Pricing Mortgage-Backed Securities in a Multifactor Interest Rate Environment: A Multivariate Density Estimation Approach

Jacob Boudoukh, Matthew Richardson, Richard Stanton, and Robert F. Whitelaw

Pricing Mortgage-Backed Securities in a Multifactor Interest Rate Environment: A Multivariate Density Estimation Approach

Jacob Boudoukh, Matthew Richardson, Richard Stanton and Robert F. Whitelaw[*]

January 24, 1995

Abstract

This paper develops a nonparametric, model-free approach to the pricing and hedging of mortgage-backed securities (MBS), using multivariate density estimation procedures to investigate the relation between MBS prices and interest rates. While the usual methods for valuing MBSs are highly dependent on specific assumptions about interest rates and prepayment, this method will yield consistent results without requiring such assumptions. Weekly MBS prices from January 1987 to May 1994 can be well described as a function of the level and slope of the term structure. Using this estimated relation to hedge the interest rate risk of MBS gives results that compare favorably with other commonly used hedging methods.

1   Introduction

The mortgage-backed security (MBS) market plays a special role in the U.S. economy. Originators of mortgages (S&Ls, savings and commercial banks) can spread risk across the economy by packaging these mortgages into investment pools through a variety of agencies, such as the Government National Mortgage Association (GNMA), Federal Home Loan Mortgage Corporation (FHLMC), and Federal National Mortgage Association (FNMA). Purchasers of MBSs are given the opportunity to invest in default-free, interest-rate contingent claims which offer different payoff structures from U.S. treasury bonds. These combined elements lower an individual's cost of obtaining a mortgage, creating welfare gains. Due to these gains, the MBS market has been one of the fastest growing, as well as one of the largest financial markets, in the United States. For example, in 1993, the face value of these securities outstanding was 1.5 trillion dollars, in comparison to approximately 100 million outstanding in 1980. The magnitude and growth of the MBS market demonstrate how important it is to the financial and real sectors of the economy.

With increased holdings of MBSs, there have been well-documented cases of huge monetary losses incurred by financial institutions and investment groups. The risk management (or lack thereof) of S&Ls' mortgage portfolios is one example of financial institutions' vulnerability to price variations in the mortgage market. This vulnerability is partly due to the complexity of MBS pricing. On one level, pricing appears to be fairly simple. Fixed-rate mortgages offer fixed nominal payments; thus, fixed-rate MBS prices will be governed by pure discount bond prices. However, the mortgage holder has the option to prepay the existing mortgage and refinance the property with a new mortgage; hence, MBS investors are implicitly writing a call option on a corresponding fixed-rate bond. Furthermore, prepayment of mortgages (and thus the timing and magnitude of the MBS's cash flows) can also take place for reasons not related to the option component.

The most common approach to valuing an MBS is through the development of theoretical models with specific parameterizations (see, for example, Dunn and McConnell (1981), Schwartz and Torous (1989), Kau, Keenan, Muller and Epperson (1992) and Stanton (1994)). While these models provide considerable insights into the pricing of MBSs, there are several reasons why they are not necessarily the best vehicle for determining the relation between MBS prices, interest rates and prepayment rates. First, the theoretical approach requires specification of all the features of the model. If this specification is incorrect, then it is unclear how to interpret the MBS valuation. This is especially important given that many

1

of the models rely on unreasonable assumptions about the process generating interest rates and prepayment behavior. Second, the applications of these models rely on parameters of the model that must be determined, either through estimation or posited from "thin air".

Recent research indicates that nominal prices of fixed income securities are governed by both real and nominal factors, indicating that models of interest rates (and prepayment) should contain at least two factors. As the dimensionality of the models increases, so does the likelihood of both model misspecification and incorrect assumptions about parameter values. An investor who wishes to hedge the interest rate risk contained in an MBS must therefore be especially cautious in applying MBS models in practice.

An alternative approach to understanding the cross-relations between mortgage-backed security prices, prepayment rates and interest rates is via a model-free methodology. In this paper, we employ multivariate density estimation (MDE) procedures to estimate the functional relation between MBS prices and their fundamentals.[1] The MDE procedure is well suited to analyzing MBSs because, while financial economists have good intuition for what the MBS pricing fundamentals are, the exact form is too complex (or assumption-specific) to be determined precisely. Furthermore, in contrast to parametric or semi-parametric econometric techniques, consistency of the MDE procedure (i.e., the MBS model specification) is assured.

An added benefit of the MDE methodology is that it is straightforward to analyze multifactor models. Current empirical evidence favors a multifactor approach to fixed-income pricing (e.g., Stambaugh (1988), Litterman and Scheinkman (1991) and Pearson and Sun (1989)), pointing to at least two factors. Given the results of this research, we assume that the interest rate level and the slope of the term structure span (possibly nonlinearly) the two pricing factors. In this two-factor setting, we apply the MDE approach to the pricing and hedging of mortgage-backed securities. We estimate the joint density between interest-rate levels, the slope of the term structure, and a cross-section of prices of MBSs with different coupons. This joint density implies pricing functionals (in terms of the available information) and reveals important characteristics of the distribution of MBS prices.

In order to study the small sample properties of the MDE method, we first examine a simulated model of MBS prices. In this model, the economy is governed by a two-factor Cox, Ingersoll and Ross (CIR) (1985a,b) model. Mortgage prepayments are introduced using the Schwartz and Torous (1989) model, which captures some of the salient features of prepayment

---

[1] For other empirical studies of MBS pricing and hedging see Breeden (1991), Harvey (1991), Richard and Roll (1989), Schwartz and Torous (1989), and Stanton (1992).

behavior. The MDE approach is then applied to the simulated economy. For this particular model, the MDE approach approximates well the functional form of MBS prices.

The MDE method is then applied to GNMA securities of various coupons over the period 1987-1994. The data are prices of weekly TBA (to be announced) GNMAs with coupons ranging from 7.5% to 10.5%.[2] A within-sample and out-of-sample analysis is provided for the pricing and hedging of these securities. From a pricing perspective the MDE methodology captures the negative convexity of MBS prices. Of particular importance, the relation between prices and the level of interest rates is also shown to be dependent on the slope of the term structure. Consistent with economic intuition, two factors are necessary to fully describe the effects of the prepayment option on prices. The analysis also reveals cross-sectional differences, across GNMAs with different coupons, especially with regard to their sensitivities to movements in the two interest rate factors. These sensitivities are used in the out-of-sample hedging analysis, and the MDE methodology compares favorably to both a linear hedge and an alternative nonparametric technique. As expected, the MDE methodology works especially well in low interest rate environments when the GNMAs behave less like fixed maturity bonds.

The paper is organized as follows. Section 2 discusses the pricing of MBSs in a multifactor framework. In Section 3, we describe the MDE methodology and its ability to explain MBS prices using data from simulated economies. Section 4 provides a detailed description of the data used in the study. Section 5 analyzes the pricing and hedging performance of the MDE methods and theoretical models throughout the 1987-1994 period. In Section 6, we make some suggestions for future research and conclude the study.

2   Mortgage-Backed Security Pricing: Preliminaries

Mortgage-backed securities represent claims on the cash flows from mortgages which have been pooled together and packaged as a financial asset. Investors in an MBS receive all payments (principal plus interest) made by mortgage holders in a particular pool (less some servicing fee). For many of these securities, the payments are guaranteed by government or private agencies so there is no question of default. In the case of a household default, the agency pays the remaining principal of that mortgage in the pool. Thus, the pricing of an MBS can be reduced to valuing the mortgage pool's cash flows at the appropriate discount

---

[2]A TBA contract is just a forward contract, trading over the counter. More details are provided in Section 4.

rate. MBS pricing then is very much an issue of estimating the magnitude and timing of the pool's cash flows.[3]

However, pricing an MBS is not a straightforward discounted cash flow valuation. This is because the timing and nature of a pool's cash flows depends on the prepayment behavior of the holders of the individual mortgages within the pool. For example, independent of interest rates, mortgages might be prepaid by individuals with enhanced wealth or who relocate. These events will lead to early payments of principal to the MBS holders. In addition, MBSs contain an embedded interest rate option. Mortgage holders have an option to prepay their existing mortgage and refinance their property at the lower interest rate. They are likely to do so as interest rates and hence refinancing rates decline to a certain point below the rate of their current coupon. This refinancing incentive tends to lower the value of the mortgage to the MBS investor because the mortgages' relatively high expected coupon payments are replaced by an immediate payoff of the principal. The equivalent investment alternative now available to the MBS investor is, of course, at the lower coupon rate. Therefore, the price of an MBS with an $X\%$ coupon is roughly equivalent to owning a default-free $X\%$ coupon-bearing bond and writing a call option on that bond (with an exercise price of par). This option component induces a concave relation between the price of MBSs and the price of default-free bonds (the so called "negative convexity").

Modeling and pricing MBSs, hence, involves two layers of complexity: (i) modeling the dynamic behavior of the term structure of interest rates, and (ii) modeling prepayment behavior of mortgage holders. Below, we review the theoretical approach to pricing MBSs and later contrast that approach with a nonparametric method for pricing MBSs.

2.1 Term Structure Dynamics and the Theoretical Approach to MBS Pricing

In analyzing the term structure behavior, two related approaches are usually taken: (i) the "technical" approach, and (ii) the "economic" approach. Technical studies, using such tools as factor analysis, conclude that two to three factors suffice in order to explain most of the variation in interest rates of various maturities.[4] "Economic" studies are concerned with

---

[3]For a description of MBSs and their relevant characteristics, see *The Handbook of Mortgage-Backed Securities*, Fabozzi, editor, 1993.

[4]Litterman and Scheinkman (1991), for example, use three observable factors, all extracted from the current term structure of interest rates: the level of the short rate, the spread between the short rate and the long rate, and the curvature of the term structure. These three factors are shown to explain approximately 98% of the variation in interest rates.

linking the dynamics of the term structure to the properties of relevant fundamental factors, namely, real interest rates (and other real variables), and inflation (and other monetary variables). Some of the studies in this class also provide more detail on the underlying economy and agents' preferences. Such an extension is particularly relevant when studying time varying premia in different maturity bonds.[5] Put together, these studies provide both the empirical and economic motivation for specifying multiple factors when describing the dynamics of the term structure. They also provide insight into which time series model is most appropriate.

In the academic literature, however, most theoretical models of MBS pricing ignore multi-factor pricing issues, and instead posit a one-factor interest rate model, preferring to focus on the prepayment characteristics of MBSs.[6] In particular, in one of the earliest academic studies in the area, Dunn and McConnell (1981a,1981b) apply an option pricing approach to the valuation of MBSs. This approach also determines the optimal prepayment strategy as part of the MBS valuation process. The Dunn-McConnell model, however, has some unattractive implications. First, the price of an MBS can never exceed par. Second, in their model all mortgage holders refinance at the same time, as soon as interest rates fall below a critical level. To correct the first problem, Timmis (1985), Dunn and Spatt (1986) and Johnston and Van Drunen (1988) add transaction costs which must be paid by borrowers should they decide to refinance their loans early. To relax the second restriction, Stanton (1994) extends the model to include heterogeneous transaction costs across mortgage holders. Thus, there is no longer a particular interest rate level which induces uniform prepayment and the exercise price of the prepayment option is uncertain. This uncertainty may be related not only to interest rates, but also to changes in the status of households within the mortgage pool (e.g., marital, relocation, and wealth). While these later models are theoretically appealing, they still have trouble capturing prepayment characteristics.

As an alternative, Schwartz and Torous (1989) use a Monte Carlo approach to price

---

[5]The best example is the seminal work of CIR (1985a, 1985b), and some of the related empirical work (e.g., Pearson and Sun (1989)), and multifactor extensions (Chen and Scott (1993)). The CIR model has two crucial elements: (i) the distinction between the real economy and the role of inflation (although without any interaction between the two), and (ii) stochastic volatility. Hence, it can be viewed as a "second generation" model, accommodating the empirical evidence which emerges from the data more precisely than previous log-linear models (see Singleton (1989) for a complete survey).

[6]Exceptions include Brennan and Schwartz (1985) who develop a multifactor MBS valuation model. Interest rate movements are assumed to be governed by the Brennan and Schwartz two-factor model, and prepayment is determined as in Dunn and McConnell (1981). Also, Waldman (1992) defines a notion of "partial duration" - the sensitivity of an asset price to shifts in specific points in the yield curve, keeping the rest constant. This is similar in spirit to our goals here; however, Waldman calculates the partial durations using some (unspecified) parametric prepayment model, and numerically takes derivatives with respect to variables of interest.

MBSs. While this method cannot be easily applied to determining optimal prepayment behavior, it can use empirical prepayment models to determine the timing of the cash flows. These cash flows can then be discounted at the appropriate risk-adjusted rate. This enables the researcher to incorporate the mortgage's *seasoning*[7] and a mortgage pool's *burnout*[8] (see Richard and Roll (1989) for a discussion of various other factors affecting prepayment rates). Of course, the impact of these factors will be somewhat related to interest rates; however, the Monte Carlo approach gives the researcher more discretion in modeling prepayment behavior than the theoretical approach.

