Competing in Markets with Digital Convergence: Product Differentiation, Platform Scope and Equilibrium Structure

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Abstract: The incorporation of digital technologies into the products of diverse industries, accompanied by a shift to von-Neumann-like platform architectures, while resulting in substantially more valuable and flexible products, also leads to increased substitutability across previously distinct markets. This paper analyzes the economic implications of this trade-off in technology markets subject to digital convergence. We present a new model of imperfect competition that captures flexible platform scope, variability in consumer requirements, and multiple product purchases. We specify four types of equilibrium configurations – local monopoly, kinked, competitive and non-exclusive – that emerge as outcomes of the model, and describe how each equilibrium structure characterizes a distinct stage of digital convergence.

Our analysis establishes that as markets converge, prices always rise initially even as competing products become less differentiated. However, when platform scope is largely dictated by exogenous factors, prices and profits eventually fall as the stage of convergence progresses, though consumer surplus and total surplus rise. Furthermore, while convergence has the expected effect of shifting consumption patterns from purchasing multiple specialized products to buying a single general-purpose product, we describe examples of equilibria in which consumers may buy multiple general-purpose products, using each for a specialized subset of their requirements. Pricing responses to changes in variable costs and consumer functionality needs are also discussed.

When firms can make strategic choices of platform scope, we show that in any subgame perfect equilibrium, duopoly prices are always higher than monopoly prices, and industries may sustain high levels of profitability even when their boundaries blur. We also establish that as technological progress lowers fixed costs, a natural outcome is for unregulated firms to over-invest in platform scope relative to the social optimum, and that this outcome is true under both monopoly and duopoly market structures

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1 Overview and motivation

Over the last few years, there has been a widespread incorporation of digital technologies into the products of a number of diverse industries, a phenomenon that has been referred to as *digital convergence* (Yoffie 1997). As product technologies in these industries become increasingly digital, there is typically an accompanying shift towards von-Neumann-like architectures, characterized by the use of powerful general-purpose hardware, and the reliance on a software platform to implement the actual functionality. This often leads to more valuable and flexible products, each offering a broader range of features. However, this use of common underlying digital technology can also result in overlapping sets of functionalities being provided by products in previously distinct industries. As a result, consumers begin to view products across these industries as substitutes (Greenstein and Khanna 1997), which causes boundaries between these industries to blur (Yoffie 1997). The primary objective of our paper is to analyze the economic implications of this trade-off – between increased value and increased substitutability – in markets subject to digital convergence.

As an illustration, consider the evolution of the mobile telephone and handheld computing industries. Cellular handsets used to be designed around fairly rigid and highly optimized product architectures that were capable of little more than wireless voice communication. Recently, the Symbian-OS platform has emerged as the cornerstone of an 'application software–operating system–hardware' architecture for mobile telephony, and in parallel, Microsoft has developed the competing Smartphone platform. The primary objective of these platforms is to establish an operating system designed specifically for mobile telephony, enabling powerful and flexible IP-based communications applications which are implemented using general-purpose digital hardware. However, the architecture shift also makes it feasible to easily add other applications suitable for mobile device users. For instance, Nokia's Series 60 software platform facilitates the seamless development and deployment of Symbian-OS based applications for personal information management, multimedia, and content browsing.

Correspondingly, in the handheld computing industry, hard-wired devices such as the Sharp Wizard have evolved into more sophisticated PalmOS and Microsoft PocketPC-based von-Neumann-like architectures. These systems are designed primarily for personal information management tasks – maintaining schedules, creating address and contact lists, and recording memos – and provide some support for PC-based applications as well. However, the shift to a platform-based architecture enables new mobile communications functionality in handheld computers, often implemented through application software. Converged devices based on these platforms, such as Handspring's Treo, Samsung's Nexio, and Danger Research's Sidekick have recently been launched.

At their core, the former set of devices are mobile telephones and the latter are handheld computers. However, the bilateral shift to a platform architecture has expanded the breadth of functionality each set of devices provides. As a consequence, they become more valuable, but simultaneously, consumers start viewing cell-phone and PDA products as gross substitutes, which affects pricing and profitability across both industries.

Another recent example of the expansion of the scope of a familiar von-Neumann architecture based system is the Media Center PC from Hewlett-Packard, based on the Media Center edition of Windows XP, which aims to integrate a competitive level of home-theater functionality (including television viewing, a DVD player, and sophisticated digital audio) into a high-end PC. This product is a recent addition to a growing set of first-generation converged home entertainment platforms, such as the Moxi Media Center from Digeo, and Microsoft's X-Box, that are blurring the boundaries between the home computing and consumer electronics industries. Similar convergence trends are prevalent in the cable television, residential Internet access and wireline telephony industries. In each of these cases, technological convergence (the shift to digital technology and a platform architecture) results in product convergence (an overlap in the functionality of products and services provided by distinct industries), which may eventually lead to industry convergence.

While each of these examples illustrates convergence driven by digital technologies, the phenomenon itself is not unique to the information age. Ames and Rosenberg (1977) document a similar transformation in manufacturing processes across a wide range of industries at the turn of the twentieth century. Convergence in that case was at the *production-process* level, wherein similar production technologies such as machine tools were being employed across several industries. A sizeable literature discusses the effect of the new generation of such general purpose production technologies, now termed flexible manufacturing systems, on technology adoption patterns, and on firms' product and manufacturing strategies (Roller and Tombak 1990, Eaton and Schmitt 1994, Johansen et. al. 1995, among others).

In contrast, the kind of convergence we model is at the *product* level, wherein products in distinct industries are increasingly designed around a common set of general purpose digital technologies – microprocessors, memory chips, hard drives, networking chipsets and operating systems. Bresnahan and Trajtenberg (1995) point out a number of important economic and structural features of industries affected by these kinds of general purpose technologies. Consistent with their theory, we treat the underlying digital technologies as being supplied by an exogenous upstream technology sector, and focus on competition between firms in the downstream converging product markets.

Thus far, research in information systems and industrial organization economics that studies competition between firms in converging industries is fairly limited. The collection of essays in Yoffie (1997) provide some insight into the effect of product convergence on competitive strategy and industry structure – in particular, Greenstein and Khanna (1997) predict that convergence at the product level will lead to a higher intensity of competition. Katz and Woroch (1997), who focus on the telephone and cable markets, observe that convergence can be a source of increased competition if it creates new entry incentives and opportunities in each others' markets, and that reduced concentration may result if there are significant economies of scale or scope from entering multiple markets.

However, there have been no substantive models that capture the effects unique to digital convergence, or that investigate how this phenomenon affects competition across industries. There are no clear theory-based strategic guidelines for firms in these industries, or prescriptions for what their optimal choices of platform scope should be. It is not clear how digital convergence affects pricing, demand, profitability or welfare. These are the questions that our paper addresses, in the sections that follow.

In Section 2, we provide an overview of our model, first explaining why digital convergence and varying platform scope necessitates a new representation of product differentiation, and then outlining the basic building blocks of the model. Section 3 analyzes the case of a single-product monopoly, deriving the optimal pricing, production levels and investments in platform scope. The value functions and monopoly demand structure derived in this section form the basis for the models of duopoly studied in the subsequent sections. In Section 4, we characterize the nature of price equilibria in a converging duopoly model, describe the four kinds of equilibrium configurations, and establish the conditions under which each is feasible. Section 5 analyzes duopoly with exogenously-specified platform scope, examines how the nature of competition, prices, profits and surplus vary as platform scope changes, and studies their sensitivity to other exogenous model parameters at each feasible level of platform scope. In section 6, we analyze duopoly with endogenous platform scope in a two stage-game, and characterize the subgame perfect Nash equilibrium outcomes when firms make costly technology investments in scope. Section 7 examines how the model's outcomes are affected when consumers' breadth of functionality requirements is high, and discusses some unique outcomes that arise in this case. Section 8 concludes and outlines current research.

2 Model

2.1 Overview and general framework

Traditional economic models of imperfect competition (Dixit and Stiglitz 1976, Salop 1979, Shaked and Sutton 1983, Economides 1984) were developed to analyze strategic interaction among functionally similar products in relatively static environments. In the standard models, products are differentiated 'spatially'/horizontally with respect to abstract attributes (Salop 1979), or vertically with respect to quality (Shaked and Sutton 1983). A primary drawback of applying these models to converging technology markets is that they are based on the assumption that all products are similar in terms of the functionality or set of functionalities they provide to consumers.

Some extensions of these basic models have been developed to study technology markets. von Ungern-Sternberg (1988) allows firms to choose the level of transportation costs in a Salop-type model of monopolistic competition. Low levels of transportation costs in this model are synonymous with a higher degree of scope. Banker et al. (1998) take a different and interesting approach – instead of an explicit specification of consumers' product preferences, they extend Dixit (1979) by incorporating product quality into the aggregate demand function, which simplifies the model considerably and lends itself to an elegant analysis. Barua et al. (1991) use a similar approach to study information technology investments in a duopoly model with indirect sources of revenue.

However, when applied to converging markets, these extensions have the same drawbacks as their underlying basic models – they restrict products to providing a similar set of functionalities, and there is typically an assumption of exclusionary choice. To get around these problems, and to capture the salient characteristics of digital convergence most effectively, we develop a new model of product differentiation, which generalizes standard approaches to modeling imperfect competition in a number of important ways.

First, the standard approach of defining products as points in a product space is inadequate when applied to converging platforms of varying scope and effectiveness – it is more natural to think about a *functionality space* rather than a product space. This is because as device hardware becomes more powerful, and operating systems more sophisticated, the scope and flexibility of platforms increases, and a *single product* provides consumers with an increasing range of functionality. In evaluating these increasingly general-purpose products, each consumer ascertains how effectively a product satisfies the set of functionalities they require. These sets themselves may not change – they are merely fulfilled in a different way, possibly using fewer products. To capture this new aspect of converging technology products, we model the space of desired functionalities directly, with consumers requiring different sets of functionalities, and with different products satisfying different sets of functionalities with varying effectiveness. Our approach bears some resemblance to the *product characteristics* approach pioneered by Lancaster[1] (1966, 1975), though we take care to ensure that our model is robust to some standard criticism levied against this approach.

Second, translated into our context, the single exogenous parameter of *misfit or transportation cost* in a standard spatial model may represent both the breadth/variety of customer functionality requirements, as well as the implied level of scope. In other words, a change in transportation cost could represent a change in how effectively a product fulfils a range of customer requirements, or a change in the range of requirements themselves. The former is a product characteristic, often influenced by the state of technological progress, and sometimes explicitly chosen by a firm. The latter is typically an exogenous characteristic of customers in the market. To separate the two effects, we need two variables – an exogenous parameter representing breadth of customer requirements, and a possibly endogenous variable representing platform scope – both of which directly influence the extent of product differentiation.

Third, we depart from the traditional assumption of exclusionary choice. As observed earlier, digital convergence results in customers evaluating a single product with higher scope in lieu of multiple products with more focused functionality. Consequently, a robust model needs to admit both possibilities. We assume that a consumer who owns multiple products fulfils a specific requirement by using only the *most effective* product that she owns, rather than by combining the capabilities of *all relevant* products. This approach is consistent with a 'quality' or effectiveness interpretation of the value derived from functionality (rather than an additive 'quantity of functionality' interpretation), and insulates one from the standard criticism levied against the Lancaster-type 'products as a bundle of attributes' models[2].

Beyond this, our model preserves other standard aspects of models of horizontal differentiation. Products are symmetrically located, 'distance' between a product's core functionality and a consumer's set of requirements affects product effectiveness and value, and distance between products continues to contribute to the level of product differentiation, thus enabling one to separate decisions about platform scope and product similarity[3].

2.2 Basic model

The basic model is described in four parts: the functionality space, the nature of the product, the distribution and value functions of consumers, and the production technology available to firms.

Functionality space: The basis for products and customer preferences is a set of *functionalities*, which are distributed around the circumference of a unit circle. Every point on the circle represents a specific functionality. Two functionalities which are closer to each other on the circle are more similar in terms of the technology needed to realize them, as compared to two functionalities that are further apart. Illustrations of points in a sample functionality space are depicted in Figure 1(a). The functionality space is exogenous, and is not altered by the choices of the firms or consumers.

Products: Each product has a core functionality, which is the functionality it provides most effectively. In addition, each product has a level of platform scope ($1/t$), which may be endogenously chosen, and which in conjunction with a market-specific exogenous loss function $g(.)$, determine the effectiveness of a product in providing each non-core functionality. Specifically, the effectiveness with which a product provides a functionality located at a distance $x \in [0, \frac{1}{2}]$ from its core functionality is given by $u(x) = \max\{1 - tg(x), 0\}$. Consequently, an increase in a product's scope (a decrease in t) increases the product's effectiveness at providing functionalities away from its core functionality[4]. This is illustrated in Figure 1(b).

The loss function $g(.)$ is assumed to be non-negative and strictly increasing. Furthermore, we assume that $1 - tg(\frac{1}{2}) \geq 0$. This ensures that every product has non-negative effectiveness on every functionality, and imposes a lower bound on scope. We discuss an example that relaxes this assumption – that is, the case of very low scope – in section 7.

Consumers: Each consumer is characterized by an arc on the unit circle. This arc represents the set of functionalities that the customer requires. Different customers may require different sets of functionalities, and consequently, the arcs of functionality requirements of these different customers are located on different segments of the unit circle. We assume that these requirements arcs are uniformly distributed around the functionality space, and are all of constant length $r \in (0, 1]$ .The consumer whose arc of functionality requirements is centered at $y \in [0, 1]$ on the unit circle is referred to as 'the consumer located at y'. The density of consumers[5] on the unit circle is $\frac{n}{2}$.

The value obtained by a consumer from a product is computed by summing the effectiveness of the product over the set of functionalities that the consumer requires. The gross value that can be obtained by a consumer located at y from a product located at z is therefore:

$$\int_{y-\frac{r}{2}}^{y+\frac{r}{2}} u(|x - z|)dx, \tag{1}$$

measured in monetary units. This value function is illustrated graphically in Figure 2(a), for two different arcs of functionality requirements, and for a larger range of requirements in Figure 2(b).

Consumers can buy none, one or multiple products. If a customer buys no products, they obtain a reservation value of zero. If a consumer buys just one product, then they obtain a gross value as specified by (1). If a consumer buys two or more products, each functionality that the customer requires is fulfilled by the product that provides that functionality *most effectively*. In all cases, consumers choose the product(s) that maximize their surplus – the total value from the products, minus the total price paid.

Technology: All firms have the same cost of production $C(q, t) = cq + F(t)$, where q is quantity produced, and t is the inverse of platform scope. Recall that an increase in t represents a decrease in scope. This separable form implies the following assumptions about $C(q, t)$.

1. Marginal cost of production is non-negative and constant: $C_1(q,t) > 0$, $C_{11}(q,t) = 0$.

2. Variable cost is independent of scope $C_{12}(q,t) = 0$.

Numbered subscripts to functions represent partial derivatives with respect to the corresponding argument. For instance $C_1(q,t)$ is the partial of C with respect to its first argument, and $C_{12}(q,t)$ is the cross partial of C with respect to its first and second arguments. This notation will be preserved throughout the paper.

In addition, the cost function is assumed to have the following properties:

3. Fixed cost is non-decreasing and convex in scope: $F_1(t) \leq 0, F_{11}(t) > 0$.

4. Marginal cost is not too high: for all $t \leq \frac{1}{g(0.5)}$, $c < 2\int_{\frac{1}{2}-\frac{r}{2}}^{\frac{1}{2}} u(x)dx$.

The fact that marginal cost is constant is a recognition of the fact that in typical electronics and computer devices, the production of the hardware component is a commodity that is outsourced to a specialist contractor like Solectron or Flextronics. We sometimes consider the case of zero marginal cost as a specific example. An increase in platform scope or flexibility is typically achieved by a superior product design, an increase in the functionality of the software component, or a combination of the two. Both of these may result in increased fixed costs, and do not affect variable production cost substantially, as captured in Properties 2 and 3. In section 6.2, we consider the case of $F_1(t) = 0$, or when the choice of scope is a costless strategic variable. Property 4 simply states that in the feasible range of scope, marginal cost c is lower than the total value a consumer located diametrically opposite a product derives from it. This makes it possible for firms to profitably sell their product to all consumers in the market.

3 Monopoly

In this section, we model a monopolist producing a single product. The monopolist chooses platform scope by choosing t, and then sets the profit-maximizing monopoly price for the chosen level of t. We provide a general characterization of the demand function, and then solve the monopolist's problem with a linear loss function. In this section, as in sections 4 through 6, we address the case of $r \leq \frac{1}{2}$ in detail. The analysis for $r > \frac{1}{2}$, while similar, has some unique aspects which we explore further in Section 7.

Without loss of generality, fix the location of the core functionality of the product at $\frac{1}{2}$. This divides the consumers into two identical and symmetric halves $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. At any price, the demand from each segment will be identical. We focus our analysis on consumers in one of these segments $[\frac{1}{2}, 1]$. To facilitate simpler analysis, we 'unfurl' the unit circle, and replicate the interval $[0, \frac{1}{2}]$ as $[1, \frac{3}{2}]$, as shown in Figures 2(a) and 2(b).

Given this depiction, the value obtained by a consumer located at $y \in [\frac{1}{2}, 1]$ (that is, a consumer whose arc of functionality requirements is centered at y) is given by:

$$U(y,t) = \int_{y-\frac{r}{2}}^{y+\frac{r}{2}} (1 - t\hat{g}(x))dx, \tag{2}$$

where

$$\hat{g}(x) = \begin{array}{ll} g(\frac{1}{2} - x) & \text{for } 0 \le x \le \frac{1}{2}; \\ g(x - \frac{1}{2}) & \text{for } \frac{1}{2} \le x \le 1; \\ g(\frac{3}{2} - x) & \text{for } 1 \le x \le \frac{3}{2}. \end{array} \tag{3}$$

Refer to Figure 2(a) or 2(b) for an intuitive understanding of the function $\hat{g}(.)$.

3.1 Consumer value functions

Based on (2) and (3), the value function $U(y,t)$ in the interval $y \in [\frac{1}{2}, 1]$ has a different functional form in each of three successive ranges of y, which is:

$$U(y,t) = \begin{array}{ll} r - t\left[G(\frac{1+r}{2} - y) + G(y - \frac{1-r}{2})\right] & \text{for } \frac{1}{2} \le y \le \frac{1+r}{2}; \\ r - t\left[G(y - \frac{1-r}{2}) - G(y - \frac{1+r}{2})\right] & \text{for } \frac{r}{2} \le y \le \frac{2-r}{2}, \text{ and} \\ r - t\left[2G(\frac{1}{2}) - G(y - \frac{1+r}{2}) - G(1 - (y - \frac{1-r}{2}))\right] & \text{for } \frac{2-r}{2} \le y \le 1, \end{array} \tag{4}$$

where $G(x) = \int g(x)dx$ is defined as the cumulative loss function. Refer to Figure 3 for a graphical sketch of the integrals, as well as the integral expressions based on (2). The following lemma describes some properties of $U(y,t)$:

Lemma 1 *For all $y \in [\frac{1}{2}, 1], t \in [0, \frac{1}{g(0.5)}]$*

(a) $U(y,t)$ is continuous and decreasing in both y and t

(b) $U_1(y,t)$ is continuous and is piece-wise differentiable in both y and t.

(c) $U_{11}(y,t) \le 0$ for $\frac{1}{2} \le y \le \frac{r}{2}$, $U_{11}(y,t) \ge 0$ for $\frac{1-r}{2} \le y \le 1$, and $sign[U_{11}(y,t)] = -sign[g_{11}(x)]$ for $\frac{r}{2} \le y \le \frac{1-r}{2}$.

(d) $U_2(y,t)$ is continuous and decreasing in y.

