Dynamic Pricing of Network Goods with Boundedly Rational Consumers

Roy Radner* and Arun Sundararajan

Stern School of Business, New York University

Working Paper CeDER-05-08, Center for Digital Economy Research

First Draft: January 2005

Abstract

An important simplifying assumption made when analyzing goods that display positive network effects is that potential consumers can form a rational expectation of the equilibrium demand for the good, and that they all form the same expectation, which is then fulfilled based on their consumption choices - sometimes called a *rational expectations equilibrium* (REE). We examine whether the results of these models are robust to the relaxation of this assumption. In our model, consumers differ in their marginal utility of total demand (intensity of the network effect), which varies according to a given distribution (the distribution of consumer "types"), and are boundedly rational in two ways. First, only a fraction of consumers "pay attention" to price announcements in any interval of time. Second, those consumers who pay attention make their consumption choices based on a boundedly rational expectation of future demand. Our benchmark model is of myopic expectations, although we show how our results generalize (1) to a case in which the population of consumers contains both those who are myopic and those who are "fully rational," and (2) to a case in which consumers have expectations that are partly "stubborn". We base our analysis on a continuous-time approximation of an underlying

*We thank Philipp Afeche, Joseph Harrington and workshop participants at Boston University, New York University and the Sixteenth Workshop on Information Systems and Economics for their feedback.

discrete-time model. Under this approximation, the instantaneous choices of consumers continuously influence the rate at which demand adjusts over time, and a monopolist chooses a price trajectory to maximize profit. First, we show that, under fairly general assumptions about the distribution of types, the profit-maximizing rational expectations equilibrium is not a steady state of the optimal trajectory with boundedly rational consumers. Our second theorem shows that if consumer types are uniformly distributed and consumers form myopic (or more rational) expectations, the monopolist's optimal pricing trajectory is generated by a "target policy" with the following properties: when current demand is below the target, the price is zero; when current demand is above the target, the price is the maximum possible, and when current demand is at the target, the price is chosen to keep demand stationary. We also show that the optimal demand target with boundedly rational consumers is always strictly lower than the equilibrium level of demand predicted by a model with rational expectations. Furthermore, the difference between the target demand and the rational expectations demand is higher when consumers pay attention to the monopolist's price announcements at a lower rate. We generalize the results from this example in two ways. Our third theorem examines the case of myopic consumers and strictly concave distributions of consumer types. To find an optimal policy one must expand the set of controls to include measure-valued controls. The optimal policy is similar to the target policy of Theorem 2, except that when current demand is at the target, the monopolist chooses the "mixture" between a price of zero and the maximum possible price that keeps demand stationary. For convex consumer type distributions in the neighborhood of the uniform distribution, we give a heuristic argument to support a conjecture that the monopolist continues to choose a demand target lower than the rational expectations demand, but varies price gradually in the neighborhood of the demand target. Finally, for uniformly distributed types and consumer expectations that are both myopic and "stubborn", we show that the monopolist's optimal pricing trajectory is generated by a target policy with the same properties as those in Theorem 2, although with a target that is strictly lower, and that increases as consumers become progressively less stubborn.

(This paper is part of a program of research whose broad objective is to explore the conditions under which the assumption of unbounded rationality in economic models is a reasonable one).

1 Introduction

The theory of rational expectations is widely used to describe economic situations in which the outcome depends partly upon what people expect to happen. This idea is especially popular in models of network effects or externalities, wherein the value of a good or service (henceforth referred to as a "network good") to each consumer is influenced by the consumption choices made by some or all other consumers. In this literature (for instance, in the seminal papers by Katz and Shapiro, 1985, and by Farrell and Saloner, 1985), potential consumers form a common expectation of demand, and make their consumption choices unilaterally based on this expectation, which is then realized in equilibrium; this solution is commonly referred to as satisfying "fulfilled expectations."

An important assumption underlying this solution is that consumers can in fact all "guess" the correct equilibrium level of demand. This assumption is consistent with a model of perfectly (or unboundedly) rational consumers. Some notion of perfect rationality is at the base of most current economic analysis, even though most economists accept that agents are not in reality unboundedly rational. Such models continue to be used, perhaps because there is an implicit belief that the 'output' of economic analysis based on the approximation of unboundedly rationality agents is (reasonably) correct.

In the specific case of network goods, however, the predictions of rational expectations models do not appear to be a good description of economic reality. For example, it has been observed that outcomes in markets for network goods are often path-dependent. This would not be the case were consumers able to form rational expectations; it also highlights the importance of the dynamics of the adoption process for eventual outcomes.

Our objective in this paper is to present an alternative model of demand for a network good, in which consumers are not "rational enough" to be able to compute equilibrium demand. Instead, they make their consumption choices based on an (imperfect) assessment of expected demand, which may depend on the current price, the current level of demand, and/or an exogenously specified "stubborn" expectation of equilibrium demand. The intensity of the network effect is heterogeneous across consumers. Different consumers therefore have a different marginal utility of total demand, which varies according to a given distribution, referred to as the distribution of consumer types. The adoption choices of the consumers continuously influence the rate at which demand adjusts over time. A monopoly seller therefore chooses the price *trajectory* that maximizes her discounted stream of profits. The rate at which demand adjusts over time is also affected by an exogenous parameter λ, which is proportional to the fraction of consumers in each period who "pay attention" to the current price.

Our first set of results uses a model of *myopic* consumers whose expectation of future demand is simply current demand. Our first theorem establishes that under fairly general assumptions about the distribution of types, the profit-maximizing rational expectations equilibrium is *never* a steady state of the optimal demand trajectory with boundedly rational consumers. Specifically, starting from the profit-maximizing rational expectations price, the monopolist can always improve her profits by charging the highest possible price for some positive length of time, and then switching to some other price trajectory.

Our second theorem shows that when consumer types are uniformly distributed, the monopolist's optimal pricing trajectory is generated by a "target policy" with the following properties: when current demand is below the target, the price is zero; when current demand is above the target, the price is the maximum possible; and when current demand is at the target, the price is chosen to keep demand stationary. The target could be interpreted as the level of adoption below which the monopolist invests in building a user base, and above which the monopolist profits from her installed base.

This theorem also shows that the optimal demand target with myopic consumers is always strictly lower than the equilibrium level of demand predicted by a model with rational expectations. As the rate at which future profits are discounted tends to zero, the optimal demand target converges to the profit-maximizing rational expectations equilibrium level of demand. Furthermore, the difference between the target demand and the REE demand is a decreasing function of λ, and tends to zero as λ increases without bound, i.e., when all consumers react to price changes infinitely fast.

Our subsequent results extend this theorem along two directions. First, we examine the variation of the optimal policy derived with myopic consumers for non-uniformly distributed consumer types. Our third theorem establishes that when one permits measure-valued controls, the optimal generalized policy for *any strictly concave* distribution of consumer types is very similar to the target policy of Theorem 2. Specifically, the monopolist chooses a demand target that is independent of the actual distribution function, and whose value is the same as the demand target when consumer types are uniformly distributed. When current demand is below the target, the price is zero; when current demand is above the target, the price is the maximum possible, and when current demand is at the target, the monopolist chooses the "mixture" between a price of zero and the maximum possible price that keeps demand stationary. This theorem therefore establishes that the optimal price trajectory suggested by considering uniformly distributed types is robust (in a sense) across a large class of distributions of consumer types, as is the specific deviation from the rational expectations outcome that the example established.

We conjecture that for *strictly convex* consumer type distributions in the neighborhood of the uniform distribution, the monopolist continues to choose a demand target lower than the rational expectations demand, but varies price gradually in the neighborhood of the demand target. We present a heuristic argument for this conjecture, but do not have a complete proof. These last two results would together indicate the extent of robustness of the model based on the uniform distribution.

We also generalize the above results to the case in which the population of consumers contains both those who are myopic and those who are "fully rational." For this model, the results are qualitatively the same as for the case in which all consumers are myopic.

