figure 1 plots the estimated $\hat{\theta}$ together with the one-month rate. The visual evidence of Figure 1 confirms that $\hat{\theta}$ behaves as a central tendency and it is somewhat smoother than the level of the one-month rate of interest.

We turn next to compare the series for $\hat{\theta}$ estimated using different maturity pairs. If the mean and volatility of the second factor were linear in θ, the normalizations imposed at the estimation stage would imply that the series $\hat{\theta}$ estimated from different maturity pairs should be the *same*. The top three panels of Figure 2 presents bivariate scatterplots of series estimated from the one- and two-year maturities against the series estimated using different

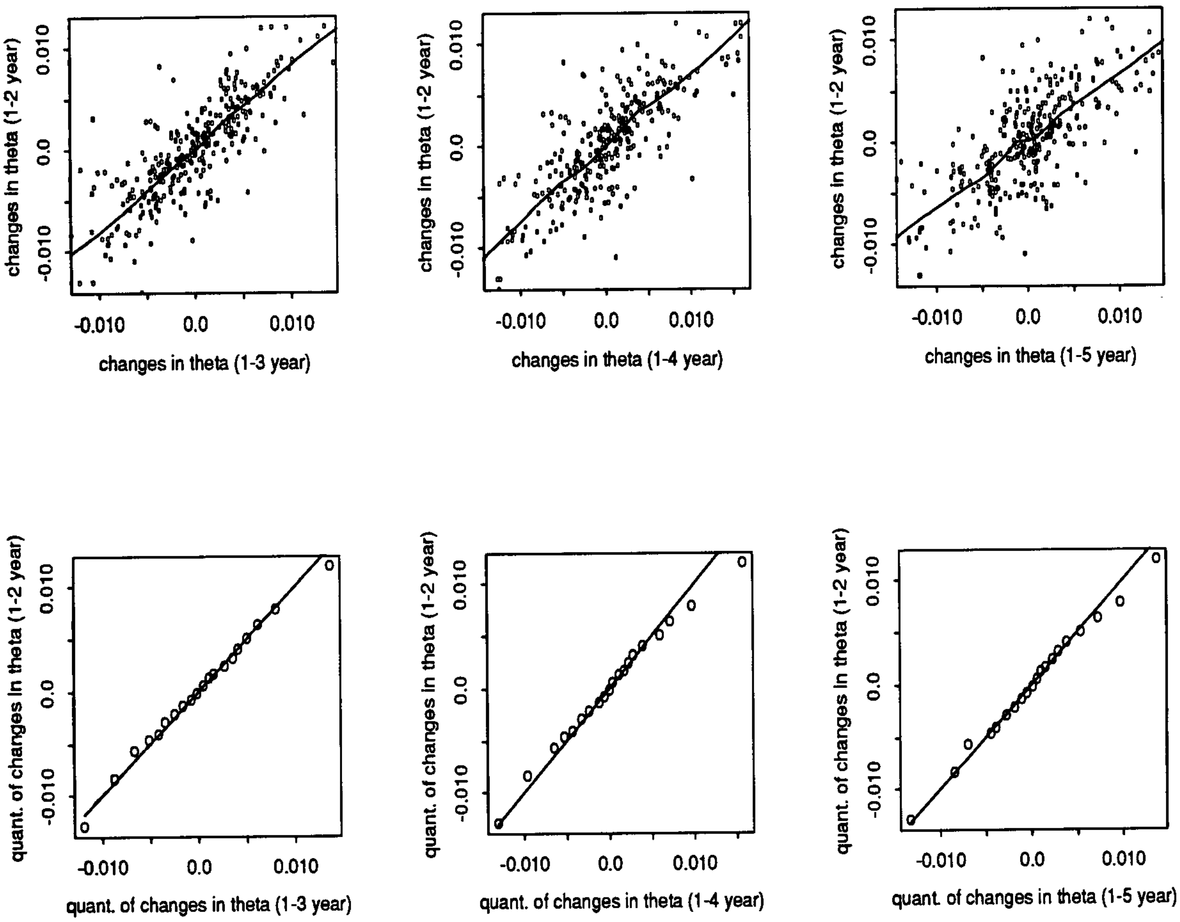

Figure 2: $\hat{\theta}$ **generated from pairs of bonds of different maturities**

maturity pairs.

In plotting each variable we symmetrically trim 2.5 percent of the data at both ends of the distribution, to eliminate the visual effect of a few extreme outliers. To help visualizing the relation between central-tendency proxies estimated using different bond-maturity pairs, we add a locally linear scatterplot smooth estimated by Cleveland's (1979) *loess* method, with span covering 40 percent of the data. Below each scatterplot, we show the corresponding quantile-quantile plot. Under the null that $m(\theta)$ and $s(\theta)$ are linear, the distribution of different central-tendency proxies should be identical, and their quantiles should pair on the 45 degree line displayed in the picture.

The Figure suggests that changes in the proxy estimated using different bond-maturity pairs are very similar. This result supports somewhat the notion that θ affects *linearly* both conditional mean and volatility in equations (1) and (2).

Finally, we consider the properties of the unconditional distribution of the one-month rate and of $\hat{\theta}$. The level of the one-month rate is *leptokurtic* (kurtosis=1.16, p-value=0.000) and normality can be strongly rejected; this is consistent with the finding that σ_2 is non-zero (see Table). In fact, in a Cox, Ingersoll, and Ross (1985) single-factor model, where the conditional volatility of the short-term rate depends on its level, the unconditional distribution for r is of the gamma type which has fatter tails than the normal. On the other hand, the estimated series for $\hat{\theta}$ displays a kurtosis of -0.199, with a p-value of 0.43, which is consistent with that of a normal variable. This suggests that the excess kurtosis in the one-month rate may be due mainly to short-lived departures of the one-month rate from its time-varying central tendency.

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