All proofs are provided in Appendix A. Part (a) of the lemma establishes that for a general loss function $g(.)$, consumers located closer to the product derive more value from it and that an increase in scope increases value for all consumers. Parts (b) and (c) establish some properties about the continuity and curvature of the value function. The final part of the lemma indicates that the marginal benefit from an increase in scope is *more* for consumers located farther away from the product. Stated differently, the marginal consumer (that is, the one who gets least value from the product) benefits more than any of the inframarginal consumers (the ones who get higher value from the product) when scope is increased. This is an important property of the value function for our subsequent analysis.

3.2 Demand and optimal prices

The properties derived in Lemma 1 become more significant when one recognizes that the value function is identical to a rescaled *inverse demand function* faced by the monopolist. The reasoning is as follows – if the monopolist were to set price $p = U(y,t)$, where $y \in [\frac{1}{2}, 1]$, all consumers located between $\frac{1}{2}$ and y would purchase, resulting in a demand of $(y - \frac{1}{2})\frac{n}{2}$ from consumers located in $[\frac{1}{2}, 1]$, and an identical demand of $(y - \frac{1}{2})\frac{n}{2}$ from consumers located in segment $[0, \frac{1}{2}]$, for a total demand of $n(y - \frac{1}{2})$.

Define $P(q,t) = U(q + \frac{1}{2}, t)$ as the rescaled inverse demand function – the argument q is the length of customers in $[\frac{1}{2}, 1]$ who purchase the product[6], and the gross profit function $\pi(q,t) = nq(P(q,t) - c)$ as the corresponding total *profits* before accounting for the fixed costs of platform scope. Also define the net profit function $\Pi(t)$ as the fixed cost of scope subtracted from the optimal gross profits at scope level t:

$$\Pi(t) = \pi(q^*(t), t) - F(t), \tag{5}$$

where $q^*(t) \in \arg\max\limits_{q} \pi(q,t)$. $P(q,t)$ is the actual price charged by the firm, and the monopolist chooses price, not quantity. However, maximizing profits by choosing $q \in [0, \frac{1}{2}]$ is mathematically identical to maximizing profits by choosing a price that results in demand $nq \in [0, \frac{n}{2}]$, based on the bijection defined by $P(q,t)$; the former approach is adopted for mathematical convenience. For a linear loss function $g(x) = x$, using (4), the gross profit function reduces to the following functional form:

$$\pi(q,t) = \begin{array}{ll} nq(r-c) - ntq(q^2 + \frac{r^2}{4}) & \text{for } 0 \leq q \leq \frac{r}{2}; \\ nq(r-c) - ntrq^2 & \text{for } \frac{r}{2} \leq q \leq \frac{1-r}{2}; \\ nq(r-c) - ntq(q(1-q) - \frac{(1-r)^2}{4}) & \text{for } \frac{1-r}{2} \leq q \leq \frac{1}{2}. \end{array} \tag{6}$$

Figures 4(c) and 4(d) depict the typical shape of $\pi(q,t)$. Finally, define the gross surplus under monopoly as $s^M(q,t) = n\int\limits_0^q (P(x,t) - c)dx$. This is the total of firm profits and consumer surplus (that is, the total surplus) before accounting for the fixed cost of scope.

The following properties are useful in establishing the profit-maximizing choices of q:

Lemma 2 *(a) For a linear loss function $g(x) = x$, $\pi(q,t)$ is strictly concave in q for $0 \leq q \leq \frac{1-r}{2}$, and therefore has no more than one interior maximum in this interval.*

(b) In the interval $\frac{1-r}{2} \leq q \leq \frac{1}{2}$, $\pi(q,t)$ is always maximized at one of its two end-points. That is, either $\pi(\frac{1-r}{2},t) \geq \pi(q,t)$ for all $q \in [\frac{1-r}{2},\frac{1}{2}]$, or $\pi(\frac{1}{2},t) \geq \pi(q,t)$ for all $q \in [\frac{1-r}{2},\frac{1}{2}]$.

Based on Lemma 2, for a fixed level of t, we can characterize the optimal choice of price and profits, and the resulting optimal demand:

Proposition 1 *For a linear loss function $g(x) = x$:*

(a) If $2 \geq t \geq \frac{(r-c)}{r^2}$, then $q^*(t) = \sqrt{\frac{4(r-c)-r^2t}{12t}}$, $P(q^*(t),t) = \left(\frac{2r+c}{3} - \frac{r^2t}{6}\right)$ and $\pi(q^*(t),t) = \frac{n}{6}\sqrt{\frac{(4(r-c)-r^2t)^3}{12t}}$.

(b) If $\frac{(r-c)}{r^2} \geq t \geq \frac{2(r-c)}{r(2-\sqrt{2r})}$, then $q^*(t) = \frac{r-c}{2rt}$, $P(q^*(t),t) = \frac{r+c}{2}$ and $\pi(q^*(t),t) = \frac{n(r-c)^2}{4rt}$, and

(c) If $\frac{2(r-c)}{r(2-\sqrt{2r})} \geq t \geq 0$, then $q^*(t) = \frac{1}{2}$, $P(q^*(t),t) = \left(r - \frac{tr(2-r)}{4}\right)$, and $\pi(q^*(t),t) = n\left(\frac{r-c}{2} - \frac{tr(2-r)}{8}\right)$

A number of results are established in Proposition 1. Firstly, as the scope of the product $(1/t)$ increases, the quantity supplied by the monopolist increases steadily upto a threshold value $\hat{t} = \frac{2(r-c)}{r(2-\sqrt{2r})}$, at which point it increases discontinuously to $\frac{n}{2}$, and all consumers buy the product. It remains at this level for further increases in scope. On the other hand, the corresponding optimal price rises steadily in scope upto a threshold value of t, then remains constant at a value of $\frac{r+c}{2}$ until the point $\hat{t} = \frac{2(r-c)}{r(2-\sqrt{2r})}$. At this point, it is profit-maximizing for the monopolist to drop the price to the point where all consumers buy the product. A further increase in scope does not change demand, but results in a steady increase in price. The variation of price and quantity as scope varies are depicted in Figures 5(a) and 5(b), and the corresponding levels of profit and total surplus at the monopolist's profit maximizing choice $q^*(t)$ are depicted in Figures 5(c) and 5(d).

3.3 Profitability and welfare analysis

Figure 5(c) depicts that the level of gross profits increases continuously with scope, and the rate of increase in profits with scope jumps substantially when $q^*(t)$ transitions to $\frac{1}{2}$. Correspondingly, total surplus increases monotonically upto $\hat{t}$, at which point it increases discontinuously (since the entire set of consumers now consume the product), and then continues to increase, albeit at a slower rate than profits, as scope increases.

Consumer surplus, which is the difference between total surplus $s^M(q^*(t),t)$ and gross profits $\pi(q^*(t),t)$, increases monotonically with scope upto the threshold $\hat{t}$, and is maximum immediately after $q^*(t)$ changes discontinuously to $\frac{1}{2}$. It subsequently decreases rapidly as platform scope increases. This pattern is both intuitive and consistent with previous research. As platform scope increases, this makes consumers more homogeneous with respect to the value they place on the product. This facilitates higher surplus extraction by the monopolist – in fact, at $t = 0$, the monopolist's profits are equal to the entire total surplus, since the product provides identical value to all consumers. This is similar to results derived in the limit for the monopoly bundling of information goods (Bakos and Brynjolfsson 1999). Analogous reasoning leads one to expect consumer surplus to decrease as the breadth of functionality requirements r increases, and this is in fact the case.

The preceding analysis is with respect to an exogenously specified level of platform scope. Alternately, if scope is endogenous, the monopolist will choose t to maximize the function $\Pi(t)$ as defined in (5). Correspondingly, a social planner who chooses the socially efficient level of scope, but lets the market set prices, would choose $t^* = \arg\max_t \left(s^M(q^*(t), t) - F(t)\right)$.

Define t_p^* as the optimal level of t chosen by the monopolist when $q^*(t) < \frac{1}{2}$ (that is, under *partial market coverage*) and t_f^* as the level of scope chosen by the monopolist when $q^*(t) = \frac{1}{2}$ (under *full market coverage*). The following proposition benchmarks monopoly choices with the social optimum.

Proposition 2 *For a linear loss function $g(x) = x$, $t_p^* > t^*$ and $t_f^* < t^*$. That is, under partial market coverage, the monopolist under-invests in platform scope, and under full market coverage, the monopolist over-invests in platform scope as compared to the socially optimal level.*

The results of Proposition 2 are illustrated in Figure 6. Recall that a lower value of t corresponds to a higher value of scope. When maximizing net profits $\Pi(t)$, the monopolist equates the marginal increase in gross profits $\pi_2(q^*(t), t)$ to the marginal increase in fixed cost of scope $F_1(t)$. Correspondingly, the socially optimal level of scope is where the marginal increase in gross total surplus $s^M(q^*(t), t)$ equals the marginal increase in fixed cost of scope $F_1(t)$.

The welfare analysis above is based on prices being chosen by the monopolist after a social planner mandates t^*. However, even if we consider the first best solution, in which the social planner always mandates full market coverage at the socially-optimal level of platform scope, the results of Proposition 2 still hold. The full-market coverage result is intuitive when one recognizes that when one increases scope, Lemma 1(d) ensures that the value of the product to the marginal consumer (which determines price) increases faster than the value to the average consumer (which determines marginal total surplus). The monopolist therefore over-invests in scope.

There has been a substantial amount of regulatory activity in technology industries subject to convergence, such as the telecommunications, cable television and software industries. The results of Proposition 2 may be useful in determining the effectiveness of policies (like mandating universal coverage) prescribed to a regulated monopolist. This issue is discussed further in Section 8.

4 Duopoly: equilibrium structure

We next consider two firms A and B, whose products' core functionalities are located diametrically opposite each other[7], at $\frac{1}{2}$ and 1 respectively. In this section, we characterize demand and payoffs, describe the different kinds of feasible price equilibria, and show when each kind is feasible. In Sections 5 and 6, we determine and analyze these equilibria when scope is exogenously specified, and when it is endogenously chosen. Consistent with our monopoly analysis, we consider the case of $r \leq \frac{1}{2}$ in detail, and discuss some unique and interesting aspects of the case of $r > \frac{1}{2}$ in Section 7.

4.1 Demand and profit functions

We refer to a firm's own choices with subscript i and those of its opponent with subscript j. Given an opponent price p_j, the inverse demand of duopolist i, denoted $P_i(q_i, t_i, t_j, p_j)$, generally takes one of two functional forms. The first form is the *monopoly inverse demand function* $P^M(q,t) = U(q + \frac{1}{2}, t)$, which is simply the inverse demand function of the monopolist The second is the *competitive inverse demand function*, which has the following form, and will be explained shortly:

$$P^C(q, t_i, t_j, p_j) = U(q + \frac{1}{2}, t_i) - U(1-q, t_j) + p_j. \tag{7}$$

When firm j prices its product higher than the reservation value of the customer who values it most, the inverse demand function of firm i, at a competitor price p_j, is simply the monopoly inverse demand function. That is, if $p_j \geq U(\frac{1}{2}, t_j)$:

$$P_i(q_i, t_i, t_j, p_j) = P^M(q_i, t_i) \quad \text{for} \;\; 0 \leq q_i \leq \frac{1}{2}. \tag{8}$$

Next, if $U(\frac{1}{2}, t_j) \geq p_j \geq U(1, t_j)$:

$$\begin{aligned} P_i(q_i, t_i, t_j, p_j) &= P^M(q_i, t_i) \quad \text{for} \;\; 0 \leq q_i \leq 1 - U^{-1}(p_j, t_j); \\ &= P^C(q_i, t_i, t_j, p_j) \quad \text{for} \;\; 1 - U^{-1}(p_j, t_j) \leq q_i \leq \frac{1}{2}, \end{aligned} \tag{9}$$

where the inverse of U is with respect to its first argument (that is, $U(U^{-1}(u,t),t) = u$).

This is illustrated in Figure 7(a). For values of $q_i \leq 1 - U^{-1}(p_j, t_j)$, $P_i(q_i, t_i, t_j, p_j)$ is identical to the monopoly demand function, since these consumers get no surplus from the purchase of product j. We call this the 'monopoly region'. For values of $q_i > 1 - U^{-1}(p_j, t_j)$, which we term the 'competitive region', the demand function becomes less elastic than the corresponding monopoly demand function. This is because in order to sell to each new customer, the duopolist must compensate not only for the reduction in value due to increased distance from her own product's core functionality, but also for an increase in value from the competing product, which is now positive and increasing in q_i. Accounting for this effect yields inverse demand of the form (7).

The demand function therefore has a kink at the point $q_i = 1 - U^{-1}(p_j, t_j)$. As the competitor lowers price p_j, the monopoly region of $P_i(q_i, t_i, t_j, p_j)$ shrinks. Finally, when all customers have positive value for firm j's product, the monopoly region completely disappears: If $p_j \leq U(1, t_j)$:

$$P_i(q_i, t_i, t_j, p_j) = P^C(q_i, t_i, t_j, p_j) \quad \text{for} \quad 0 \leq q_i \leq \frac{1}{2}. \tag{10}$$

There is also a third kind of region on the duopolist's inverse demand function (the non-exclusive region), which occurs only if $r > \frac{1}{2}$, is illustrated in Figure 7(b), and is elaborated on in Section 7.

Let $\pi^i(q_i, t_i, t_j, p_j)$ represent the gross profit function of the duopolist:

$$\pi^i(q_i, t_i, t_j, p_j) = nq_i(P_i(q_i, t_i, t_j, p_j) - c), \; i = A, B, \tag{11}$$

which has the following useful property:

Lemma 3 *The function $\pi^i(q_i, t_i, t_j, p_j)$ has either no interior maximum, or a unique interior maximum in q_i.*

We subsequently refer to the portion of the gross profit function which corresponds to the monopoly region of $P_i(q_i, t_i, t_j, p_j)$ as the *monopoly portion* of the duopolist's gross profit function, and the portion of the gross profit function which corresponds to the competitive region of $P_i(q_i, t_i, t_j, p_j)$ as the *competitive portion* of the duopolist's gross profit function

4.2 Equilibrium configurations

Depending on the position of each duopolist's equilibrium choice of q on their inverse demand function, four different equilibrium configurations are possible for the second-stage pricing game:

1. **Local monopoly equilibrium:** At this equilibrium, each firm behaves like a local monopolist and prices in the monopoly region of their demand curves. Some customers do not purchase either product, and the sum of the equilibrium demand of both firms is less than $\frac{n}{2}$ (that is, at equilibrium, $q_A + q_B < \frac{1}{2}$). This is illustrated in Figure 7(c). The prices and firm profits are generally identical to those in the single product monopoly case, but in some instances, one of the prices may be higher.

2. **Kinked equilibrium**: At the kinked equilibrium, both firms price at the kink between the monopoly and competitive regions of their respective demand curves, and total equilibrium demand is $\frac{n}{2}$. This is illustrated in Figure 7(d). At first glance it might seem as if this equilibrium configuration is a knife's-edge case, but it is feasible across a whole range of scope and demand parameters.

3. **Competitive equilibrium**: This equilibrium configuration is analogous to the 'standard' equilibrium that one comes across in Salop-like models of imperfect competition. The firms price in the competitive region of their demand curves, wherein the firms compete for each others' marginal consumer. In equilibrium, customers still purchase only a single product, and the total equilibrium demand is still $\frac{n}{2}$. This is illustrated in Figure 7(e)

4. **Non-exclusive equilibrium**: In this equilibrium, a subset of customers purchase both products, and the total equilibrium demand is greater than $\frac{n}{2}$. This is illustrated in Figure 7(f) – since this kind of equilibrium occurs only if $r > \frac{1}{2}$, we discuss it further in Section 7.

4.3 Feasible pure-strategy price equilibria

Given the values of scope $(\frac{1}{t_i}, \frac{1}{t_j})$ and its opponent's price p_j, firm i chooses a price p_i that maximizes its payoff. As in section 3, since choosing q_i is mathematically identical to choosing p_i, and is analytically more convenient, we use q_i as the optimization variable. This implies that given p_j, t_i, and t_j, firm i chooses q_i to maximize $\pi^i(q_i, t_i, t_j, p_j)$. We establish necessary and sufficient conditions for the existence of the different equilibrium configurations and then proceed to compute them[8].

The regions of the parameter space (t_A, t_B) in which each equilibrium configuration is feasible are mapped out by identifying where pairs of actions (i.e., choices of q_i) are *locally* optimal, a necessary condition for a candidate action pair to be feasible as an equilibrium. Given t_i, we define $Q^M(t_i)$ as the interior local maximum, if it exists, on the monopoly profit function of firm i, for each $i = A, B$. Lemma 2 ensures that there is at most one interior local maximum. Where an interior local maximum does not exist, we set $Q^M(t_i) = \frac{1}{2}$. Consequently:

$$\begin{aligned} Q^M(t_i) &= x : \frac{U(x+\frac{1}{2}, t_i) - c}{-U_1(x+\frac{1}{2}, t_i)} = x, \text{ if such an } x \text{ exists in } [0, \frac{1}{2}] \\ &= \frac{1}{2} \text{ otherwise.} \end{aligned} \tag{12}$$

Analogously, given t_i and t_j, we define $Q^C(t_i, t_j)$ as the interior local maximum of the duopoly profit function (which, from Lemma 3, is unique if it exists), in the competitive region of the demand function,

Range of t_i	Value of $Q^M(t_i)$	Bounds on $Q^M(t_i)$
$2 \geq t_i \geq \frac{r-c}{r^2}$	$Q^M(t_i) = \sqrt{\frac{4(r-c)-r^2t_i}{12t_i}}$	$\sqrt{\frac{2(r-c)-r^2}{12}} \leq Q^M(t_i) \leq \frac{r}{2}$
$\frac{r-c}{r^2} \geq t_i \geq \frac{r-c}{r(1-r)}$	$Q^M(t_i) = \frac{r-c}{2rt_i}$	$\frac{r}{2} \leq Q^M(t_i) \leq \frac{1-r}{2}$
$\frac{r-c}{r(1-r)} \geq t_i \geq \frac{12(r-c)}{4-3(1-r)^2}$	$Q^M(t_i) = \frac{1}{3} - \sqrt{\frac{4-3(1-r)^2}{36} - \frac{r-c}{3t_i}}$	$\frac{1-r}{2} \leq Q^M(t_i) \leq \frac{1}{3}$
$t_i \leq \frac{12(r-c)}{4-3(1-r)^2}$	$Q^M(t_i) = \frac{1}{2}$	–

Table 1: Functional form of $Q^M(t_i)$ in different ranges of t_i for a linear loss function

for a value of $p_j = U(1 - Q^C(t_i, t_j), t_j)$:

$$\begin{aligned} Q^C(t_i, t_j) &= x : \frac{U(x + \frac{1}{2}, t_i) - c}{-[U_1(x + \frac{1}{2}, t_i) + U_1(1 - x, t_j)]} = x, \text{ if such an } x \text{ exists in } [0, \frac{1}{2}] \\ &= \frac{1}{2} \text{ otherwise.} \end{aligned} \tag{13}$$

$Q^C(t_i, t_j)$ is the interior local maximum of the competitive profit function $q_i(P^C(q_i, t_i, t_j, p_j) - c)$, if it exists, at a level of p_j such that its maximizing value intersects the monopoly profit function. Since $U_1(q, t) < 0$ for all q and t, (12) and (13) imply that $Q^C(t_i, t_j) < Q^M(t_i)$ for all $Q^C(t_i, t_j) \neq \frac{1}{2}$.

The quantities $Q^M(t_i)$ and $Q^C(t_i, t_j)$ are of interest because they completely define when each of the first three equilibrium configurations are feasible:

Proposition 3 *For any t_A and t_B:*

(a) A local monopoly equilibrium is feasible only if $Q^M(t_A) + Q^M(t_B) \leq \frac{1}{2}$;

(b) A kinked equilibrium is feasible only if $Q^M(t_A) + Q^M(t_B) \geq \frac{1}{2}$, and $Q^C(t_A, t_B) + Q^C(t_B, t_A) \leq \frac{1}{2}$, and

(c) A competitive equilibrium is feasible only if $Q^C(t_A, t_B) + Q^C(t_B, t_A) \geq \frac{1}{2}$.