2.2   A Nonparametric Approach to Valuing MBSs

Both the rational and empirical approaches to prepayment modeling and MBS valuation depend crucially on the correct parameterization of prepayment behavior and on the correct model for interest rates. Despite the widespread implementation of these models, they have had limited success in pricing MBSs. In this project, we take a different approach by estimating *nonparametrically* the relation between MBS prices and fundamental factors related to the term structure. Specifically, we estimate the joint density of MBS prices and term-structure factors. Given the estimated density, we can then calculate directly the pricing functional for MBSs.

We do not necessarily advocate density estimation as a substitute for theoretical modeling of MBS prices, but more as a complementary method of analysis. Specifically, the advantage of using a multivariate density estimation (MDE) approach for pricing fixed-income securities is that it is model-free. Apart from weak distributional assumptions (such as stationarity of the inter-relations between the interest rate variables), no assumptions about functional forms are needed. Thus, model specification plays a much smaller role with the MDE approach than with the approaches discussed above. Of course, an empirical approach like MDE introduces estimation error. While this is different from model misspecification, it can have similar effects on pricing. This aspect of the MDE methodology is studied in the next

---

[7]That is, prepayment rates on mortgages will initially tend to increase with the age of the mortgage, since there are frictions to household changes. For example, brand new mortgages are unlikely to have been taken out if the holders thought they were to get divorced, relocate or default.

[8]That is, for aged (and substantially prepaid) pools in a positive coupon spread environment, there is a tendency for low future prepayments. The intuition is that if a mortgage holder were going to prepay, then he/she would have already done so. This *burnout* effect could reflect nonoptimal behavior on the part of some mortgage holders, or frictions they face in trying to refinance their property (e.g., the value of the house may have fallen by such an amount that refinancing is no longer possible, yet there are sufficient costs to defaulting).

section.

The MDE approach can also incorporate multiple factors in a way which is internally consistent and straightforward to implement. For example, although it is necessary to choose the number of interest rate factors, the factors themselves can be unobservable. All that is required is that the interest rate variables used in the estimation are invertible to the true factors.[9]

If two factors can explain (i) the term structure (straight-bond component of MBSs), (ii) the mortgage rate (refinancing incentive), and (iii) prepayment characteristics (economic conditions), then, in the absence of estimation error, two interest rate variables should explain most of the important features of MBS pricing. In practice (i.e., in small samples), these variables should be chosen to maximize the information content of the two factors.

There are good reasons to choose the 10-year yield and the spread between this yield and the 3-month T-bill rate for capturing the salient features of MBSs. The MBSs analyzed in this paper have 30 years to maturity; however, due to potential prepayments and scheduled principal payments, the old rule-of-thumb is to treat them as if they have maturities of 12 years (see, for example, *The Handbook of Mortgage-Backed Securities*, 1993). Thus, the 10-year yield should approximate the level of interest rates at which to discount the MBS's cash flows. Further, the 10-year yield has a correlation of .98 with the mortgage rate. Since the spread between the mortgage rate and the MBS's coupon determines the refinancing incentive, the 10-year yield should prove useful when valuing the option component.

The second variable, the slope of the term structure (in this case, the spread between the 10-year and 3-month rates) provides the market's expectations about the future path of interest rates, which helps determine variation in the discount rate over short and long horizons. Thus, steep term structure slopes (relative to the 10-year) will discount short-term and long-term bonds at lower and higher rates, respectively. Further, steep term structures may imply increases in future mortgage rates, which impacts the market for mortgage refinancing.

Other factors, of course, may play an additional role in the valuation of MBSs. For example, if the level of interest rates is not the sole determinant of interest rate volatility (in contrast to the CIR model), then volatility should directly impact the value of the

---

[9]In a related empirical investigation, Harvey (1991) also uses a density estimation procedure to estimate MBS prices. Harvey (1991), however, only considers one-factor models and focuses in particular on the ability of T-bond futures to hedge various GNMA prices during the 1986 period of volatile interest rates. Breeden (1992) provides a description of the MBS market during the 1980's, documenting *negative convexity* of MBS prices and the effects of prepayments and aggregate economic conditions. Of particular interest, he also uses a different nonparametric approach (based on the prevailing MBS market) to hedge GNMAs.

prepayment option embedded in the MBS. Similarly, perhaps the slope and level of the term structure are not enough to explain interest-rate movements; for instance, Litterman and Scheinkman (1991) argue that the curvature of the term structure is also relevant (albeit less important). Finally, prepayments themselves may be related to factors other than those associated with interest-rate movements. Market factors or structural factors may cause prepayment rates to change, affecting the pricing of MBSs. The importance of other factors is an empirical question that is left for future research. We concentrate on the two interest rate factors discussed above, and, in the next section, we describe the MDE approach and evaluate its ability to estimate the pricing functional for MBSs in a two-factor economy.

3   The MDE Methodology

3.1   Nonparametric Density Estimation

We employ a kernel estimation procedure for estimating the relation between mortgage-backed prices and components of the term-structure of interest rates.[10] Kernel estimation is a nonparametric method for estimating the joint density of a set of random variables. Specifically, given m-dimensional vectors $\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_T$ (e.g., MBS prices, the interest rate level, and the slope of the term structure) from an unknown density $f(\mathbf{z})$, then a kernel estimator of this density is

$$\hat{f}(\mathbf{z}) = \frac{1}{Th^m} \sum_{t=1}^{T} K\left(\frac{\mathbf{z} - \mathbf{z}_t}{h}\right), \tag{1}$$

where $K(\cdot)$ is a suitable kernel function and h is the window width or smoothing parameter. This fixed window width estimator is often called the Parzen estimator. The density at any point is estimated as the average of densities centered at the actual data points. The further away a data point is from the estimation point, the less it contributes to the estimated density. Consequently, the estimated density is highest near high concentrations of data points and lowest when observations are sparse.

The econometrician has at his discretion the choice of $K(\cdot)$ and h. It is important to point out, however, that these choices are quite different than those faced by researchers employing parametric methods. Here, the researcher is not trying to choose functional forms or parameters that satisfy some goodness-of-fit criterion (such as minimizing squared errors

---

[10]For examples of MDE methods for approximating functional forms in the empirical asset pricing literature, see Pagan and Hong (1991), Harvey (1991) and Ait-Sahalia (1994).

in regression methods), but is instead characterizing the joint distribution from which the functional form will be determined.

One popular class of kernel functions is the symmetric beta density function, which includes the normal density, the Epanechnikov (1969) "optimal" kernel, and the commonly used biweight kernel as special cases. Results in the kernel estimation literature suggest that any reasonable kernel gives almost optimal results, though in small samples there may be differences (see Epanechnikov (1969)). In this paper, we employ the multivariate normal kernel, and leave for future research the issue of a particular kernel's optimality for our application.

The other parameter, the window width, is chosen based on the dispersion of the observations. For the multivariate normal kernel, Scott (1992) suggests the window width,

$$\hat{h}_i = \hat{\sigma}_i T^{\frac{-1}{m+4}},$$

where $\hat{\sigma}_i$ is the standard deviation estimate of each variable z_i, T is the number of observations, and m is the dimension of the variables. This window width has the appealing property that, for certain joint distributions of the variables, it minimizes the asymptotic mean integrated squared error of the estimated density function. Though the necessary distributional properties are not satisfied within our sample, Scott's rule provides an objective starting point for the MDE procedure.[11]

Consider the relation between three variables: MBS prices (P_{mb}), the level of long-term interest rates (i_l), and the slope of the term structure ($i_l - i_s$). Given the kernel estimate of the joint density of these variables, $\hat{f}(P_{mb}, i_l, i_l - i_s)$, the price at time t of an MBS at any interest rate level and term structure slope can be estimated by

$$P_{mb,t}(i_{l,t}, i_{l,t} - i_{s,t}) = E[P_{mb,t}|i_{l,t}, i_{l,t} - i_{s,t}] = \int P_{mb,t} \frac{\hat{f}(P_{mb,t}, i_{l,t}, i_{l,t} - i_{s,t})}{\hat{f}(i_{l,t}, i_{l,t} - i_{s,t})} dP_{mb,t}, \quad (2)$$

which is readily calculated from the kernel estimation of the joint density and the prices at

---

[11]There is some evidence that, in finite samples, the fixed-kernel estimators perform poorly in regions where the observations are sparse, producing spurious peaks at these points, and where they are dense, producing too little resolution around these points. A potential solution to this problem is the variable kernel estimator (VKE), proposed by Breiman, Meisel and Purcell (1977), as it allows the window width to vary across the data points. These window widths (though computationally intensive) lead to the smoothing (unsmoothing) of the density where the data are sparse (dense). The extension to VKEs for pricing MBSs will be considered in future research.

each data point in the sample. Specifically,

$$\hat{P}_{mb}(i_l, i_l - i_s) = \frac{\sum_{t=1}^{T} P_{mb,t} K\left(\frac{i_l - i_{l,t}}{h_{i_l}}\right) K\left(\frac{[i_l - i_s] - [i_{l,t} - i_{s,t}]}{h_{i_l - i_s}}\right)}{\sum_{t=1}^{T} K\left(\frac{i_l - i_{l,t}}{h_{i_l}}\right) K\left(\frac{[i_l - i_s] - [i_{l,t} - i_{s,t}]}{h_{i_l - i_s}}\right)}, \tag{3}$$

where $K(z) = 2\pi^{-\frac{1}{2}} e^{-\frac{1}{2}z^2}$.

Thus, for any given long rate i_l^* and a given short rate i_s^*, there is a mapping to the MBS price $P_{mb}(i_l^*, i_l^* - i_s^*)$. These prices can then be used to evaluate how MBS prices move with fundamental interest rate factors.

3.2 MBS Pricing in a Simulated Economy

In this section, we examine the extent to which multivariate density estimation can uncover the relation between MBS prices and interest rates. As a first pass, we judge the MDE's ability to capture the salient features of MBS pricing in a simulated economy. While this economy is simple in structure, it provides a useful benchmark by which to judge the effectiveness of the MDE procedure. In particular, we wish to answer the following question: under which circumstances (i.e., number of observations, range of data) does the MDE perform well, that is, is the estimation error sufficiently small?

In the appendix, we describe a simple multifactor MBS pricing model, based on the Schwartz and Torous (1989) model, which exhibits many of the commonly noted features of mortgage prepayment. The likelihood of prepayment increases as current interest rates fall relative to the coupon rate on the mortgage. In addition, the higher the fraction of a mortgage pool which has already prepaid, the lower the prepayment speed of the remaining loans in the pool (a simple form of burnout). Interest rates are described by a two-factor CIR model.

For this simulated model, Figure 1 graphs the price of a 13% MBS (with 30 years to maturity) against the 10-year yield. Note that a 13% coupon is chosen to capture the expected prepayment features of MBSs. The prices reflect both the negative convexity of MBSs and a second (albeit small) interest rate factor. At high interest rates, MBS prices behave much like a straight bond. They become concave in the interest rate level only at low interest rates when the refinancing incentive takes hold. The thickness of the pricing line, most evident at low interest rates, implies that there are multiple MBS prices for a given interest rate level, which can be explained by variation in the simulated model's second factor.

10

Since the pricing functional only has two sources of uncertainty, a two-factor MDE should explain MBS prices perfectly if there is no estimation error. Since the sample sizes are finite, however, estimation error is clearly present. In order to document the estimation error, we simulated 50 independent economies each with 500 observations on interest rates and MBS prices. From these simulated economies, we chose 10 economies with relevant interest rate ranges (i.e., $i_l \in (6\%, 12\%)$).[12]

For each economy, we estimated the MDE-implied pricing functionals for MBS prices using data on the first 50 and 150 observations, as well as using the full sample of 500 observations. In order to make the pricing functionals comparable across different sample sizes, the MBS prices were calculated using MDE for each sample size over the range of interest rates of the first 50 observations.[13] Pricing errors were then calculated by subtracting the MDE's MBS price from the model's true MBS price over this interest rate range for a cross-section of term structure spreads. Of particular interest, note that the MBS price is estimated from the MDE's pricing functional for sets of i_ls and $i_l - i_s$s, which may or may not have occurred in the sample. The only requirement is that these sets lie within the relevant interest rate and spread ranges.

Figure 2 documents the average pricing error of the MDE procedure across the simulated economies and across several interest rate spreads. Note that the x-axis measures the level of the interest rate relative to the maximum and minimum observed in the first 50 observations. The points "0" and "100" correspond to the minimum and maximum, respectively. For a large range of interest rates in the sample (i.e., the lowest 60%), the MDE procedure's estimate of the MBS price coincides very closely with the true model price. For example, at the mid range of interest rates, the pricing errors are approximately 10 cents. Since the 13% MBS is a premium bond over this range (with prices of $110 to $120), this represents less than a .1% pricing error. As we look to the higher interest rate levels within the sample, however, the pricing error increases.