In parallel, we examine how the monopolist's optimal price trajectory varies when the expectation of demand formed by each consumer who pays attention is a weighted average of the myopic expectation and an exogenously specified "stubborn" expectation. Our final theorem establishes that the monopolist's optimal pricing trajectory continues to be generated by a target policy with the same properties as the one derived in Theorem 2, although with a *lower* target demand level. The target increases as consumers become less stubborn, eventually converging to the target demand level of the policy for purely myopic consumers.

2 Overview of our Model

2.1 Models of Bounded Rationality

We derive our continuous-time formulation as the limiting case of a discrete-time model. A network good is provided by a monopolist, who sells the good one period at a time. (Think of the good as a service.) The length of each period is h, and therefore time is indexed as $t = 0, h, 2h, ...$ A unit mass of a continuum of consumers is indexed by a "type" parameter θ in the unit interval. Let F be the cumulative distribution function of θ, i.e., the fraction of consumers with type less than or equal to θ is $F(\theta)$. For simplicity, we assume F to be absolutely continuous and strictly increasing on the the unit interval. The monopolist announces a price $p(t)$ in each period. We assume that the price, $p(t)$, is constrained to be nonnegative, and is bounded above (more on this later).

The first aspect in our model of rationality is that of *bounded attention.* In each period, a random fraction λh of consumers of each type "pay attention to" the price $p(t)$. Correspondingly, the remaining fraction $(1 - \lambda h)$ of consumers of each type do not notice the monopolist's price announcement, and their choice remains unchanged in period t (i.e., from time t to time $(t + h)$). Notice that an equal fraction λh of consumers of each type "pay attention" in each period, and that the magnitude of this fraction depends on the length of the interval h. One might therefore interpret λ as measuring a "rate of attention" of consumers to price changes, or a "rate of adjustment." Thus the average time between successive price checks by a consumer is $(1/\lambda)$. The latter interpretation implies that as λ tends to infinity, every consumer reacts instantaneously to each price announcement, which is one way of interpreting a model of unboundedly rational consumers.

The second aspect in our model of bounded rationality specifies how consumers who are paying attention form their *expectation of demand* for period t. Specifically, each consumer who notices the price $p(t)$ at the beginning of period t makes the same prediction, $q_E(t,h)$, of the total demand in period t. Therefore, a consumer of type θ who notices $p(t)$ will buy the good if and only if $\theta q_E(t,h) \geq p(t)$. We specify a few simple models of $q_E(t,h)$ in subsequent sections.

2.2 A Continuous Time Approximation

Our analysis uses a continuous-time version of the model described above, which is derived as a limiting case of the discrete-time model, as $h \to 0$. Define

$$q_E(t) = \lim_{h \to 0} q_E(t,h),$$

and assume that $q_E(t)$ is well-defined, and depends at most on the current demand and price, $q(t)$ and $p(t)$, respectively. The resulting time-rate of change of demand is described in our first lemma.

Lemma 1 *If at time $t \quad (\geq 0)$ the demand and price are $q(t)$ and $p(t)$, respectively, then the time-rate of change of demand is specified by:*

$$q'(t) = \begin{cases} 0, \quad q(t) = 0, \\ \lambda \left\{ Q[q_E(t), p(t)] - q(t) \right\}, \quad 0 < q(t) \leq 1, 0 \leq p(t) \leq q_E(t), \\ -\infty, \quad 0 < q(t) \leq 1, p(t) > q_E(t), \end{cases} , \tag{1}$$

where

$$Q(x,p) \equiv 1 - F\left(\frac{p}{x}\right), \quad 0 < x \leq 1. \tag{2}$$

Proof. First, when $q(t) = 0$, the product is of no value to all consumers, which yields the first line of (1). Next, suppose the demand from consumers of type θ in period t is denoted by $w(\theta,t)$. Recall that a fraction λh of consumers of type θ notice $p(t)$, form a shared expectation of demand $q_E(t,h)$, and decide whether or not to adopt the product for period t. Therefore, if

$\theta \geq [p(t)/q_E(t,h)]$, each consumer in this fraction λh adopts the product, and if $\theta < [p(t)/q_E(t,h)]$, then none of these consumers adopt the product. Since the remaining fraction $(1-\lambda h)$ continue to do in period t what they were doing in period $t-h$, it follows that:

$$w(\theta,t) = \begin{cases} \lambda h + (1-\lambda h) w(\theta, t-h), & \theta \geq p(t)/q_E(t,h), \\ (1-\lambda h) w(\theta, t-h) & \theta < p(t)/q_E(t,h), \end{cases}, \tag{3}$$

and therefore,

$$w(\theta,t) - w(\theta,t-h) = \begin{cases} \lambda h[1 - w(\theta, t-h)], & \theta \geq p(t)/q_E(t,h), \\ -\lambda h[w(\theta, t-h)], & \theta < p(t)/q_E(t,h). \end{cases}. \tag{4}$$

Dividing both sides by h and letting h tend to zero yields the time rate of change of demand for consumers of type θ:

$$\frac{dw(\theta,t)}{dt} \equiv \lim_{h\to 0}\left(\frac{w(\theta,t)-w(\theta,t-h)}{h}\right) = \begin{cases} \lambda[1 - w(\theta, t-h)], & \theta \geq p(t)/q_E(t), \\ -\lambda[w(\theta, t)], & \theta < p(t)/q_E(t). \end{cases} \tag{5}$$

Since

$$q(t) = \int_0^1 w(\theta,t)dF(\theta), \tag{6}$$

it follows that

$$q'(t) = \int_0^1 \left(\frac{dw(\theta,t)}{dt}\right) dF(\theta), \tag{7}$$

so by (6) and (7),

$$q'(t) = \int_0^{p(t)/q_E(t)} -\lambda[w(\theta,t)]dF(\theta) + \int_{p(t)/q_E(t),}^{1} \lambda[1-w(\theta,t)]dF(\theta), \tag{8}$$

which simplifies to

$$q'(t) = \lambda\left[1 - F\left(\frac{p(t)}{q_E(t)}\right)\right] - \int_0^1 \lambda w(\theta,t)dF(\theta), \tag{9}$$

and using (3), this yields the second line of (1). ■

The third line of (1) expresses the assumption that, if $p(t) > q_E(t)$, every consumer will expect all the subscribers to unsubscribe immediately. This has the effect of imposing on the monopolist the constraint that the current price must not exceed the current expectation of total demand . Furthermore, we impose the constraint that the price must be nonnegative; other lower bounds could be imposed without changing the qualitative nature of our results. Thus we assume:

$$0 \leq p(t) \leq q_E(t). \tag{10}$$

The monopolist wants to choose a price trajectory $p(t)$ to maximize her profit. Assume, for simplicity, that her (marginal) cost of providing the service is zero; then her total discounted profit is

$$\int_0^\infty e^{-rt} p(t) q(t) dt, \tag{11}$$

where $r > 0$ is her given discount rate, and $q(t)$ evolves according to (1).

We shall analyze the maximization problem using the method of dynamic programming, in which the state variable is the current demand. First, by Blackwell's Theorem, there is no loss in restricting our attention to *stationary* policies, i.e., policies for which the current price at any time is a function of the current state only:

$$p(t) = \alpha[q(t)]. \tag{12}$$

Note that the function α does not change in time. Of course, a policy is admissible only if the differential equation (1) has a unique solution starting from any initial state $q(0)$.

The value of a policy α at an initial state $q(0) = q$ is the corresponding profit,

$$V_\alpha(q) = \int_0^\infty e^{-rt} \alpha[q(t)] q(t) dt. \tag{13}$$

Define

$$V(q) = \sup_\alpha V_\alpha(q), \tag{14}$$

where the supremum is over all admissible policies α. A policy is *optimal* if its profit attains the supremum at every state q.