For the linear loss function $g(x) = x$, Table 1 summarizes the values of $Q^M(t_i)$, along with bounds on its value, for different ranges of t_i. The derivation of these values is based on (12) above. Analogously, Table 2 summarizes the computed values of $Q^C(t_i, t_j)$, along with bounds on its value, if any, for different ranges of t_i and t_j: These values are derived based on (13).

Based on Table 1 and Table 2, we identify the different regions of the (t_A, t_B) space in which different pure strategy price equilibria are feasible. Proposition 3 ensures that a maximum of one equilibrium configuration is feasible in each region. Figure 8(a) illustrates this partition, some mathematical details of which are presented in Appendix B.

Figure 8(a) shows that when both products have low scope (high t), then the feasible equilibrium

Range of t_j in terms of t_i	Value of $Q^C(t_i,t_j)$	Bounds on $Q^C(t_i,t_j)$
$t_j \geq 2(\frac{r-c}{r^2} - t_i)$	$Q^C(t_i,t_j) = \sqrt{\frac{4(r-c)-r^2 t_i}{4(3t_i+2t_j)}}$	$Q^C(t_i,t_j) \leq \frac{r}{2}$
$2(\frac{r-c}{r^2} - t_i) \geq t_j \geq 2(\frac{r-c}{r(1-r)} - t_i)$	$Q^C(t_i,t_j) = \frac{r-c}{r(2t_i+t_j)}$	$\frac{r}{2} \leq Q^C(t_i,t_j) \leq \frac{1-r}{2}$
$2(\frac{r-c}{r(1-r)} - t_i) \geq t_j$ $\geq (A - 2t_i + \sqrt{A(A-t_i)})$	$Q^C(t_i,t_j) = \frac{2t_i+t_j-\sqrt{(2t_i+t_j)^2 - A(3t_i+2t_j)}}{2(3t_i+2t_j)}$	$\frac{1-r}{2} \leq Q^C(t_i,t_j) \leq \frac{2t_i+t_j}{2(3t_i+2t_j)}$
$t_j \leq (A - 2t_i + \sqrt{A(A-t_i)})$	$Q^C(t_i,t_j) = \frac{1}{2}$	–
Note: $A = 4(r-c) + t_i(1-r)^2$		

Table 2: Functional form of $Q^C(t_i,t_j)$ in different ranges of t_i and t_j

configuration is local monopoly. Both products provide positive but minimal effectiveness on the core functionality of their rival product. Therefore, there ends up being no real strategic interaction between the price and demand choices of the two firms, and each firm prices like a local monopolist, leaving a portion of the market unserved by either. As either firm increases its scope, thereby reducing t_i, the gross value of the product increases for the non-adopting customers increases faster than price. This gradually increases adoption, to the set of points represented by the curve KM in Figure 8(a), where all consumers purchase one or the other product.

The feasible equilibrium configuration now transitions to a kinked equilibrium. The market is fully covered, yet neither firm finds it profitable to try and gain market share when their product's scope increases – the benefit of stealing an opponent's customer by dropping prices is outweighed by the corresponding loss in profit from their existing customer base. This behavior persists across an entire range of (t_A, t_B) values.

Finally, when scope is high enough, firms start to compete for each others' customers, thereby entering the competitive equilibrium configuration region. Their markets overlap in a strategically relevant way, since it is profitable to increase demand at the margin, even after accounting for the losses in revenue from one's own customer base. Consequently, under this equilibrium configuration, prices are bilaterally reduced, and this is reinforced as platform scope increases, moving one away from the curve CK.

4.4 Equilibrium demand and prices

Having identified the feasible pure-strategy Nash equilibria in each region, we now specify their corresponding equilibrium q pairs. Denote the equilibrium choice of q by firm i as $q_i^*(t_i,t_j)$. Actual realized demand will therefore be $nq_i^*(t_i,t_j)$.

For the linear loss function $g(x) = x$, the derived equilibrium values are summarized in Table 3. These

Equilibrium configuration: Local Monopoly	
Range of values of t_i	**Equilibrium $q_i^*(t_i, t_j)$**
$2 \geq t_i \geq \frac{(r-c)}{r^2}$	$q_i^*(t_i, t_j) = \sqrt{\frac{4(r-c)-r^2 t_i}{12 t_i}}$
$\frac{(r-c)}{r^2} \geq t_i \geq \frac{(r-c)}{r(1-r)}$	$q_i^*(t_i, t_j) = \frac{n(r-c)}{2rt_i}$
$\frac{(r-c)}{r(1-r)} \geq t_i \geq \frac{12(r-c)}{4-3(1-r)^2}$	$q_i^*(t_i, t_j) = \frac{1}{3} - \sqrt{\frac{4-3(1-r)^2}{36} - \frac{(r-c)}{3t_i}}$

Equilibrium configuration: Kinked	
Relative values of $Q^M(t_i), Q^M(t_j), Q^C(t_i, t_j)$ and $Q^C(t_j, t_i)$	**Range of equilibrium $q_i^*(t_i, t_j)$**
$\frac{1}{2} - Q^C(t_j, t_i) \geq Q^M(t_i) \geq \frac{1}{2} - Q^M(t_j) \geq Q^C(t_i, t_j)$	$\frac{1}{2} - Q^M(t_j) \leq q_i^*(t_i, t_j) \leq Q^M(t_i)$
$Q^M(t_i) \geq \frac{1}{2} - Q^C(t_j, t_i) \geq \frac{1}{2} - Q^M(t_j) \geq Q^C(t_i, t_j)$	$\frac{1}{2} - Q^M(t_i) \leq q_i^*(t_i, t_j) \leq \frac{1}{2} - Q^C(t_j, t_i)$
$\frac{1}{2} - Q^C(t_j, t_i) \geq Q^M(t_i) \geq Q^C(t_i, t_j) \geq \frac{1}{2} - Q^M(t_j)$	$Q^C(t_i, t_j) \leq q_i^*(t_i, t_j) \leq Q^M(t_i)$
$Q^M(t_i) \geq \frac{1}{2} - Q^C(t_j, t_i) \geq Q^C(t_i, t_j) \geq \frac{1}{2} - Q^M(t_i)$	$Q^C(t_i, t_j) \leq q_i^*(t_i, t_j) \leq \frac{1}{2}$
Note: at any kinked equilibrium, $q_A^*(t_A, t_B)$ and $q_B^*(t_B, t_A)$	sum to $\frac{1}{2}$ (see Figure 12)

Equilibrium configuration: Competitive	
Range of values of t_i	**Equilibrium $q_i^*(t_i, t_j)$**
All feasible competitive equilibrium values	$q_i(t_i, t_j) = \frac{t_i + 2t_j}{6(t_i + t_j)}$

Table 3: Equilibrium quantity values for each equilibrium configuration

expressions are obtained by deriving the form of the actual Nash equilibrium (pairs of q that are best responses to each other), for each region in the (t_A, t_B) space, given the feasible equilibrium configuration in this region[9] depicted by Figure 8(a). When the local monopoly configuration is feasible, Table 3 illustrates that there are three sub-segments, each with its corresponding unique optimal $q_i(t_i, t_j)$. Similarly, when the competitive equilibrium configuration is feasible, there is a unique equilibrium $q_i(t_i, t_j)$ for each firm i. On the other hand, for each pair of t values at which a kinked equilibrium configuration is feasible, there is a continuum of equilibrium q pairs, as illustrated (in one case) in Figure 8(b).

This is easily explained if one refers back to Proposition 3(b). A kinked equilibrium configuration is feasible only if $Q^M(t_i) + Q^M(t_j) \geq \frac{1}{2}$, and $Q^C(t_i, t_j) + Q^C(t_j, t_i) \leq \frac{1}{2}$. When these inequalities are strict, there is a continuum of pairs $q_A^*(t_A, t_B), q_B^*(t_B, t_A)$ which sum to $\frac{1}{2}$, at which payoffs are strictly increasing along the monopoly portions of the duopoly gross profit curves of both firms (because $Q^M(t_i) + Q^M(t_j) > \frac{1}{2}$), and at which payoffs are strictly decreasing along the competitive portion of the duopoly gross profit curves of both firms (because $Q^C(t_i, t_j) + Q^C(t_j, t_i) < \frac{1}{2}$). Each of these pairs is consequently a Nash equilibrium.

Equilibrium configuration	Range of values of t_i	Equilibrium price $P_i^*(t_i,t_j)$
Local monopoly	$2 \geq t_i \geq \frac{(r-c)}{r^2}$	$\frac{2r+c}{3} - \frac{r^2 t_i}{6}$
Local monopoly	$\frac{(r-c)}{r^2} \geq t_i \geq \frac{(r-c)}{r(1-r)}$	$\frac{r+c}{2}$
Local monopoly	$\frac{(r-c)}{r(1-r)} \geq t_i \geq \frac{12(r-c)}{4-3(1-r)^2}$	$\frac{(2r+c)}{3} + \frac{t_i(1-3r(2-r)+\sqrt{(1+3r(2-r))-\frac{12(r-c)}{t_i}}}{18}$
Kinked	As specified in Table 3	$P^M(q_i^*(t_i,t_j),t_i)$
Competitive	All feasible values	$c + \frac{r(t_i+2t_j)}{6}$

Table 4: Equilibrium prices pairs under different equilibrium configurations

Equilibrium prices $P_i^*(t_i,t_j)$ under each equilibrium configuration are summarized in Table 4.

5 Duopoly: exogenous platform scope

This section analyzes the duopoly model with an exogenously specified and symmetric scope level $\frac{1}{t}$. The results of this section apply to a scenario in which the effectiveness of products in fulfilling different consumer requirements is not explicitly chosen by the duopolists, but is largely determined by an exogenous factor (such as progress in technology in an upstream industry, for instance, as in the case of semiconductor-based devices, or in a downstream industry, as in the case of operating system or application software). The analysis allows us to examine how the nature of competition, prices, profits and surplus vary as scope changes, and their sensitivity to other exogenous model parameters at each feasible level of scope. It also serves as a useful benchmark for the results of section 6.

5.1 Equilibrium

In each of the equilibria derived in this section, $nq_i^*(t)$, $P_i^*(t)$ and $\pi_i^*(t)$ denote the equilibrium demand, price and payoffs to each of the firms $i = A, B$. The diagonal of the (t_A, t_B) graph in Figure 8(a) represents the set of feasible equilibria when constrained to symmetric choices of scope. This suggests that as scope increases, and t falls, the equilibrium configuration will move from being local monopoly, to kinked, to competitive. These equilibria are characterized in the following three propositions:

Proposition 4 *If both products have identical and exogenously specified scope $\frac{1}{t}$, a local monopoly equilibrium configuration is feasible if and only if $t > 2(\frac{r-c}{r})$. The unique equilibrium outcomes are:*

(a) If $2 \geq t \geq \frac{(r-c)}{r^2}$: demand is $nq_i^*(t) = n\sqrt{\frac{4(r-c)-r^2 t}{12t}}$, prices are $P_i^*(t) = \left(\frac{2r+c}{3} - \frac{r^2 t}{6}\right)$, and payoffs are $\pi_i^*(t) = \frac{n}{6}\sqrt{\frac{(4(r-c)-r^2 t)^3}{12t}}$, for $i = A, B$

(b) If $\frac{(r-c)}{r^2} \geq t \geq 2\frac{(r-c)}{r}$*: demand is* $nq_i^*(t) = \frac{n(r-c)}{2rt}$*, prices are* $P_i^*(t) = \frac{r+c}{2}$*, and payoffs are* $\pi_i^*(t) = \frac{n(r-c)^2}{4rt}$*, for* $i = A, B$*.*

As one would expect, these equilibrium pairs are identical to the corresponding monopoly outcomes under partial market coverage. The variation of demand, price and profits for each firm is as discussed in section 3.2. Figure 9(a) and 9(b) plot the variation in demand and price with scope $\frac{1}{t}$. The first segment of the graphs depicts that in the local monopoly equilibrium, both demand and prices increase as scope is increased.

As t falls below $2(\frac{r-c}{r})$, the outcome transitions to one of a continuum of kinked equilibria:

Proposition 5 *If both products have identical and exogenously specified scope* $\frac{1}{t}$*, a kinked equilibrium configuration is feasible if and only if* $\frac{4}{3}(\frac{r-c}{r}) \leq t \leq 2(\frac{r-c}{r})$*. For each value of* t*, there is a continuum of equilibrium pairs* $(q_A^*(t), q_B^*(t))$*, as specified in Table 3.*

(a) In the unique symmetric equilibrium, $q_A^*(t) = q_B^*(t) = \frac{1}{4}$*. Therefore, demand is* $nq_i^*(t) = \frac{n}{4}$*, prices are* $P_i^*(t) = r(1 - \frac{t}{4})$*, and payoffs are* $\pi_i^*(t) = n(\frac{r-c}{4} - \frac{rt}{16})$*, for* $i = A, B$*.*

(b) For each asymmetric pair $(q_A^*(t), q_B^*(t))$*, equilibrium demand is* $nq_i^*(t)$*, equilibrium prices are* $P_i^*(t) = U(\frac{1}{2} + q_i^*(t), t)$*, and equilibrium payoffs are* $\pi_i^*(t) = nq_i^*(t)[P_i^*(t) - c]$*, for* $i = A, B$*.*

We choose the symmetric equilibrium $q_A^*(t) = q_B^*(t) = \frac{1}{4}$ for further discussion. Apart from being symmetric, it is also feasible in all four ranges of t_i and t_j specified in Table 3. Moreover, among all the kinked equilibria, it is the one that maximizes total surplus. As depicted by Figure 9(b), under this kinked equilibrium, prices continue to *increase* as t falls and scope $\frac{1}{t}$ increases. In other words, as the products become *less* differentiated, prices *rise* rather than fall. Moreover, these prices are not only higher than the prices charged when the firms are each local monopolists, but they are higher than the corresponding single-firm profit-maximizing prices.

This unusual equilibrium response is a consequence of the relative slopes of the monopoly and competitive regions of the duopolist's profit function. In a kinked equilibrium, both duopolists are at the transition point between the monopoly and the competitive portions of their respective gross profit functions. The slope on the monopoly portion is positive, which would induce a monopolist to increase realized demand. However, the presence of the rival duopoly firm makes this demand increase impossible without reducing profits (because the slope of the competitive portion of the gross profit function is negative). As a consequence, the equilibrium response to this tension ends up being an increase in price, rather than in realized demand[10].

The equilibrium increase in prices in response to a reduction in t is not restricted to the symmetric kinked equilibrium. Note that for any kinked equilibrium, $P_i^*(t) = U(\frac{1}{2} + q_i^*(t), t)$, and since $U_2(y, t)$ is negative, the equilibrium response to an increase in scope $\frac{1}{t}$ is an increase in price, so long as the firms stay at the same equilibrium pair[11] $(q_A^*(t), q_B^*(t))$.

Finally, when scope is increased to the point where t falls below $\frac{4}{3}(\frac{r-c}{r})$, the equilibrium configuration becomes competitive:

Proposition 6 *If both products have identical and exogenously specified scope $\frac{1}{t}$, a competitive equilibrium configuration is feasible if and only if $t < \frac{4}{3}(\frac{r-c}{r})$. The unique equilibrium outcomes are: demand $nq_i^*(t) = \frac{n}{4}$, price $P_i^*(t) = c + \frac{rt}{2}$, and payoff $\pi_i^*(t) = \frac{nrt}{8}$, for $i = A, B$.*

This equilibrium outcome is similar, though not identical, to that of the standard circular differentiated-products duopoly. The firms always split the market; as t decreases and scope rises, prices fall as depicted in Figure 9(b). The decline in prices is proportionate to t, and rapidly pulls the duopoly prices below the corresponding single-firm price.

5.2 Sensitivity to costs and breadth of consumer requirements

Inspection of the price expression in Proposition 6 indicates that prices *rise* as the breadth of functionality requirements of customers r *increases.* This is interesting, because much like an increase in platform scope $\frac{1}{t}$, an increase in r has two effects – it increases the value of each product for all consumers, but simultaneously reduces the level of product differentiation. Proposition 6 shows that in a competitive equilibrium configuration, the former effect dominates the latter, which is in stark contrast to the effect of increasing platform scope. This effect persists across the other two equilibrium configurations as well – any marginal increase in r always raises prices and profits. This is not surprising for local monopoly, and complements a similar scope effect in the case of the kinked equilibrium. More importantly, it highlights the importance of separating changes in product differentiation that are a result of platform scope choices, and those that are a result of changes in consumer needs.

Reducing the unit variable cost c generally does prices – however, the impact varies substantially with equilibrium configuration. In the competitive equilibrium configuration, when unit costs fall, the entire cost reduction is transferred to the customers, which is reflective of a high degree of competition. Under local monopoly, there is a downward price adjustment which raises realized demand, results in the benefits being shared by the buyers and sellers, and is consistent with the monopolistic nature of the equilibrium. However, in a kinked equilibrium configuration, the entire benefit of the cost reduction is absorbed by the

Range of values of t	Equilibrium configuration	Total surplus	Consumer surplus
$2 \geq t \geq \frac{r-c}{r^2}$	Local monopoly	$\frac{2n}{9}\sqrt{\frac{(4(r-c)-r^2t)^3}{3t}}$	$\frac{n}{18}\sqrt{\frac{(4(r-c)-r^2t)^3}{3t}}$
$\frac{r-c}{r^2} \geq t \geq 2(\frac{r-c}{r})$	Local monopoly	$\frac{n}{4}[\frac{3(c-r)^2}{rt} - \frac{r^3t}{3}]$	$\frac{n}{4}[\frac{(c-r)^2}{rt} - \frac{r^3t}{3}]$
$2(\frac{r-c}{r}) \geq t \geq \frac{4}{3}(\frac{r-c}{r})$	Kinked	$\frac{n}{2}[(r-c) - \frac{rt(3+4r^2)}{24}]$	$\frac{nt[3-2r(4r^2-6r+3)]}{96}$
$\frac{4}{3}(\frac{r-c}{r}) \geq t \geq 0$	Competitive	$\frac{n}{2}[(r-c) - \frac{rt(3+4r^2)}{24}]$	$\frac{nt[3-2r(4r^2-6r+3)]}{96}$

Table 5: Gross total surplus and consumer surplus with exogenous t

duopolists, with no transfer of surplus to the buyers. Beyond these local effects within an equilibrium configuration, changes in r and c also affect the relative sizes the of the regions of t under which each equilibrium is feasible. It is easily seen that $(\frac{r-c}{r})$ is increasing in r and decreasing in c. Consequently, an increase in the breadth of functionality requirements, or a reduction in unit costs both decrease the range of scope values in which both local monopoly equilibria and kinked equilibria are feasible, and increases the ranges of scope values for which the outcome is a competitive equilibrium.

5.3 Profits and welfare

Consistent with a scenario where scope is exogenous, we examine comparative statics of the *gross* profits and surplus as scope varies. As shown in Figure 9(c), gross profits are strictly increasing in scope in the local monopoly and kinked equilibrium regions. This reflects increases in both equilibrium prices and demand in the local monopoly region, and increases in price at a constant demand of $\frac{n}{4}$ in the kinked region. However, in the competitive regions, increases in scope reduce profits – an expected outcome, since demand continues to be $\frac{n}{4}$, and prices decrease[12]. In addition, per-firm duopoly gross profits are lower that the corresponding single-firm gross profits in the kinked[13] and competitive regions (and substantially so in the latter case).