Several observations on this pricing error increase are in order. First, the increase is worse for the smaller sample sizes. For example, the 50-observation sample has pricing errors of $1.50 (almost 1.5%), whereas the 500-observation sample has pricing errors of less than $1 (.9%). Second, for the 50-observation sample, the pricing error increases dramatically

---

[12]*Ex ante* we chose the economies with tight interest rate ranges for two reasons: (i) the simulated data coincides more closely with actual data we analyze in Section 5, and (ii) the interest rate levels are small enough to provide us with enough observations in the MBS's concave pricing region.

[13]Whereas the i_l's range over the simulations is between 6% and 12%, the range over the first 50 observations is tighter.

outside of the interest rate range (i.e., above 100%). This shows that MDE does not work well outside the data range; that is, the MDE interpolates the functional relation quite well, but does not extrapolate at the tails of the data.[14] Third, note that the 150- and 500-observation samples perform better than the 50-observation sample in the tails of the data. Recall that the interest rate range is chosen based on the 50-observation sample; thus, the other sample sizes may contain some observations in the 100%+ range of the data and the MDE will therefore not have to extrapolate MBS prices.

The overall conclusion from this particular simulated model is that the MDE works well, especially within the range of interest rates and spreads observed in the data. At the tails, however, the performance of the estimated pricing functional worsens. The ensuing errors seem to be monotonically related to the number of observations used in estimation. Nevertheless, even in the extremes of the data, the pricing errors are still on average less than 1%.

4 Data Description

4.1 Data Sources

Mortgage-backed security prices are obtained from Bloomberg Financial Markets and cover the period January 1987 to May 1994. Specifically, we collected weekly data on 30-year fixed-rate Government National Mortgage Association (GNMA) MBSs, with coupons ranging from $7\frac{1}{2}\%$ to $10\frac{1}{2}\%$. The prices represent dealer-quoted bid prices on $X\%$ coupon-bearing GNMAs traded for delivery on a *to be announced* (TBA) basis.

The TBA market is most commonly employed by mortgage originators who have a given set of mortgages that have not yet been pooled. However, trades can also involve existing pools, yet on an unspecified basis. Rules for the delivery and settlement of TBAs are set by the Public Securities Association (PSA) (see, for example, Bartlett (1989)). For example, an investor might purchase $1 million worth of 8% GNMAs for forward delivery next month. The dealer is then required to deliver 8% GNMA pools within 2.5% of the contracted amount (i.e., between $975,000 and $1,025,000), with specific pool information to be provided on a TBA basis (just prior to settlement). This means that, at the time of the agreed-upon-transaction, the characteristics of the mortgage pool to be delivered (e.g., the age of the

---

[14]This problem is less apparent at the left tails of the interest rate distribution. Given the parameter values described in the appendix, the two-factor CIR model bounds i_l at approximately 6%. Thus, the MDE's failure to extrapolate is not an issue.

pool and its prepayment history) are at the discretion of the dealer. Nevertheless, for the majority of the TBA's, the delivered pools represent newly issued pools.[15]

With respect to the interest rate series, weekly data for the 1987-1994 period were collected on the average rate for 30-year mortgages (collected from Bloomberg Financial Markets),[16] the prices of the 10-year Treasury note futures (representing the futures on the cheapest-to-deliver security), and the yield on the 3-month Treasury bill.

4.2 Characteristics of Interest and Mortgage Rates (1987-1994)

Before describing the pricing and hedging results for MBSs using the MDE approach, we briefly describe the environment for interest rates and mortgage rates during the sample period, 1987-1994.

Table 1 provides ranges, standard deviations and cross-correlations of GNMA prices (Table 1A), mortgage and interest rates (Table 1B), and rates of return (Table 1C) during this period. The average price of the 9% and 10% GNMAs is above par (100.08 and 104.35 respectively), while that of the 8% GNMA is below par (95.58). The average 10-year rate over the period is 7.78%, which, prepayments aside, may give us an idea about the source of such price levels. The yield curve is upward sloping on average, as is the case in most sample periods. The average spread is 2.12%, and the yield curve is rarely inverted (with the spread obtaining its minimum at -.19%, and its maximum at 3.84%). Notice that the correlation between the 10-year rate and the three month rate is .85, considerably higher in absolute terms than the correlation of the 10-year rate with the spread, which is -.45. This finding provides some additional motivation, from a purely empirical standpoint, for our choice of

---

[15]There are several reasons for choosing the TBA market and the post 1986 time period to investigate MBS pricing using the MDE methodology. First, during 1985 and 1986, interest rates dramatically declined, leading to mortgage originations for a wide variety of coupon rates. Thus, the GNMA TBAs in 1987-1994 correspond to mortgage pools with little prepayment history (i.e., no *burnout*) and long maturities. In contrast, prior to this period, the $7\frac{1}{2}$% to $10\frac{1}{2}$% GNMAs were backed by mortgages originated in the 1970's and thus represented a different security (in both maturity and prepayment levels). Second, MDE pricing requires joint stationarity between MBS prices and the interest rate variables. This poses a potential problem in estimating the statistical properites of any fixed maturity security, since the maturity is changing over the life of that security. Note that the TBA market refers to unspecified mortgage pools available in the marketplace. Thus, to the extent that there are originations of mortgages in the GNMA coupon range, the maturity of the GNMA TBA is less apt to change from week to week. Of course, if no originations occur in the coupon range, then the maturity of the available pool will decline. In this case, the researcher may need to add variables to capture the maturity effect and possibly any prepayment effects. In our analysis, we chose to limit the dimensionality of the multivariate system, and instead focus on the relation between MBS prices and the two interest rate factors.

[16]Bloomberg's source for this rate is "Freddie Mac's Primary Mortgage Market Survey", which reports the average rate on 80% of newly originated 30-year, first mortgages on a weekly basis.

pricing factors to be the 10-year rate and the spread (see Section 2.2).

Going back to the MBS data, we find that the 8% GNMA is more volatile than the 10% GNMA, both in terms of price levels and in terms of returns, and behaves more like the 10-year T-note futures. For example, with respect to return series (Table 1C), the standard deviation of returns on the 8% GNMA is .927 (in percent-per-week terms), 40% higher than that of the 10% GNMA, yet similar to that of the T-note futures (which is .977).[17] Also, the correlation between the return series of the 8% GNMA and the T-note futures is .92, higher than that between the 10% GNMA and the T-note futures, which is .85. It seems reasonable to come to the preliminary conclusion that the 8% GNMA resembles the 10-year note, more so than does the 10% GNMA. This can be explained as a result of the option component being in the money for the 10% GNMA, that is, the "flattening" of the pricing relation between interest rates and the 10% GNMA in the relevant range of interest rates over our sample period.

The important element of the option component for MBS valuation is the refinancing incentive. Since the mortgage rate represents the available rate at which homeowners can refinance, it plays an especially important role with respect to this incentive. For example, consider a 10% GNMA security. Note that this 10% GNMA is backed by 10.5% 30-year mortgages since there is a .5% servicing fee associated with GNMA pools. Figure 3 graphs the mortgage rate for 1987 through 1994. For most of the sample (especially 1990 on), the existing mortgage rate lies below 10.5% and the prepayment option is at- or in-the-money.[18] Historically, given the costs associated with refinancing, a spread of approximately 150 basis points between the old mortgage rate and the existing rate is required to induce rapid prepayments.[19] Under this assumption, a 10% GNMA first becomes in-the-money

---

[17]Return data calculations in Table 1C, as well as the return series used throughout the paper, are adjusted series, with adjustments being made once a month when the prices of different contracts are spliced together. The adjustment of the TBA GNMA price series is made during the splice week using a version of the "Cost of Carry" model, modified for prepayments, known as the "Dollar Roll Breakeven" method (see Askin and Meyer (1986)). We use the adjusted series for the hedging exercise (in Section 5.2), while the pricing regressions are performed straight off the unadjusted price series. We use unadjusted prices because the economics of comovements of rates and prices are most naturally captured and interpreted using the original price series of the GNMA TBAs. On the other hand, hedging results using an unadjusted return series would be difficult to interpret since every month the underlying asset switches to a new forward TBA contract.

[18]Figure 3 also graphs one of the interest rate factors, the 10-year yield. The correlation between the 10-year rate and the mortgage rate over our sample period is .980. However, there is a difference in the level between the two series (i.e, on average 1.56%), representing the cost of origination and bank profits, among other factors.

[19]See Bartlett (1993) and Breeden (1991) for some historical evidence of the relation between prepayment rates and the mortgage spread. Note that in the 1990's the 150 basis point spread has been somewhat lower — in some cases, 75 to 100 basis points. Some have argued that this is due to the proliferation of new types of mortgage loans (and ensuing marketing efforts by the mortgage companies) (Bartlett (1993)), though it

in September of 1991 and remains there throughout the rest of sample. In contrast, a 9% GNMA is in-the-money briefly in January 1992, July 1992 to December 1992, and January 1993 to April 1994.

5   Empirical Results

We describe the functional relation between GNMA prices and two interest-rate factors, the level of interest rates (the 10-year yield) and the slope of the term structure (the spread between the 10-year yield and the 3-month yield).

5.1   Pricing

We estimate the pricing functional given in equation (3) for each of the GNMA coupons. As an illustration, Figure 4 graphs the 9% GNMA against the 10-year yield and the spread between the 10-year and 3-month yield. The figure illustrates the well-known negative convexity of MBSs. Specifically, the MBS price is convex in interest-rate levels at high interest rates (when it behaves more like a straight 9% bond), yet concave at low interest rates (as the prepayment option becomes in-the-money).

This functional form does not hold in the northwest region of the figure, that is, at low spreads and low interest rates. However, recall from Section 3 that the MDE approach works well in the regions of the available data, but extrapolates poorly at the tails of the data and beyond. Figure 5 graphs a scatter plot of the 10-year yield against the spread between the 10-year yield and the 3-month bill. As evident from the figure, there are periods in which large slopes (3%-4%) are matched with both low interest rates (in 1993-1994) and high interest rates (in 1988). However, few observations are available at low spreads joint with low interest rates. Thus, the researcher needs to be cautious when interpreting MBS prices in this range.

Within the sample period, the largest range of 10-year yields occurs around a spread of 2.70%. Therefore, we take a slice of the pricing functional for the 8%, 9% and 10% GNMAs, conditional on this level of the spread. Figure 6 graphs the relation between GNMA prices for each of these coupons against the 10-year yield. Several observations are in order. First, the negative convexity of each MBS is very apparent. Second, the price differences between the various GNMA securities narrows as interest rates fall. This just represents the fact that

---

may also be related to aggregate economic factors.

15

higher coupon GNMAs are expected to prepay at faster rates. As GNMAs prepay at par, their prices fall because they are premium bonds, thus reducing the differential between the various coupons. Third, the GNMA prices change as a function of interest rates at different rates depending on the coupon level, i.e., on the magnitude of the refinancing incentive.

Figure 7 graphs the derivatives of the pricing functional, $\frac{\partial P_{mb}}{\partial i_l}$, for the 8%, 9% and 10% GNMAs.[20] The derivatives follow a U-shaped pattern for each GNMA, which is consistent with an MBS having both a straight bond and an option component. Specifically, the derivatives of the straight bond component should become more negative as i_l falls, while the derivatives of the option component should be increasing and positive. This combination is what causes the U-shaped form. Moreover, the overall magnitudes of the derivatives are smaller for the 10%s than the 8%s, since the prepayment option becomes in-the-money at higher interest rate levels for higher coupon MBSs. The fact that the derivatives are close for 8%s and 9%s at $i_l = 9.5\%$, while the derivatives are close for 9%s and 10%s at $i_l = 6.0\%$, is further evidence of the option component. That is, at $i_l = 9.5\%$, the 8% and 9% GNMAs are clearly both out-of-the-money in contrast to the 10% GNMA; while at $i_l = 6.0\%$, the 9% and 10% GNMAs are clearly both way in-the-money in contrast to the 8% GNMA.

Within the sample period, the widest array of spreads occur around 10-year yields of 8.9%. We therefore consider a slice of the pricing functional for the 8%, 9% and 10% GNMAs, conditional on this level of interest rates. Figure 8 then graphs the relation between MBS prices and the term structure slope, conditional on 8.9% yields. As is clear from Figure 8, the slope plays a smaller role than the level in pricing MBSs. For example, over spreads of 0.0% to 4.0%, there is less than a $2 change in the price of the MBS for any of the coupons. Nevertheless, for each GNMA security, high slopes tend to correspond with slightly lower prices. For the higher coupon GNMAs, low slopes also suggest lower prices. This is not the case, however, for the lower coupon GNMAs.

The above results suggest the presence of a second factor for pricing MBSs. To understand the impact of the term structure slope, Figure 9 graphs the various GNMA prices against interest rate levels, conditional on two different spreads (2.70% and .30%). Recall that the slope of the term structure is defined using the yield on a full-coupon note, not a ten-year zero-coupon rate. As a result, positive spreads imply upward sloping full-coupon yield curves and even more steeply sloping zero-coupon yield curves. In contrast, when the spread is close to zero, both the full-coupon and zero-coupon yield curves tend to be flat.[21]

---

[20]Price elasticities should be interpreted with some caution because it is difficult to obtain precise estimates using the MDE approach (see Scott(1992)).