3 Myopic Consumers

This section describes the monopolist's optimal price trajectory for a class of models of bounded rationality in which, in our underlying discrete-time model, the consumers' expectation of total demand during period t equals the current demand, $q(t)$, at the beginning of period t. Thus, in the discrete-time model,

$$q_E(t,h) = q(t). \tag{15}$$

Corresponding to Lemma 1, for the continuous-time approximation we have, trivially,

Lemma 2 *In the continuous-time model, with myopic consumers,*

$$q_E(t) = q(t),$$

and therefore, if at time t, *the demand and price are* $q(t)$ *and* $p(t)$, *respectively, then the time-rate of change of demand is specified by* $q'(t) = m(q(t), p(t))$, *where:*

$$m(q,p) = \begin{cases} 0, \quad q = 0, \\ \lambda \left[F\left(1 - \dfrac{p}{q}\right) - q \right], \quad 0 < q \le 1, 0 \le p \le q, \\ -\infty, \quad 0 < q \le 1, p > q. \end{cases}, \tag{16}$$

In the first subsection, we give a precise definition of a Rational Expectations Equilibrium (REE), and show that a REE cannot be a steady state of the system with myopic consumers and a profit-maximizing monopolist. In the subsequent three subsections we discuss the special cases in which the cumulative distribution of types is, respectively, (1) uniform, (2) concave, and (3) convex.

3.1 Rational-Expectations Equilibrium is not Optimal

An alternative theory of consumer behavior is embodied in the concept of "rational-expectations equilibrium." Imagine that, when faced with a price p, each consumer correctly predicts the total demand at that price, and decides whether or not to subscribe on the basis of that prediction. Thus the total demand at that price must satisfy

$$Q(q,p) = q,$$

where $Q(q,p)$ was defined in (2). Following standard terminology, we shall call such a pair (q,p), a *rational-expectations equilibrium* (REE). We shall call a REE *optimal* (for the monopolist) if it maximizes the product pq in the set of REEs. Thus, an implication of the REE hypothesis is that a monopoly market equilibrium must be an optimal REE. We shall show that, under wide range of conditions, an optimal REE that is feasible in the preceding model with boundedly rational consumers cannot be a steady state for an optimal policy . In fact, under such conditions we can expect the REE demand to be larger, and the REE price to be smaller, than the respective demand and price in a (nondegenerate) steady state of an optimal policy.

Note that from the first line of (1), for any price p, the pair $(0,p)$ is a REE; such a pair will be called *degenerate*. Additionally, since λ is strictly positive, (q,p) is a REE if and only if

$$q'(t) = 0,$$

i.e., q is a steady state of the demand process given the constant price p. For $q > 0$ define $P(q)$ implicitly by

$$Q[q, P(q)] - q = 0,$$

or

$$F\left[\frac{P(q)}{q}\right] = 1 - q. \tag{17}$$

Of course, this equation may have no solution or multiple solutions. In the latter case, take $P(q)$ to be the largest solution. Let $\boldsymbol{Q}$ be the set of demands q such that $P(q)$ exists, and for q in $\boldsymbol{Q}$ define

$$v(q) = qP(q). \tag{18}$$

In what follows we make various assumptions about the regularity of the functions P and v. In subsequent sections we shall show that these assumptions are satisfied in a "robust" set of cases.

Define

$$q^* = \arg\max \{v(q) | q \in \boldsymbol{Q}\},$$

and suppose that the usual first-order condition is satisfied at q^* namely,

$$v'(q^*) = 0. \tag{19}$$

Recall that F is strictly increasing. Therefore, (17) implies that

$$P(q^*) < q^*. \tag{20}$$

Theorem 1 *The optimal REE cannot be a steady state of an optimal dynamic price policy with boundedly rational consumers who are myopic.*

Proof. Suppose that $q(0) = q^* > 0$, and that for some $u \geq 0$ the monopolist charges the price $p(t) = q_E(t)$ for $0 \leq t < u$, and then $P[q(u)]$ thereafter. Hence

$$q'(t) = -\lambda q(t), \quad 0 \leq t < u. \tag{21}$$

The resulting total discounted profit is

$$f(u) = \int_0^u e^{-rx} [q(x)]^2 \, dx + \frac{e^{-ru} v[q(u)]}{r}, \quad u \geq 0, \tag{22}$$

Differentiating f, one gets

$$f'(u) = e^{-ru} \left\{ [q(u)]^2 - v[q(u)] + \left(\frac{1}{r}\right) v'[q(u)] q'(u) \right\}. \tag{23}$$

Setting $u = 0$ one gets

$$f'(0) = (q^*)^2 - v(q^*) + \left(\frac{1}{r}\right) v'(q^*) q'(0). \tag{24}$$

By (18) and (19), this last equation becomes

$$f'(0) = q^*[q^* - P(q^*)]. \tag{25}$$

But

$$P(q^*) < q^*, \tag{26}$$

and so $f'(0) > 0$. Hence, for some sufficiently small ϵ,

$$f(\epsilon) > f(0). \tag{27}$$

However,

$$f(0) = \frac{v(q^*)}{r}, \tag{28}$$

so that setting $u = \epsilon$ yields the monopolist a larger total discounted profit than charging the price $P(q^*)$ and keeping the demand at q^* forever. QED.

3.2 Uniformly Distributed Types

We now give a complete solution of the optimal dynamic price problem for the special case in which θ is distributed uniformly on the unit interval; thus

$$F(\theta) = \theta, \quad 0 \le \theta \le 1; \tag{29}$$

$$Q(q,p) = 1 - \frac{p}{q}, \tag{30}$$

the law of motion is

$$m(q,p) = \begin{cases} 0, \quad q = 0, \\ \lambda[1 - (p/q) - q], \quad 0 < q \le 1, 0 \le p \le q, \\ -\infty, \quad 0 < q \le 1, p > q. \end{cases}, \tag{31}$$

and the price path is constrained by

$$0 \le p(t) \le q(t). \tag{32}$$

Recall that $P(q)$ is the solution of the equation,

$$Q(q, P(q)) - q = 0; \tag{33}$$

for $0 \leq q \leq 1$. (30) implies that

$$P(q) = q(1-q), \quad 0 < q \leq 1. \tag{34}$$

If the current demand is q, and the monopolist charges $P(q)$, then the demand will remain unchanged. In other words, the pair $[q, P(q)]$ is a stationary point for the demand-price process.

We now consider a family of policies called *target policies.* Define the (stationary) target policy with target s $(0 < s < 1)$ by

$$\mu(q) = \begin{cases} 0, & q < s, \\ P(q), & q = s, \\ q, & q > s. \end{cases} \tag{35}$$

If the initial state is less than the target s, then the demand will increase until it reaches the target, during which time the profit is zero. After than, the demand will remain at the target, and the profit will be $\sigma P(\sigma)$. Hence the total discounted profit will be

$$\int_T^\infty e^{-rt} sP(s)dt, \tag{36}$$

where T is the time at which demand reaches the target s. Note that there is a tradeoff between reaching a higher target and getting there sooner. Let π be the "optimal target policy," i.e., the one that maximizes the last expression. We shall prove that π *is optimal among all policies.*

Theorem 2 *The optimal target policy π is optimal among all policies, and the optimal target is*

$$\sigma \equiv \frac{2\lambda}{3\lambda + r}.$$

Proof. Consider an arbitrary target policy, μ, and let s denote the corresponding target. Suppose that the initial state q is less than s. We shall first calculate the optimal target, s, starting from $q(0) = q$. Until it reaches s, $q(t)$ solves

$$q'(t) = \lambda[1 - q(t)]. \tag{37}$$

The unique solution to the differential equation in (37) with the initial condition $q(0) = q$ is:

$$q(t) = 1 - e^{-\lambda t}(1 - q). \tag{38}$$

As a consequence, if the initial state is $q < s$, the time T at which $q(T) = s$ solves

$$s = 1 - e^{-\lambda T}(1 - q), \tag{39}$$

and therefore

$$T = \frac{1}{\lambda} \log\left(\frac{1 - q}{1 - s}\right). \tag{40}$$

Therefore, under the policy μ, the value function is:

$$V_\mu(q) = sP(s)\left(\int_T^\infty e^{-rt}dt\right), \tag{41}$$

which simplifies to:

$$V_\mu(q) = \frac{1}{r}\left(\frac{1 - q}{1 - s}\right)^{-\left(\frac{r}{\lambda}\right)} s^2(1 - s). \tag{42}$$

