Nonconvex Production Technology and Price Discrimination

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Abstract

We revisit the issue of product line design by a monopolist and extend the model of Mussa and Rosen (1978) in two ways. First, we consider the case in which the unit cost is a nonconvex function of product quality. We show that the firm does not offer those qualities where the unit cost is linear or exceeds its lower convex envelope. Consequently, there are "gaps" in its optimal quality choice. Second, when the firm can offer only a limited number of quality levels (due to possible fixed costs), we characterize the optimal location of these finitely many quality levels. This characterization again has the property that none of these qualities will lie within an interval where the unit cost is linear or exceeds its lower convex envelope. Several implications of the above results are discussed.

Keywords: Product Line Design, Price Discrimination, Product Quality.

JEL Classification: D42, L12, L15.

1 Introduction

Consider a monopolist firm that provides a product at different quality levels. Consumers differ in their willingness to pay for quality, but they all prefer a product of a higher quality. In such a familiar context of vertical differentiation, we extend the seminal Mussa and Rosen (1978) model in two directions. First, the firm's unit cost is a strictly convex function of quality in Mussa and Rosen (1978) (and also the remaining literature on product line design). A central task of the current paper is to relax this assumption and to examine the effects of non-convexity in the firm's unit cost function on its price discrimination and product choice. We show that the firm's optimal price policy is what it would be if the unit cost function were the lower convex envelope of the firm's true cost function. In particular, the firm will not sell in those "anomalous" quality intervals where the unit cost function exceeds its convex envelope. That is, its optimal quality choice contains "holes", which contrasts with the standard conclusion that the firm's quality choice is a continuum (p. 310-311 of Mussa and Rosen 1978, Proposition 6 of Rochet and Chone 1998).

Second, we also examine the firm's quality location problem when it can offer only a limited number of quality levels. Mainly for analytical convenience, the extant literature on product line design suppresses the likely fixed costs associated with offering each quality level, and assumes that the firm offers all possible varieties in a quality continuum (i.e., infinitely many quality levels). Without a constraint on the number of its quality levels, the firm's quality choice is merely an outcome of consumers' self-selection under the optimal price policy. However, casual observations reveal that firms often can afford to offer only a limited number of quality varieties. In such a case, how to locate these quality levels in the firm's quality space becomes an imperative question. We characterize the optimal location of these quality levels, and show that none of them will be located in an anomalous quality interval.

One factor that may lead to nonconvexity in a firm's unit cost function is its installation of multiple distinct types of production technologies. For a given technology, it might be true that the associated unit cost rises more rapidly as product quality increases, and an increasing, convex cost function effectively captures such decreasing returns in quality provision. Nevertheless, firms in many industries frequently employ multiple types of technology to produce the same generic kind of goods. Examples include using digital (analog) machines to make products with high (low) precision requirements, using flexible systems (dedicated assembly lines) in settings requiring high (low) degrees of customization, and using liquid crystal display (i.e., LCD) and cathode ray tube (i.e., CRT) to build displays with high and low degrees of steadiness and clarity, respectively. Based on a distinct engineering principle, each type of technology has its own cost advantage within a certain range of product performance parameters. Even though the cost function associated with each individual technology may be convex, the combination of multiple convex technologies may give an overall unit cost function that is no longer convex over the entire quality domain.

The printer industry is another well known example for utilizing multiple types of technology, such as ink-jet, laser, and dye sublimation. Currently, ink-jet has the lowest cost and is most suitable for low-to-intermediate qualities of black-and-white printing. Laser printers require higher unit costs than ink-jet and are ideal for high-end black-and-white printing. Dye sublimation is the most expensive but provides the best quality, especially for color or photographic printing. Many printer manufacturers install all three types of technology (plus possibly others) to meet the demand for various printing qualities. The readers are referred to Sutton (1998) for additional historical episodes of firms' pursuing multiple "technological trajectories".

Our modeling framework is almost the same as Mussa and Rosen's, except that in our model the unit cost function may exhibit nonconvexity. To ease exposition, we also assume that the distribution of consumer types satisfies a familiar hazard-rate condition so that "bunching" does not arise.

When the unit cost function is nonconvex, a direct analysis of both the pricing and quality location problems will prove very complex. In this paper we use a simple shortcut to tackle both problems. The crux throughout is to regard the lower convex envelope of the true unit cost as a "virtual" unit cost function, and to consider the hypothetical problem in which the firm could produce according to this virtual unit cost function. Since this virtual unit cost function is convex by construction, this hypothetical problem can be readily solved via standard techniques, and its solution is then shown to be optimal for the firm's true problem as well.

Our present paper is related to two streams of literature: product line design and vertical differentiation. Product line design by a monopolist is a widely studied topic in economics (e.g., Mussa and Rosen 1978, Itoh 1983, Maskin and Riley 1984, Gabszewicz et al 1986, Wilson 1993). A key contribution of Mussa and Rosen (1978) is to realize the optimality of inducing different consumers to purchase the same product (also called "bunching") under certain conditions on the distribution of consumer types, and to devise an "ironing" procedure to deal with bunching. Gabszewicz et al (1986) examine how a "natural" monopolist's product line choice may critically depend on the scope of the consumer income distribution. The natural monopolist prices its product line in a manner that keeps out potential entry, and its output is thus fixed. Much of the subsequent research effort down this line has focused on higher dimensional spaces of product attributes and consumer types (e.g., Matthews and Moore 1987, McAfee and McMillan 1988, Wilson 1993, Armstrong 1996, Sibley and Srinagesh 1997, and Rochet and Chone 1998).