If both firms choose the same q value, then the duopoly gross total surplus under this choice is:

$$s^D(q,t) = 2n\int_0^q (P^M(x,t) - c)dx, \tag{14}$$

and consumer surplus is simply the difference between $s^D(q,t)$ and total profits. Table 5 summarizes the functional forms of gross total surplus and consumer surplus for the different equilibrium outcomes. As illustrated in Figure 9(d), gross total surplus in a duopoly is strictly increasing as scope increases. Equilibrium consumer surplus increases with scope across the local monopoly and competitive configurations, but falls in the kinked region. Moreover, gross total surplus is strictly higher than the corresponding surplus under monopoly. Of course, this comes at the cost of two investments in the fixed cost of scope, rather than just one, and our welfare analysis of duopoly with endogenous scope, in Section 6, explores this issue further.

6 Duopoly: endogenous platform scope

We now analyze the game where firms make simultaneous first stage investments in platform scope, followed by a simultaneous choice of prices in the second stage. While we characterize some of the asymmetric subgame-perfect equilibria that exist, we fully specify and discuss only the pure strategy equilibria which involve symmetric choices of scope. To ensure that the payoff functions for the game are well-defined for all out-of-equilibrium action pairs, we restrict the first-stage action spaces of the firms to those values of scope for which second-stage pure strategy pricing equilibria exist[14].

6.1 Marginal incentives for scope investments

In this sub-section, we characterize the marginal impact of a first-stage change in scope, when this change leaves firms within the same equilibrium configuration in the second stage. The main result is that within an equilibrium configuration, the marginal effect of changing t_i on firm payoffs is always strictly monotonic.

Lemma 4 *Consider any pair (t_A, t_B) for which a pure-strategy price equilibrium exists. If (t_A, t_B) is such that small changes in t_i do not change the second-stage equilibrium configuration, then:*

(a) If the equilibrium configuration is local monopoly, $\frac{d}{dt_i}\pi^i(q_i^*(t_i, t_j), t_i, t_j, P_j^*(t_j, t_i)) < 0$.

(b) If the equilibrium configuration is kinked, $\frac{d}{dt_i}\pi^i(q_i^*(t_i, t_j), t_i, t_j, P_j^*(t_j, t_i)) < 0$.

(c) If the equilibrium configuration is competitive, $\frac{d}{dt_i}\pi^i(q_i^*(t_i, t_j), t_i, t_j, P_j^*(t_j, t_i)) > 0$.

Note that the derivatives in Lemma 4 are total derivatives (which include adjustments for the equilibrium second-stage changes in $q_i^*(t_i, t_j)$ and $P_j^*(t_j, t_i)$ that occur when one changes t_i). The lemma therefore shows that under any first stage choice (t_A, t_B) for which small changes in t_i do not change the resulting equilibrium configuration of the second-stage pricing game, each firm can increase their gross profits by making appropriate changes in their choice of scope.

6.2 Equilibrium with zero marginal cost of platform scope

In this section, we solve for and discuss the subgame perfect pure strategy equilibria for the endogenous scope game when firms can costlessly choose any level of scope – that is, when $F_1(t) = 0$ for all t. This can be viewed as the limiting case in a scenario where technology progress reduces the cost of platform scope. The following proposition characterizes the subgame perfect pure strategy equilibria:

Proposition 7 *(a) When $F_1(t) = 0$, there is a unique symmetric subgame perfect equilibrium, in which:*

(i) $t_A^* = t_B^* = \frac{4}{3}(\frac{r-c}{r})$;

(ii) $q_A^*(t_A^*, t_B^*) = q_B^*(t_B^*, t_A^*) = \frac{1}{4}$;

(iii) $P_A^*(t_A^*, t_B^*) = P_B^*(t_B^*, t_A^*) = \frac{c+2r}{3}$, *and*

(iv) $\pi^A(q_A^*(t_A^*, t_B^*), t_A^*, t_B^*, P_A^*(t_A^*, t_B^*)) = \pi^B(q_B^*(t_B^*, t_A^*), t_B^*, t_A^*, P_B^*(t_B^*, t_A^*)) = \frac{1}{6}(r - c)$.

(b) In addition, there is a continuum of subgame perfect pure strategy Nash equilibria with asymmetric choices of scope. In each of these equilibria, first-stage choices (t_A^*, t_B^*) result in kinked equilibrium configurations in the second stage pricing game, and are on the boundary between the regions where kinked equilibria and competitive equilibria are feasible – that is, on the curve CK in Figure 8(a).

Proposition 7 implies that when firms can endogenously and costlessly choose platform scope, the choices are such that the equilibrium level of competition is not particularly intense. It is somewhat striking (though not surprising) that even when scope is costless, firms limit the value provided by their products. An increase in platform scope makes the products more valuable to their consumers and hence allows them to charge a higher price. Simultaneously, however, a high level of scope also reduces the level of differentiation between the two products, which can result in intense price competition if the second-stage equilibrium configuration is competitive. This is the central trade-off presented by digital convergence, and the firms' choices are driven by the need to balance these two conflicting incentives.

Interestingly, the equilibrium prices are higher than they would be under monopoly provision of a product with the same scope. In fact, if one refers back to Figure 9(b), the equilibrium choice of t is at the point where prices are as high as is possible under any symmetric second-stage equilibrium. Moreover, a reduction in marginal costs lead to lower equilibrium choices of platform scope. As a consequence, reduced variable costs c do lead to lower prices[15], although firms pass on only a small fraction of the savings to the consumers. Recall from Section 5 that under a kinked equilibrium configuration with exogenous platform scope, the duopolists did not reduce prices when c was reduced – the effect of changes in c on price in this endogenous case is therefore entirely indirect, through the equilibrium change in the choice of scope.

When $F_1(t) = 0$, the welfare maximizing choice is to provide infinite scope, or to set $t = 0$. The equilibrium level of scope chosen by the duopolists is thus socially insufficient. At the socially efficient level of platform scope, each product satisfies every functionality requirement of the consumer perfectly, thus resulting in the highest amount of surplus possible, but also making the products perfect substitutes.

and leading to Bertrand competition. Interestingly, a single product (or multiproduct) monopolist would find it optimal to provide an infinite level of scope in her products, achieving the first best outcome. However, the monopolist would also end up appropriating the entire value created, leaving no surplus to the consumers.

6.3 Equilibrium with costly platform scope

When $F_1(t) < 0$, firms may find it profitable to choose a level of scope lower than the equilibrium value derived in Proposition 7. In this section, we discuss some such cases in which a unique subgame perfect symmetric equilibrium exists.

In the first stage, firm i's net profit function, after accounting for the cost of scope, and the second-stage equilibrium choices of q and price, is:

$$\Pi^i(t_i, t_j) = \pi^i(q_i^*(t_i, t_j), t_i, t_j, P_j^*(t_j, t_i)) - F(t_i).$$

Since $F_1(t) < 0$, it is clear from Lemma 4(c) that $\Pi_1^i(t_i, t_j) > 0$ for any pair (t_A, t_B) that results in a competitive equilibrium configuration in the second stage. As a consequence, such a pair can never be an equilibrium choice. Since we look for equilibria with symmetric choices of scope, the problem therefore reduces to finding values of t such that for $i = A, B$:

$$\frac{4(r-c)}{3r} \leq t \leq 2, \tag{15}$$

$$\Pi_1^i(t,t) = 0, \text{ and} \tag{16}$$

$$\Pi_{11}^i(t,t) < 0. \tag{17}$$

If such values of t do not exist, we examine the value of $\Pi_1^i(t,t)$ at the end-points $t = \frac{4(r-c)}{3r}$ and $t = 2$.

In general, the outcome of the game is crucially dependent on the slope of the fixed cost curve $F_1(t)$. If there is a unique point t_d^* which satisfies (15) - (17), then this is the unique symmetric equilibrium. The curves *(b)* and *(c)* in Figure 10(a) illustrate this situation, when this point of intersection is in the local monopoly and kinked equilibrium regions respectively. Alternately, the curves *(a)* and *(d)* in Figure 10(a) illustrate the outcome if there is no point satisfying (15) and (16). If $F_1(t)$ lies entirely above the second-stage equilibrium marginal profit curve, then the unique symmetric subgame perfect equilibrium is $t_d^* = \frac{4}{3}(\frac{r-c}{r})$. This is the same outcome as obtained in section 6.2, and strengthens that result, to the extent that Proposition 7 continues to hold if the cost of scope is non-zero but increases slowly with scope.

On the other hand, if $F_1(t)$ lies entirely below the second-stage equilibrium marginal profit curve, then the unique symmetric subgame perfect equilibrium is $t_d^* = 2$. This outcome is driven entirely by our restriction on the permissible range of t, and so we do not interpret it further. If there is a unique value t_d^* that satisfies conditions (15) and (16), but not (17), then it is a local minimum. In this case, both end-points are candidate equilibria. Very little can be said in the case where there are multiple points which satisfy (15) and (16).

In a duopoly market structure, the socially optimal level of scope t^* (with the firms then choosing equilibrium prices at this level of scope) satisfies:

$$t^* = \arg\max_t s^D(q^*(t), t) - 2F(t). \tag{18}$$

Under each of the interior outcomes depicted in Figure 10(a), the following proposition holds:

Proposition 8 *If there exists a unique symmetric equilibrium value t_d^* satisfying (15) - (17), then*

(i) If $t_d^* > \frac{2(r-c)}{r}$ then $t_d^* > t^*$.

(ii) If $t_d^* \leq \frac{2(r-c)}{r}$ then $t_d^* < t^*$

Proposition 8 establishes that for any unique interior solution, the duopolists under provide scope as compared to the social optimum if the equilibrium configuration in the second stage is local monopoly, and over-provide scope if it is a kinked equilibrium. The similarity of this result to the one established in Proposition 2 becomes obvious once one recognizes that under a local monopoly equilibrium in the second stage of the game, the market is only partially served, while under a kinked equilibrium regime, the market is fully served. This is discussed further in Section 8.

7 Higher breadth of functionality requirements

The analysis in sections 3 through 6 assumed that $r \leq \frac{1}{2}$ – in this section, we outline some unique features of the model when $r > \frac{1}{2}$. We describe the changes in the inverse demand function and the resulting variation in monopoly outcomes. Next, we discuss the existence and properties of the fourth kind of duopoly equilibrium configuration – the non-exclusive equilibrium. Finally, we illustrate how when platform scope is low and r is close to 1, outcomes where consumers purchase both products can occur even when the equilibrium configuration is local monopoly.

Range of values of t	Equilibrium configuration	Demand per firm	Equilibrium prices
$4 \geq t \geq \frac{4(r-c)}{4r^2-6r+3}$	Local monopoly	$nq^*(t) = n\sqrt{\frac{r-c-\frac{r^2t}{4}}{3t}}$	$P^*(t) = \frac{2r+c}{3} - \frac{r^2t}{6}$
$\frac{4(r-c)}{4r^2-6r+3} \geq t \geq 4(r-c)$	Local monopoly	$nq^*(t) = \frac{n(4(r-c)-t(2r-1))}{8t(1-r)}$	$P^*(t) = \frac{4(r+c)-t(2r-1)}{8}$
$4(r-c) \geq t \geq \frac{4(r-c)}{2-r}$	Kinked	$nq^*(t) = \frac{n}{4}$	$P^*(t) = r(1-\frac{t}{4})$
$\frac{4(r-c)}{2-r} \geq t \geq 0$	Competitive	$nq^*(t) = \frac{n}{4}$	$P^*(t) = c + \frac{t(1-r)}{2}$
$\frac{4(r-c)}{2-r} \geq t \geq \frac{128}{9}c$	Non-exclusive (for high r)	see Table 8	see Table 8

Table 6: Duopoly equilibrium configurations and outcomes for $r > 0.5$

7.1 Monopoly

When one admits higher values of r, the value function derived in (4) changes. For $r > \frac{1}{2}$, this function $U(y)$ is:

$$U(y,t) = \begin{array}{ll} r - t\left[G(\frac{1+r}{2} - y) + G(y - \frac{1-r}{2})\right] & \text{for } \frac{1}{2} \leq y \leq \frac{2-r}{2}; \\ r - t\left[2G(\frac{1}{2}) + G(\frac{1+r}{2} - y) - G(1 - (y - \frac{1-r}{2}))\right] & \text{for } \frac{2-r}{2} \leq y \leq \frac{1+r}{2} \\ r - t\left[2G(\frac{1}{2}) - G(y - \frac{1+r}{2}) - G(1 - (y - \frac{1-r}{2}))\right] & \text{for } \frac{1+r}{2} \leq y \leq 1 \end{array} \quad (19)$$

This new value function shares all the properties presented in Lemma 1, except that $sign[U_{11}(y,t)] = sign[g_{11}(x)]$ for $\frac{1-r}{2} \leq y \leq \frac{r}{2}$. Proceeding as in section 3.1, the gross profit function with a linear loss function can be derived:

$$\pi(q,t) = \begin{array}{ll} nq(r-c) - ntq(q^2 + \frac{r^2}{4}) & \text{for } 0 \leq q \leq \frac{1-r}{2}; \\ nq(r-c) - ntq(q(1-r) + \frac{2r-1}{4}) & \text{for } \frac{1-r}{2} \leq q \leq \frac{r}{2}; \\ nq(r-c) - ntq(q(1-q) - \frac{(1-r)^2}{4}) & \text{for } \frac{r}{2} \leq q \leq \frac{1}{2}. \end{array} \quad (20)$$

While the ranges of q values are different, reflecting the fact that $\frac{r}{2} > \frac{1-r}{2}$ for $r > \frac{1}{2}$, the actual functional form of $\pi(q,t)$ is the same in two of the three segments. Results analogous to Lemma 2 and Proposition 1, are obtained for the optimal demand, price and profits. The main effect of increasing r beyond $\frac{1}{2}$ is that full market coverage becomes increasingly more likely. This is illustrated most starkly when marginal costs are zero. In this case, full market coverage is *always* optimal. That is, when $c = 0$, for any $r > \frac{1}{2}$, and for any value of $t \leq 2$, $q^*(t) = \frac{1}{2}$, $P(q^*(t),t) = \left(r - \frac{tr(2-r)}{4}\right)$, and $\pi(q^*(t),t) = n\left(\frac{r}{2} - \frac{tr(2-r)}{8}\right)$.

7.2 Duopoly: non-exclusive and incremental purchases

For high enough price levels, the structure of the inverse demand curves remain as described in (8), (9), and (10) of Section 4, though the expressions are now based on the $U(y,t)$ functions in (19), since $r > \frac{1}{2}$.

Range of values of r	Incremental value function $I(y,t)$	
$\frac{1}{2} \leq r \leq \frac{3}{4}$	$I(y,t) = \frac{t}{8}$	for $\frac{1}{2} \leq y \leq \frac{1+2r}{4}$
	$I(y,t) = t[\frac{1}{8} - t(y - \frac{1+2r}{4})^2]$	for $\frac{1+2r}{4} \leq y \leq \frac{1+r}{2}$
	$I(y,t) = t(\frac{3+2r}{4} - y)^2$	for $\frac{1+r}{2} \leq y \leq \frac{5-2r}{4}$
	$I(y,t) = t[(\frac{3+2r}{4} - y)^2 + (y - \frac{5-2r}{4})^2]$	for $\frac{5-2r}{4} \leq y \leq 1$
$\frac{3}{4} \leq r \leq 1$	$I(y,t) = \frac{t}{8}$	for $\frac{1}{2} \leq y \leq \frac{1+2r}{4}$
	$I(y,t) = t[\frac{1}{8} - t(y - \frac{1+2r}{4})^2]$	for $\frac{1+2r}{4} \leq y \leq \frac{5-2r}{4}$
	$I(y,t) = t(\frac{3+2r}{4} - y)^2$	for $\frac{5-2r}{4} \leq y \leq \frac{1+r}{2}$
	$I(y,t) = t[(\frac{3+2r}{4} - y)^2 + (y - \frac{5-2r}{4})^2]$	for $\frac{1+r}{4} \leq y \leq 1$

Table 7: Incremental value function $I(y,t)$

The first three equilibrium configurations – local monopoly, kinked, and competitive – continue to be feasible, and when a symmetric level of t is exogenously specified to both firms, exactly one of these three configurations occur in each segment of the permissible range of t value. Though the ranges themselves are different, the basic structure derived in Section 5 is preserved – as t decreases, the equilibrium transitions from local monopoly to kinked, and then from kinked to competitive. Table 6 summarizes the relevant ranges of t, and the corresponding outcomes. At $r = \frac{1}{2}$, the results coincide with the corresponding results derived in Section 5, and there is therefore no discontinuity. Directionally, the comparative statics discussed in Section 5 are preserved, except those relating to the impact of changing r.

More importantly, the fourth kind of equilibrium configuration described in Section 4.2 is feasible for $r > \frac{1}{2}$. It arises when prices fall to the point where, rather than trying to get consumers to switch products, incremental demand is driven by inducing them to buy a second product. As a consequence, the inverse demand function of firm i is determined not by the difference in total value between the two products, but by the *incremental value* that product i provides to a consumer who already owns product j.

Under symmetric scope across products, let $I(y,t)$ be the *incremental value function*, which specifies the additional value that the consumer located at a distance $q \in [0, \frac{1}{2}]$ from product i obtains from product i if she already owns product j. The function is defined recursively as:

$$I(q + \frac{1}{2}, t) = Max[0, U(q + \frac{1}{2}, t) - U(1-q, t) + I(1-q, t)].$$

Figures 11 illustrate the function $I(y,t)$. The function becomes economically relevant when neither product dominates the other on all of a consumer's desired functionalities[16].

Table 7 summarizes the algebraic expressions of $I(y,t)$ for the linear loss function. For $r \geq \frac{1}{2}$, $I(y,t)$ is monotonically decreasing in y, but positive everywhere. Under symmetric scope, the consumer who

Range of values of r	Equilibrium demand per firm	Equilibrium prices
$\frac{3-\sqrt{3}}{2} \leq r \leq \frac{9-\sqrt{5}}{8}$	$nq^*(t) = n(\frac{2r-1}{6} + \frac{\sqrt{4r^2-4r+7}}{12})$	$P^*(t) = \frac{t}{8} - \frac{(\sqrt{t(1-2r)^2}+\sqrt{(7-4r(1-r))t})^2}{144}$
$\frac{9-\sqrt{5}}{8} < r \leq \frac{7}{8}$	$nq^*(t) = \frac{n(5-4r)}{32(1-r)}$	$P^*(t) = \frac{(5-4r)t}{16}$
$\frac{7}{8} < r \leq 1$	$nq^*(t) = \frac{n}{2}$	$P^*(t) = \frac{t1-2r)^2}{8}$

Table 8: Equilibrium outcomes under the non-exclusive equilibrium configuration

has the highest incremental value from a second product will be located at $y = \frac{3}{4}$. This consumer will be indifferent between the two products when their prices are equal. If $p_j \leq I(\frac{3}{4}, t)$, the inverse demand function changes slope discontinuously from its value in the competitive region and a third region, which we term the non-exclusive region, emerges. The demand function in this non-exclusive region is more elastic than in the corresponding competitive region, and actually has the same slope as the monopoly inverse demand function. This is depicted in Figure 7(b). In summary, for $p_j \leq I(\frac{3}{4}, t)$:

$$\begin{aligned} P_i(q_i, t, t, p_j) &= U(q_i + \frac{1}{2}, t) - U(1 - q_i, t) + p_j \quad \text{for } 0 \leq q_i \leq q_{NE}; \\ &= I(q_i + \frac{1}{2}, t_i, t_j) \quad \text{for } q_{NE} \leq q_i \leq \frac{1}{2}, \end{aligned} \tag{21}$$

where q_{NE} is defined using the equation $I(1 - q_{NE}, t) = p_j$.

For a high enough value of r, a symmetric non-exclusive equilibrium always exists. Table 8 summarizes the equilibrium outcomes for different values of r, when marginal costs are zero. Figure 11(c) and 11(d) illustrate these outcomes, depicting the equilibrium demand $nq^*(t)$ and prices $P^*(t)$ as a function of r. Given the diametrically opposite location of firms at $\frac{1}{2}$ and 1, an equilibrium demand of $nq > \frac{n}{4}$ implies that the markets start overlapping in the middle, and that consumers located at $y \in [1-q, \frac{1}{2}+q]$ buy both products.