Equation (42) can be rewritten as:

$$V_\mu(q) = \frac{1}{r}\left[(1 - q)^{-\left(\frac{r}{\lambda}\right)}\right]\left[s^2(1 - s)^{\left(1+\frac{r}{\lambda}\right)}\right]. \tag{43}$$

Differentiating (43) with respect to s yields:

$$\frac{dV_\mu(q)}{ds} = \frac{1}{r}\left[(1 - q)^{-\left(\frac{r}{\lambda}\right)}\right]\left[2s(1 - s)^{\left(1+\frac{r}{\lambda}\right)} - \left[\left(1 + \frac{r}{\lambda}\right)s^2(1 - s)^{\left(\frac{r}{\lambda}\right)}\right]\right]. \tag{44}$$

For $0 < s < 1$, the right-hand side of (43) is strictly quasiconcave in s. Additionally, $\frac{dV_\mu(q)}{ds} = 0$ at $s = 0$ and $s = 1$, which are minima for which $V_\mu(q) = 0$ (In fact, both these statements are true for any function of the form $Kx^a(1 - x)^b$ for $a, b \geq 1$).

As a consequence, the value $\sigma \in [0, 1]$ that maximizes $V_\mu(q)$ with respect to s solves

$$2(1 - \sigma) = \left(1 + \frac{r}{\lambda}\right)\sigma, \tag{45}$$

which yields

$$\sigma = \frac{2\lambda}{3\lambda + r}. \tag{46}$$

The corresponding price is

$$P(\sigma) = \phi \equiv \frac{2\lambda(\lambda + r)}{(3\lambda + r)^2}. \tag{47}$$

(Note that when $r = 0$, this yields the rational expectations equilibrium quantity $\sigma = 2/3$ and price $\phi = 2/9$. Also note that the price ϕ in (47) satisfies (**??**) for each $r \geq 0, \lambda > 0$.)

Correspondingly, if the initial state is $q > s$, until it reaches s, $q(t)$ solves

$$q'(t) = -\lambda q(t)], \tag{48}$$

which corresponds uniquely to the demand trajectory:

$$q(t) = qe^{-\lambda t}, \tag{49}$$

and a similar computation yields the value function:

$$V(q) = \frac{1}{2\lambda + r}\left(q^2 + q^{-\frac{r}{\lambda}} s^{(2+\frac{r}{\lambda})} \left[\frac{2\lambda(1-s) - rs}{r}\right]\right) \tag{50}$$

which is also maximized with respect to s by the value of σ in (46).

Denote by π the target policy with target σ. We shall now show that the target policy π is optimal. For a given policy μ, if its value function is continuously differentiable, then the corresponding *Bellmanian Functional* is defined by

$$B_\mu(q, p) = pq - rV_\mu(q) + V_\mu'(q)\, m(q, p). \tag{51}$$

According to a well-known proposition, a policy μ is optimal if it satisfies the Hamilton-Jacobi-Bellman condition:

$$B_\mu(q, p) \leq 0 \text{ for all } q, p. \tag{52}$$

An alternative form for the last condition is

$$\mu(q) = \arg\max_p B_\mu(q, p). \tag{53}$$

This follows from the fact that, for all q,

$$B_\mu[q, \mu(q)] = 0, \tag{54}$$

which is readily verified. (In fact, this identity is true for any stationary policy whose value function is C^1.) Hence, from the above,

$$B_\pi[q, \pi(q)] = \pi(q)q - rV_\pi(q) + \lambda\left(1 - q - \frac{\pi(q)}{q}\right)V'_\pi(q) = 0. \tag{55}$$

It will be useful to define

$$G(q) \equiv q^2 - \lambda V'_\pi(q).$$

and write $B_\pi(q, p)$ in the form,

$$\begin{aligned} B_\pi(q, p) &= pq - rV_\pi(q) + \lambda\left(1 - q - \frac{p}{q}\right)V'_\pi(q) &(56)\\ &= -rV_\pi(q) + \lambda(1 - q)V'_\pi(q) + \frac{p}{q}G(q). &(57)\end{aligned}$$

Thus $B_\pi(q, p)$ is linear in p, and the coefficient of p is $\dfrac{G(q)}{q}$. Hence

$$\arg\max_p B_\pi(p, q) = \begin{cases} 0, & \text{if } G(q) < 0,\\ q, & \text{if } G(q) > 0.\end{cases} \tag{58}$$

It will also be useful to define the *stay-where-you-are* policy by

$$p(t) = P[q(t)]. \tag{59}$$

With slight abuse of notation, we denote this policy by P. With this policy, $q(t) = q(0)$ for all $t > 0$ (see (33)), and its value function is

$$V_P(q) = \frac{P(q)q}{r} = \frac{q^2(1-q)}{r}. \tag{60}$$

Case 1. $0 < q < \sigma$**:** In this case $\pi(q) = 0$, and

$$B_\pi[q, \pi(q)] = -rV_\pi(q) + \lambda(1 - q)V'_\pi(q) = 0, \tag{61}$$

so

$$\begin{aligned} \lambda V'_\pi(q) &= \frac{rV_\pi(q)}{1-q}, &(62a)\\ G(q) &= q^2 - \frac{rV_\pi(q)}{1-q}, &(62b)\end{aligned}$$

and from (60), it follows that

$$G(q) < 0 \Leftrightarrow V_\pi(q) > V_P(q). \tag{63}$$

Suppose that the monopolist uses the policy π for $0 \leq t < u$, and then switches to the "stay-where-you-are" policy P, from then on. Since her price is zero for $0 \leq t < u$, her resulting profit will be

$$g(u) \equiv e^{-ru}V_P[q(u)] = \left(\frac{1}{r}\right) e^{-ru}[q(u)]^2[1-q(u)], \tag{64}$$

where $q(t)$ is determined by the differential equation $q'(t) = 1 - q(t)$ on the interval $[0, T)$, with $q(0) = q$. Note that

$$\begin{aligned} g(0) &= V_P(q), & (65)\\ g(T) &= V_\pi(q), & (66)\end{aligned}$$

where, as before, T is the time at which $q(t)$ reaches the target σ under the policy π. Differentiating (64) with respect to u, and simplifying the resulting expression yields

$$g'(u) = \left(\frac{1}{r}\right) e^{-ru}[q(u)][1-q(u)][2\lambda - (3\lambda + r)q(u)] > 0 \quad \text{for } 0 \leq u < T, \tag{67}$$

since

$$q(u) < \sigma = \frac{2\lambda}{3\lambda + r} \text{ for } 0 \leq u < T. \tag{68}$$

Hence, $g(u)$ is strictly increasing in u and so

$$V_\pi(q) = g(T) > g(0) = V_P(q), \tag{69}$$

and using (63), $B_\pi(q, p)$ is maximized at $p = 0$.

Case 2. $q > \sigma$**:** In this case $\pi(q) = q$. Using an analogous argument, we find that

$$\lambda V_\pi'(q) = \frac{-rV_\pi(q) + q^2}{q}, \tag{70}$$

which leads to a condition similar to (63),

$$G(q) > 0 \Leftrightarrow V_\pi(q) > V_P(q). \tag{71}$$

The analogous expression for g is

$$\begin{aligned} g(u) &\equiv q \int_0^u e^{-rt} q(t) dt + e^{-ru} V_P[q(u)] && (72) \\ &= q \int_0^u e^{-rt} q(t) dt + \left(\frac{1}{r}\right) e^{-ru} [q(u)]^2 [1-q(u)], && (73) \end{aligned}$$

where $q(t)$ is defined by the differential equation

$$q'(t) = -\lambda q(t), \quad q(0) = q \tag{74}$$

in $[0, T)$. Differentiating (73) with respect to u yields:

$$g'(u) = e^{-ru} \left([q(u)]^2 - [q(u)]^2 [1-q(u)] + \frac{1}{r} \left(2q(u) - 3[q(u)]^2 \right) q'(u) \right),$$

which simplifies to:

$$g'(u) = \frac{q(u)}{r} e^{-ru} \left[(3\lambda + r) q(u) - 2\lambda \right] q(u), \tag{75}$$

which is strictly positive, since

$$q(u) > \sigma = \frac{2\lambda}{3\lambda + r} \text{ for } 0 \le u < T. \tag{76}$$

Therefore, $g(u)$ is strictly increasing in u and so

$$V_\pi(q) = g(T) > g(0) = V_P(q), \tag{77}$$

and therefore, $B_\pi(q, p)$ is maximized at $p = q$.