The extant papers on product line design usually adopt specific forms of unit cost functions and thus have not fully considered the role of production in designing an optimal screening procedure. For example, the unit cost is an increasing, strictly convex function of quality in Mussa and Rosen (1978) and Rochet and Chone (1998), and is zero for all qualities in Gabszewicz et al (1986).

There also exist oligopoly models in the context of vertical differentiation, e.g., Gabszewicz and Thisse (1979), Shaked and Sutton (1982), Gal-Or (1983), Moorthy (1985), De Fraja (1996), Johnson and Myatt (2003), and Jing (2004). Here a paper more closely related to our current analysis is Johnson and Myatt (2003), which examines how an incumbent firm adjusts the structure of its product line in response to entry. In particular, they identify respective conditions under which the incumbent will add a "fighting brand" to intensify product competition and withdraw a brand too close to the entrant's to avoid head-on product competition.

Section 2 presents the model. Sections 3 derives the optimal price and product policies when the firm does not face a variety constraint. In Section 4, we characterize the optimal location of a finite number of quality levels in a continuous quality domain. Section 5 contains concluding remarks.

2 Model

In this market a monopolist firm provides a product at various quality levels. The firm's quality space is unidimensional and represented by an interval $Q = [0, b]$. The marginal cost of providing a product of quality s is $c(s)$, and is independent of the amount produced at this or any other quality level. There is no fixed cost. We assume that c is nonnegative and twice continuously differentiable, except possibly at finitely many points. We also assume, without essential loss of generality, that

$$c(0) = 0, \quad c(b) < b. \tag{1}$$

We now turn to the model of demand. There is a continuum of consumers, indexed by the real variable θ, called the *consumer's type*. Each individual consumer either does not purchase the product, or purchases exactly one unit. If consumer θ purchases a product of quality s at price $p(s)$, her net utility is

$$U[s, p(s); \theta] = \theta s - p(s). \tag{2}$$

This is the familiar Mussa-Rosen utility function, where θ measures the consumer's marginal valuation of an additional unit of quality. Through proper rescaling, (2) can accommodate the class of utility functions that are multiplicatively separable in consumer type and product quality. Given the price function p, each consumer chooses a quality level that maximizes her utility. In particular, the choice of quality zero generates zero utility, and thus represents not purchasing this product. Assume that consumer type θ is distributed on the unit interval $[0, 1]$ according to a strictly positive probability density function $f(\theta)$. Let

$$J(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)},$$

where $F(\theta) = \int_0^\theta f(x)dx$. Assume that

$$J'(\theta) > 0, \tag{3}$$

which is the familiar hazard-rate condition under which bunching does not arise. This assumption also implies that $J^{-1}(\cdot)$ exists and has a positive first derivative.

3 Exogenous Quality Space

In this Section we assume that the monopolist does not face a variety constraint. The problem of the monopolist is to choose a subset of Q and a price schedule to maximize its profit. In what follows, we shall use the so-called "direct" approach (due to Wilson (1993)) that works with the price schedule, as opposed to the indirect approach that deals with the quality assignment. Once the optimal price policy is characterized, the firm's quality choice is then an immediate outcome of consumer self-selection. In the body of the text, we only sketch the formulation of the firm's problem and state its solution. A precise and expanded treatment of this Section is given in the Appendix.

The firm's price policy is a real-valued function p on Q, which we assume satisfies the following conditions:

$$\begin{aligned} p(s) &\geq 0 \text{ and is nondecreasing,} && (4) \\ p(0) &= 0 \text{ and } p(b) \leq 1. && (5) \end{aligned}$$

Because each consumer's utility is linear in quality, one can show that, without loss of generality, *we can limit our attention to price functions that are convex* (see the Appendix.)

Suppose, for the moment, that p is twice differentiable and its derivative is nondecreasing. From the consumer utility function (2), the optimal quality choice of consumer θ, $\sigma(\theta)$, is simply the solution of the first order condition

$$p'[\sigma(\theta)] = \theta, \tag{6}$$

provided that

$$0 \leq \sigma(\theta) \leq b \text{ and } \theta\sigma(\theta) - p[\sigma(\theta)] > 0.$$

If the consumer is indifferent between multiple qualities, we make the convention that she will choose the lowest such quality. Then since $p'(s)$ is nondecreasing,

$$\sigma(t) \leq \sigma(\theta) \text{ if and only if } t = p'[\sigma(t)] \leq p'[\sigma(\theta)] = \theta.$$

Hence, the mass of consumers who purchase a quality not exceeding s is $F(p'(s))$, and the density of consumers who purchase quality s is $f(p'(s))p''(s)$.

Therefore, the price function $p(s)$ yields the firm a profit

$$\pi(p) = \int_0^b [p(s) - c(s)] f(p'(s)) p''(s) ds. \tag{7}$$

In fact, one cannot generally assume that the optimal price function will have the nice properties we have just assumed for it. Essentially, one can only require that the price function be sufficiently "regular" so that the profit is well defined. A precise statement of the class of "admissible" price functions is given in the Appendix.