[21]The spreads and interest rate ranges are chosen to coincide with the appropriate ranges of available

At high interest rate levels, the option to prepay is out-of-the-money. Consequently, many of the cash flows are expected to occur as scheduled, and GNMAs have long expected lives. The appropriate discount rates for these cash flows are the longer-term zero-coupon rates. When the spread is high, the term structure is sharply upward sloping and long-term zero-coupon rates are high. In contrast, holding the 10-year full-coupon yield constant, short-term zero-coupon rates are lower for high spreads than when the term structure spread is low. Consider first the effects on the price of an 8% GNMA. Since this security has its cash flows concentrated at long maturities, its price should be lower for higher spreads, just as we observe in Figure 9. On the other hand, the option component of the 10% GNMA is much closer to being at-the-money, even for the highest interest rates shown in the figure. Hence, at these interest rates, 10% GNMA prices do not follow the same ordering as 8% GNMAs vis-a-vis the level of the spread. However, for much higher rates, one can expect the same pattern. To see this, note that the 9% GNMA, falling between these two securities in terms of expected cash flow life, has relative prices for the two spreads that lie between the cases commented on above.

As interest rates fall, prepayments become more likely, and the expected life of the MBS falls for GNMAs of all coupons. As this life declines, the levels of the shorter-term zero-coupon rates become more important for pricing. In this case, high spreads imply lower discount rates at the relevant maturities, for a fixed 10-year full-coupon yield. Consequently, when the GNMAs are priced as shorter-term securities due to high expected prepayments, high spreads imply higher prices for all coupons. This implication is illustrated in Figure 9 by the fact that, while prices always increase for declining long rates, the increase is much larger when spreads are high. For the 8% GNMA, this effect causes the prices to cross at a long rate of approximately 8.4%, while for the 10% GNMA it causes the pricing functionals to diverge further as rates decrease.

The effect in Figure 9 is primarily driven by changes in expected cash flow life. The 10-year yield proxies for the moneyness of the option, the expected level of prepayments, and the average life of the cash flows. The addition of the second factor, the term structure slope, also controls for the average rate at which these cash flows should be discounted.

Figure 10 illustrates a similar, albeit slightly more complex, duration effect. This figure graphs GNMA prices at low interest rate levels, conditional on spreads of 2.70% and 3.50%. Again, these particular spreads are chosen to correspond to regions in which there is an adequate amount of data. Following the reasoning above, low interest rates imply that

---

data, to insure that the MDE approach works well.

GNMAs have short expected lives; hence higher spreads, which imply lower short-term rates, should generate higher prices. Exactly this phenomenon is apparent for rates of 7% and above. Below this level, however, the price functional for high spreads crosses the price functional for lower spreads for each of the GNMAs.

There are three possible explanations for this effect. First, the effect may be spurious and the crossover is due to estimation error. An examination of the price data suggests estimation error is not the problem. Average prices for this interest rate range are, in fact, higher for lower spreads. Second, there may be a missing factor for which we have not controlled. For example, the spread and the long rate may not be sufficient to describe the full term structure. Variations in intermediate rates not capture by the two factors could explain the price variations. Such an explanation seems unlikely, although not impossible. Third, the duration effect may be more complex than described previously. In particular, the long rate may not be sufficient to proxy for expected prepayments and the expected life of the cash flows. Consider that steeper term structures imply expectations of greater increases in future rates. Under these circumstances, steeper term structures may imply higher prepayments for a given long rate. This intuition is consistent with the prices in Figure 10. For very low long rates, steeper slopes imply higher prepayments, shorter expected life and lower prices for premium GNMAs.

5.2   Hedging MBSs

If, as above, we assume that there are two pricing factors for fixed-income securities, then it can be shown that a position in any three assets (in our case, one being the MBS) is fully hedged if

$$
\omega_1 \frac{\partial P_1}{\partial i_l} + \omega_2 \frac{\partial P_2}{\partial i_l} = -\frac{\partial P_{mb}}{\partial i_l}
$$

$$
\omega_1 \frac{\partial P_1}{\partial (i_l - i_s)} + \omega_2 \frac{\partial P_2}{\partial (i_l - i_s)} = -\frac{\partial P_{mb}}{\partial (i_l - i_s)},
$$

where ω_i is the position in the two assets (#1 and #2), and P_i represents the price of asset i (e.g., a 3-month T-bill, or a 10-year Treasury note).

Solving for ω_1 and ω_2 gives

$$
\omega_1 \quad = \quad \frac{-\frac{\partial P_{mb}}{\partial (i_l - i_s)} \frac{\partial P_2}{\partial i_l} + \frac{\partial P_{mb}}{\partial i_l} \frac{\partial P_2}{\partial (i_l - i_s)}}{\frac{\partial P_1}{\partial i_l} \frac{\partial P_2}{\partial (i_l - i_s)} - \frac{\partial P_1}{\partial (i_l - i_s)} \frac{\partial P_2}{\partial i_l}}, \tag{4}
$$

$$\omega_2 = \frac{-\frac{\partial P_{mb}}{\partial i_l}\frac{\partial P_1}{\partial(i_l-i_s)} + \frac{\partial P_{mb}}{\partial(i_l-i_s)}\frac{\partial P_1}{\partial i_l}}{\frac{\partial P_1}{\partial i_l}\frac{\partial P_2}{\partial(i_l-i_s)} - \frac{\partial P_1}{\partial(i_l-i_s)}\frac{\partial P_2}{\partial i_l}}. \tag{5}$$

Over small movements in i_l and i_s, this result holds essentially for any continuously differentiable process (i.e., for all diffusion processes). Thus, a portfolio will be hedged if the investor holds one MBS (at a cost of P_{mb}) against ω_1 of fixed-income asset #1 and ω_2 of fixed-income asset #2. Using equations (4) and (5), these hedged portfolios then can be constructed *ex ante* based on the econometrician's estimate of the partial derivatives of the three fixed-income assets with respect to the two factors. These estimates can be generated from historical data (prior to the forming of the hedge) using kernel estimation. For example, an estimate of $\frac{\partial P_{mb}}{\partial i_l}$ can be calculated from equation (3) using

$$\frac{\partial P_{mb}}{\partial i_l} = \frac{\sum_{t=1}^{T} P_{mb,t} K'\left(\frac{i_l - i_{l,t}}{h_{i_l}}\right) K\left(\frac{[i_l - i_s] - [i_{l,t} - i_{s,t}]}{h_{i_l - i_s}}\right)}{\sum_{t=1}^{T} K\left(\frac{i_l - i_{l,t}}{h_{i_l}}\right) K\left(\frac{[i_l - i_s] - [i_{l,t} - i_{s,t}]}{h_{i_l - i_s}}\right)} -$$

$$\frac{\sum_{t=1}^{T} P_{mb,t} K\left(\frac{i_l - i_{l,t}}{h_{i_l}}\right) K\left(\frac{[i_l - i_s] - [i_{l,t} - i_{s,t}]}{h_{i_l - i_s}}\right) \sum_{t=1}^{T} K'\left(\frac{i_l - i_{l,t}}{h_{i_l}}\right) K\left(\frac{[i_l - i_s] - [i_{l,t} - i_{s,t}]}{h_{i_l - i_s}}\right)}{\left[\sum_{t=1}^{T} K\left(\frac{i_l - i_{l,t}}{h_{i_l}}\right) K\left(\frac{[i_l - i_s] - [i_{l,t} - i_{s,t}]}{h_{i_l - i_s}}\right)\right]^2},$$

where $K'(z) = -2\pi^{-\frac{1}{2}} z e^{-\frac{1}{2}z^2}$. Unfortunately, it is well known that it is extremely difficult to estimate the derivative accurately (see Scott (1992)). Therefore, we average the estimated derivative with price sensitivities estimated over a range of long rates or slopes. For example, we calculate the elasticity

$$\frac{\Delta P_{mb}}{\Delta i_l} = \frac{P_{mb}(i_l^a) - P_{mb}(i_l^b)}{i_l^a - i_l^b}$$

for two different pairs of interest rates, (i_l^a, i_l^b), and average these values with the kernel derivative. The points are chosen to straddle the interest rate of interest. Specifically, we use the 10th and 20th nearest neighbors along the interest rate dimension within the sample, if they exist, and the highest or lowest interest rates within the sample if there are not 10 or 20 observations with higher or lower interest rates. The return on the hedged portfolio is then given by

$$\frac{P_{mb,t+1} + \hat{\omega}_1(P_{1,t+1} - P_{1,t}) + \hat{\omega}_2(P_{2,t+1} - P_{2,t})}{P_{mb,t}},$$

where it is assumed that the investor starts with one unit of GNMAs at time t. The hedged portfolio can then be followed through time and evaluated based on its volatility and correlation with the fixed-income factors, as well as other factors of interest.

5.2.1 Hedging Analysis

We performed an out-of-sample hedging exercise over the period January 1990 to May 1994. Starting in January 1987, approximately three years of data (150 weekly observations) are used on a weekly rolling basis to estimate the joint density of the fixed-income instruments (MBS, 3-month T-bill, and futures on the 10-year T-note) and the two interest rate factors (level (10-year yield), and slope (10-year yield minus 3-month yield)). For each rolling period, several different hedges were formed for comparison purposes:

1. To coincide with existing practice, a linear hedge of the GNMAs against the T-note futures was estimated using rolling regressions.

2. Breeden (1991) suggests a roll-up/roll-down approach to computing hedge ratios. Specifically, the hedge can be formed for an $X\%$ GNMA by computing the ratio between the T-note futures price elasticity and the GNMA price elasticity. (The GNMA price elasticity is calculated from the difference between GNMA prices of $X + \frac{1}{2}\%$ and $X - \frac{1}{2}\%$ coupons for a 1% interest rate change. We investigate hedging of 8%, 9% and 10% GNMAs using GNMAs with 7.5% through 10.5% coupons).

3. We investigate the two-factor MDE hedge described by the portfolio weights given in equations (4) and (5).

4. To the extent that the second factor (the slope) seems to play a small role in pricing, we employ a one-factor MDE hedge using the T-note futures and GNMA as a function of only the 10-year yield.[22]

Table 2 compares the performance of the four hedges for the 8% (Table 2A), 9% (Table 2B) and 10% GNMAs (Table 2C) over the 1990 to 1994 sample period. Consider first the 10% GNMA. The unhedged GNMA return has a volatility of .414% (41.4 basis points) on a weekly basis. The two-factor MDE hedge reduces the volatility of the portfolio to 26.1 basis points weekly. In contrast, the one-factor MDE hedge, the roll-up/roll-down hedge and linear hedge manage only 30.0, 29.4 and 34.9 basis points, respectively. As described in Section 5.1, the 10% GNMA is the most in-the-money in terms of the refinancing incentive. It is comforting to find that, in the GNMA's most nonlinear region, the MDE approach works well.

---

[22]The second factor should play an even lesser role in hedging weekly changes to the extent that the range of slopes over any given period is much smaller than that over the entire sample.

Figure 11 illustrates how the volatility of the hedged and unhedged returns move through time. While the volatility of the unhedged returns declines over time, this pattern is not matched by the hedged returns. To quantify this evidence Table 2C breaks up the sample into four subperiods: January 1990 – February 1991, March 1991 – April 1992, May 1992 – June 1993, and July 1993 – May 1994. The most telling fact is that the MDE approach does very well in the last subperiod relative to the other hedges (19.2 versus 39.4 basis points for the roll-up/roll-down approach). This is a period in which massive prepayments occurred in the first part of the period. Due to these prepayments, 10% GNMAs are much less volatile than in previous periods. Thus, the linear and roll-up/roll-down approaches tended to overhedge MBSs, resulting in large exposures to interest rate risks. This might explain some of the losses suffered by Wall Street during this period.

On the other hand, the MDE approach does not fare as well in the first two subperiods. For example, the one- and two-factor hedges have 38.8 and 29.6 basis points of volatility respectively versus the unhedged GNMA's volatility of 48.1 basis points in the second subperiod. In contrast, the roll-up/roll-down hedge has only 26.4 basis points of volatility. We can explain the poorer performance of the MDE approach based on our simulation results. Recall that the MDE procedure does not extrapolate well beyond the tails of the data. During the first and second subperiod, the rolling estimation period faces almost uniformly higher interest rate levels than the out-of-sample forecast. Thus, hedge ratios were calculated for sparse regions of the data.