Finally, note that, from (62a) and (70),

$$\begin{aligned} V_\pi'(\sigma^-) &= V_\pi'(\sigma^+) = V_\pi'(\sigma) = \frac{\sigma^2}{\lambda}, && (78) \\ G(\sigma) &= 0, && (79) \end{aligned}$$

so V_π is continuously differentiable for all q, and $B_\pi(\sigma, p)$ is independent of p. Hence B_π satisfies the Bellman Optimality Condition, which completes the proof.

3.3 Concave Distributions of Customer Types

We now extend the example above for any cumulative distribution F of consumer types that is *strictly concave.* While there is no optimal stationary policy in the classical sense, we show that there is an optimal measure-valued policy which is independent of F, and whose structure is very similar to that of the optimal target policy derived in Theorem 2.

Assume that the cumulative distribution function, F, of consumer types is strictly increasing and strictly concave on the unit interval. Recall that the law of motion is

$$m(q,p) = \begin{cases} 0, \quad q=0, \\ \lambda[1-F(p/q)-q], \quad 0<q\leq 1, 0\leq p\leq q, \\ -\infty, \quad 0<q\leq 1, p>q. \end{cases} , \tag{80}$$

We extend the set of permissible controls to include those that specify a probability measure over the set of feasible prices at each time t. Therefore, at any time t, define a *generalized control* as a probability measure $\psi(\cdot;t)$ on $[0,q(t)]$, and Ψ as the set of all such generalized controls. The immediate return from ψ is

$$q(t)\int p d\psi(p;t), \tag{81}$$

the extended law of motion is

$$\overline{m}[q(t),\psi(\cdot;t)] = \begin{cases} 0, \quad q(t)=0, \\ \lambda\left[1-q(t)-\int F[\frac{p}{q(t)}]d\psi(p;t)\right], \quad 0<q(t)\leq 1, \end{cases} \tag{82}$$

and the monopolist's expected profits are

$$\tilde{V}(q,\psi) = \int_0^\infty e^{-rt}q(t)\left(\int p d\psi(p;t)\right)dt, \tag{83}$$

where $q'(t)$ solves

$$q'(t) = m[q(t),\psi(\cdot;t)],\ q(0)=q. \tag{84}$$

Based on Gamkrelidze (1978), we assert that for any $q\in[0,1]$, there exists an optimal generalized control such that

$$V(q) = \sup_{\psi\in\Psi}\tilde{V}(q,\psi), \tag{85}$$

where $V(q)$ maximizes the monopolist's profits. Recall that the Bellmanian function for a deterministic control was

$$B(q,p) = pq - rV(q) + \lambda V'(q)\, 1 - q - F\left(\frac{p}{q}\right). \tag{86}$$

Correspondingly, the Bellmanian function for a measure-valued control μ is:

$$\hat{B}(q,\mu) = q\int p d\mu(p) - rV(q) + \lambda V'(q)\left[1 - q - \int F\left(\frac{p}{q}\right) d\mu(p)\right]. \tag{87}$$

(87) can be rewritten as

$$\hat{B}(q,\mu) = \int B(q,p) d\mu(p). \tag{88}$$

By Blackwell's Theorem, we can restrict our attention to controls that are stationary. Let $\mathcal{M}(q)$ be the set of measures over $[0,q]$, with elements $M[\cdot;q]$, and let $\mu(\cdot;q)$ be a stationary generalized policy that maps the state q to a measure $M[\cdot;q] \in \mathcal{M}(q)$. If $V(q)$ is continuously differentiable, the optimal stationary generalized policy $\mu^*(\cdot;q)$ satisfies the Hamilton-Jacobi-Bellman condition:

$$\mu^*(\cdot;q) = \arg\max_{\mu\in\mathcal{M}(q)} \overline{B}(q,\mu). \tag{89}$$

Let $\Phi(q)$ be the subset of measures in $\mathcal{M}(q)$ such that the probability is concentrated on the endpoints of the interval $[0,q]$, i.e., for which

$$M[\{q\},q] = \phi(q), \tag{90}$$

$$M[\{0\},q] = [1-\phi(q)] \tag{91}$$

Finally, define $\Sigma(s,q)$ to be the subset of $\Phi(q)$ determined by functions $\phi_s(q)$ of the form

$$\phi_s(q) = \begin{cases} 0, & q < s; \\ (1-s), & q = s, \\ 1, & q > s. \end{cases} \tag{92}$$

Call a stationary generalized policy a *generalized target policy with target* s if $\mu(\cdot;q) \in \Sigma(s,q)$ for each q.

Theorem 3 *If the cumulative distribution of consumer types F is strictly concave, then the optimal stationary generalized policy is a generalized target policy with target:*

$$\sigma = \frac{2\lambda}{3\lambda + r}. \tag{93}$$

Proof. Consider any generalized target policy with target s, and let $V_s(q)$ be the corresponding value function. For $q = s$, (82) and (84) yield:

$$q'(t) = \lambda \left[1 - q(t) - \left(sF\left[\frac{0}{q(t)}\right] + (1-s)F\left[\frac{q(t)}{q(t)}\right]\right)\right], \tag{94}$$

or $q'(t) = 0$, and (81) yields an instantaneous payoff of $s^2(1-s)$. Therefore,

$$V_s(s) = \int_0^\infty e^{-rt}s^2(1-s)dt = \frac{s^2(1-s)}{r}. \tag{95}$$

When $q < s$, from (82) and (84):

$$q'(t) = \lambda[1 - q(t)], \tag{96}$$

with initial condition $q(0) = q$, yielding a solution

$$q(t) = 1 - e^{-\lambda t}(1-q). \tag{97}$$

The time T to get to s is therefore

$$T = \frac{1}{\lambda}\log\left(\frac{1-q}{1-s}\right), \tag{98}$$

and the value function is:

$$V_s(q) = s^2(1-s)\left(\int_T^\infty e^{-rt}dt\right), \tag{99}$$

or

$$V_s(q) = \frac{1}{r}\left[(1-q)^{-\left(\frac{r}{\lambda}\right)}\right]\left[s^2(1-s)^{\left(1+\frac{r}{\lambda}\right)}\right], \quad q < s. \tag{100}$$

Similarly, when $q > s$, (82) and (84) yield $q'(t) = -\lambda q(t)$, and a similar sequence of steps yields the value function for $q > s$:

$$V_s(q) = \frac{1}{2\lambda + r}\left(q^2 + q^{-\frac{r}{\lambda}}s^{\left(2+\frac{r}{\lambda}\right)}\left[\frac{2\lambda(1-s) - rs}{r}\right]\right), \quad q > s. \tag{101}$$

As in Section 3.2, that the value of s that maximizes $V_s(q)$ for both $q < s$ and $q > s$ is

$$\sigma = \frac{2\lambda}{3\lambda + r}. \tag{102}$$

After some elementary simplification, the corresponding value function $V_\sigma(q)$ is

$$V_\sigma(q) = \begin{cases} \frac{1}{r}\left[(1-q)^{-\left(\frac{r}{\lambda}\right)}\right]\left[\sigma^2(1-\sigma)^{\left(1+\frac{r}{\lambda}\right)}\right], & q < \sigma, \\ \frac{\sigma^2(1-\sigma)}{r}, \quad q = \sigma, \\ \frac{1}{2\lambda+r}\left(q^2 + \frac{\lambda}{r} q^{-\frac{r}{\lambda}} \sigma^{\left(3+\frac{r}{\lambda}\right)}\right), & q > \sigma. \end{cases} . \tag{103}$$

It is easily verified that $V_\sigma(q)$ is continuous at $q = \sigma$, and

$$V_\sigma'(\sigma^-) = V_\sigma'(\sigma^+) = \frac{\sigma^2}{\lambda}, \tag{104}$$

which verifies that V_σ is continuously differentiable. One also verifies that $V_\sigma'(q) > 0$ for all q (see below).