Of particular interest is the case of $r \geq \frac{7}{8}$, where every consumer in the market buys both products, and prices drop to the incremental value of the consumer who values the product the least. This price is far lower than the actual value that any consumer gets from the product she buys, and is also lower than the corresponding competitive equilibrium price at the same level of scope. As a result, the consumer surplus will be much higher under the non-exclusive equilibrium. Interestingly, total surplus is also strictly higher than under the competitive equilibrium configuration at the same level of scope, because a fraction (sometimes all) of consumers buy both products, and use the *more efficient* of the two products to satisfy their functionality requirements.

Range of values of r	Equilibrium demand per firm	Equilibrium prices
$1-\frac{1}{t} \leq r \leq \frac{5t-2-\sqrt{t^2-4t+20}}{4t}$	$nq^*(t) = \frac{n(2rt-4+\sqrt{rt(rt-4)-28})}{6t}$	$P^*(t) = \frac{24-(rt-2)^2+(rt-2)\sqrt{rt(rt-4)+28}}{36t}$
$\frac{5t-2-\sqrt{t^2-4t+20}}{4t} \leq r \leq \frac{t^2-2}{t^2}$	$nq^*(t) = \frac{2+t(1-r)(t-2)}{4t^2(1-r)}$	$P^*(t) = \frac{2+t(1-r)(t-2)}{4t}$
$\frac{t^2-2}{t^2} \leq r \leq 1$	$nq^*(t) = \frac{n}{2}$	$P^*(t) = \frac{(2-t(1-r))^2}{4t}$

Table 9: Equilibrium demand and prices for very low scope levels

7.3 Duopoly: very low platform scope

Finally, we discuss a market in which products have very low scope – when $t > 4$ With a linear loss function, the fraction of functionalities a product covers is $\frac{2}{t}$ and hence $t > 4$ implies that each product covers less than half the functionalities demanded in the market. Clearly, there is no overlap in the set of functionalities provided by the two products. Each firm is a local monopolist, and prices accordingly. However, at very high levels of r, both products may provide some consumers with positive value, on distinct subsets of their desired functionalities.

We consider a specific example, where $t > 4$, $1 - \frac{1}{t} \leq r \leq 1$, and the loss function $g(x) = x$. Under these constraints, the consumer's value function is[17]:

$$U(y,t) = \begin{cases} \frac{1}{t} & \text{for } \frac{1}{2} \leq y \leq \frac{1+r}{2} - \frac{1}{t}; \\ \frac{1}{2t} - (y - \frac{1+r}{2})(1 + \frac{t}{2}(y - \frac{1+r}{2})) & \text{for } \frac{1+r}{2} - \frac{1}{t} \leq y \leq \frac{3-r}{2} - \frac{1}{t} \\ \frac{1}{t} - (1-r)(1 + t(1-y)) & \text{for } \frac{3-r}{2} - \frac{1}{t} \leq y \leq \frac{1+r}{2} \\ \frac{1}{t} - (1-r) + t((1-y)^2 + (\frac{1-r}{2})^2) & \text{for } \frac{1+r}{2} \leq y \leq 1 \end{cases} \tag{22}$$

Refer to Figure 12 for the intuition underlying the derivation of $U(y,t)$.

The profit function is similar to the monopoly profit function – there is at most one interior maximum, and profits are maximized either at this interior point, or at full market coverage. For $c = 0$, the optimal (local monopoly equilibrium) choices for the symmetric duopolists are summarized in Table 9. Even though the firms are local monopolists and do not influence each others prices or profits, it is clear that in equilibrium, there is some overlap in the consumers served by both products[18]. As the span of consumers' functionality requirements increases, a larger fraction of the consumers buy both products. For values of r sufficiently close to 1, all consumers in the market buy both products.

This example is particularly interesting when contrasted with the other non-exclusive equilibrium described in Section 7.2. It suggests that as digital technology progresses, equilibrium outcomes may be such that consumers start by buying multiple specialized products, then switch to a single general-purpose product, and then finally buy multiple products again, using the additional powerful general-purpose products as if they were specialized. We discuss this further in Section 8.

8 Discussion

We have developed and presented a model that provides a careful and detailed representation of how technology platforms fulfil diverse consumer functionality requirements, how the effectiveness of these products varies with platform scope, and how this affects product value and differentiation. The model is a new contribution to the economics of information technology, and we hope it will enable deeper analysis into some of the issues we have raised.

Moreover, since we have solved the model comprehensively, characterized the structure of the equilibria, and derived the usual measures of interest (prices, demand, profits and surplus), this enables us to explain some unique aspects of digital convergence in product markets, to predict likely outcomes, and to provide prescriptions for the relevant firms, and for policy makers.

8.1 Product differentiation, platform scope and price trends

Any standard model of product differentiation would suggest that as differentiation falls, so do prices. Our analysis shows that in the case of converging technology products, this is not always true. In fact, as markets that were distinct begin to overlap, prices rise in response to reduced differentiation. Subsequently, at a critical level of platform scope, the cost of increased substitutability outweighs the benefit of increased value, and prices do fall. This trend is consistent with the recent convergence in the mobile telephony and handheld computing industries. Following the launch of flagship and mainstream converged devices from both industries (the Palm-based Handspring Treo, and the Symbian-based Nokia Communicator), average prices for these devices are actually higher than those that prevailed when the flagship products from each of these device makers was more specialized. This is despite an evident increase in the overlap in their functionality. Our model predicts this phase, and indicates that while it will sustain for a while, it will be followed by rapid price declines across both industries, when technology progresses to the point where they enter the competitive equilibrium configuration.

8.2 Strategic choices of platform choice

If firms in converging technology markets can decide the level to which they should expand their platform scope, our analysis prescribes that after a point, their choices should be determined purely by strategic considerations. As described in section 6, even when expanding platform scope is costless, firms that make strategically sound decisions will limit the extent to which they fulfill customer needs, even if more extensive fulfillment is technologically viable.

In addition, a firm's choice of platform scope may also be influenced by competition from within one's industry (rather than the fear of overlapping with an adjacent provider). A shortcoming of our existing model is that we do not explicitly consider this effect. However, our analysis in Section 5, which characterizes outcomes when firms cannot explicitly control the scope of their platforms, becomes more relevant in this context. By explaining how firms will behave when they are forced to vary scope, and when the only strategic variable they can control is price, we can predict that if there are intra-industry competitive factors that dictate platform scope choices, converging markets will eventually transition into the competitive equilibrium configuration, with prices falling as platform scope increases.

In fact, converging into a different industry may be a strategic response to increased competition (this may explain, for instance, why Handspring chose to incorporate voice communication into their product via the Springboard). A more precise analysis of competition within each duopolist's location, as well as the threat of convergence-driven overlap from a neighboring industry, remains research in progress.

8.3 Welfare and policy implications

We find that the market generally does not provide the socially efficient level of scope. What is surprising is the direction in which the market errs. Intuitively, under complete market coverage, when there is no incentive to recruit new consumers, one would expect firms would slacken on their provision of platform scope. However, we find that this is precisely the scenario under which firms provide a level of scope which is socially excessive. Correspondingly, when only a subset of the consumers in the market purchase the product, the firms underprovides scope. This result is independent of market structure – it persists under both single-product monopoly and duopoly, and we can easily show that it holds for multi-product monopoly as well.

It is likely that universal access will become a social priority for mobile telephony and Internet access, as the use of these services supersede wireline telephony as the primary mode of access to emergency police or medical services, or simply if public policy dictates equitable access to electronic forms of commerce and work. Our results establish that as progress in the underlying digital technologies reduces the marginal cost of platform scope, these social objectives can often be achieved without resorting to regulatory intervention

8.4 Non exclusive choice and technology cycles

The analysis of Section 7 suggests an interesting trajectory of consumption that accompanies the steady increase in platform scope driven by progress in the underlying digital technologies. For a sufficiently wide span of functionality requirements, consumers initially purchase multiple products to satisfy their needs, since each product offers low scope. As platform scope increases, firms increase prices and consumers shift to buying a single general-purpose product, as expected. However, at very high levels of scope, consumers may switch to buying multiple products again. The first instance of non-exclusive purchases is driven by the inability of a single product to fulfill the entire span of consumer functionality requirements satisfactorily. At the high end of scope, while each product is very effective at satisfying a consumer's entire span of desired functionalities, consumers still purchase multiple products, and employ these general-purpose products as if they were specialized.

If one examines the Windows-Mac market, there are indications that the prices have dropped to the point where something resembling a non-exclusive equilibrium is emerging. Dedicated Mac users have often bought PC's for their 'office' needs. Currently, more dedicated Windows users are buying Macs for specific functionality needs – DVD viewing, digital photography, and video editing being common examples. Both products are powerful, general-purpose machines, capable of fulfilling all the functionality requirements described above. Clearly, these multi-product purchases are being driven by incremental value.

The trajectory we have described may in fact be cyclical. As the power of any generation of technology-based products increases, and consumers get used to the level of performance provided by the general-purpose products they are using as specialized devices, they may 'revise' their partial-equilibrium utility functions to ones that expect a higher level of effectiveness. This could result in firms' platform scope choices as being perceived as specialized again, after which when the next generation of technology emerges, the cycle repeats. Incorporating this kind of effect into a formal model remains work in progress. We have also made some progress on a more precise representation of how intra-industry competition may dictate platform scope choices. We hope to complete this work in the near future.

Notes

[1]In Lancaster's models, consumption is represented by a process in which products are inputs, characteristics are outputs, and the preferences of consumers are specified over these characteristics. Preferences over products are derived from these preferences over characteristics, depending on the specific characteristics provided by each product.

[2]However, this precludes value from a lower-quality 'backup product', or from combining the capabilities of multiple products for a specific functionality requirement. It also does not capture possible technological complementarities from co-location of functionality within a device, or performance degradation from having multiple tasks competing for the same device resources. We analyze some of these issues explicitly elsewhere (Mantena and Sundararajan, 2002).

[3]For a fixed set of customer needs, a product may be more differentiated from its competitors due to the fact that its scope (and that of its competitors) is lower, or because its core functionality is more distinct (further away) from that of its competitors.

[4]To simplify exposition, we sometimes use the phrase 'a product located at z' rather than the more cumbersome 'a product whose core functionality is located at z'.

[5]Assuming a density of $\frac{n}{2}$ and not n leads to simpler algebraic expressions.

[6]Therefore, the realized demand corresponding to an inverse demand of $P(q,t)$ is nq – which is why we define it as the 'rescaled' inverse demand function, and is also why we choose $\frac{n}{2}$ rather than n as the market size. For expositional simplicity, we subsequently drop 'rescaled', and refer to it as the inverse demand function.

[7]Since we focus on the effect of product scope, the products' locations are fixed by assumption. The diametrically opposite location (or equivalently maximal differentiation) assumed here is standard in spatial models (Salop, 1979, Grossman and Shapiro, 1984).

[8]We adopt this approach because the best response functions involve a number of different cases, are often discontinuous and provide little intuition or tractability in terms of either proving the existence of equilibria or characterizing them.

[9]This analysis, while straightforward, is very lengthy – a copy is available on request.

[10]This effect is analogous to Salop's result of prices increasing as costs reduce – the source of tension is similar (an inability to increase realized demand as a consequence of the presence of the rival firm).

[11]For small changes in t, this pair will continue to be a feasible kinked equilibrium so long as it is not at one of the end-points of the $q_i^*(t)$ intervals specified in Table 5.

[12]This also suggests that under endogenous and costless choices of scope, the level of scope at the boundary between the kinked and competitive regions is a likely symmetric equilibrium. This is explored further in Section 6.

[13]The fact that profits are strictly lower is due to our choosing the kinked equilibrium with symmetric scope. However, one can use Proposition 3(b) to show that in the kinked equilibrium most favorable to a firm, that firm's profits are bounded above by (though sometimes equal to) the corresponding monopoly profits.

[14]More precisely,there are portions of the (t_A, t_B) space (where the values of t_A and t_B are substantially different) in which no pure strategy equilibrium pairs of q may exist. It is possible that mixed-strategy equilibria do exist, but their derivation is beyond the scope of this paper. A simple way around this problem (used by Economides 1984, for instance) is to set the payoffs of both firms to zero for these pairs of values. Note that pure-strategy equilibria always exist for all symmetric pairs $t_A = t_B$, and these are the equilibria we discuss in most detail.

[15]This is in contrast with the unusual result obtained under the kinked equilibrium in Salop (1979), where prices rose as marginal costs fell.

[16]If product j provides a higher quality than product i on the entire range of functionalities that a consumer cares about, then this incremental value is simply zero. If on the other hand, product i is superior to product j on the entire relevant range of functionalities, the incremental value is simply the difference between the gross values provided by the two products. In either of these cases, under symmetric prices, consumers have no incentive to buy both products.

[17]Interestingly, with non-overlapping functionalities provided by the two products, a product's value function is identical to the incremental value function that was used in deriving the non-exclusive equilibrium in the last section.

[18]At the margin, with $t = 4$ and $r = 1 - \frac{1}{4} = \frac{3}{4}$, $q^*(t) \simeq 0.29 > \frac{1}{4}$.

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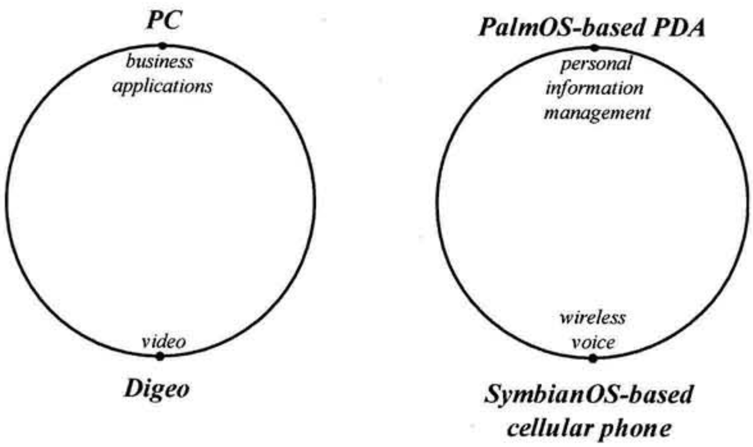
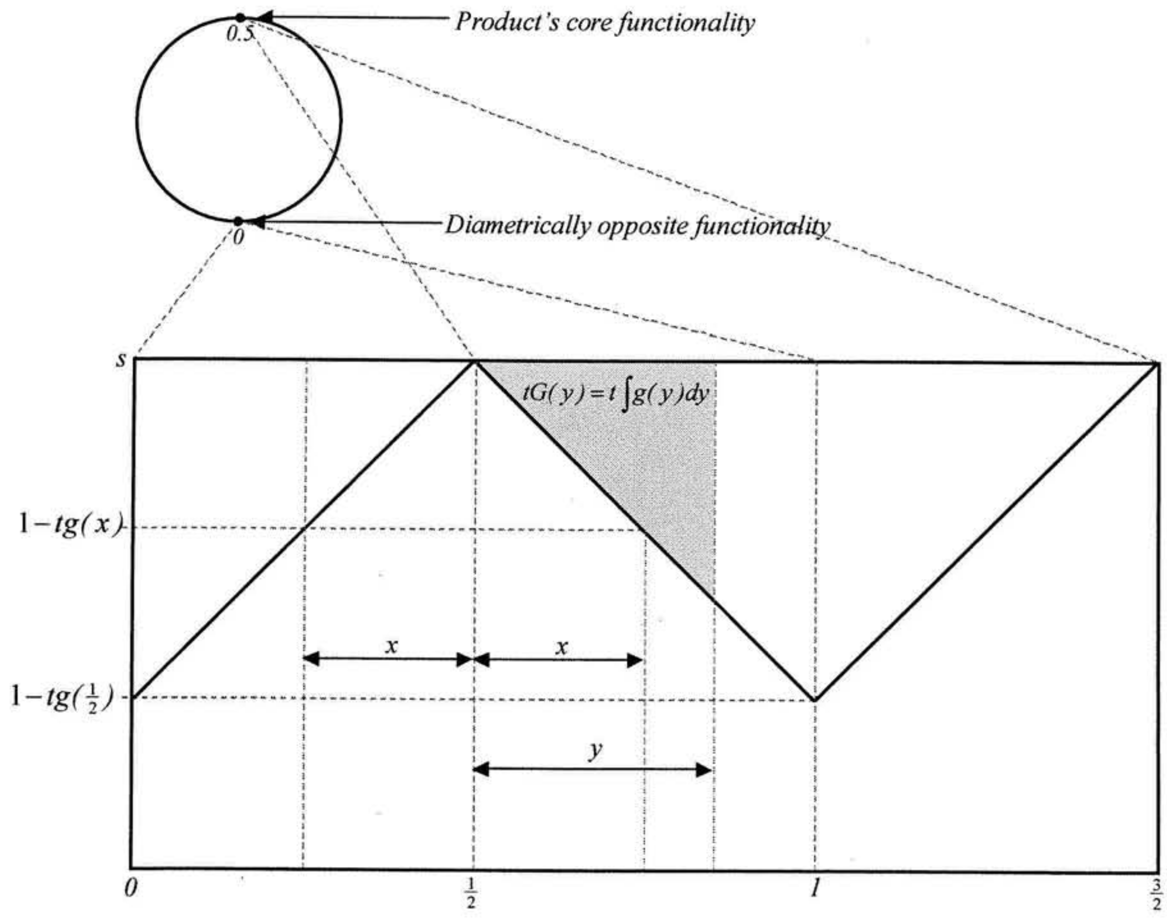


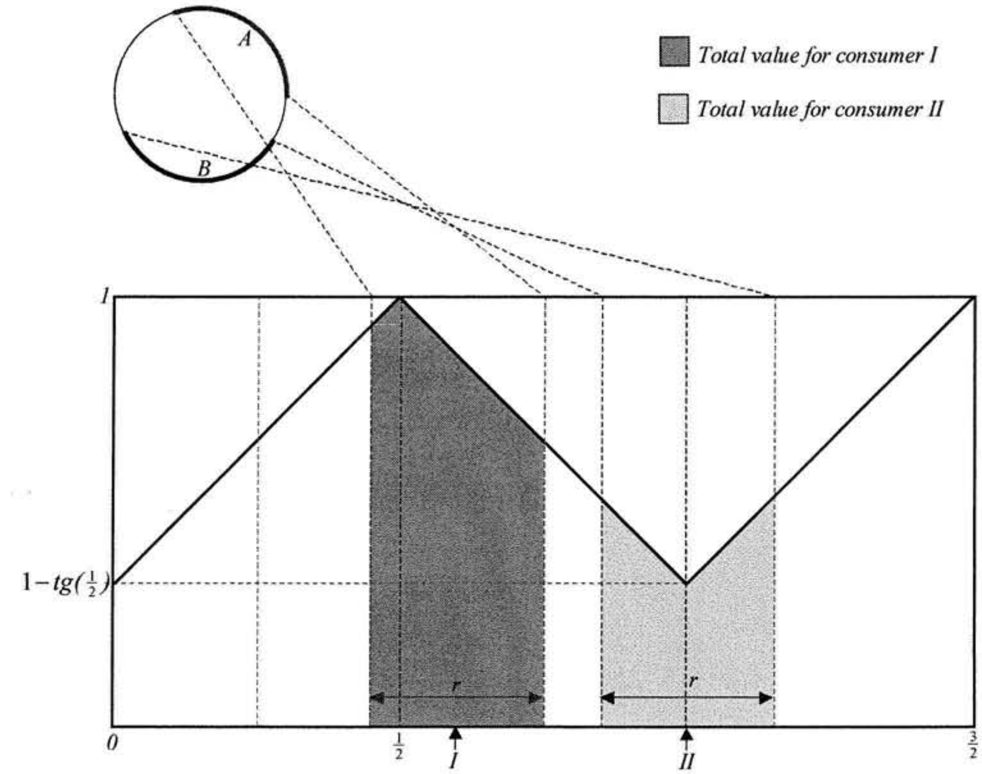
(a) Examples of products and their core functionalities