Recall that the MDE two-factor hedge reduces the volatility to 65% of the unhedged GNMA's volatility. Since the hedging was performed on an out-of-sample basis, there is no guarantee that the remaining variation of the GNMA's return is free of interest-rate exposure. Table 2C provides results from a linear regression of the GNMA unhedged and hedged portfolio's return on changes in the interest rate level (i.e., $\Delta i_{l,t}$) and movements in the terms structure slope (i.e., $\Delta(i_{l,t} - i_{s,t})$). It gives the volatility of each portfolio due to interest rate and term structure slope movements. For example, the volatility of the explained portion of the 10% GNMA due to the interest rate level and slope is 28.6 basis points a week; in contrast, the MDE two-factor hedged 10% GNMA's interest rate risk exposure is only 5.4 basis points. Note that the roll-up/roll-down and linear hedges face much more exposure — 11.3 and 16.4 basis points, respectively. To make matters worse for these hedges, note that this measure of exposure can be misleading. The regression method is only strictly valid in a linear setting; and, thus, the volatilities are only a lower bound on the true volatility due to interest rate exposure. To the extent that the MDE approach explicitly accounts for

21

nonlinearities, the other approaches may actually face even more exposure to interest rate risk than implied by Table 2C.[23]

So far, we have described the results for hedging the 10% GNMA. Tables 2A and 2B provides results for the 8% and 9% GNMAs. Essentially, the patterns are very similar to the 10%, except that the MDE approach fares less well relative to the roll-up/roll-down approach. To understand why this is the case, note that the 8% and 9% GNMAs have a lower refinancing incentive. The bonds therefore behave more like a straight bond, and are more volatile (see Table 1). Thus, because the negative convexity of the GNMAs is less prevalent for the 8% and 9% coupons, one explanation for why the MDE approach to hedging GNMAs fares relatively less well with lower coupons is that estimation error is more important. In fact, the roll-up/roll-down method actually produces a lower volatility of the hedged GNMA portfolio than the MDE two-factor approach for both the 8% and 9% GNMAs (27.6 versus 29.4 basis points for the 8%s and 24.6 versus 25.6 basis points for the 9%s).

Multiple factors become less important from a hedging perspective as the GNMA coupon falls (e.g., compare the 8% to 10%). This is to be expected, since we argued that the term structure slope plays a role in pricing as the moneyness of the prepayment option changes through time. The subperiod analysis confirms the intuition based on our findings for the 10% GNMAs. While the relative hedging performance of the various approaches is still related to the subperiods, it is less prevalent for the lower coupon GNMAs. The MDE approach fares relatively best in periods with substantial nonlinearities, e.g., the 10% GNMAs during July 1993 to May 1994. The large prepayments which induced 10% GNMA prices to fall (ceteris paribus) did not occur for the 8% GNMAs. After all, the 8% GNMAs are backed by 8.5% mortgages, and the lowest 30-year fixed-rate mortgage only briefly dropped below 7%.

Of particular interest, both the MDE approach and the roll-up/roll-down hedges substantially reduce the interest rate exposure of their 8% and 9% GNMA hedge portfolios. For example, for the 8% (9%) GNMA, the unhedged GNMA has 59.0 (41.1) basis point of volatility due to the interest rate factors, while the MDE and roll-up/roll-down approaches have only 4.3 (3.9) and 6.8 (1.2) basis points respectively.

---

[23]For completeness we also report the volatility of the returns due only to movements in the long rate. These results are very similar to those discussed above, suggesting that most of the volatility on a weekly basis is attributable to variation in the long rate.

6 Conclusion

In this paper we develop a model-free, nonparametric methodology for valuing mortgage-backed securities. Instead of postulating and estimating parametric models for both interest rate movements and prepayments, as in previous approaches to mortgage-backed security valuation, we instead estimate directly the functional relation between mortgage-backed security prices and the level of economic fundamentals. We do this using multivariate density estimation (MDE) to estimate the joint distribution of interest rate levels, the slope of the term structure, and MBS prices. This approach yields consistent prices without the need to make the strong assumptions about the processes governing interest rates and prepayment required by previous approaches.

Using simulated data, we confirm that the MDE procedure works well except when trying to extrapolate beyond the range of the data. Using weekly prices for GNMA MBSs between 1987 and 1994, we find that these prices can be well described as a function of the level of interest rates and the slope of the term structure. A single interest rate factor, as used in most previous mortgage valuation models, is insufficient. The relation between prices and interest rates displays the usual stylized facts, such as negative convexity in certain regions, and a narrowing of price differentials as interest rates fall. Most interesting, the term structure slope plays an important role in valuing MBSs via its relation with the interest rate level and the refinancing incentive associated with a particular MBS.

Using the estimated relation between fixed-income security prices and interest rates to construct hedged portfolios, we find that our methodology compares favorably with other commonly used hedging methods for MBS. While the two-factor MDE hedge consistently reduces the volatility of the GNMA portfolio, it performs especially well relative to the other methods in periods for which the option component is important (such as 1993-1994).

In general, the MDE procedure will work well (in a relative sense) under the following conditions. First, since density estimation is data intensive, the researcher either needs a large data sample or an estimation problem in which there is little disturbance error in the relation between the variables. Second, the problem should be described by a relative low dimensional system, since MDE's properties deteriorate quickly when variables are added to the estimation. Third, and especially relevant for comparison across methods, MDE will work relatively well for highly nonlinear frameworks. As it happens, these features also describe derivative pricing. Hence, while the results we obtain here for GNMAs are encouraging, it is likely that the MDE approach we develop would fare even better for other,

more complex derivative securities. An example in the mortgage-backed area is the pricing of interest only (IO) and principal only (PO) strips, and collateralized mortgage obligations (CMOs), since the relation between the prices of these securities and interest rates is more highly nonlinear than that of a GNMA. The advantage of the MDE approach is its ability to capture arbitrary nonlinear relations between variables, making it ideally suited to capturing the extreme convexity exhibited by many derivative mortgage-backed securities.

Appendix: Theoretical MBS Pricing Model

Interest Rates

Assume interest rates are described by the two-factor interest rate model estimated and tested by Pearson and Sun (1989). The two factors are the instantaneous riskless real interest rate, r, and the expected inflation rate, y. The real interest rate is given by $r_t = r_t' + \bar{r}$, where $\bar{r}$ is a constant, and

$$dr_t' = \kappa_1(\theta_1 - r_t')\,dt + \sigma_1\sqrt{r_t'}\,dZ_t^1. \tag{6}$$

The expected inflation rate moves according to the equation

$$dy_t = \kappa_2(\theta_2 - y_t)\,dt + \sigma_2\sqrt{y_t}\,dZ_t^2, \tag{7}$$

where the two Brownian motions dZ_t^1 and dZ_t^2 are uncorrelated. The price level, p, moves according to the equation

$$dp_t = y_t p_t\,dt + \sigma_p p_t\sqrt{y_t}\,dZ_t^3,$$

where $E(dZ_t^2 dZ_t^3) = \rho\,dt$. When $\bar{r} = 0$, this reduces to the standard 2 factor CIR model. The equilibrium risk premium for real bonds is $\lambda r'$. Pearson and Sun estimated the parameter values $\bar{r} = -10$, $\sigma_p = 0$, $\rho = 0$, $\kappa_1 = 7.4525$, $\sigma_1 = 0.0197$, $\theta_1 + \bar{r} = 0.0264$, $\lambda = -0.0048$, $\kappa_2 = 0.0797$, $\sigma_2 = 0.1170$, $\theta_2 = 0.093$.

Prepayment and Calculation of Cash Flows

To value mortgage-backed securities, we need a model which specifies the cash flows each period as a function of the history of interest rates.[24] We use a model based on that of Schwartz and Torous (1989). Prepayment is governed by a hazard function π_t,[25] defined by

$$\pi_t = \frac{\gamma p(\gamma t)^{p-1}}{1 + (\gamma t)^p}\exp(\beta v).$$

Here, γ and p are constants, and v is a vector of explanatory variables, defined by

$$v_1(t) = c - l_t,$$
$$v_2(t) = \ln(\text{proportion of pool not yet prepaid}),$$

---

[24]Or a model of the expected cash flows each period, as long as the risk of deviation from this expected value is not priced.

[25]In other words, as Δt approaches zero, the probability of prepayment occurring in a time interval of length Δt approaches $\pi_t \Delta t$.

where c is the coupon rate on the mortgage, and l_t is the yield on a long term government bond. We assume a one year bond, and use parameters based on those estimated by Stanton (1992), $\gamma = 0.00864$, $p = 1.71724$, $\beta_1 = 0.49108$, $\beta_2 = 0.14890$.

Given π_t, the expected cash flow in month t, per dollar of initial principal, is given by

$$C_t = \text{SF}_{t-1}\left(X + (1 - e^{\pi_t/12})\text{BAL}_{t-1}\right),$$

where X is the scheduled monthly payment, given by

$$X = \frac{c/12}{1 - (1 + c/12)^{-360}},$$

BAL_t is the scheduled balance remaining on the loan at the end of month t,

$$\text{BAL}_t = \frac{X}{c/12}\left[1 - (1 + c/12)^{-(360-t)}\right],$$

and SF_t is the probability that the mortgage has not prepaid prior to t, given by $\text{SF}_0 = 1$, and

$$\text{SF}_t = (1 - e^{\pi_t/12})\,\text{SF}_{t-1}.$$

Valuation

Assets whose value depends only on current values of r and y can be valued by writing down and solving a partial differential equation with appropriate boundary conditions (see Cox, Ingersoll and Ross (1985a,b)). This approach cannot easily handle path dependence of the sort we have described, where an asset's cash flows depend on the entire history of interest rates, rather than just the current values. An alternative approach is based on the fact that, given the interest rate model described above, we can write V, the value of an asset which pays out nominal cash flows at a (possibly path dependent) rate C_t, in the form

$$V_t = E\left[\int_t^T e^{-\int_t^s (\widehat{r}_u + y_u)\,du} C_s\,ds\right], \tag{8}$$

where $\widehat{r}_\tau = \widehat{r'}_\tau + \overline{r}$, and $\widehat{r'}$ follows the "risk adjusted" process,

$$\begin{aligned}
d\widehat{r'}_\tau &= \left[\kappa_1(\theta_1 - \widehat{r'}_\tau) - \lambda\widehat{r'}_\tau\right]dt + \sigma_1\sqrt{\widehat{r'}_\tau}\,dZ_\tau^1 \qquad \text{for all } \tau \geq t, \tag{9} \\
\widehat{r'}_t &= r'_t.
\end{aligned}$$

This says that the value of the asset equals the expected sum of discounted cash flows paid over the life of the asset, except that it substitutes the risk adjusted process $\widehat{r'}$ for the true process r'_t for r.

26

This representation leads directly to a valuation algorithm based on Monte Carlo simulation. For each (r_t, y_t) pair (simulated using the model described in equations (6) and (7)), 500 paths for $\hat{r}$ and y were simulated using equations (9) and (7). Along each path, the cash flows C_t were calculated as above, then discounted back along the path followed by the instantaneous nominal riskless rate $\hat{r} + y$. The average of the sum of these values taken over all simulated paths is an approximation to the value V. The more paths simulated, the closer this approximation.

TABLE 1: SUMMARY STATISTICS

Table 1A – Price Levels

	8% GNMA	9% GNMA	10% GNMA	T-Note Futures	3m T-Bill
Mean	95.578	100.084	104.347	100.713	98.607
Min	81.625	86.531	92.688	86.000	97.759
Max	106.563	108.281	110.937	116.906	99.334
Std.Dev.	6.287	5.260	4.294	6.905	0.461
Correlation Matrix					
8% GNMA	1.000	0.992	0.987	0.982	0.860
9% GNMA	0.992	1.000	0.995	0.954	0.868
10% GNMA	0.987	0.995	1.000	0.951	0.887
T-Note Futures	0.982	0.954	0.951	1.000	0.839
3m T-Bill	0.860	0.868	0.887	0.839	1.000

Table 1B – Rates

	i_{3m}	i_{10y}	$i_{10y} - i_{3m}$	Mortgage Rate
Mean	5.659	7.779	2.119	9.337
Min	2.680	5.170	-0.190	6.740
Max	9.170	10.230	3.840	11.580
Std.Dev.	1.894	1.123	1.101	1.206
Correlation Matrix				
i_{3m}	1.000	0.855	-0.848	0.882
i_{10y}	0.855	1.000	-0.450	0.980
$i_{10y} - i_{3m}$	-0.848	-0.450	1.000	-0.518
Mortgage Rate	0.882	0.980	-0.518	1.000

Table 1C – Adjusted Returns

	8% GNMA	9% GNMA	10% GNMA	T-Note Futures	3m T-Bill
Mean	0.065	0.066	0.059	0.054	0.106
Max	7.810	8.306	5.866	7.849	0.495
Min	-3.409	-3.333	-2.914	-3.312	-0.006
95 percentile	1.362	1.136	0.952	1.414	0.178
5 percentile	-1.447	-1.245	-1.052	-1.574	0.042
avg($|x|$)	0.653	0.536	0.438	0.708	0.106
Std.Dev.	0.927	0.830	0.664	0.977	0.049
Correlation Matrix					
8% GNMA	1.000	0.969	0.923	0.917	0.377
9% GNMA	0.969	1.000	0.962	0.884	0.398
10% GNMA	0.923	0.962	1.000	0.845	0.368
T-Note Futures	0.917	0.884	0.845	1.000	0.377
3m T-Bill	0.377	0.398	0.368	0.377	1.000

Prices and returns of TBA contracts on 8%, 9% and 10% GNMAs, 10-year T-note futures, and 3-month T-bills (returns are adjusted for splicing); and long rates (10-year, i_{10y}), short rates (3-month, i_{3m}), their difference ($i_{10y} - i_{3m}$), and the average mortgage rate. All data are weekly from January 1987 through May 1994. Returns are in percent per week, and interest rates are in percent per year.