Next, we only describe the firm's optimal price policy; the detailed analysis is relegated to the Appendix. Let w denote the largest convex function that lies below the cost function c. Formally, let $\ominus$ denote the set of all convex functions φ on $[0, b]$ such that

$$\varphi(s) \leq c(s) \text{ for all } s \text{ in } [0, b],$$

and define w by

$$w(s) = \max \{\varphi(s) | \varphi \in \ominus\}. \tag{8}$$

We call w the *lower convex envelope of* c. One can show that w is convex and nondecreasing, and that $w(0) = 0$. We call a quality level s *extreme* if it is not contained in an open interval on which w is linear. We also call an interval on which $c(s) > w(s)$ *anomalous*. Note that by construction $w(s)$ is linear in an anomalous quality interval, and that at the extreme quality levels, $w(s) = c(s)$.

Theorem 1 *The optimal price function is given by*

$$p^*(s) = \int_0^s J^{-1}(w'(x)) dx. \tag{9}$$

Furthermore, demand is zero except at the extreme quality levels.

Because of the consumer utility function (2), by choosing a price function that is linear on a particular interval, the firm has effectively chosen not to offer the quality levels in that interval. Clearly, the optimal price function is linear where $w(s)$ is linear. Theorem 1 thus implies that, when the unit cost is not a convex function of quality, the monopolist will choose not to offer one or more intervals of quality. This directly contrasts with the previously known result that the quality choice of a monopolist with a strictly convex cost function has no gaps (Mussa and Rosen 1978, and Rochet and Chone 1998).

When the consumer types are uniformly distributed on $[0, 1]$, the monopolist's optimal price schedule simply becomes

$$p^*(s) = \frac{s + w(s)}{2}.$$

A second implication of Theorem 1 relates to technology utilization by a firm employing multiple individually convex production technologies.

Corollary 2 *Suppose the monopolist employs I production technologies, each represented by a convex unit cost function c_i on $(0, b]$. Then its optimal quality choice is a collection of intervals such that on each interval, for some technology i, $c_i(s) = w(s)$, where w is the lower convex envelope of $c(s) = \min\{c_i(s), i = 1, ..., I\}$.*

The proof is immediate from Theorem 1. According to this Corollary, the firm should use technology i to produce those quality intervals where $c_i(s) = w(s)$. A technology for which no such interval exists may be called *obsolescent*. Note that, in our model a technology may become obsolescent even if it still possesses a cost advantage over certain quality levels; this happens for technology i if $c_i(s) > w(s)$ holds for all s.

Theorem 1 also implies a fairly general sufficient condition for price discrimination to be suboptimal for the monopolist. When $c(s) > w(s)$ on $(0, b)$, the monopolist's quality choice degenerates to a singleton $\{b\}$, namely the upper bound of the quality space. Alternatively, a corner solution is optimal as long as the cost function $c(s)$ lies above the straight line connecting $(0, 0)$ and $(b, c(b))$ (Here the straight line is the lower convex envelope $w(s)$), *regardless of the curvature of* $c(s)$. This corresponds to the previous results by Stokey (1979), Salant (1989), and Johnson and Myatt (2003). In a intertemporal monopoly that can potentially discriminate on delivery time (with a sooner delivery meaning a higher quality), Stokey (1979) shows that under certain cost conditions the firm may choose only its earliest feasible delivery date and thus does not invoke price discrimination. Also in settings of vertical differentiation, Salant (1989) and Johnson and Myatt (2003) show that the monopolist will sell only its highest feasible quality when there are increasing returns to quality (i.e., when the unit cost function is sufficiently flat).

4 Locating a Finite Number of Qualities

In Section 3, we have examined the monopolist's quality choice when it does not face a variety constraint. To benchmark with the extant literature, the preceding analysis has ignored any likely fixed costs required for offering each quality. When such fixed costs are taken into account, however, the firm can only afford to offer a limited number of qualities, despite the fact that it is technologically feasible to produce at any level in the quality continuum. The following question then arises naturally : How should the firm locate a finite number of qualities in the continuous quality space? The current Section aims to answer this question.

The monopolist has to locate K (a finite number) distinct quality levels labeled $0 < s_1 < ... < s_K$ in Q. We proceed in two stages. First, we determine the optimal pricing policy for any K given quality levels.

4.1 Pricing a Finite Set of Qualities

Suppose for now that the quality levels $s_1, ..., s_K$ are fixed. Without any confusion, let c_k denote the marginal cost at quality s_k, $c(s_k)$. To ease exposition, we introduce a zero-quality product $s_0 = 0$, which costs the firm nothing to produce, i.e., $c_0 = 0$. Let $\mathbf{s} = \langle s_0, s_1, ..., s_K \rangle$ and $\mathbf{c} = \langle c_0, c_1, ..., c_K \rangle$. Define

$$d_k = \frac{c_k - c_{k-1}}{s_k - s_{k-1}}, \quad 1 \leq k \leq K. \tag{10}$$

These d_k are the "slopes" of the cost vector. Since the unit cost function $c(s)$ need not be convex in our model, the cost vector $\mathbf{c}$ may be such that d_k is not nondecreasing in k.

The firm charges a price, p_k, for each product of quality s_k. We make the convention that s_0 is offered for free, i.e., $p_0 = 0$. Given the price vector $\mathbf{p} = \langle p_0, p_1, ..., p_K \rangle$, each consumer chooses a quality level that maximizes her net utility. Again, a consumer choosing s_0 does not purchase.