The Bellmanian (87) corresponding to $V_\sigma(q)$ is

$$\overline{B}(q,\mu) = q \int p d\mu(p) - rV_\sigma(q) + \lambda V_\sigma'(q)\left[1 - q - \int F\left(\frac{p}{q}\right) d\mu(p)\right]. \tag{105}$$

Since F is strictly concave and $V_\sigma'(q) > 0$, for each q the Bellmanian is maximized in μ by taking a $\mu \in \Phi(q)$. By a slight abuse of notation, denote a measure in $\Phi(q)$ by the corresponding probability $\phi(q)$. Thus for a measure $\phi \in \Phi(q)$, the Bellmanian equals

$$\overline{B}(q,\phi) = G(q)\phi - rV(q) + \lambda V'(q)(1-q), \tag{106}$$

where

$$G(q) \equiv q^2 - \lambda V_\sigma'(q). \tag{107}$$

The Bellmanian is therefore linear in ϕ, and consequently, is maximized by $\phi(q) = 0$ for $G(q) < 0$, and by $\phi(q) = 1$ for $G(q) > 0$.

Case 1. $q < \sigma$**:** Differentiating line 1 of (103) with respect to q and simplifying yields:

$$V_\sigma'(q) = \frac{\sigma^2}{\lambda}\left(\frac{1-\sigma}{1-q}\right)^{\left(1+\frac{r}{\lambda}\right)}, \; q < \sigma, \tag{108}$$

and therefore,

$$G(q) = \frac{q^2(1-q)^{(1+\frac{r}{\lambda})} - \sigma^2\,(1-\sigma)^{(1+\frac{r}{\lambda})}}{(1-q)^{(1+\frac{r}{\lambda})}} \text{ for } q < \sigma. \tag{109}$$

Since $q < \sigma$, it follows that $G(q) < 0$, and therefore, $\phi(q) = 0$.

Case 2. $q > \sigma$**:** Differentiating line 3 of (103) with respect to q and simplifying yields:

$$V'_\sigma(q) = \frac{1}{2\lambda + r}\left[2q - \left(\sigma^2\left(\frac{\sigma}{q}\right)^{(1+\frac{r}{\lambda})}\right)\right], q > \sigma, \tag{110}$$

and therefore,

$$G(q) = q^2 - \frac{\lambda}{2\lambda + r}\left[2q - \left(\sigma^2\left(\frac{\sigma}{q}\right)^{(1+\frac{r}{\lambda})}\right)\right] \text{ for } q > \sigma, \tag{111}$$

which simplifies to

$$G(q) = q^2\left(\left[1 - \left(\frac{\sigma}{q}\right)^{3+\frac{r}{\lambda}}\right] - \left[\left(\frac{3+\frac{r}{\lambda}}{2+\frac{r}{\lambda}}\right)\left(\frac{\sigma}{q}\right)\left(1 - \left(\frac{\sigma}{q}\right)^{2+\frac{r}{\lambda}}\right)\right]\right) \text{ for } q > \sigma. \tag{112}$$

This is of the form

$$K[\left(1 - x^{(a+1)}\right) - \frac{a+1}{a}x(1 - x^a)], \text{ with } x = \frac{\sigma}{q},\ a = 2 + \frac{r}{\lambda}. \tag{113}$$

The identity

$$\frac{1 - x^{(a+1)}}{1 - x^a} > \frac{(a+1)}{a}x \text{ for } a > 0,\ 0 < x < 1 \tag{114}$$

establishes that $G(q) > 0$ and therefore, $\phi(q) = 1$.

Finally, from (107) and (104),

$$G(\sigma) = 0, \tag{115}$$

and therefore, $\overline{B}(\sigma, \phi)$ is constant in ϕ. The generalized target policy with target σ therefore satisfies the Hamilton-Jacobi-Bellman condition, which completes the proof. ■

3.4 Convex Distributions of Customer Types: A Conjecture

This section presents a conjecture about the monopolist's optimal price trajectory when the cumulative distribution function, F, of consumer types is strictly convex, and presents a heuristic argument that supports this conjecture. Assume that F is continuously differentiable and strictly increasing on the unit interval. Recall that the law of motion is

$$m(q,p) = \begin{cases} 0, \quad q = 0, \\ \lambda[1 - F(p/q) - q], \quad 0 < q \leq 1, 0 \leq p \leq q, \\ -\infty, \quad 0 < q \leq 1, p > q. \end{cases} \tag{116}$$

and the price path is constrained by

$$0 \leq p(t) \leq q(t). \tag{117}$$

Recall also that Bellmanian Functional for a policy α is defined by

$$B(q,p) = pq - rV_\alpha(q) + V_\alpha'(q)\, m(q,p). \tag{118}$$

(For notational simplicity, we suppress the subscript α on the symbol B.) Also recall that, for all q,

$$B[q, \alpha(q)] = 0, \tag{119}$$

and α is optimal if

$$\alpha(q) = \arg\max_p B(q,p). \tag{120}$$

In the light of (119), an alternative form for the last condition is

$$B(q,p) \leq 0 \text{ for all } q, p. \tag{121}$$

From (116) and (118),

$$B(q,p) = pq - rV_\alpha(q) + \lambda[1 - F(p/q) - q]V_\alpha'(q)\,. \tag{122}$$

The partial derivative of B with respect to p is

$$B_2(q,p) = q - \left(\frac{\lambda}{q}\right) F'\left(\frac{p}{q}\right) V'(q). \tag{123}$$

Hence

$$B_2(q,0) \quad < \quad 0 \Leftrightarrow \frac{V'(q)}{q^2} > \frac{1}{\lambda F'(0+)} \equiv c_0, \tag{124}$$

$$B_2(q,q) \quad > \quad 0 \Leftrightarrow \frac{V'(q)}{q^2} < \frac{1}{\lambda F'(1-)} \equiv c_1. \tag{125}$$

Now *assume that F is convex.* It follows that

$$c_0 \geq c_1. \tag{126}$$

Note that if F is uniform, then $c_0 = c_1$. Let V be the value function of an optimal policy for F.