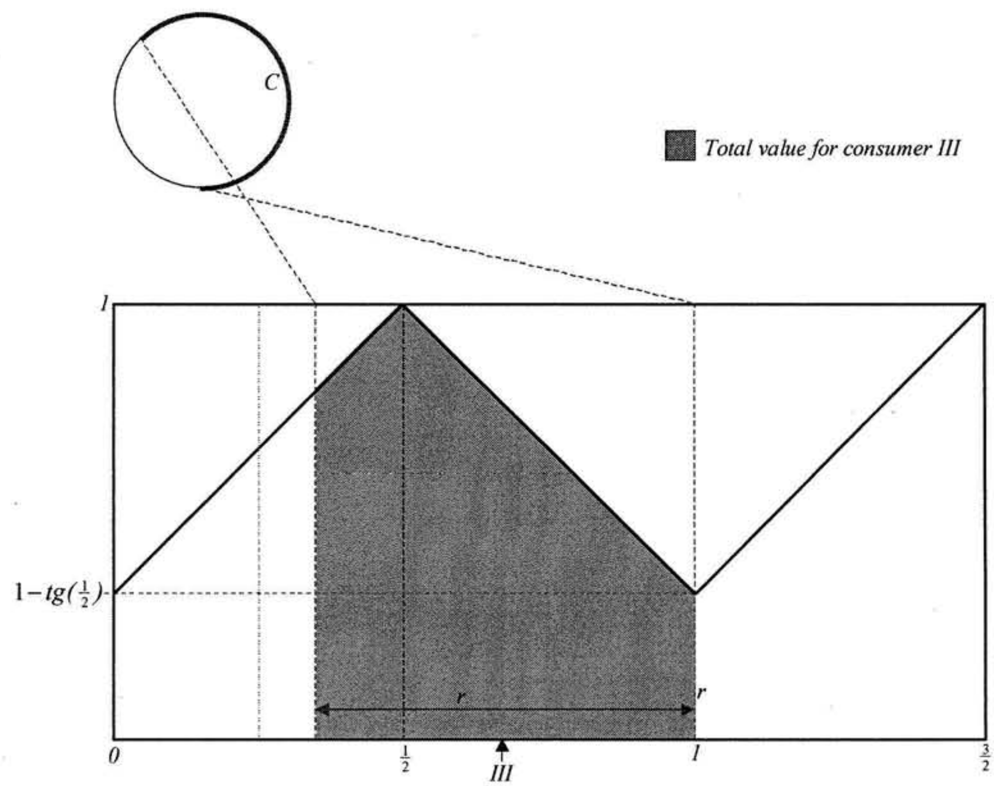
(b) The effectiveness of a product across its different functionalities

Figure 1: Depicts typical products, their core functionalities, and the effectiveness of the product across its functionalities. The circle is 'unfurled' in figure 1(b), illustrating how product effectiveness degrades as one moves from the core functionality at $\frac{1}{2}$ to the opposite functiona

G(y) is also illustrated. For mathematical convenience, th

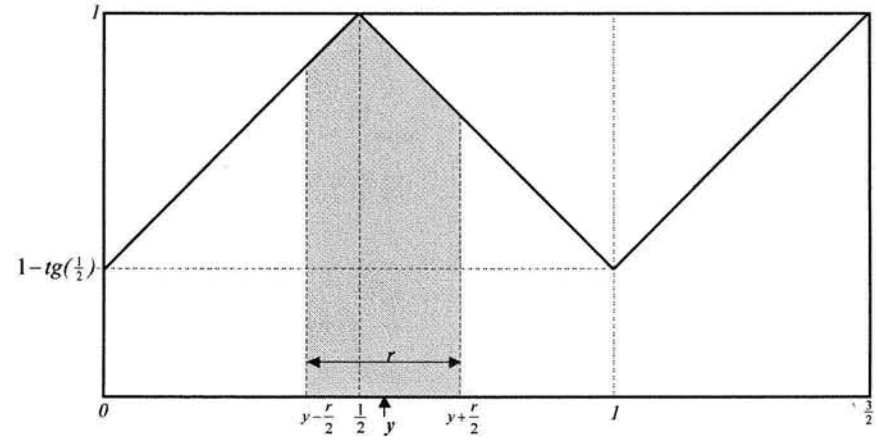


(a) Total value obtained by two candidate consumers, for *r* < *0.5*

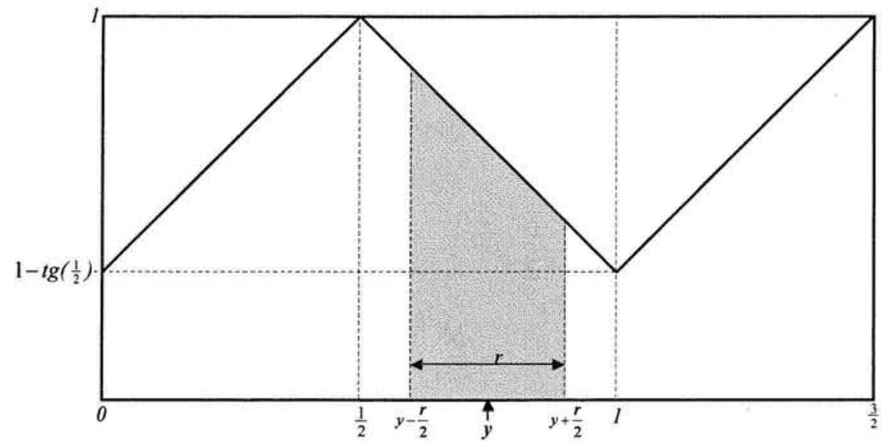


(b) Total value obtained by a candidate consumer, for *r* > *0.5*

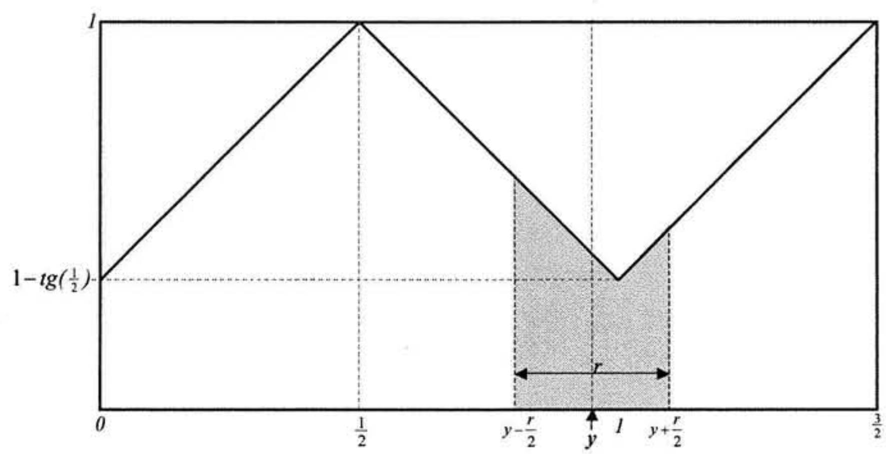
Figure 2: Depicts the value obtained by consumers located at different points in the market, for different values of r. A consumer is identified by the midpoint of their interval of desired functionalities, which then span $r/2$ on either side of this midpoint. The total value therefore depends on the location of this midpoint, relative to that of the product's core functionality, and is equal to the shad



$$U(y,t)=\int_{y-\frac{r}{2}}^{\frac{1}{2}}(1-tg(\tfrac{1}{2}-x)dx+\int_{\frac{1}{2}}^{y+\frac{r}{2}}(1-tg(x-\tfrac{1}{2})dx$$

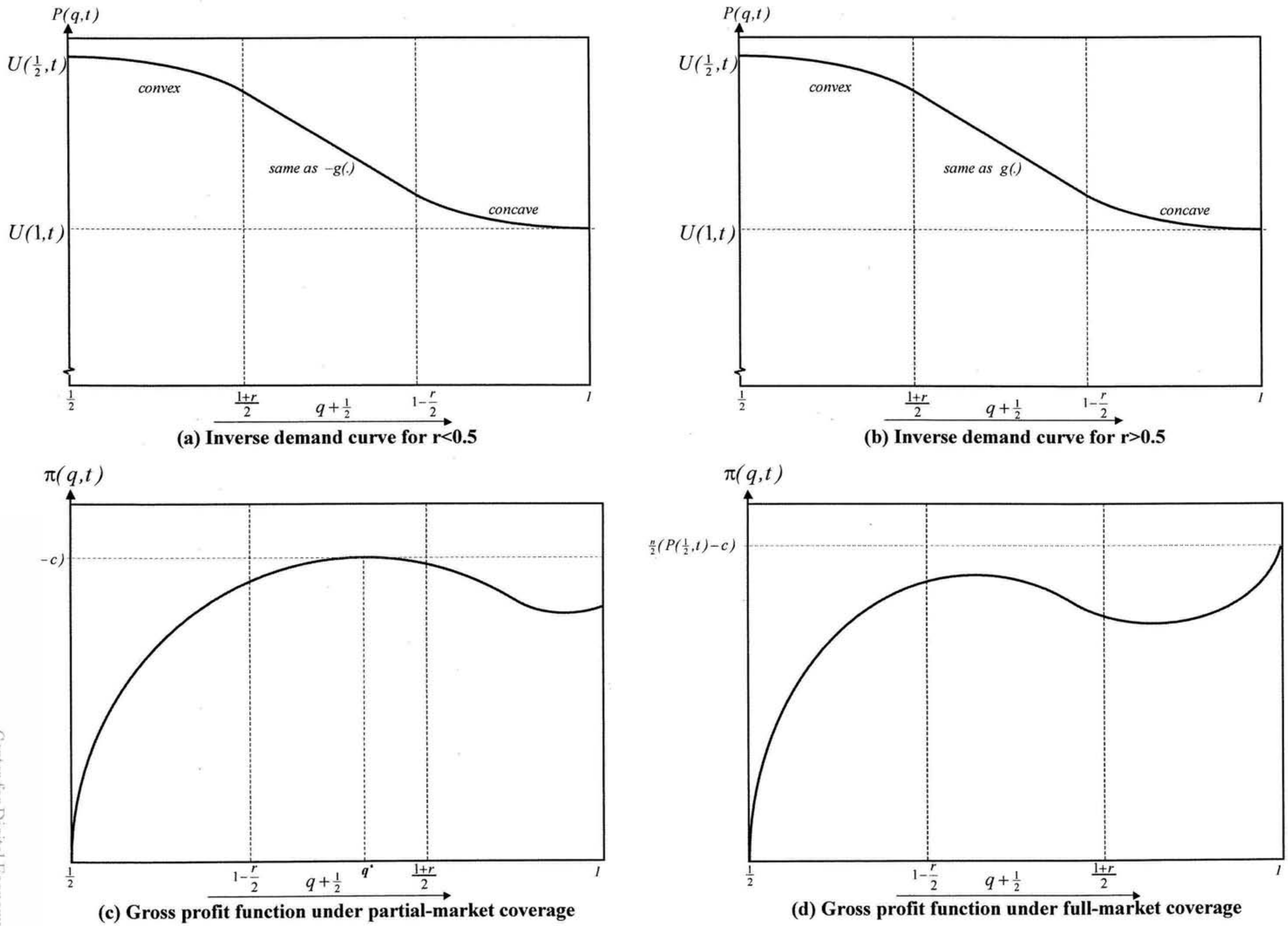


$$U(y,t)=\int_{y-\frac{r}{2}}^{y+\frac{r}{2}}(1-tg(x-\tfrac{1}{2})dx$$

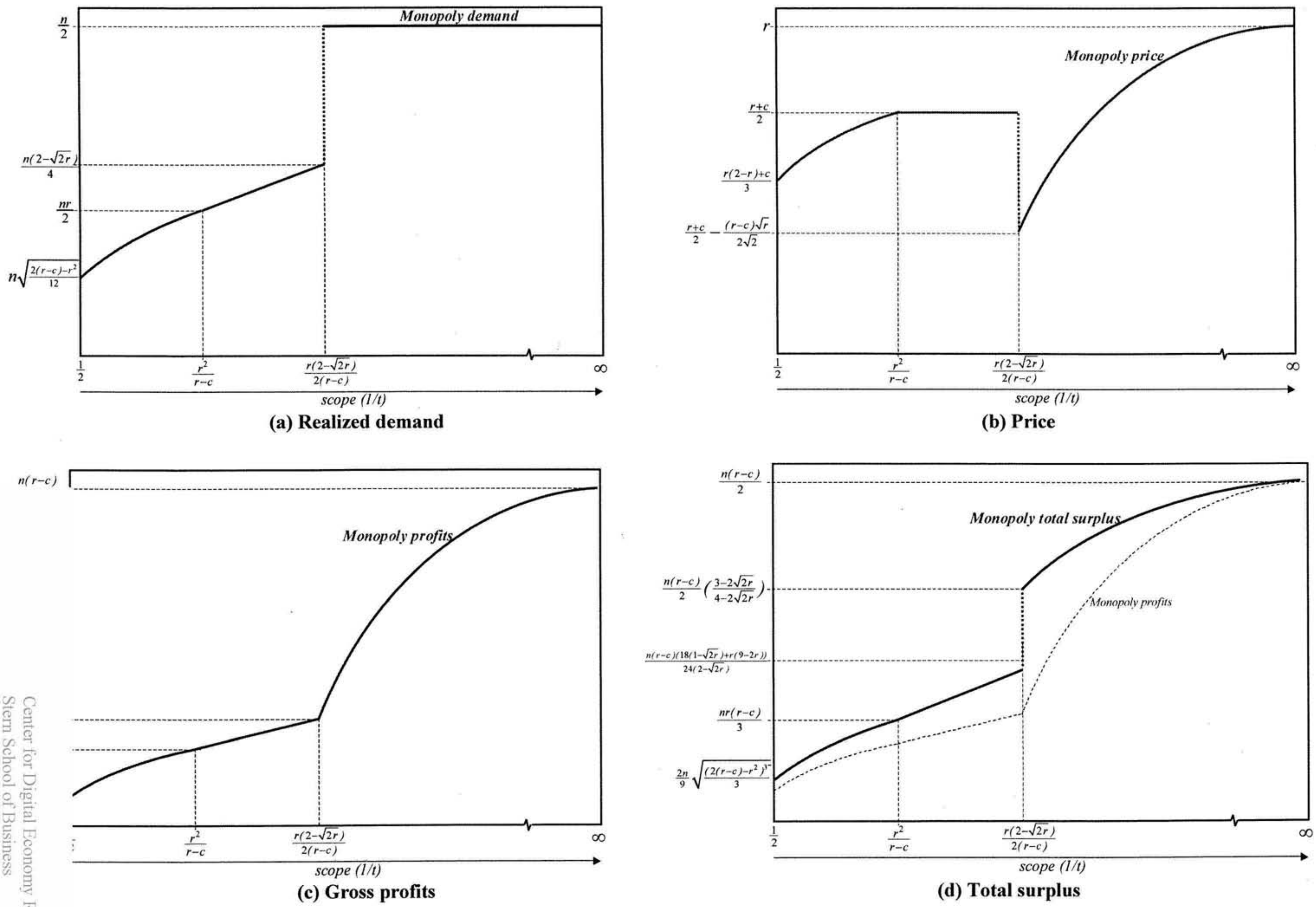


$$U(y,t)=\int_{y-\frac{r}{2}}^{1}(1-tg(x-\tfrac{1}{2})dx+\int_{1}^{y+\frac{r}{2}}(1-tg(\tfrac{3}{2}-x)dx$$

Figure 3: This illustrates the derivation of the consumer
the product covers segments of consumer functionality r
above), thus resulting in three different integral expressic



ıre 4: Inverse demand and revenue functions for a single-product monopolist. The structure of the demand curve is largely independent of the shape of – it starts out convex and ends up concave. This results in two candidate maxima for the profit function, as depicted in (c) and (d). Again, independent of the profit function has, at most, one interior local maximum.



re 5: Depicts the monopoly outcomes as a function of scope (1/t). Demand and prices increase steadily up to a threshold, at which point the monopolist s it optimal to lower prices substantially and sell to all consumers in the market. While profits and total surplus also increase monotonically, consumer lus (the difference between total surplus and profits in (d)) increases up to the threshold value of scope, and then after a substantial jump, falls rapidly.

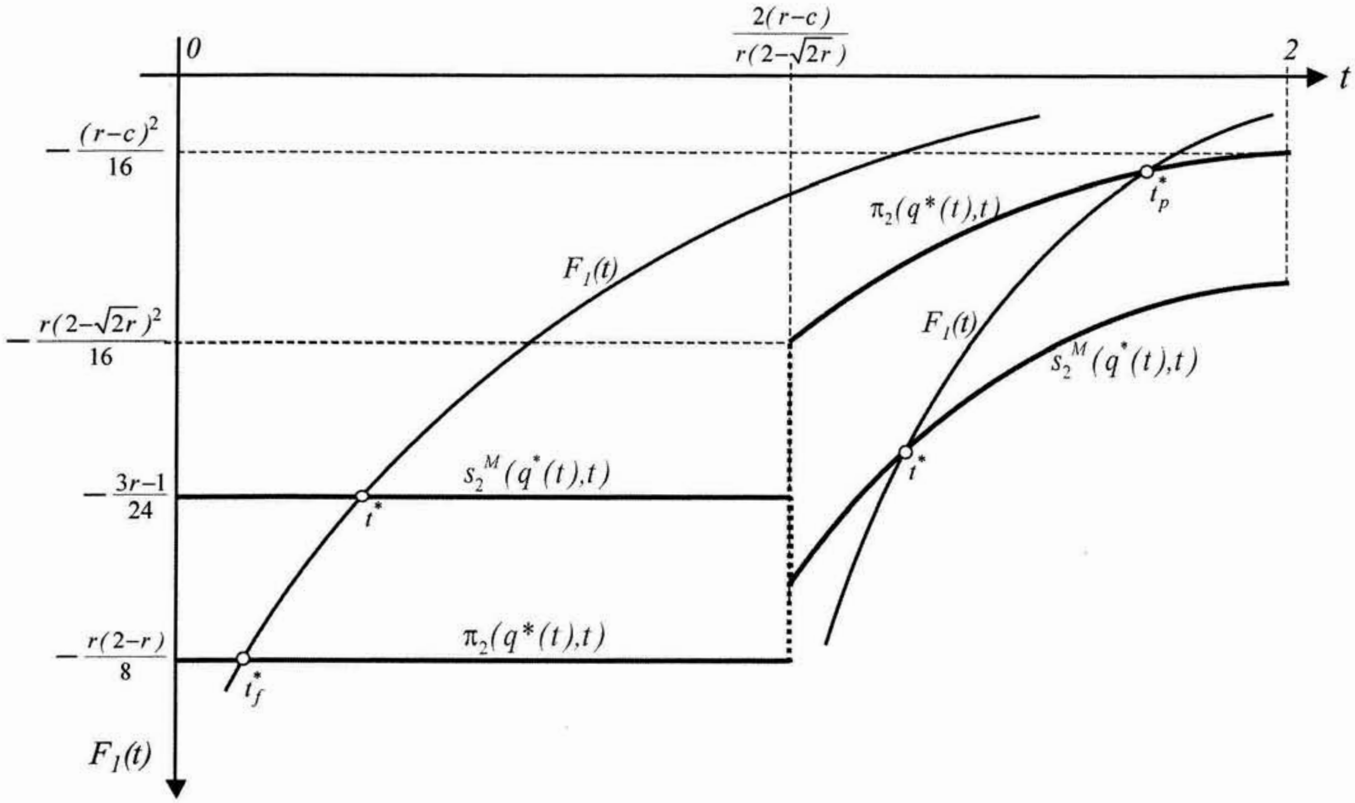
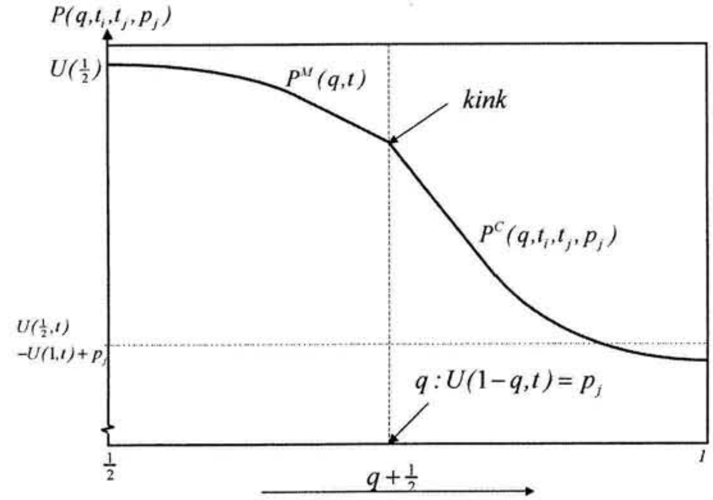
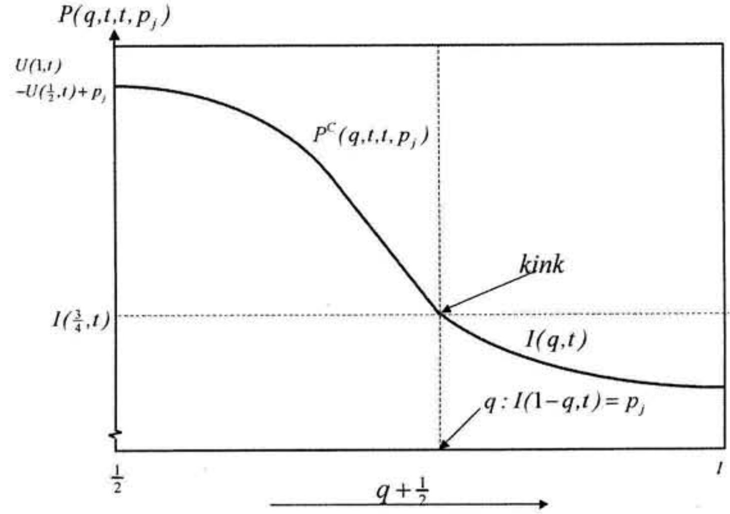