TABLE 2: HEDGING RESULTS

Table 2A – 8% GNMA

Period	GNMA	Linear	Roll-Up Roll-Down	MDE 1-fctr	MDE 2-fctr
1/90-5/94	68.3	35.0	27.6	30.0	29.4
1/90-2/91	85.5	26.9	27.6	27.8	30.1
3/91-4/92	72.2	30.5	31.7	34.8	32.1
5/92-6/93	61.3	37.7	25.9	29.3	27.8
7/93-5/94	45.5	43.2	24.8	26.9	27.2
$\sigma_{\Delta i_l, \Delta(i_l-i_s)}$	59.0	15.0	6.8	6.1	4.3
$\sigma_{\Delta i_l}$	59.0	15.0	6.8	6.1	4.2

Table 2B – 9% GNMA

Period	GNMA	Linear	Breeden	MDE 1-fctr	MDE 2-fctr
1/90-5/94	53.0	36.8	24.6	29.3	25.6
1/90-2/91	73.9	24.3	23.5	26.0	27.2
3/91-4/92	55.2	32.3	25.8	38.1	28.2
5/92-6/93	43.8	46.4	25.3	29.3	25.3
7/93-5/94	23.8	39.6	23.7	19.7	20.8
$\sigma_{\Delta i_l, \Delta(i_l-i_s)}$	41.1	18.7	1.2	5.5	3.9
$\sigma_{\Delta i_l}$	41.1	18.6	0.7	5.3	0.1

Table 2C – 10% GNMA

Period	GNMA	Linear	Breeden	MDE 1-fctr	MDE 2-fctr
1/90-5/94	41.4	34.9	29.4	30.0	26.1
1/90-2/91	58.2	24.0	22.3	27.6	27.8
3/91-4/92	48.1	33.8	26.4	38.8	29.6
5/92-6/93	34.8	44.6	29.2	29.5	27.8
7/93-5/94	20.3	32.2	39.4	18.8	19.2
$\sigma_{\Delta i_l, \Delta(i_l-i_s)}$	28.6	16.4	11.3	5.9	5.4
$\sigma_{\Delta i_l}$	28.6	16.4	11.3	5.7	1.4

Results of hedging the 8%, 9% and 10% GNMAs with various methods. Each method's hedge ratios are calculated using the past 150 weeks, for the next week. Hence the hedging period is January 1990 through May 1994. The methods are (i) GNMA – the total volatility of an open position (no hedging), in basis points, (ii) linear – hedging via linear regression on T-note futures returns, (iii) roll-up/roll-down – a method which infers hedge ratios from contemporaneous market prices of near coupon MBSs, (iv) MDE – hedge ratios determined via a one factor (long rate only) and two factor (long rate and spread) models, trading in T-note futures and T-bills in the corresponding hedge ratios. The last two rows provide a measure of the quantity of interest rate risk (two factor risk or one factor risk), which remains using each method's hedging results. In all cases the numbers in the tables represent the standard deviation of weekly returns in basis points.

Figure 1: Simulated MBS Data
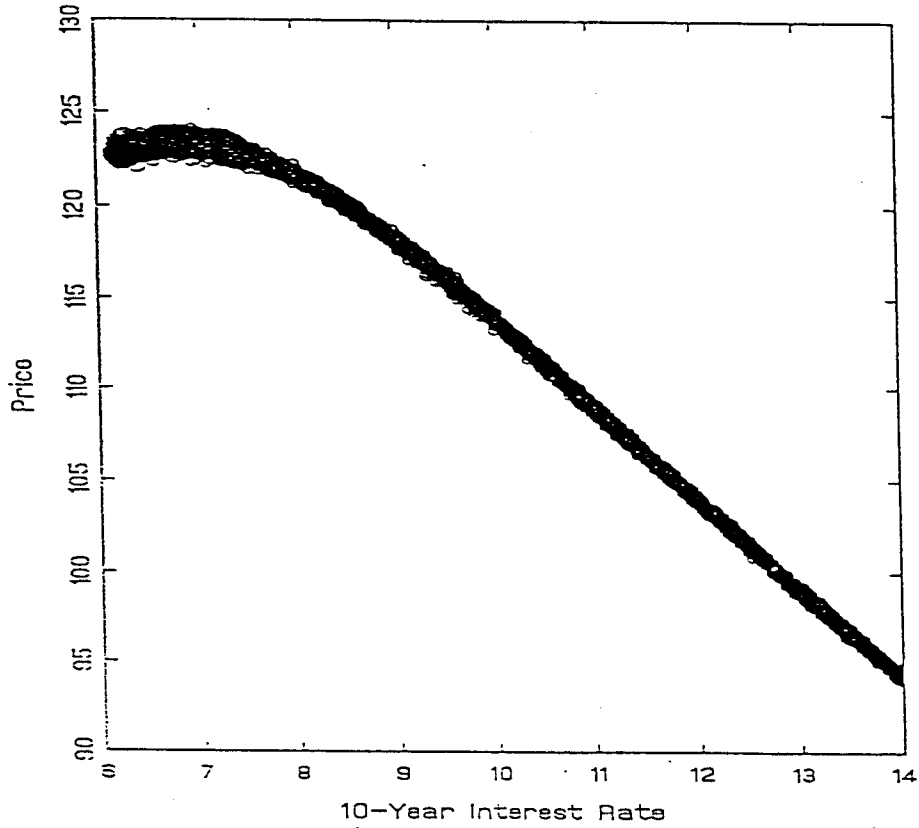
Scatter plot of simulated 13% GNMA prices in a two factor economy. The model used is discussed in detail in the Appendix.

30

Figure 2: Average MDE Pricing Errors
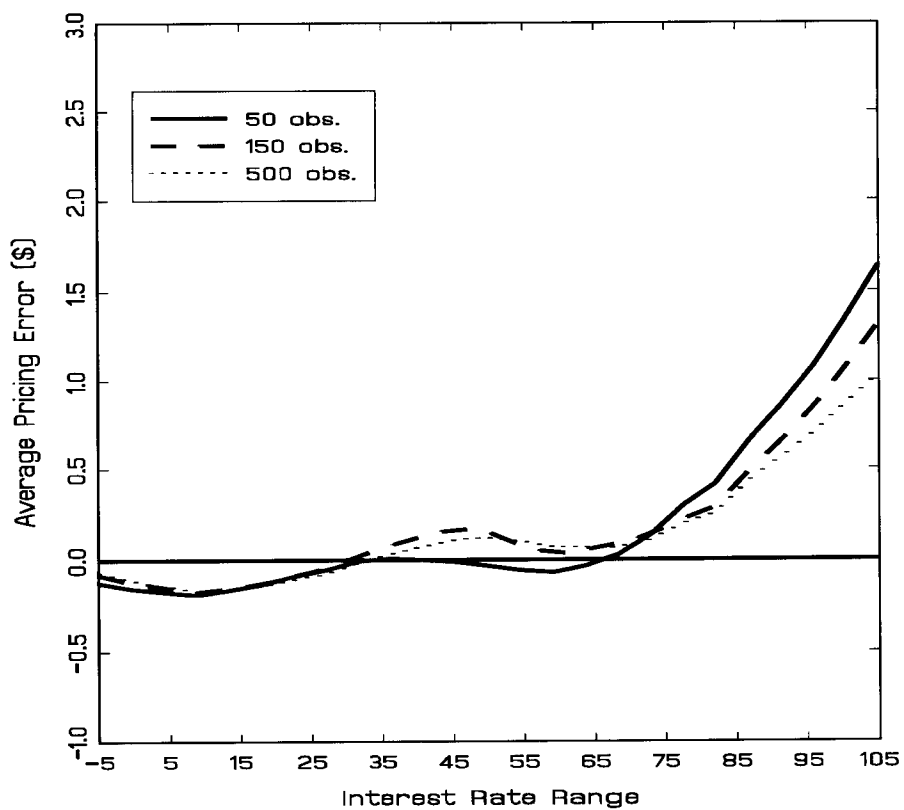
Average pricing errors resulting from applying the MDE approach to the simulated price data for a 13% GNMA. Note that the x-axis measures the level of the interest rate relative to the maximum and minimum in the first 50 observations. The points "0" and "100" correspond to the minimum and maximum, respectively.

Figure 3: The 10-Year Yield and the Mortgage Rate
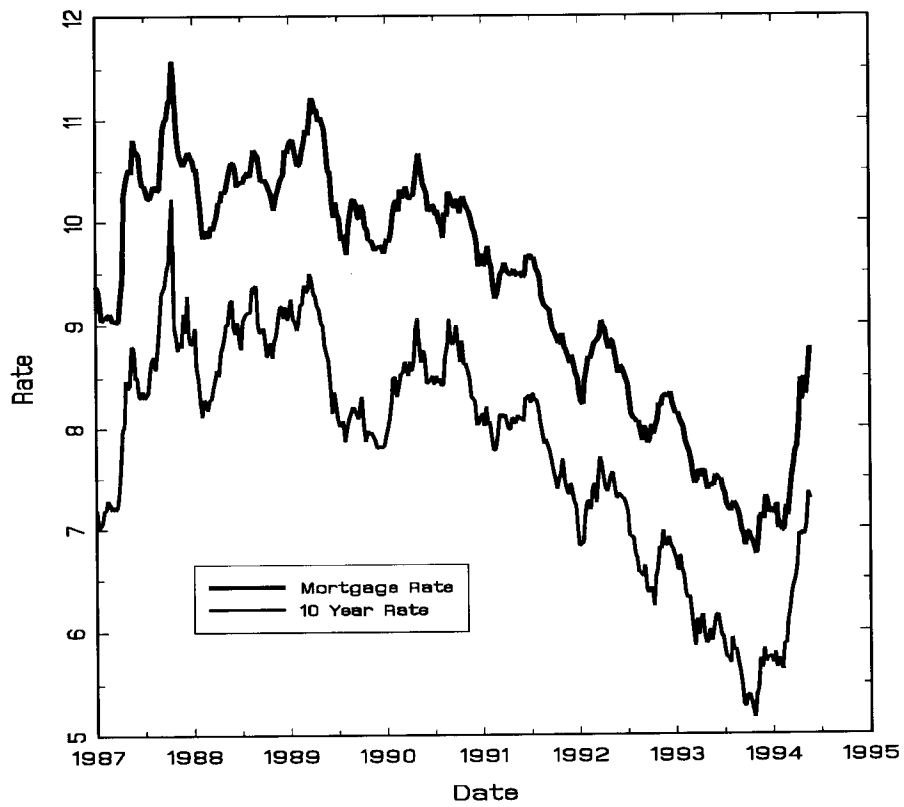
The yield on the "on-the-run" 10-year Treasury note and the average 30-year mortgage rate, from January 1987 to May 1994.

Figure 4: Price of a 9% GNMA as a Function of the Long Rate and the Spread
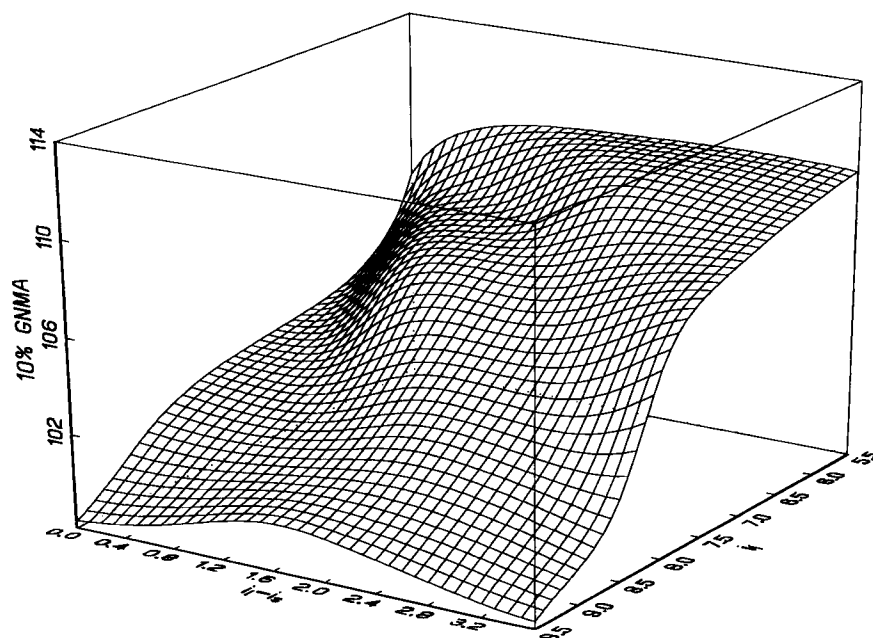
The price of a 9% GNMA as a function of the pricing factors: the long rate and the spread. The pricing functional is estimated using the MDE approach and weekly data from January 1987 to May 1994.