For two C^2 functions. F and G, on the unit interval, define the distance between them as

$$\rho(F, G) = \max_{n=0,1,2} \max_q |F^{(n)}(q) - G^{(n)}(q)|, \tag{127}$$

where $F^{(n)}$ denotes the n^{th} derivative of F. From (42) and (50), we see that for F uniform,

$$\frac{V'(q)}{q^2} \text{ is strictly decreasing in } q. \tag{128}$$

We conjecture that for F sufficiently close to uniform in the metric ρ, its value function also satisfies (128). Hence, the optimal policy for F, say α, would satisfy

$$\begin{aligned} \alpha(q) &= 0, \quad 0 < q \leq c_0, \\ B_2[q, \alpha(q)] &= 0, \quad c_0 \leq q \leq c_1, \\ \alpha(q) &= q, \quad c_1 < q \leq 1. \end{aligned} \tag{129}$$

We can write the second line of this condition in the form

$$F'\left[\frac{\alpha(q)}{q}\right] = \frac{q^2}{\lambda V'(q)}. \tag{130}$$

Since F is convex, and V satisfies (128), it follows that

$$\frac{\alpha(q)}{q} \text{ is strictly increasing in } q \text{ for } c_0 \leq q \leq c_1. \tag{131}$$

Furthermore,

$$\frac{\alpha(c_0)}{c_0} = 0, \quad \frac{\alpha(c_1)}{c_1} = 1. \tag{132}$$

On the other hand, recall that $P(q)$ is the solution of

$$m[q, P(q)] = 0, \tag{133}$$

or

$$F\left[\frac{P(q)}{q}\right] = 1 - q. \tag{134}$$

Hence

$$\frac{P(q)}{q} \text{ is strictly decreasing in } q. \tag{135}$$

It would now follow from (78), (80), and the continuity of α and P, that, in the nonuniform case ($c_0 < c_1$) there is a unique σ such that $c_0 < \sigma < c_1$,

$$\alpha(q) \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} P(q) \text{ according as } q \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} \sigma, \tag{136}$$

$$m[q, \alpha(q)] \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0 \text{ according as } q \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} \sigma. \tag{137}$$

Hence:

Conjecture 1 *If F is convex, and sufficiently close to uniform in the metric ρ, then there is an optimal policy α that satisfies (129), (136), and (137). In particular, there is a "target demand," σ, such the demand increases (decreases) to σ if the initial demand is less than (greater than) σ, and reaches σ in finite time. Once the demand reaches σ it is stabilized by charging the price $P(\sigma)$. If F is not uniform, then there is a nondegenerate interval $[c_0, c_1]$ such that σ is in the interior of the interval, and when the demand $q(t)$ is in the interior of the interval, the optimal price $\alpha[q(t)]$ is strictly between 0 and $q(t)$.*

4 Extensions of the Model of Myopic Customers

4.1 Myopic and Rational Consumers

We now consider a situation in which some consumers are myopic and some are fully rational. We maintain the assumption that consumers adjust only gradually to price changes in any finite period. Again, we start with a discrete-time model and move to a continuous-time approximation. We maintain the assumption that the distribution of consumer types is uniform on the unit interval.

Formally, suppose that in a period of length h a fraction λh of the consumers myopically adjust their demands, and another fraction μh of the consumers can predict the total demand in the next period. This give rise to the difference equation

$$\begin{aligned} q(t) &= \lambda h \left[1 - \frac{p(t)}{q(t-h)}\right] + \mu h \left[1 - \frac{p(t)}{q(t)}\right] + (1 - \lambda h - \mu h) q(t-h), \\ 0 &< q(t-h), \\ 0 &\leq p(t) \leq \min\{q(t-h), q(t)\}. \end{aligned} \tag{138}$$

Hence

$$q(t) - q(t-h) = \lambda h \left[1 - \frac{p(t)}{q(t-h)}\right] + \mu h \left[1 - \frac{p(t)}{q(t)}\right] - (\lambda h + \mu h) q(t-h), \tag{139}$$

$$\frac{q(t) - q(t-h)}{h} = (\lambda + \mu) - \frac{\lambda p(t)}{q(t-h)} - \frac{\mu p(t)}{q(t)} - (\lambda + \mu) q(t-h). \tag{140}$$

Letting h tend to zero, we get

$$q'(t) = (\lambda + \mu) \left[1 - \frac{p(t)}{q(t)} - q(t)\right]. \tag{141}$$

Thus we get the same law of motion as in the "purely myopic" case ($\mu = 0$), except that λ has been replace by $(\lambda + \mu)$. Hence we see that in this model what matters is the total rate of adaptation, $(\lambda + \mu)$, not how the "adapters" are distributed between myopic and fully rational consumers. This property of the model will no longer be preserved in the next case we consider, that of "stubborn" consumers.

4.2 Myopic and "Stubborn" Consumers

This section describes some properties of the monopolist's optimal price trajectory when consumers are something between being myopic and "stubborn". We assume that consumer types are uniformly distributed, and therefore $F(\theta) = \theta$. Rather than basing their expectation of total demand in the next period on the current period's demand level, consumers who adapt to the monopolist's price announcement partly base their prediction on a stubborn assessment, ω, of the total demand for the good. The extent to which they base their expectation on ω is determined by a parameter γ, where $0 \leq \gamma \leq 1$, and

$$q_E(t,h) = \gamma q(t-h) + (1-\gamma)\omega. \tag{142}$$

Proceeding as in Sections 2.1 and 3.1, it follows that (1)

$$q_E(t) = \gamma q(t) + (1-\gamma)\omega, \tag{143}$$

(2) the law of motion is

$$m(q,p) = \begin{cases} 0, \quad q = 0, \\ \lambda \left[1 - \dfrac{p}{\gamma q + (1-\gamma)\omega} - q\right], \quad 0 < q \leq 1, \quad 0 \leq p \leq \gamma q + (1-\gamma)\omega, \\ -\infty, \quad 0 < q \leq 1, p > q. \end{cases} \tag{144}$$

and (3) the "stay-where-you-are" price is

$$P(q) = (1-q)[\gamma q + (1-\gamma)\omega]. \tag{145}$$

A stationary target policy μ with target σ $(0 < \sigma < 1)$ is defined by

$$\mu(q) = \begin{cases} 0, \quad q < \sigma, \\ P(\sigma), \quad q = \sigma, \\ \gamma q + (1-\gamma)\omega, \quad q > \sigma. \end{cases} \tag{146}$$

For a given γ and ω, let π be the optimal target policy, and denote its target as $\sigma(\gamma)$

Theorem 4 *(a) The monopolist's optimal price trajectory is generated by the target policy with target $\sigma(\gamma)$.*

(b) $\sigma(\gamma)$is strictly increasing in γ, and has the following values at its end points:

$$\sigma(0) = \frac{\lambda}{2\lambda + r}, \tag{147}$$

$$\sigma(1) = \frac{2\lambda}{3\lambda + r}. \tag{148}$$

Proof. (a) Consider an arbitrary target policy with target s. Proceeding as we did in Section 3.1, we shall first characterize the optimal target, starting from $q(0) = q$.

Case 1. $q < s$. Until it $q(t)$ reaches s, $q(t)$ satisfies the differential equation,

$$q'(t) = \lambda[1 - q(t)]. \tag{149}$$

As in Section 3.1, the value function for the target policy with target s is:

$$\begin{aligned} V(q,s) &= \left(\frac{1-s}{1-q}\right)^{\rho} \frac{P(s)s}{r}, \\ \text{where } \rho &= \frac{r}{\lambda}. \end{aligned} \tag{150}$$

Using the above expression for $P(s)$, we have

$$\begin{aligned} V(q,s) &= \frac{f(s)}{r(1-q)^{\rho}}, \text{ where} \\ f(s) &= (1-s)^{\rho+1}\left[\gamma s^2 + (1-\gamma)\omega s\right]. \end{aligned} \tag{151}$$

Hence the target that maximizes $V(q,s)$ is the value of s that maximizes $f(s)$. One verifies that

$$\begin{aligned} f'(s) &= (1-s)^{\rho} G(s), \text{ where} \\ G(s) &= -\gamma(\rho+3)s^2 + [2\gamma - (\rho+2)(1-\gamma)\omega]s + (1-\gamma)\omega. \end{aligned} \tag{152}$$

Note that $f'(s)$ and $G(s)$ have the same sign. Also, G is quadratic and concave, and $G(0) = (1-\gamma)\omega > 0$. Hence f is maximized at the larger of the two roots of $G(s) = 0$. Call this root $\sigma(\gamma)$; it is the optimal target. Note that it is independent of the starting state, q.