Define

$$\theta_k = \frac{p_k - p_{k-1}}{s_k - s_{k-1}}, \quad 1 \leq k \leq K. \tag{11}$$

The variables θ_k are the "slopes" of the price vector. Observe that each product has a non-negative demand if and only if θ_k is nondecreasing in k. Hence the price vector $\mathbf{p}$ is called *admissible* if

$$0 \leq \theta_1 \leq ... \leq ...\theta_K \leq 1. \tag{12}$$

For an admissible price vector $\mathbf{p}$, the demand for each quality s_k $(1 \leq k \leq K)$ is $F(\theta_{k+1}) - F(\theta_k)$, where $\theta_{K+1} = 1$. The firm chooses an admissible price vector $\mathbf{p}$ to maximize its profit:

$$\textbf{Problem P}. \ \underset{\mathbf{p}}{\text{Max}} \ \pi_c(\mathbf{p}) = \sum_{1}^{K} (p_k - c_k)[F(\theta_{k+1}) - F(\theta_k)]. \tag{13}$$

In what follows, it will be useful to express (13) in terms of the variables θ_k.

Lemma 3 *The objective in Problem P is equivalent to*

$$\pi_c(\mathbf{p}) = \sum_1^K (s_k - s_{k-1})(\theta_k - d_k)\left[1 - F(\theta_k)\right]. \tag{14}$$

The proof of the Lemma is relegated to the Appendix. For a strictly convex cost function, Itoh (1983) first used a profit formula similar to (14) to analyze the impacts of adding a new product by the monopolist (representing finer market segmentation) on the prices of its existing products, and consequently on consumer welfare.

If the "cost slopes" d_k are non-decreasing in k, then maximizing (14) pointwise will give the optimal "price slopes" θ_k that are also non-decreasing in k, and thus the resulting price vector will be admissible. However, since the unit cost function $c(s)$ is nonconvex in our model, the cost vector $\mathbf{c}$ may be such that d_k are not non-decreasing in k. This means that maximizing (14) pointwise does not yield an admissible solution in general.

To obtain an admissible price policy for Problem P, we next construct and consider an alternative, virtual problem of the firm (Problem V), which has an admissible solution. We then show that the solution to Problem V is also optimal for Problem P.

Let H be the convex hull of the pairs (s_k, c_k), $0 \leq k \leq K$, and let v be the function whose graph is the lower boundary of H. Clearly, $v(s_0) = c_0$ and $v(s_k) \leq c_k$ for $k \geq 1$. In Problem V, the firm were able to produce according to the hypothetical unit cost $v(s_k)$, instead of c_k, and had to choose a price vector $\mathbf{p}$ for the given quality vector $\mathbf{s}$ to maximize its profit:

$$\textbf{Problem V}.\ \operatorname*{Max}_{\mathbf{p}} \pi_v(\mathbf{p}) = \sum_1^K (p_k - v(s_k))[F(\theta_{k+1}) - F(\theta_k)]. \tag{15}$$

Define

$$g_k = \frac{v(s_k) - v(s_{k-1})}{s_k - s_{k-1}}, \quad 1 \le k \le K.$$

Note that g_k is non-decreasing in k by construction of v. Using an argument similar to Lemma 3, we rewrite (15) as

$$\pi_v(\mathbf{p}) = \sum_1^K (s_k - s_{k-1})(\theta_k - g_k)\left[1 - F(\theta_k)\right]. \tag{16}$$

Maximizing (16) pointwise with respect to θ_k yields

$$\theta_k^* - \frac{1 - F(\theta_k^*)}{f(\theta_k^*)} = g_k, \tag{17}$$

or equivalently

$$\theta_k^* = J^{-1}(g_k), \tag{18}$$

from which the optimal price vector for Problem V follows:

$$p_k^* = \sum_1^k J^{-1}(g_i)(s_i - s_{i-1}), \quad 1 \le k \le K. \tag{19}$$

Since $J^{-1}(\cdot)$ is strictly increasing, θ_k^* is non-decreasing in k, and therefore p_k^* is admissible.

From (17) and (18), we also have

$$\left[J^{-1}(g_k) - g_k\right] f(J^{-1}(g_k)) = 1 - F(J^{-1}(g_k)). \tag{20}$$

We are now in a position to characterize the optimal price vector for $\mathbf{s}$.

Theorem 4 $\mathbf{p}^*$ *is an optimal price vector for Problem P. In particular, the demand for* s_k *is positive only if* $g_k < g_{k+1}$.

Proof: Denote π_c^* and π_v^* as the maximum profit attainable in Problems P and V, respectively. First, $\pi_v^* \ge \pi_c^*$. This is because, for any admissible price vector $\mathbf{p}$, $\pi_v(\mathbf{p}) \geq \pi_c(\mathbf{p})$ as $c_k \geq v(s_k)$ by construction of v.

Next, for any quality s_k where $g_k = g_{k+1}$, we have $\theta_k^* = \theta_{k+1}^*$ under the price vector $\mathbf{p}^*$ (by (18)). Therefore, in Problem V the demand for such quality levels is zero. That is, the demand for s_k is positive only if $g_k < g_{k+1}$. Note that $g_k < g_{k+1}$ implies $c_k = v(s_k)$. Therefore the profit π_v^* is also attainable with the firm's true cost vector $\mathbf{c}$ and price vector $\mathbf{p}^*$. That is, $\pi_v^* = \pi_c^*$.

This shows that $\mathbf{p}^*$ solves the monopolist's real problem, Problem P. Q.E.D.

When g_k is strictly increasing in k, we have $c_k = v(s_k)$ for all k, and thus d_k must be strictly increasing in k. Therefore, Theorem 4 implies that *d_k must be strictly increasing in k in any nondegenerate location of these K quality levels* (in the sense that each quality level attracts positive demand under optimal pricing.)