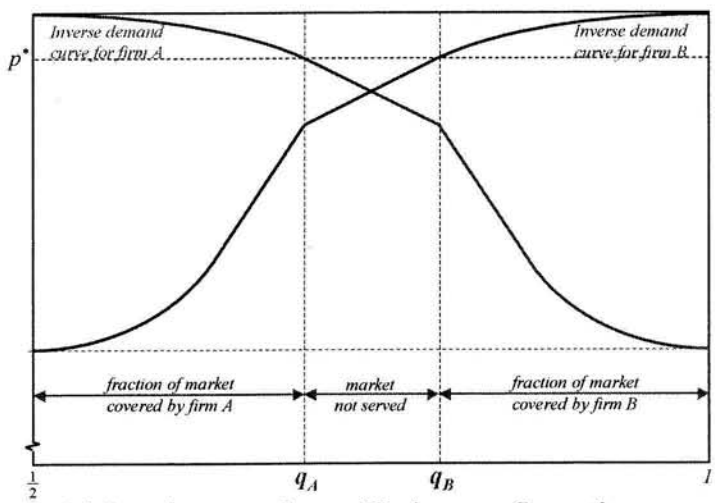
Figure 6: Depicts the relative values of the monopolist's optimal choice of t and the socially optimal choice of t for two different convex fixed cost functions. When $F(t)$ is strictly convex, the $F_1(t)$ curves slope upwards. Therefore, when the marginal cost of scope is relatively low, the monopolist's optimal choice of scope occurs in the full-market coverage region, where the marginal gross profit curve $\pi_2(q^*(t),t)$ is below the marginal gross total surplus curve $s_2^M(q^*(t),t)$, and leads to over-ir
partial-market coverage region, the marginal gross prof
marginal cost of scope is high, the monopolist's choice o



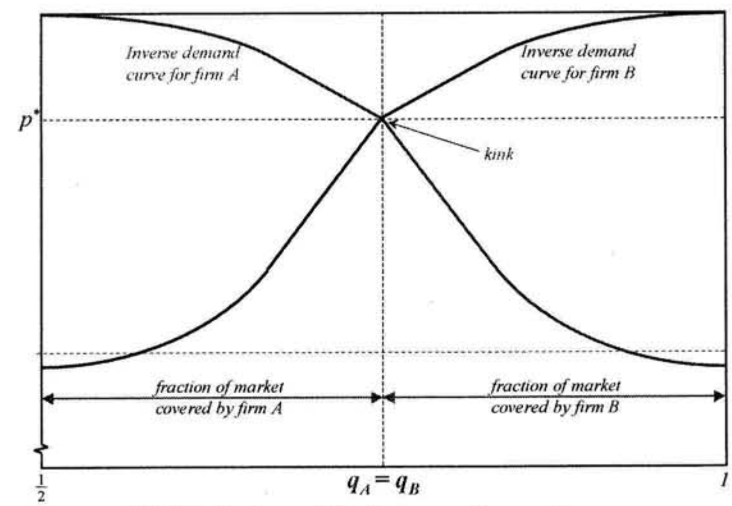
(a) Inverse demand function with higher prices



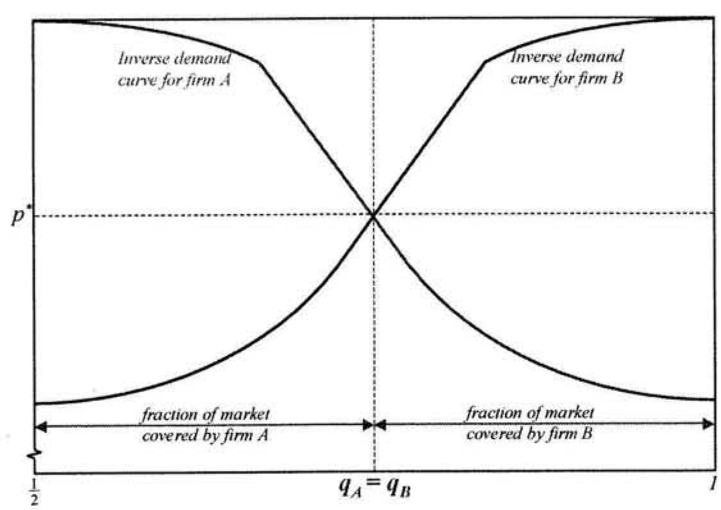
(b) Inverse demand function with very low prices



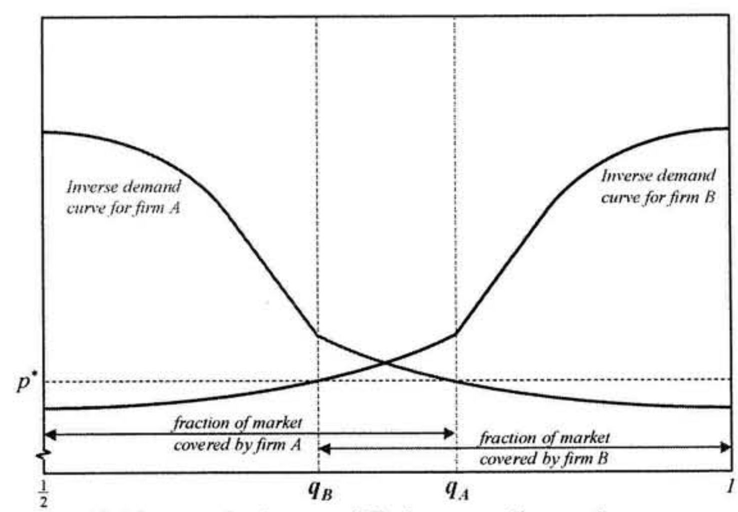
(c) Local monopoly equilibrium configuration



(d) Kinked equilibrium configuration

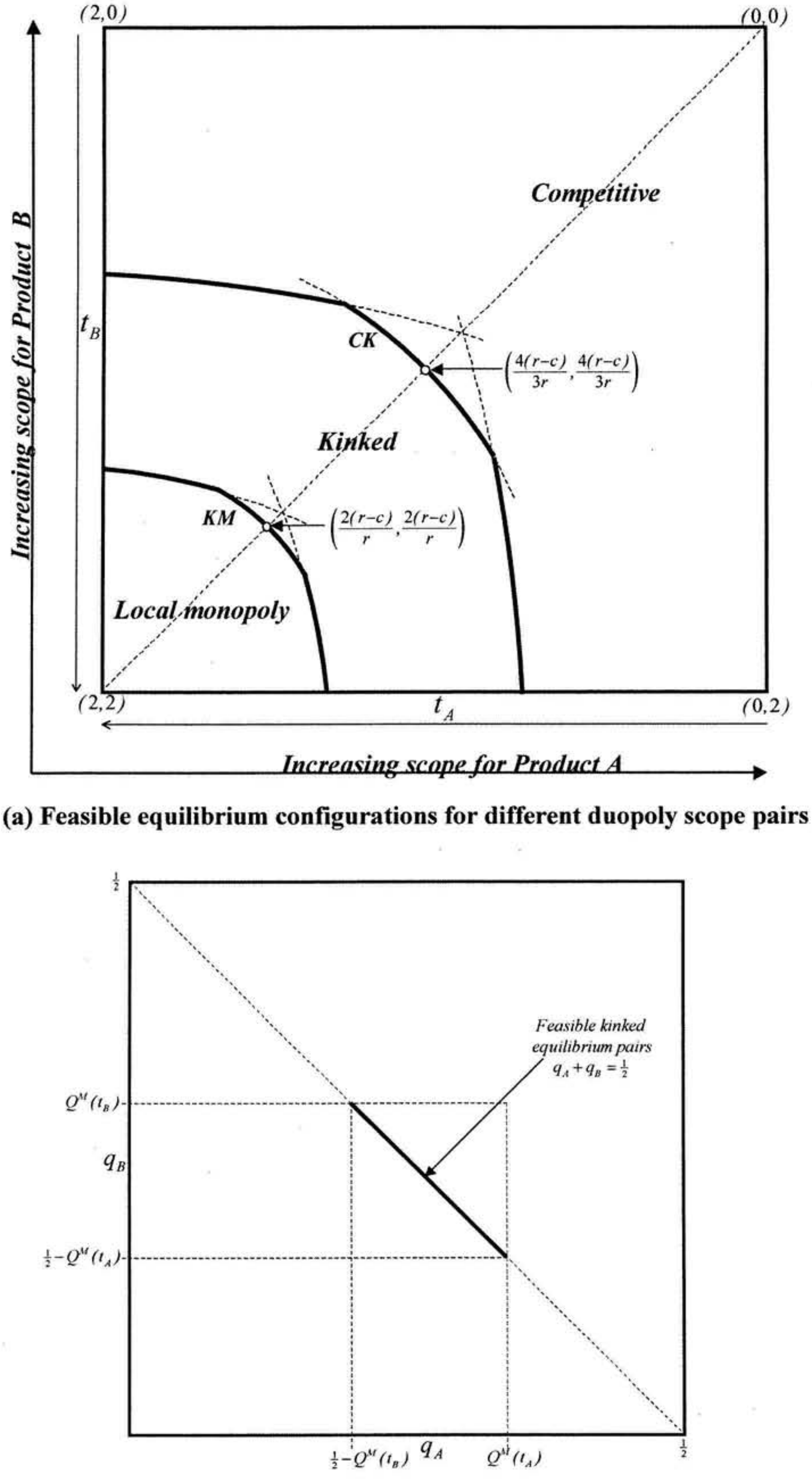


(e) Competitive equilibrium configuration



(f) Non-exclusive equilibrium configuration

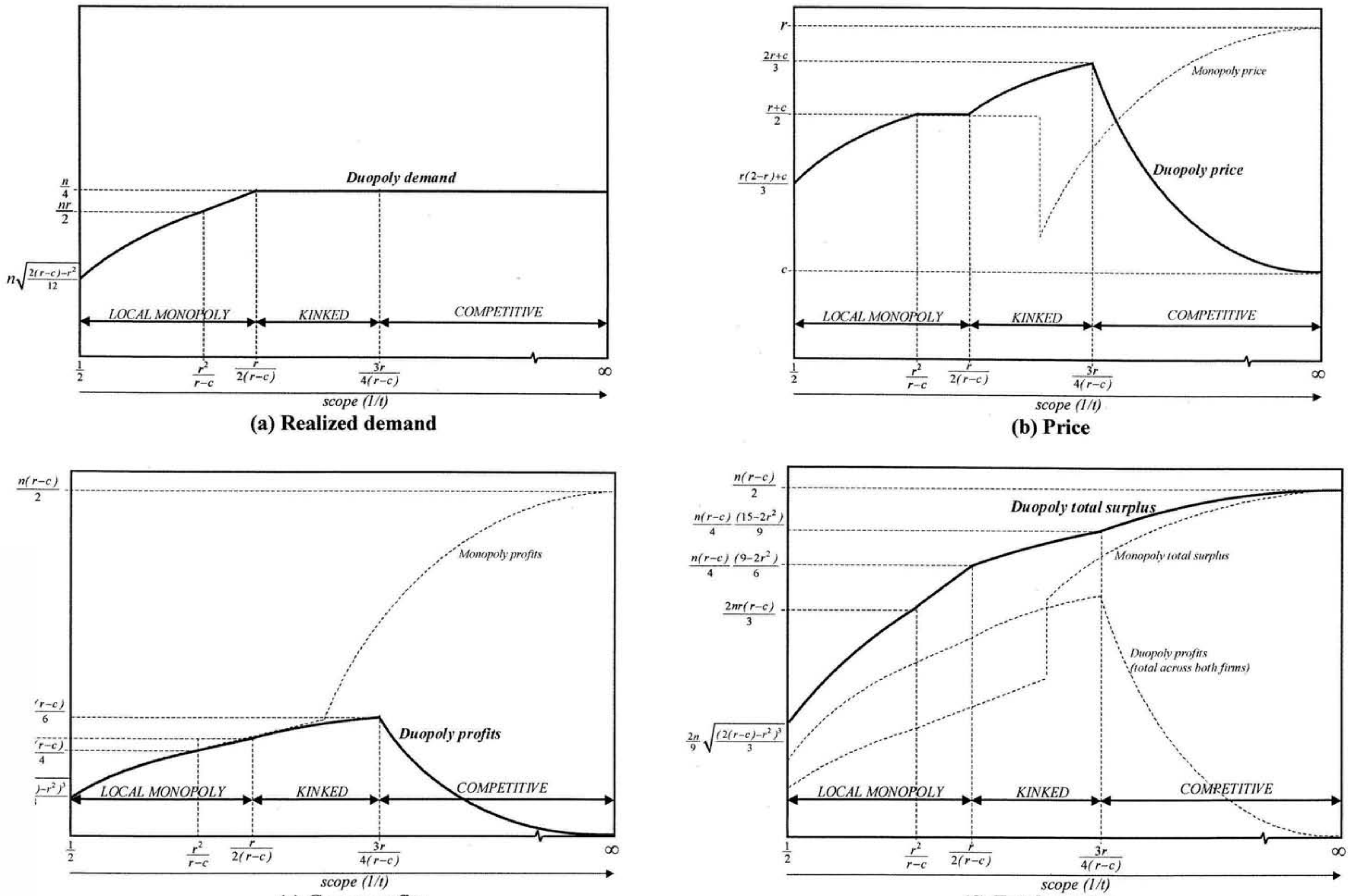
Figure 7: Inverse demand functions and equilibrium configurations under duopoly. The kinks in the inverse demand curves reflect first the influence of a competing product with non-zero surplus, and then a competing product such that the seller's product has non-zero incremental surplus. There are four equilibrium configurations feasible under these demand curves. In the local monopoly equilibrium, a pc
split evenly in the kinked and competitive equilibria,
r>0.5, a fraction of customers buy both products.



(a) Feasible equilibrium configurations for different duopoly scope pairs

(b) Illustration of multiplicity of kinked equilibrium outcomes

Figure 8: Illustrates the regions of the (t_A, t_B) space in which different equilibrium configurations occur, showing that as scope increases, the feasible configuration shifts frc
CK, which is the entire curve separating the kinked and
Proposition 7. In (b), the continuum of kinked equilibrium



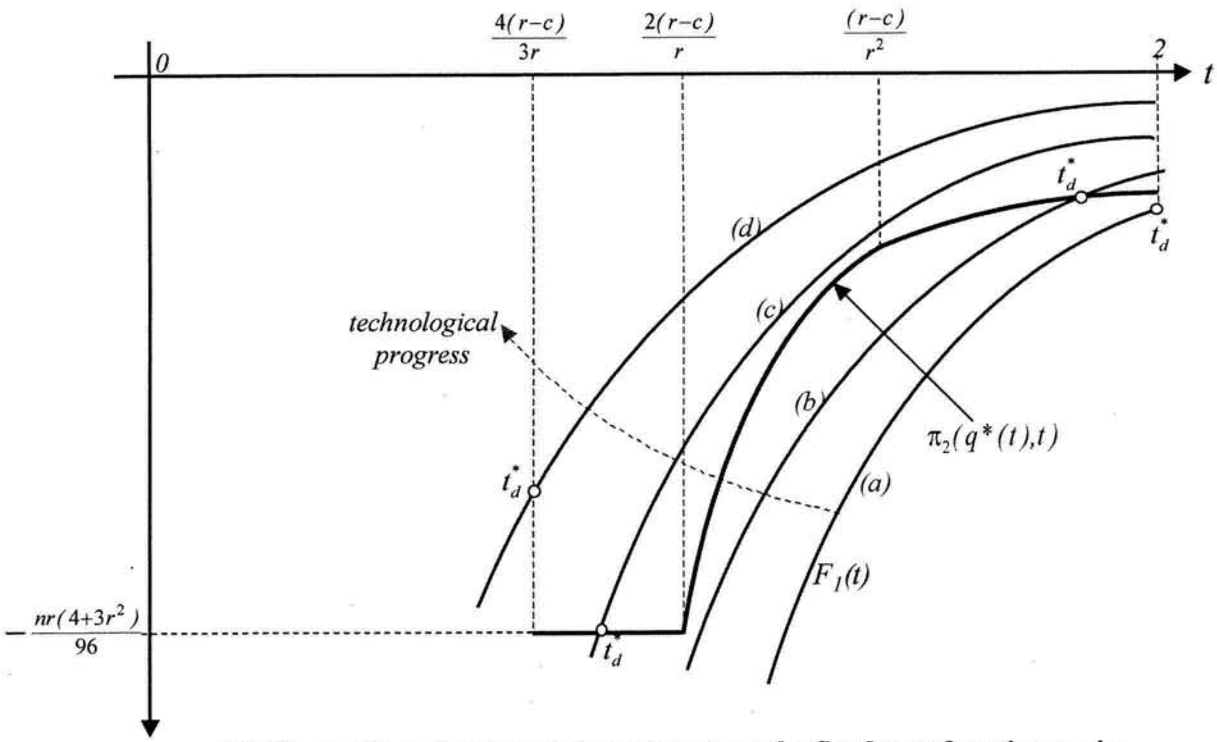
(a) Realized demand

(b) Price

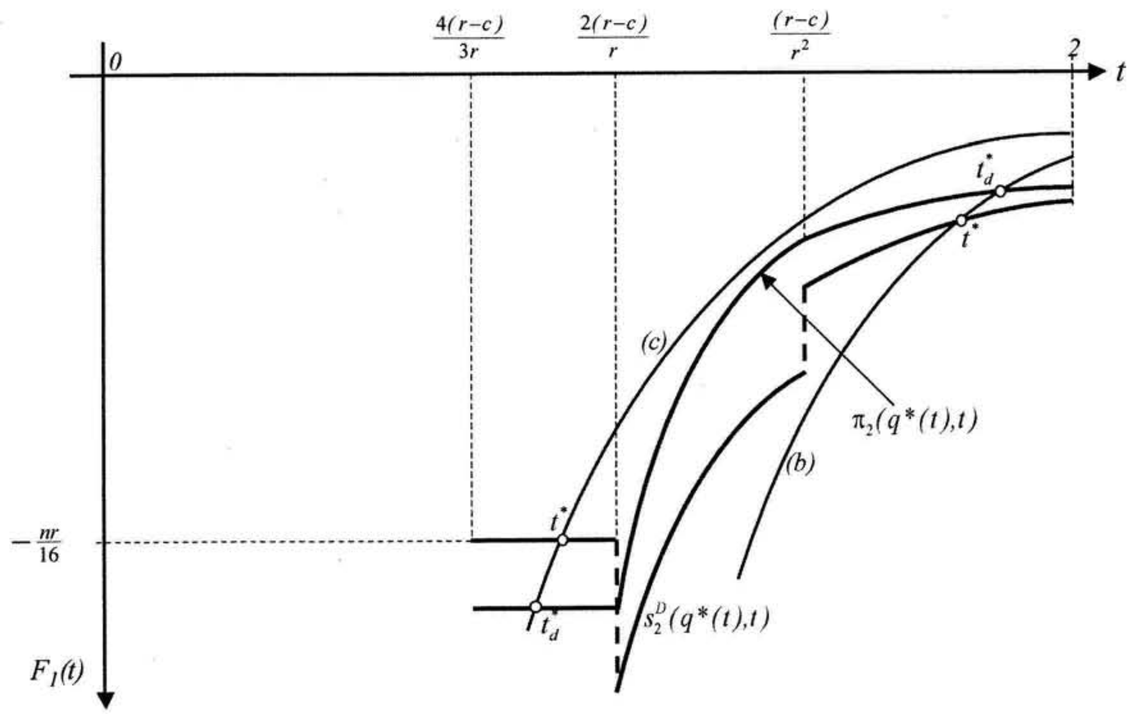
(c) Gross profits

(d) Total surplus

e 9: Depicts the duopoly outcomes as a function of scope (1/t), for symmetric scope. Demand rises steadily in the local monopoly region, after which it stant at 0.25. Prices continue to rise in the kinked region, before falling rapidly in the competitive region. Gross profits increase with scope in the ɔoly and kinked region, before falling in the competitive regions – they are always lower than the corresponding monopoly profits. Total surplus is / increasing across all scope values and equilibrium configurations. This is because product value increases for all customers as scope increases. brium consumer surplus (the difference between the duopoly total surplus and duopoly profit curves in (d)) increases across the local monopoly and :titive regions, but falls slightly in the kinked region, in response to the rapid increase in price. In addition, gross total surplus is strictly higher under ly than under monopoly – even under full market coverage in both regimes, there are two products in duopoly, and hence total value created is higher.