We now show that, for $q < \sigma(\gamma)$, the target policy with target $\sigma(\gamma)$ is optimal among all policies. For this purpose, we abbreviate $\sigma(\gamma)$ to σ. The Bellmanian functional for this policy is

$$B(q,p) = pq - rV(q,\sigma) + \lambda\left[1 - \frac{p}{\gamma q + (1-\gamma)\omega} - q\right] V_1(q,\sigma). \tag{153}$$

Differentiating with respect to p, we have

$$B_2(q,p) = q - \frac{\lambda V_1(q,\sigma)}{\gamma q + (1-\gamma)\omega}. \tag{154}$$

From (151), we have

$$V_1(q,\sigma) = \frac{f(\sigma)}{\lambda(1-q)^{\rho+1}},$$

and hence

$$B_2(q,p) = q - \frac{f(\sigma)}{(1-q)^{\rho+1}\left[\gamma q + (1-\gamma)\omega\right]}.$$

It follows that $B_2(q,p) < 0$ if and only if

$$\begin{aligned} (1-q)^{\rho+1}\left[\gamma q^2 + (1-\gamma)\omega q\right] &< f(\sigma), \text{ or} \\ f(q) &< f(\sigma), \end{aligned}$$

which is true for $q < \sigma$. This completes the proof of the optimality of the target policy with target σ in Case 1. The argument for Case 2, $q > \sigma$, is analogous, and is omitted. Finally, one can verify that $V_1(\sigma-,\sigma) = V_1(\sigma+,\sigma)$. This completes the proof of Part (a) of the theorem.

To prove Part (b), write $G(s)$ in (152) in the form

$$\begin{aligned} G(s,\gamma) &= \gamma g_1(s) + (1-\gamma)g_0(s), \text{ where} \\ g_1(s) &= -(\rho+3)s^2 + 2s, \\ g_0(s) &= -(\rho+2)\omega s + \omega. \end{aligned} \tag{155}$$

Recall that $\sigma(\gamma)$ is the larger root of

$$G[s,\gamma] = 0.$$

A standard "comparative statics" calculation yields

$$\sigma'(\gamma) = -\frac{g_1[\sigma(\gamma)] - g_0[\sigma(\gamma)]}{\gamma g_1'[\sigma(\gamma)] + (1-\gamma)g_0'[\sigma(\gamma)]}. \tag{156}$$

Let σ_1 be the positive root of $g_1(s) = 0$ (the other root is 0), and let σ_0 be the root of $g_0(s) = 0$. Then

$$\sigma_1 = \frac{2}{\rho+3}, \quad \sigma_0 = \frac{1}{\rho+2}. \tag{157}$$

Note that

$$\frac{\sigma_1}{2} < \sigma_0 < \sigma_1. \tag{158}$$

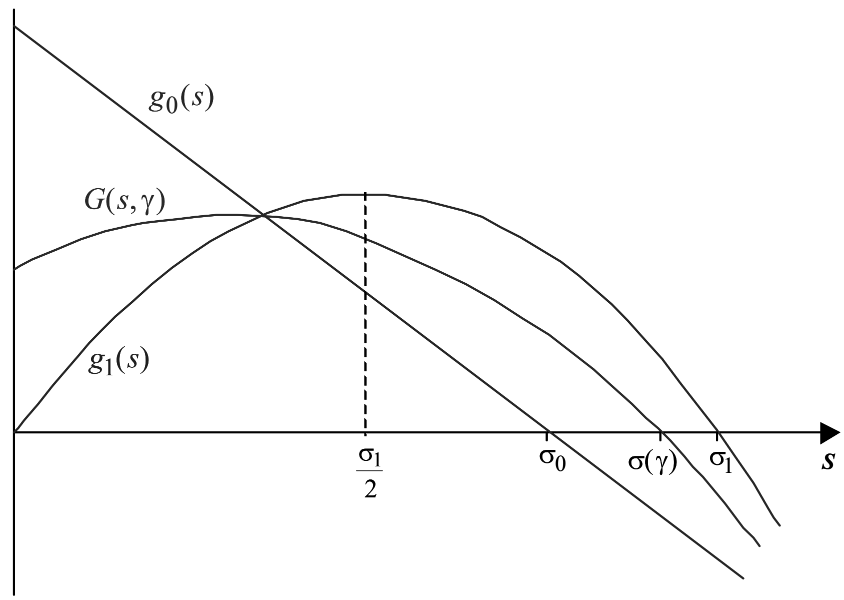

Figure 1: Illustrates the proof of part (b) of Theorem 4.

Note also that

$$\begin{aligned} g_1(s) \text{ is decreasing and positive for } \frac{\sigma_1}{2} &\leq s < \sigma_1; \\ g_0(s) \text{ is decreasing, and is negative for } \sigma_0 &< s \leq \sigma_1, \\ \sigma_0 &< \sigma(\gamma) < \sigma_1 \text{ for } 0 < \gamma < 1 \end{aligned}$$

(see Figure 1). Hence, by (156) and (158), $\sigma'(\gamma) > 0$ for $0 < \gamma < 1$, which completes the proof of Part (b) of the theorem.

It would be desirable to make ω dependent on customer type θ, or on the level of demand when the customer first chooses to subscribe. At this time, we have no comparable results for such an extension.

5 Concluding Remarks

We have explored several variations of a model of optimal dynamic monopoly pricing of a network good in which consumers are boundedly rational. In all cases, the predictions of the model differ significantly from those of the standard model with "unboundedly rational" (UR) consumers, who have "self-fulfilling rational expectations" about the demand at any given price. In particular: (1) the pricing predictions of the BR model are more realistic that those of the UR model, in that the price in the UR model is constant, whereas the BR model pricing policies are dynamic "target policies," in which the monopolist charges a low price until a target market penetration is attained, after which the price is raised to stabilize the demand at the target level; and (2) the optimal market penetration in the UR model is not a steady state of the optimal (dynamic) price trajectory of the BR model, and is typically larger than the optimal target of the BR model. The difference between the optimal target in the BR model and the optimal penetration in the UR model (as well as the corresponding difference in prices) can be significant, depending on the parameters of the model.

In future research we plan to investigate models with (1) competing network goods, (2) consumer decisions based on local network effects, and (3) consumers with noisy observations and adaptive expectations.

6 Bibliographic Notes

An exposition of the theory of rational expectations in economic analysis can be found in Radner (1982). Surveys of models of network effects can be found in Economides (1996) and Farrell and Klemperer (2001). Arthur (1989) provides a simple model of adoption choices between two competing network goods which is closely related to ours in spirit. In his model, myopic consumers make their choices based on the current market share of each good. He shows that over time, the market share of one of the goods will tend to 100%, though one cannot predict ex-ante which of the two goods it would be. Fudenberg and Tirole (2000) model dynamic pricing by a monopolist who sells a network good to overlapping generations of consumers who live for two periods, though they also assume perfect rationality on the part of their consumers. The bounded rationality of agents in our model leads to a demand adjustment process that is "viscous", and is similar in this regard to the model of Radner (2003).

There is also a literature on adaptive expectations, where people base their expectations of what will happen in the future based on what has happened in the past, which preceded (and possibly motivated) modeling rational expectations.

7 References

1. Arthur, B., 1989, Competing Technologies, Increasing Returns and Lock-in by Historical Events. *Economic Journal* 99, 106-131.

2. Economides, N., 1996, The Economics of Networks. *International Journal of Industrial Organization* 14, 673-699.

3. Farrell, Joseph, and P. Klemperer, 2001, Coordination and Lock-in: Competition with Switching Costs and Network Effects. *http://www.paulklemperer.org/*

4. Farrell, J. and G. Saloner, 1985, Standardization, Compatibility, and Innovation. *Rand Journal of Economics* 16, 70-83.

5. Fudenberg, D. and J. Tirole, Pricing a Network Good to Deter Entry. *Journal of Industrial Economics* 48, 373-390.

6. Gamkrelidze, R., 1978, *Principles of Optimal Control Theory.* Plenum, New York.

7. Katz, M. and C. Shapiro, 1985, Network Externalities, Competition and Contracting. *American Economic Review* 75, 424-440.

8. Radner, R. 1982, Equilibrium Under Uncertainty. In K. Arrow and M. Intriligator, eds., *Handbook of Mathematical Economics (*North Holland, Amsterdam), 923-1006.

9. Radner, R. 2003, Viscous Demand. *Journal of Economic Theory* 112, 189-231.