In essence, this Subsection is the discrete analogue of Section 3. Comparing Theorems 1 and 4, we see that the optimal price policies have the same spirit for both continuous and discrete quality sets.

4.2 Quality Location

We now turn to the firm's problem of locating $\mathbf{s}$ in Q. By Theorem 4, plugging $\mathbf{p}^*$ ((19)) into Problem P gives the firm's profit as a function of $\mathbf{s}$:

$$\textbf{Problem P}'\text{: } \underset{\mathbf{s}}{\mathrm{Max}}\ \Pi_c(\mathbf{s}) = \sum_1^K (p_k^*(\mathbf{s}) - c(s_k))[F(J^{-1}(g_{k+1})) - F(J^{-1}(g_k))]. \tag{21}$$

To solve Problem P$'$, we first consider the hypothetical quality location problem (Problem V$'$) in which the firm could produce according to the virtual cost function $w(s)$, the lower convex envelope of $c(s)$. Let H' denote the convex hull of the pairs $(s_k, w(s_k))$, $0 \leq k \leq K$, and let v be the function whose graph is the lower boundary of H'. Since $w(s)$ is convex by construction, we have $v(s_k) = w(s_k)$ for all k.

Define

$$h_k = \frac{w(s_k) - w(s_{k-1})}{s_k - s_{k-1}}, \quad 1 \leq k \leq K. \tag{22}$$

Therefore, if the firm were to produce according to the virtual cost function $w(s)$, then the optimal price for each quality s_k would simply be

$$p'_k(\mathbf{s}) = \sum_1^k J^{-1}(h_i)(s_i - s_{i-1}), \quad 1 \leq k \leq K,$$

(cf. (19)), and Problem V′ may be formulated as

Problem V′: $\displaystyle \operatorname*{Max}_{\mathbf{s}} \Pi_w(\mathbf{s}) = \sum_1^K (p'_k(\mathbf{s}) - w(s_k))[F(J^{-1}(h_{k+1})) - F(J^{-1}(h_k))],$ (23)

or equivalently (by an argument analogous to Lemma 3):

$$\operatorname*{Max}_{\mathbf{s}} \Pi_w(\mathbf{s}) = \sum_1^K (s_k - s_{k-1})(J^{-1}(h_k) - h_k)\left[1 - F(J^{-1}(h_k))\right]. \tag{24}$$

The following equation would also hold valid (cf. (20)):

$$\left[J^{-1}(h_k) - h_k\right] f(J^{-1}(h_k)) = 1 - F(J^{-1}(h_k)). \tag{25}$$

This entity plays a critical role in the subsequent analysis.

As we have seen in Section 3, absent the variety constraint, the firm will not sell a quality level in an anomalous interval. The next Theorem shows that this result remains valid when the firm can offer only K quality levels in the quality domain.

Theorem 5 *An optimal interior solution* $\mathbf{s}^*$ *for Problem P′ is characterized* *by: for* $1 \leq k \leq K-1$*,*

$$[1-F(\theta_k^*)]\theta_k^* - [1-F(\theta_{k+1}^*)]\theta_{k+1}^* = [F(\theta_{k+1}^*) - F(\theta_k^*)]w'(s_k^*), \tag{26}$$

and

$$\theta_K^* = w'(s_K^*), \tag{27}$$

where $\theta_k^* = J^{-1}(h_k)$*. Furthermore, each* s_k^* *must be an extreme quality.*

Proof: We first show that $\mathbf{s}^*$ as characterized in (26) and (27) is an optimal interior solution to Problem V′, and then show that it is also optimal for Problem P′.

Differentiating (24) with respect to s_k and cancelling terms with (25), we have, for $1 \leq k \leq K-1$,

$$\begin{aligned} \frac{\partial \Pi_w(\mathbf{s})}{\partial s_k} &= [1-F(J^{-1}(h_k))][J^{-1}(h_k) - w'(s_k)] \\ &\quad + [1-F(J^{-1}(h_{k+1}))][w'(s_k) - J^{-1}(h_{k+1})] \end{aligned} \tag{28}$$

and

$$\frac{\partial \Pi_w(\mathbf{s})}{\partial s_K} = [1-F(J^{-1}(h_K))][J^{-1}(h_K) - w'(s_K)]. \tag{29}$$

Since $\Pi_w(\mathbf{s})$ is continuous, an optimal solution to Problem V′, $\mathbf{s}^*$, always exists. The characterization of an interior solution $\mathbf{s}^*$ (as given in (26) and (27)) follows from rearranging the first order conditions ((28) and (29)). In fact, it has the following property.

Lemma 6 *In the optimal solution to Problem V′, no quality level is in an open interval where* w *is linear.*

The proof of the Lemma is given in the Appendix.

Denote Π_c^* and Π_w^* as the maximum profits attainable in Problems P′ and V′, respectively. We have $\Pi_w^* \geq \Pi_c^*$. This is because, for any given quality vector $\mathbf{s}$ and a corresponding price vector $\mathbf{p}$, producing with w yields a profit at least as large as producing with c does (cf. (13)), because $w(s) \leq c(s)$ by construction.

Recall that w is linear in an anomalous interval (where $c(s) > w(s)$). By Lemma 6, at each s_k^* we must have $c(s_k^*) = w(s_k^*)$. This implies $\Pi_w^* = \Pi_c^*$. Therefore, the solution to Problem V$'$, $\mathbf{s}^*$, must also be optimal for Problem P$'$. The second statement of the Theorem follows directly from Lemma 6. Q.E.D.