Figure 5: Scatter Plot of the Long Rate vs. the Spread
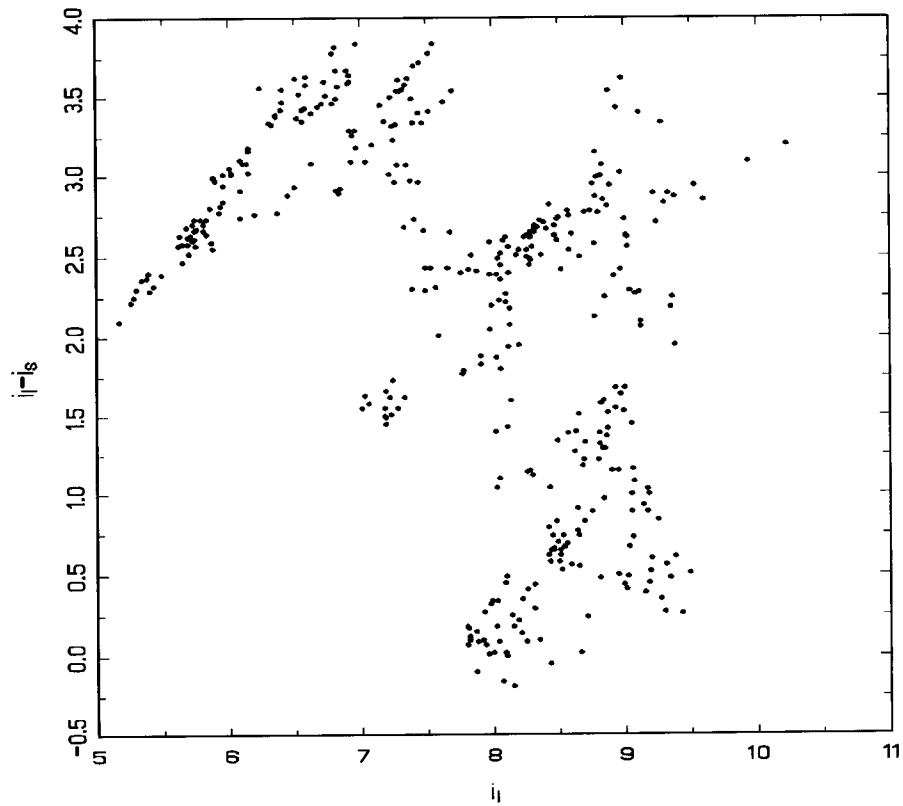
A scatter plot of the pairs of data available for the 10-year rate and the spread between the 10-year rate and the 3-month rate, from January 1987 to May 1994.

Figure 6: GNMA Prices vs. the Long Rate
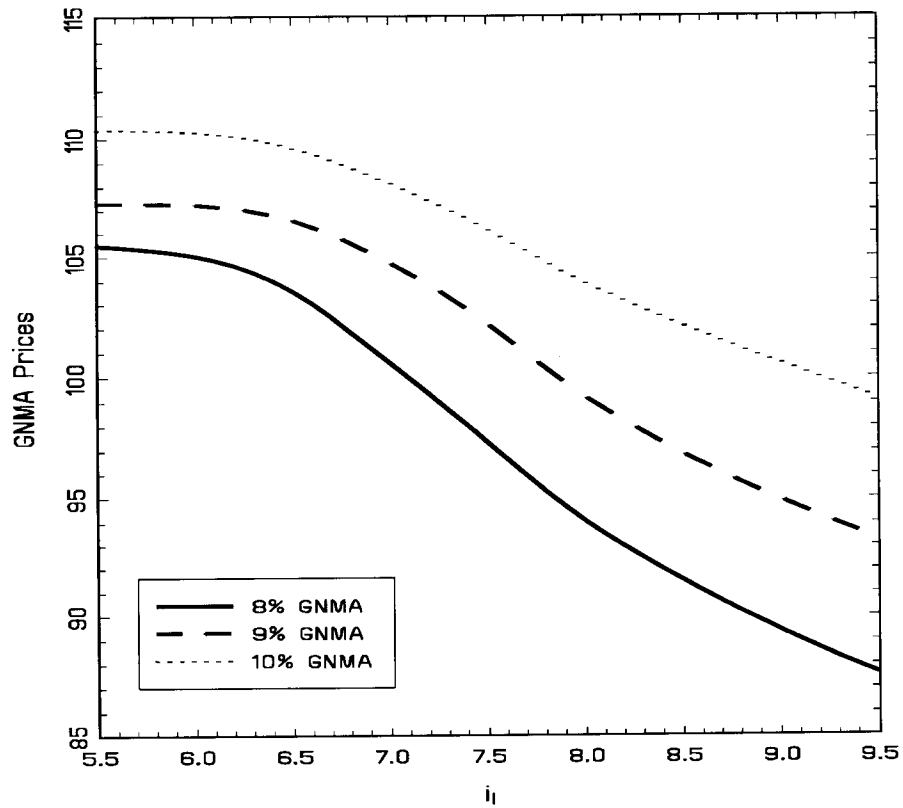
Prices of 8%, 9% and 10% GNMAs for various interest rates, with the spread fixed at 2.70%, as estimated via the MDE approach using weekly data from January 1987 to May 1994.
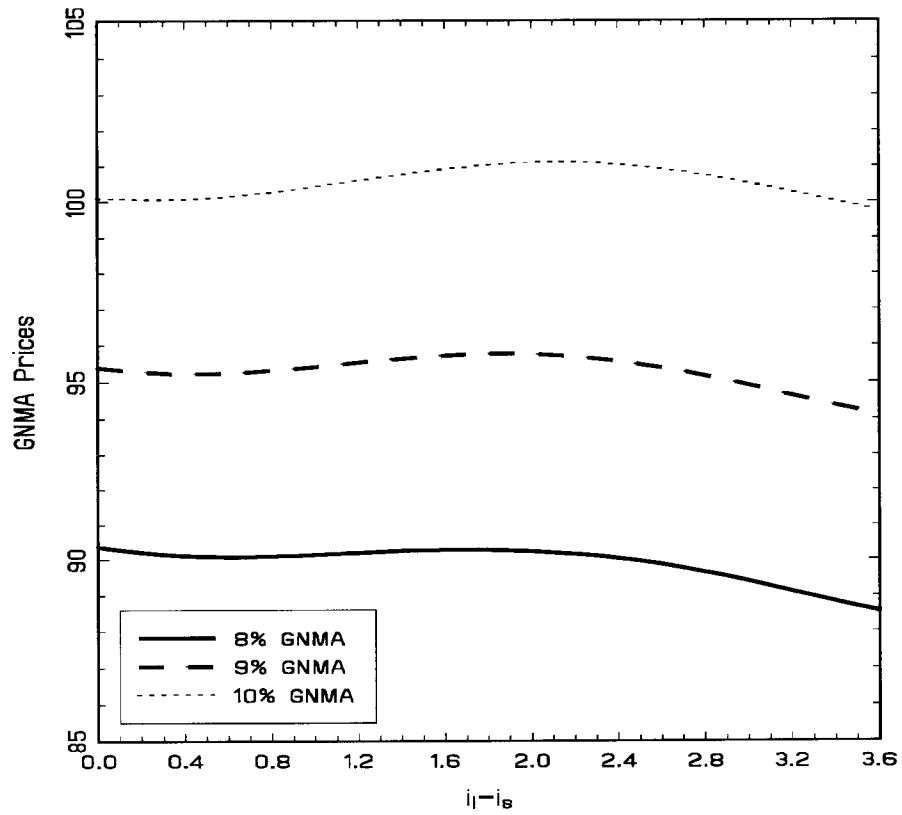
Figure 7: GNMA Price Elasticities
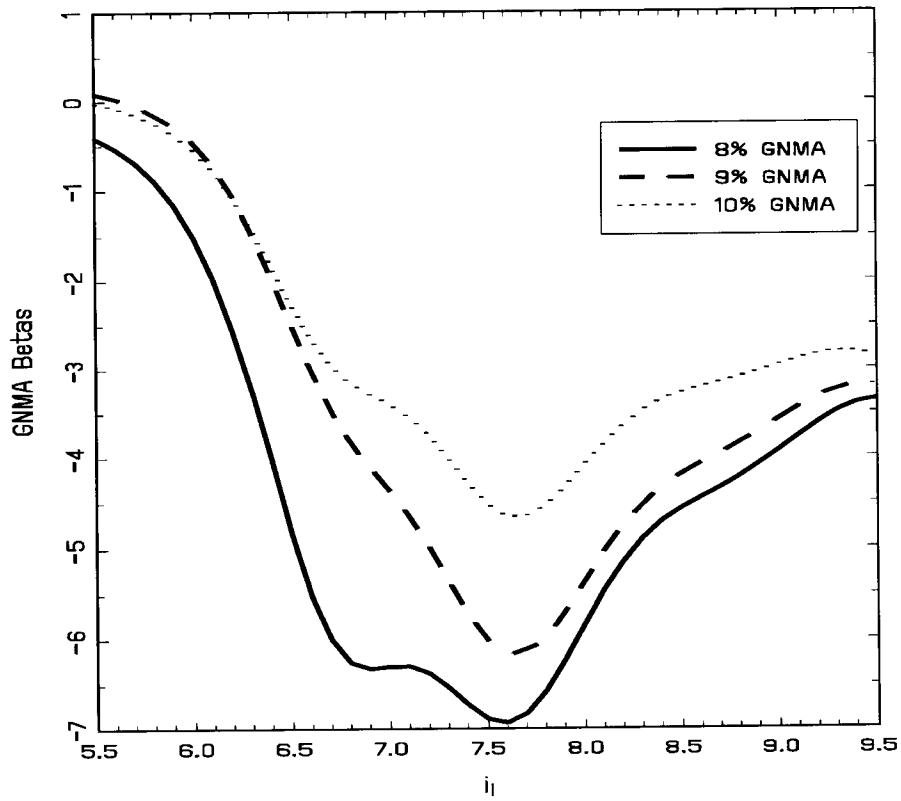
Elasticities of 8%, 9% and 10% GNMAs with respect to the long rate (i.e., $\frac{\partial P_{mb}}{i_l}$) for various interest rates, with the spread fixed at 2.70%, as estimated via the MDE approach using weekly data from January 1987 to May 1994.
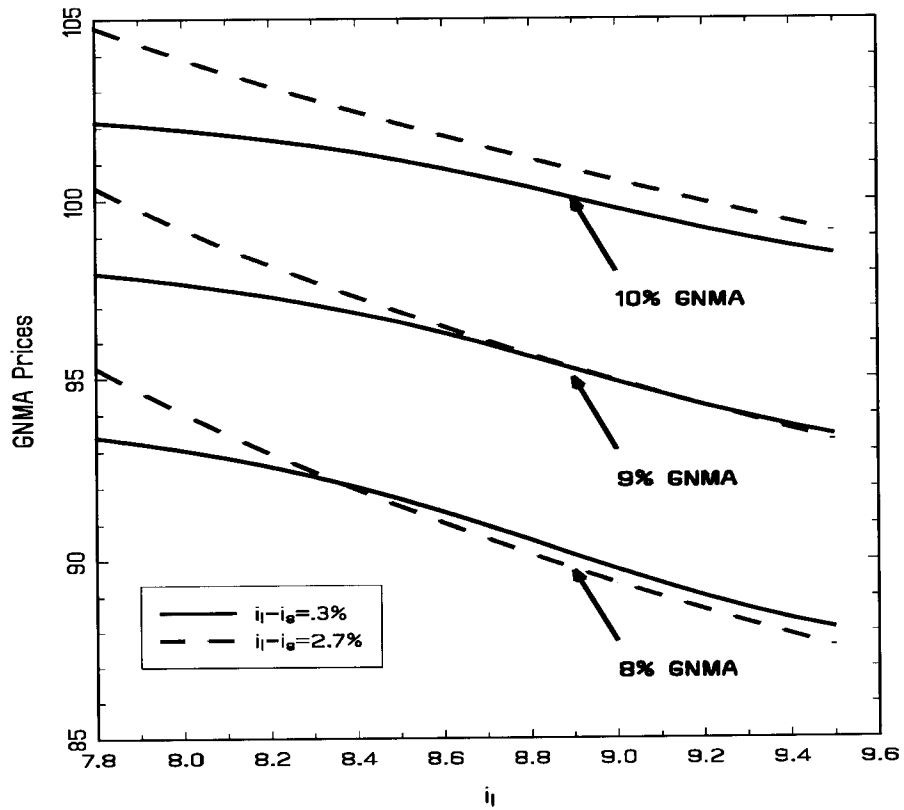
Figure 8: GNMA Prices vs. the Spread

Prices of 8%, 9% and 10% GNMAs for various spread levels, with the long rate fixed at 8.90%, as estimated via the MDE approach using weekly data from January 1987 to May 1994.