Since the demand for a product of quality s_k^* is $F(\theta_{k+1}^*) - F(\theta_k^*)$, the RHS of (26) above represents the marginal costs due to an additional increment in s_k^*. The first term on the LHS ($[1 - F(\theta_k^*)]\theta_k^*$) is the increase in revenue from consumers purchasing a quality equal to or above s_k, due to an additional increment in s_k. An increment in s_k would also make it more attractive to the original purchasers of s_{k+1},..., s_K, and thus compete away sales and reduce the revenue from these qualities above s_k. The second term on the LHS captures such a revenue reduction, and reflects the spirit of "upstream interference" as first pointed out by Mussa and Rosen (1978). Equation (26) thus equates the marginal revenue from quality s_k to its marginal cost, and (27) has a similar interpretation.

Theorem 5 has only considered the case of an interior solution ($0 < s_K^* < b$). Since the function $J(\cdot)$ is defined on $[0, 1]$, $J^{-1}(\cdot)$ is bounded between 0 and 1. An interior solution obtains when $w'(b-) \geq 1$. To see this, simply note that $\frac{\partial \Pi_w(\mathbf{s})}{\partial s_K}|_{S_K=b} \leq 0$ when $w'(b-) \geq 1$.

Note that, s_K^* may also occur at the corner (i.e., $s_K^* = b$). A sufficient condition for $s_K^* = b$ is $w'(b-) < 1$ and $J^{-1}(w(s)/s) - w'(s) > 0$ on $(0, b]$. When $w'(b-) < 1$, we always have $h_K < 1 = J(1)$ or $J^{-1}(h_K) < 1$, which implies $1 - F(J^{-1}(h_K)) > 0$. When $J^{-1}(w(s)/s) - w'(s) > 0$ on $(0, b]$, we have

$$J^{-1}(h_K) = J^{-1}\left(\frac{w(s_K) - w(s_{K-1})}{s_K - s_{K-1}}\right) > J^{-1}\left(\frac{w(s_K)}{s_K}\right) > w'(s_K).$$

We therefore have $\frac{\partial \Pi_w(\mathbf{s})}{\partial s_K} > 0$ on $(0, b]$, which implies $s_K^* = b$ must hold.

As an example, when consumer types are uniformly distributed on $[0, 1]$, (26) and (27) simply reduce to

$$w'(s_k) = \frac{1}{2}(h_{k+1} + h_k),\ \ k < K,$$

and

$$w'(s_K) = \frac{1}{2}(1 + h_K),$$

respectively.

From the proof of Theorem 5, the problem of locating K qualities for a nonconvex cost function $c(s)$ (Problem P′) is equivalent to locating K qualities for its lower convex envelope (Problem V′).

Another property of the quality location problem is its supermodularity.

Theorem 7 $\Pi_w(\mathbf{s})$ *in Problem V′ is a supermodular function.*

Proof: We only need to show that the cross partial derivatives of $\Pi_w(\mathbf{s})$ w.r.t. qualities are nonnegative (Topkis 1978): For $k < K$,

$$\begin{aligned}
\frac{\partial^2 \Pi_w(\mathbf{s})}{\partial s_k \partial s_{k+1}} &= -f(J^{-1}(h_{k+1}))(J^{-1}(h_{k+1}))' \frac{\partial h_{k+1}}{\partial s_{k+1}}[w'(s_k) - J^{-1}(h_{k+1})] \\
&\quad -[1 - F(J^{-1}(h_{k+1}))](J^{-1}(h_{k+1}))' \frac{\partial h_{k+1}}{\partial s_{k+1}} \\
&= f(J^{-1}(h_{k+1}))(J^{-1}(h_{k+1}))' \frac{\partial h_{k+1}}{\partial s_{k+1}}[h_{k+1} - w'(s_k)] \\
&> 0,
\end{aligned}$$

where the second equality is derived after collecting terms with (25), and the inequality is due to $\partial h_{k+1}/\partial s_{k+1} > 0$ and $h_{k+1} > w'(s_k)$.

For $|k-j| > 1$, we have

$$\frac{\partial^2 \Pi_w(\mathbf{s})}{\partial s_k \partial s_j} = 0.$$

Q.E.D.

An implication of Theorem 7 is that a local cost reduction around quality s_k may lead to an upward adjustment not only to s_k, but also to the remaining quality levels.

5 Conclusion

This paper has extended the theory of product line design along two directions: a nonconvex unit cost function and endogenous location of a finite number of quality levels. To the best of our knowledge, these two directions are still underexplored in the extant literature. When the monopolist does not face a variety constraint, its quality choice consists only of those "extreme" quality levels (i.e., quality levels where the lower convex envelope of the unit cost function is nonlinear.) Thus, there are gaps in its product line. This result also has implications for: (1) utilization of multiple individually convex technologies; and (2) conditions under which price discrimination is suboptimal (i.e., when the monopolist should "bunch" all participating consumers to a single quality level.)

Another contribution of this paper is characterizing how the monopolist should position a finite number (K) of products in the quality space, since most firms in reality will face such a variety constraint. We see that the optimal locations of these K products are again "extreme" qualities. The following two types of quality region are therefore excluded: (1) the anomalous intervals where the unit cost lies strictly above its lower convex envelope; and

(2) those quality intervals where the unit cost function is linear. Note that *this result does not follow directly from the pricing policy.* Recall that the pricing policy (Theorem 4) only entails that the slopes of the cost trajectory formed by the K qualities are strictly increasing. One could still obtain a cost trajectory with increasing slopes even with one or more qualities located in the anomalous intervals.