Figure 9: GNMA Prices vs. the Long Rate for Different Spreads

Prices of 8%, 9% and 10% GNMAs for various interest rates, with the spread fixed at 2.70% and 0.30%, as estimated via the MDE approach using weekly data from January 1987 to May 1994.

38

Figure 10: GNMA Prices vs. the Long Rate for Different Spreads
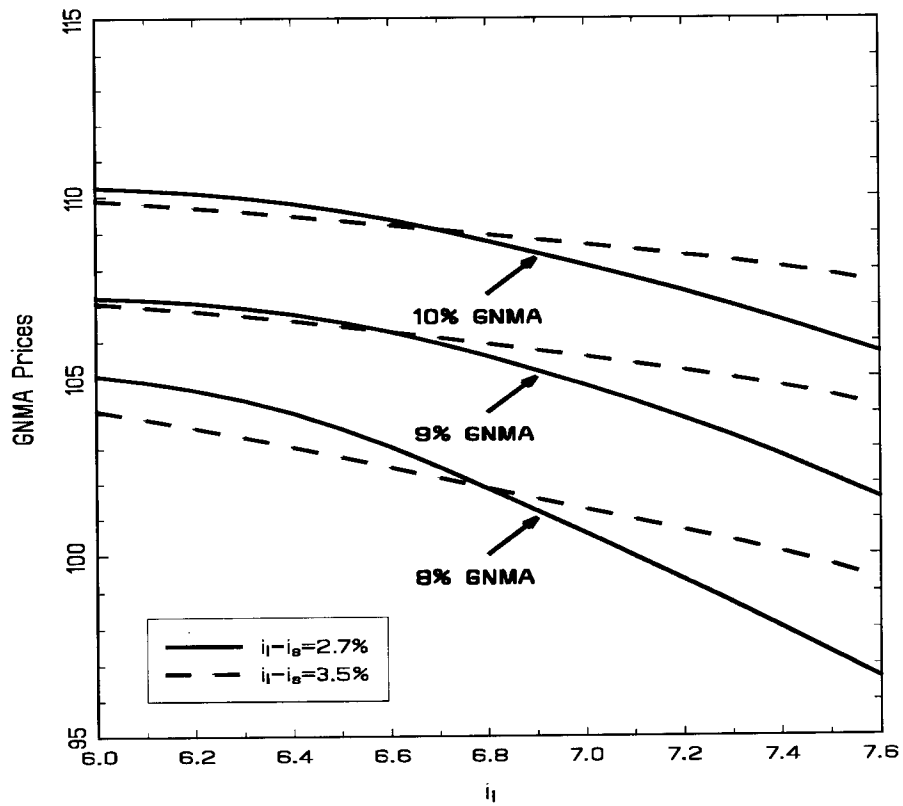
Prices of 8%, 9% and 10% GNMAs for various interest rates, with the spread fixed at 3.50% and 2.70%, as estimated via the MDE approach using weekly data from January 1987 to May 1994.

Figure 11: Hedging Errors
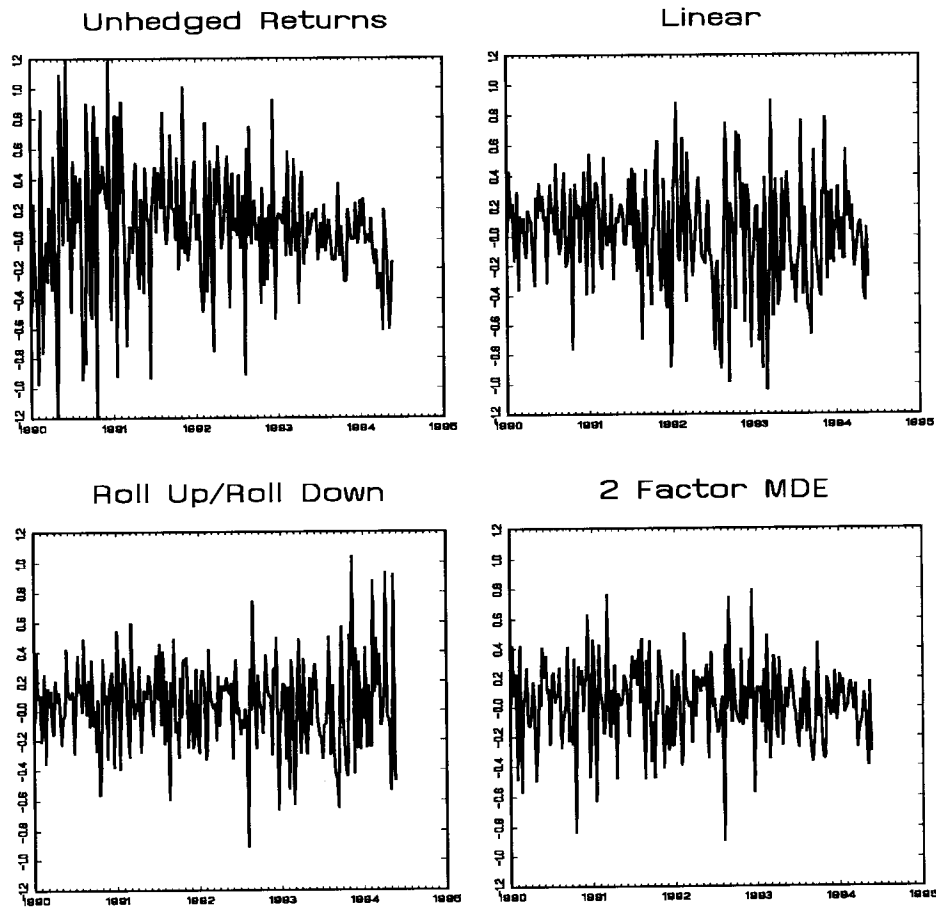
Results from hedging the 10% GNMA using a rolling regression method, where "Linear" is hedging via linear regression of returns on T-note futures, "Roll-Up/Roll-Down" infers hedge ratio from market prices of near coupon MBSs, and "2 Factor MDE" uses the two factor MDE approach.

References

[1] Ait-Sahalia, Y., 1994, "Nonparametric Pricing of Interest Derivative Securities," working paper, Chicago University.

[2] Askin, D. J., and S. D. Meyer, 1986, "Dollar Rolls: A Low-Cost Financing Technique," Mortgage-Backed Securities Research, Drexel Burnham Lambert.

[3] Bartlett, W. W., 1989, *Mortgage-Backed Securities: Products, Analysis, Trading*, New York Institute of Finance.

[4] Boudoukh, J., M. Richardson, R. Stanton, and R. F. Whitelaw, 1995, "A New Strategy for Dynamically Hedging Mortgage-Backed Securities," working paper, New York University.

[5] Breeden, D. T., 1991, "Risk, Return and Hedging of Fixed Rate Mortgages," *Journal of Fixed Income*, 1, 85-107.

[6] Breiman, L., W. Meisel and E. Purcell, 1977, "Variable Kernel Estimates of Multivariate Densities," *Technometrics*, 19, 135-144.

[7] Brennan, M. J., and E. S. Schwartz, 1979, "A Continuous Time Approach to the Pricing of Bonds," *Journal of Banking and Finance*, 3, 133-156.

[8] Brennan, M. J., and E. S. Schwartz, 1985, "Determinants of GNMA Mortgage Prices," *AREUEA Journal*, 13, 209–228.

[9] Chen, R. R., and L. Scott, 1992, "Pricing Interest Rate Options in a Two-Factor Cox-Ingersoll-Ross Model of the Term Structure," *Review of Financial Studies*, 5, 613-636.

[10] Constantinides, G. M., 1990, "Theory of the Term Structure of Interest Rates," *The Review of Financial Studies*, 5, 531-552.

[11] Cover, T., 1972, "A Hierarchy of Probability Density Function Estimates," *Frontiers of Pattern Recognition*, Academic Press.

[12] Cox, J. C., J. E. Ingersoll, and S. A. Ross, 1985a, "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica*, 50, 363-384.

[13] Cox, J. C., J. E. Ingersoll, and S. A. Ross, 1985b, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, 385-407.

[14] Duffie, D., and R. Kan, 1994, "A Yield-Factor Model of Interest Rates," working paper, Stanford Graduate School of Business.

[15] Dunn, K. B. and J. J. McConnell, 1981, Valuation of GNMA Mortgage-Backed Securities, *Journal of Finance*, 36, 599-616.

[16] Dunn, K. B., and C. S. Spatt, 1985, "An Analysis of Mortgage Contracting: Prepayment Penalties and the Due-On-Sale Clause," *Journal of Finance*, 40, 293-308.

[17] Epanechnikov, V., 1969, "Nonparametric Estimates of Multivariate Probability Density," *Theory of Probability and Applications*, 14, 153-158.

[18] Fabozzi, F., 1993, *The Handbook of Mortgage-Backed Securities*, Probus Publishing.

[19] Gallant R., 1981, "On the Bias in Flexible Functional Forms and an Essentially Unbiased Form: The Fourier Flexible Form," *Journal of Econometrics*, 15, 211-245.

[20] Gallant R., 1982, "Unbiased Determination of Production Technologies," *Journal of Econometrics*, 20, 285-323.

[21] Gallant, R., and G. Souza, 1991, "On the Asymptotic Normality of Fourier Flexible Form Estimates," *Journal of Econometrics*, 50, 329-353.

[22] Harvey, C., 1991, "The Specification of Conditional Expectations," working paper, Fuqua School of Business.

[23] Johnston, E., and L. Van Drunen, 1988, "Pricing Mortgage Pools with Heterogeneous Mortgages: Empirical Evidence," working paper, University of Utah.

[24] Kau, J. B., D. C. Keenan, W. J. Muller and J. F. Epperson, 1992, "A Generalized Valuation Model for Fixed-Rate Residential Mortgages," *Journal of Money, Credit and Banking*, 24, 279-299.

[25] Litterman, R., and J. Scheinkman, 1991, "Common Factors Affecting Bond Returns," *Journal of Fixed Income*, 1, 54-61.

[26] Pagan, A., and Y.S. Hong, 1991, "Non-parametric Estimation and the Risk Premium," in W. Barnett, J. Powell and G. Tauchen (eds), *Semiparametric and Nonparametric Methods in Econometrics and Statistics*, Cambridge University Press, Cambridge.

[27] Pearson, N. D. and T.-S. Sun, 1989, "A Test of the Cox, Ingersoll, Ross Model of the Term Structure of Interest Rates Using the Method of Maximum Likelihood," working paper, MIT and Shearson Lehman Commercial Paper, Inc.

[28] Pennacchi, G. G., 1991, "Identifying the Dynamics of Real Interest Rates and Inflation: Evidence Using Survey Data," *The Review of Financial Studies*, 4, 53-86.

[29] Richard, S. F., and R. Roll, 1989, "Prepayments on Fixed-Rate Mortgage-Backed Securities," *Journal of Portfolio Management*, 15, 74-82.

[30] Schwartz, E. S., and W. N. Torous, 1989, "Prepayment and the Valuation of Mortgage-Backed Securities," *Journal of Finance*, 44, 375-392.

[31] Scott, D. W., 1992, *Multivariate Density Estimation: Theory, Practice and Visualization*, John Wiley & Sons, Inc.

[32] Shiller, R. J., 1990, "The Term Structure of Interest Rates," The Handbook of Monetary Economics, edited by Frank Hahn and Benjamin Friedman, Vol.1, Chapter 13.

[33] Singleton, K. J., 1989, "Modeling the Term Structure of Interest Rates in General Equilibrium," in *Theory of Valuation: Frontiers of Modern Financial Theory*, edited by S. Bhattacharya and G. Constantinides.

[34] Stambaugh, R., 1988, "The Information in Forward Rates: Implications for Models of the Term Structure," *Journal of Financial Economics*, 21, 41-70.

[35] Stanton, R. H., 1992, "Essays in Financial Economics: Contingent Claims Analysis and Mortgage Prepayment," Unpublished dissertation, Stanford University Graduate School of Business.

[36] Stanton, R. H., 1994, "Rational Prepayment and the Valuation of Mortgage-Backed Securities," Forthcoming *Review of Financial Studies*.

[37] Sun, T.-S., 1987, "Empirical Evidence on the Price Level: An Application of the Generalized Method of Moments to Estimate Diffusion Processes," unpublished dissertation, Graduate School of Business, Stanford University.

[38] Tauchen, G, and R. Hussey, 1991, "Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models," *Econometrica*, 59, 371-395.

[39] Timmis, G. C., 1985, "Valuation of GNMA Mortgage-Backed Securities with Transaction Costs, Heterogeneous Households and Endogenously Generated Prepayment Rates," working paper, Carnegie Mellon University.

[40] Titman, S., and W. Torous, 1989, "Valuing Commercial Mortgages: An Empirical Investigation of the Contingent-claims Approach to Pricing Risky Debt," *Journal of Finance*, 43, 345-374.

[41] Ullah, A., 1988, "Non-Parametric Estimation of Economic Functionals," *Canadian Journal of Economics*, 21, 625-658.

[42] Waldman, M., 1992, "Beyond Duration: Risk Dimensions of Mortgage Securities," *Journal of Fixed Income*, 2, 5-15.