We conclude by discussing certain assumptions in the model. Our Section 3 uses the assumption that the cost function is twice differentiable except at (at most) finitely many points. This assumption facilitates exposition but does not appear essential. In fact, the analysis in Section 3 holds for cost functions with kinks or jumps, including step functions.

A limitation is that our results are confined subject to the class of utility functions that are multiplicatively separable in quality and consumer type. It would be useful to extend the current analysis to more general classes of utility functions. However, such an extension would undoubtedly be much more complex. Largely for the same reason, a vast majority of the literature on screening and vertical differentiation has also focused on multiplicatively separable utilities.

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6 Appendix

A1. Expanded Treatment of Section 3: Exogenous Quality Space

Here we complete the description of the firm's problem in Section 3, and give the details of the analysis. First, we assume that *the price function, p, is* *nonnegative and Lebesgue measurable on the interval* $[0, b]$*.* Given the utility functions (2) of the consumers, there is no loss of generality in assuming that *p is monotone nondecreasing, and*

$$p(0) = 0, \; p(b) \leq 1. \tag{30}$$

Furthermore, since the utility functions are linear in the quality level s, *the firm can restrict its price functions to be convex.* [To see this, let L_p denote the lower convex envelope of p; then if $p(s) > L_p(s)$, no consumer type will demand quality s.]

The following standard proposition about convex functions will be useful (see, e.g., Royden, 1988, Prop. 17, pp. 113-114).

Proposition 8 *If p is convex on $[0, b]$, then it is absolutely continuous. Its right-and left-hand derivatives exist at each point, and are equal to each other except possibly on a countable set. The left- and right-hand derivatives are monotone nondecreasing functions, and at each point the left-hand derivative is less than or equal to the right-hand derivative.*

In view of the Proposition, we shall make the convention that the derivative of the price function, which we shall denote by G, is its right-hand derivative, and so it is continuous from the right. With this convention,

$$p(s) = \int_0^s G(t)dt. \tag{31}$$

We next characterize the measure of the consumers who purchase a quality not exceeding s. For the moment, fix a convex price function, say p, and let G denote its derivative. Since p is convex, the net utility of consumer θ is concave as a function of quality s, and the right-hand derivative of her net utility function at s is

$$\theta - G(s).$$

There are three cases to consider. First, suppose that there is an interior s^* such that $G(s^*-) \leq \theta \leq G(s^*)$ and s^* is the minimum of such values. Then the consumer will purchase s^*, provided

$$\theta s^* - p(s^*) > 0.$$

Second, if $\theta \leq G(0)$, then the consumer will purchase quality level 0. Finally, if $G(b-) \leq \theta \leq 1$, then the consumer will purchase quality level b, provided

$$\theta b - p(b) > 0.$$

In this case, the mass of consumers purchasing quality level b is $1 - G(b-)$.

A price function p is called *admissible* if it satisfies

$$p(0) = 0,\ G(0) \geq 0,\ G(b) = 1,\ \text{and} \tag{32}$$

$$G(s) \text{ is nondecreasing in } s. \tag{33}$$

The assumption that $G(b) = 1$ is only a convention, since it does not affect the price function, but with this convention the mass of consumers purchasing quality level b is equal to $G(b) - G(b-)$.

It now follows that, since θ is distributed on $[0, 1]$ according to c.d.f. F, for an admissible price function the measure of the set of consumers who purchase a quality level not exceeding s is equal to $F(G(s))$. Therefore, the firm's profit is

$$\pi_c(p(s)) = \int_0^b [p(s) - c(s)]dF(G(s)). \tag{34}$$

[Cf. equation (7) of Section 3.]

We assume here that the unit cost function, c, is nonnegative and continuous. We also assume, without essential loss of generality, that

$$c(0) = 0,\ c(b) < b.$$

Observe that these assumptions are weaker than those made in Section 3. As in Section 3, let w denote the lower convex envelope of c. Let H denote the set of points (s, y) that lie above the graph of w, i.e.,

$$H = \{(s, y) | 0 \leq s \leq b \text{ and } y \geq w(s)\}. \tag{35}$$

Note that H is a closed convex set. Recall that an *extreme point of* H is a point in H that is not a nondegenerate convex combination of two other distinct points of H. The following two facts about extreme points of H will be useful, and are stated without proof.

Lemma 9 *The pair $(s, w(s))$ is an extreme point of H if and only if there exist no s', s'' such that $s' < s < s''$ and h is linear on $[s', s'']$. Furthermore, if $(s, w(s))$ is an extreme point of H, then $w(s) = c(s)$.*

Referring to the terminology of Section 3, the preceding lemma implies that a quality level s is extreme if and only if $(s, w(s))$ is an extreme point of H.

Define π_c^* to be the maximum profit the firm can obtain with the cost function c. Analogously, let π_w^* be the maximum profit the firm could obtain if its cost function were w. Since we have not yet shown that these maxima exist, to be precise we define

$$\pi_c^* = \sup\{\pi(p) | p \text{ is admissible and the cost function is } c\}, \tag{36}$$

$$\pi_w^* = \sup\{\pi(p) | p \text{ is admissible and the cost function is } w\}. \tag{37}$$

Since $w \leq c$, it follows that

$$\pi_w^* \geq \pi_c^*. \tag{38}$$

We shall now show that $\pi_w^* = \pi_c^*$. Let $\pi_w(p)$ denote the firm's profit if its price function were p and its cost function were w. Thus

$$\begin{aligned}\pi_w(p) &= \int_0^b [p(s) - w(s)]dF(G(s)) \\ . &= \int_0^b p(s)dF(G(s)) - \int_0^b w(s)dF(G(s)).\end{aligned}$$

Since w is convex and increasing, the Proposition is applicable; let its derivative be denoted by w'. Since G is right-continuous and F is continuous, $F(G(\cdot))$ is also right-continuous. Integrating by parts,

$$\begin{aligned}\pi_w(p) &= p(b)F(G(b)) - p(0)F(G(0)) - \int_0^b F(G(s))p'(s)ds \\ &\quad -w(b)F(G(b)) + w(0)F(G(0)) + \int_0^b F(G(s))w'(s)ds.\end{aligned}$$

For the validity of the formula for integration by parts in this context, see Rudin (1964). However,

$$\begin{aligned}p' &= G. \\ p(b) - w(b) &= \int_0^b [G(s) - w'(s)]ds \\ p(0) &= w(0) = 0, \\ G(b) &= 1.\end{aligned}$$

Hence, after making the appropriate substitutions and collecting terms, we arrive at:

Lemma 10

$$\pi_w(p) = \int_0^b [G(s) - w'(s)][1 - F(G(s))]ds. \tag{39}$$

Maximizing the right-hand side of (39) pointwise with respect to $G(s)$, we get

$$G^*(s) - \frac{1 - F(G^*(s))}{f(G^*(s))} = w'(s),$$

or equivalently

$$G^*(s) = J^{-1}(w'(s)). \tag{40}$$

Integrating the last gives the price function

$$p^*(s) = \int_0^s J^{-1}(w'(x))dx, \tag{41}$$

which is optimal for the cost function w. (Note that to determine the constant of integration we have used the condition that $p(0) = 0$.)

Now recall that w is linear in any anomalous interval, and hence so is p^*. Hence demand is zero except where $(s, w(s))$ is extreme. Furthermore, at any extreme point, $w(s) = c(s)$, and so

$$\pi_w^* = \pi_w(p^*) = \pi_c(p^*) \leq \pi_c^*.$$

But we previously showed that $\pi_w^* \geq \pi_c^*$ [cf. (38)], and so

$$\pi_w^* = \pi_c^*. \tag{42}$$

We thus have shown that $p^*(s)$ as given in (41) is also optimal for the cost function c, proving the first statement of Theorem 1. The second statement of Theorem 1 is obvious considering the consumer utility function (2).

A2. Proof of Lemma 3.

Lemma 3.

$$\pi_c(\mathbf{p}) = \sum_1^K (s_k - s_{k-1})(\theta_k - d_k)\left[1 - F(\theta_k)\right].$$

Proof : Let

$$x_k = p_k - c_k.$$

With this notation, (13) becomes

$$\pi(\mathbf{p}) = \sum_1^K x_k[F(\theta_{k+1}) - F(\theta_k)],$$

where $\theta_{K+1} = 1$. Rearranging terms, one has

$$\sum_1^K x_k[F(\theta_{k+1}) - F(\theta_k)] = -\sum_1^K (x_k - x_{k-1})F(\theta_k) + x_K.$$

(This is an analogue of integration by parts for finite sums.) Hence

$$\pi(\mathbf{p}) = -\sum_1^K (x_k - x_{k-1})F(\theta_k) + p_K - c_K.$$

Observe that

$$\begin{aligned} x_k - x_{k-1} &= (s_k - s_{k-1})(\theta_k - d_k), \\ p_K - c_K &= \sum_1^K (s_k - s_{k-1})(\theta_k - d_k). \end{aligned}$$

These, together with the preceding equation for the profit, lead immediately to the conclusion of the lemma. Q.E.D.

A3. Proof of Lemma 6.

Proof: We prove this Lemma by showing that $\Pi_w(\mathbf{s})$ is convex in s_k in any open interval where w is linear. Differentiating (28) with respect to s_k and simplifying with (25), we see that for $1 \le k \le K-1$, when $w''(s_k) = 0$,

$$\begin{aligned} \frac{\partial^2 \Pi_w(\mathbf{s})}{\partial s_k^2} &= (J^{-1}(h_k))' \frac{\partial h_k}{\partial s_k} f(J^{-1}(h_k))[w'(s_k) - h_k] \\ &\quad + (J^{-1}(h_{k+1}))' \frac{\partial h_{k+1}}{\partial s_k} f(J^{-1}(h_{k+1}))[h_{k+1} - w'(s_k)]. \end{aligned}$$

We can readily verify the following: $\partial h_k / \partial s_k > 0$, $\partial h_{k+1}/\partial s_k > 0$, $w'(s_k) \ge$

h_k, and $w'(s_k) \leq h_{k+1}$. From these, it is clear that $\partial^2\Pi_w(\mathbf{s})/\partial s_k^2 \geq 0$ wherever $w''(s_k) = 0$. Similarly, it can be verified that, wherever $w''(s_K) = 0$,

$$\frac{\partial^2\Pi_w(\mathbf{s})}{\partial s_K^2} = (J^{-1}(h_K))' \frac{\partial h_K}{\partial s_K} f(J^{-1}(h_K))[w'(s_K) - h_K] \geq 0.$$

This indicates that, in Problem V′ the firm would never locate any quality in an open interval where w is linear. Q.E.D.

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