(a) Change in optimal scope investments as the fixed cost function varies

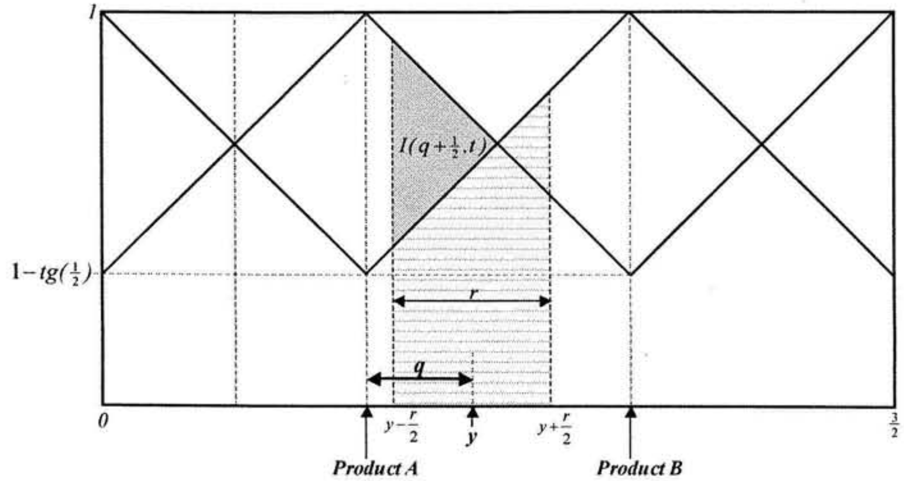


(b) Comparison with social optimum

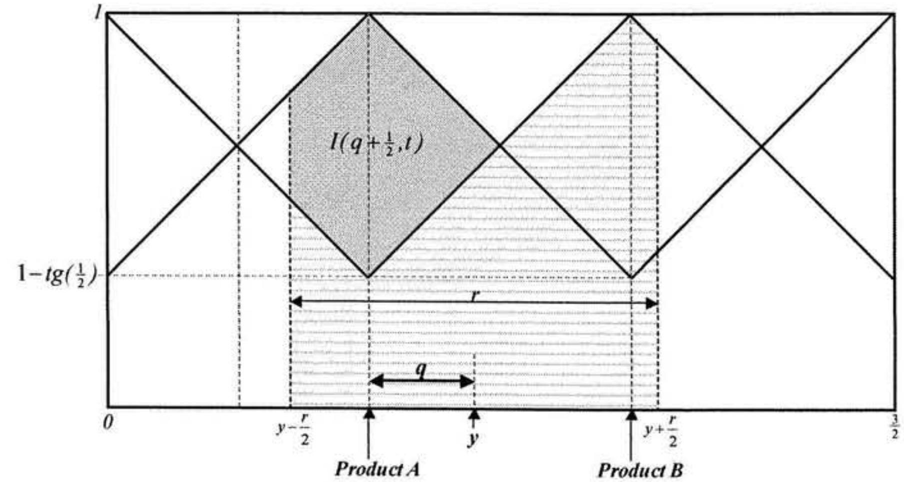
Figure 10: Depicts equilibrium scope investments for different costs of scope, when there is a unique solution.

(a) equilibrium duopoly scope increases as the marginal cost of platform scope $F_1(t)$ falls. When this curve lies entirely above the marginal gross profit curve $\pi_2(q^*(t),t)$, the result of Proposition 7 continues to hold.

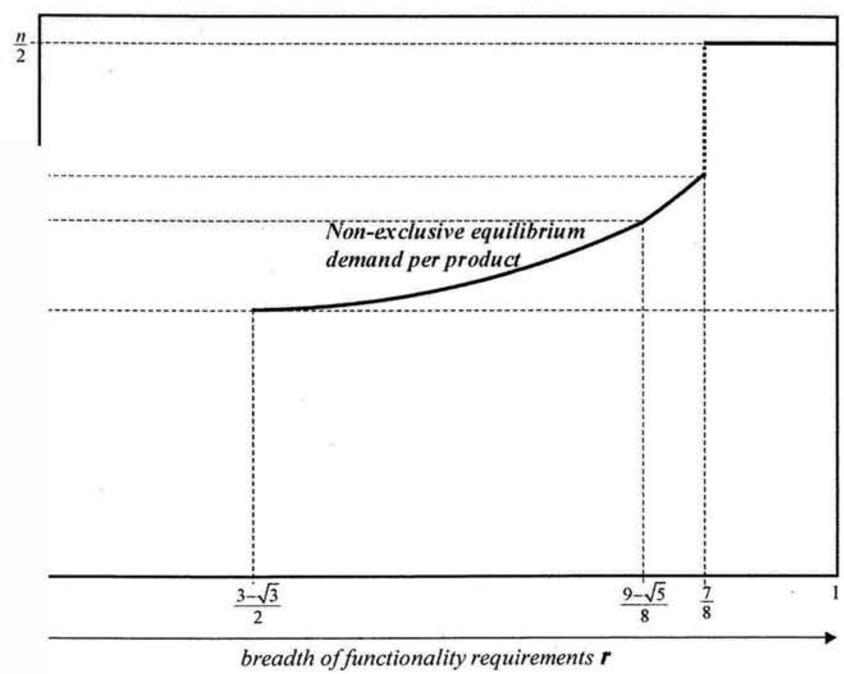
(b) When the solution is unique and in the interior, partial coverage to over-investment – which is identical to the out



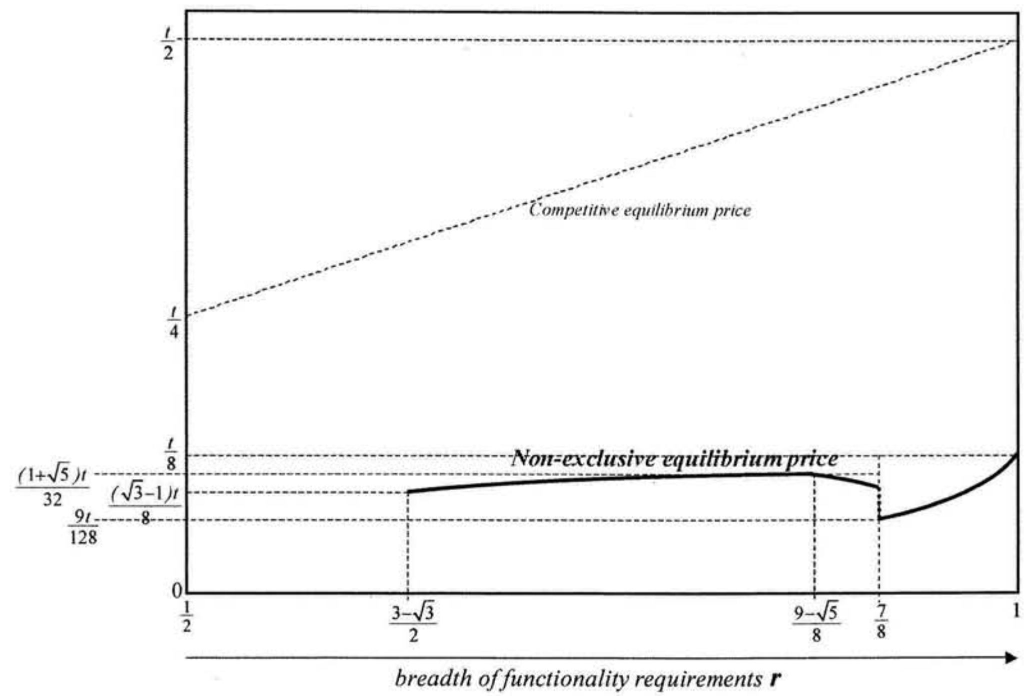
(a) Incremental value function for r<0.5



(b) Incremental value function for r>0.5

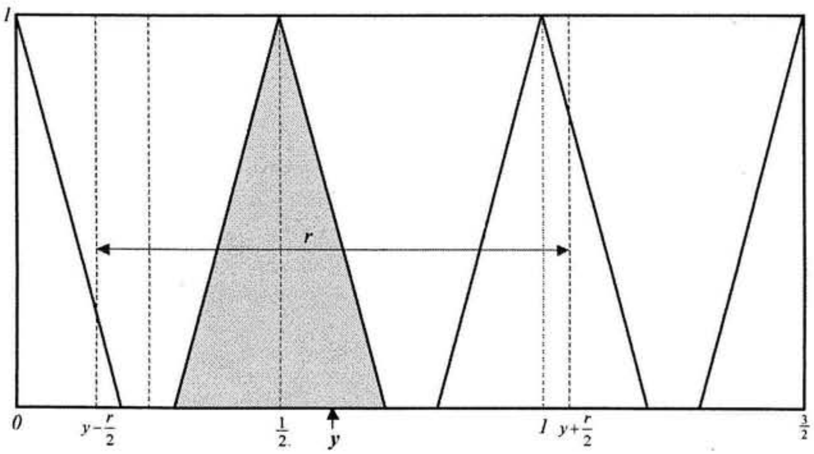


(c) Realized demand under non-exclusive equilibrium

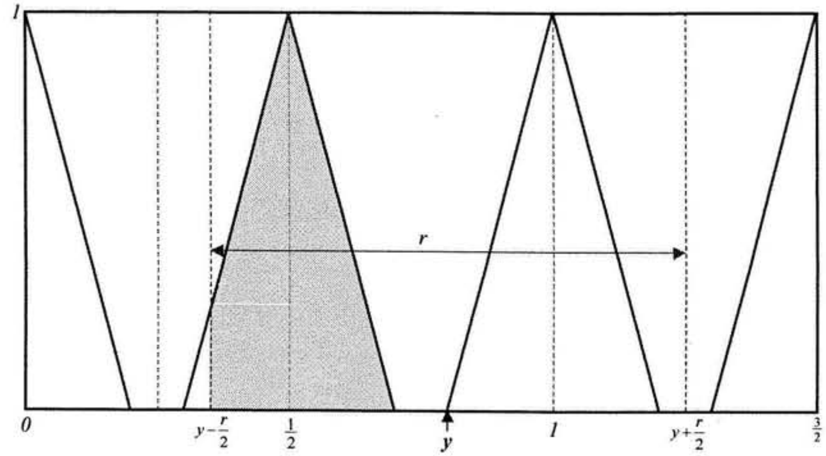


(d) Prices under non-exclusive equilibrium

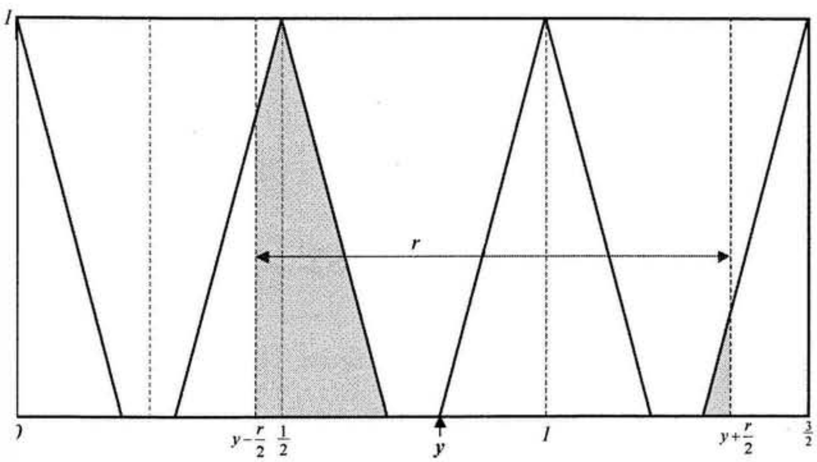
gure 11: Depicts the incremental value function and its corresponding non-exclusive equilibrium outcomes. When r is sufficiently higher than 0.5, the cremental value is high enough to induce equilibrium in which some consumers buy both products. As r increases, the number of non-exclusive purchases o increase, until after a critical value, when all consumers buy both products. However, prices are substantially lower than the corresponding competitive uilibrium prices, as depicted in (d)



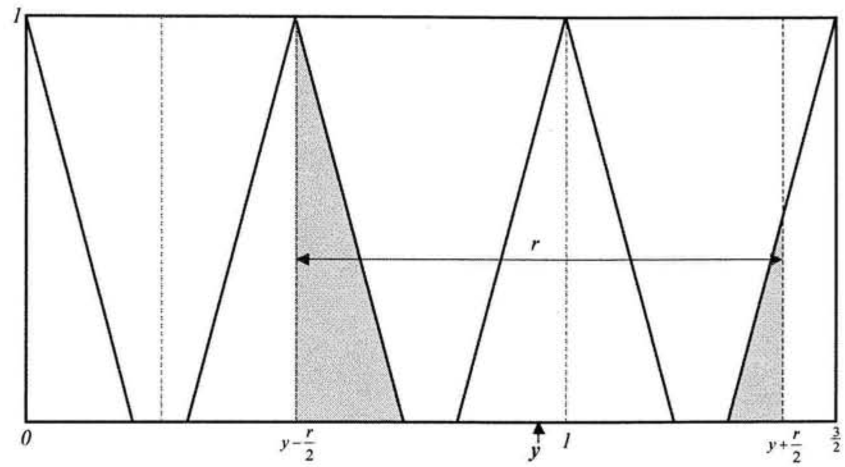
$$U(y,t)=\int_{\frac{1}{2}-\frac{1}{t}}^{\frac{1}{2}}(1-tg(\tfrac{1}{2}-x)dx+\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{t}}(1-tg(x-\tfrac{1}{2})dx$$



$$U(y,t)=\int_{y-\frac{r}{2}}^{\frac{1}{2}}(1-tg(\tfrac{1}{2}-x)dx+\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{t}}(1-tg(x-\tfrac{1}{2})dx$$



$$U(y,t)=\int_{y-\frac{r}{2}}^{\frac{1}{2}}(1-tg(\tfrac{1}{2}-x)dx+\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{t}}(1-tg(x-\tfrac{1}{2})dx+\int_{\frac{3}{2}-\frac{1}{t}}^{y+\frac{r}{2}}(1-tg(\tfrac{3}{2}-x)dx$$



$$U(y,t)=\int_{y-\frac{r}{2}}^{\frac{1}{2}+\frac{1}{t}}(1-tg(x-\tfrac{1}{2})dx+\int_{\frac{3}{2}-\frac{1}{t}}^{y+\frac{r}{2}}(1-tg(\tfrac{3}{2}-x)dx$$

re 12: Depicts the value function for products with very low scope, focusing on parameter values such that $1-\frac{1}{t}\le r\le 1$. The integral expressions are lar to those in Figure 3 (though not identical). The products are completely distinct in the set of functionalities that they cover with positive effectiveness. e r is high, while a subset of customers only derive value from one or the other product, a majority of them place a positive value on both products.

A Appendix: Proofs

Proof of Lemma 1

Define

$$
\begin{aligned}
U^1(y,t) &= r - t\left[G(\frac{1+r}{2} - y) + G(y - \frac{1-r}{2})\right], \\
U^2(y,t) &= r - t\left[G(y - \frac{1-r}{2}) - G(y - \frac{1+r}{2})\right], \\
U^3(y,t) &= r - t\left[2G(\frac{1}{2}) - G(y - \frac{1+r}{2}) - G(1 - (y - \frac{1-r}{2}))\right].
\end{aligned} \tag{23}
$$

This implies that

$$
U(y,t) = \begin{array}{ll} U^1(y,t) & \text{for } \frac{1}{2} \le y \le \frac{1+r}{2}; \\ U^2(y,t) & \text{for } \frac{r}{2} \le y \le \frac{2-r}{2}; \\ U^2(y,t) & \text{for } \frac{2-r}{2} \le y \le 1 \end{array} \tag{24}
$$

(a) It is easily verified that

$$U^1(\tfrac{1+r}{2},t) = U^2(\tfrac{1+r}{2},t) = r - tG(r), \tag{25}$$

and that

$$U^2(\tfrac{2-r}{2},t) = U^3(\tfrac{2-r}{2},t) = r - t[G(\tfrac{1}{2}) - G(\tfrac{1}{2} - r)], \tag{26}$$

which establishes that $U(y,t)$ is continuous in its arguments. Also, differentiating both sides of equation (23) with respect to y yields:

$$
\begin{aligned}
U_1^1(y,t) &= -t\left[g(y - \frac{1-r}{2}) - g(\frac{1+r}{2} - y)\right], \\
U_1^2(y,t) &= -t\left[g(y - \frac{1-r}{2}) - g(y - \frac{1+r}{2})\right], \\
U_1^3(y,t) &= -t\left[g(1 - (y - \frac{1-r}{2})) - g(y - \frac{1+r}{2})\right],
\end{aligned} \tag{27}
$$

and with respect to t yields:

$$
\begin{aligned}
U_2^1(y,t) &= -\left[G(\frac{1+r}{2} - y) + G(y - \frac{1-r}{2})\right], \\
U_2^2(y,t) &= -\left[G(y - \frac{1-r}{2}) - G(y - \frac{1+r}{2})\right], \\
U_2^3(y,t) &= -\left[2G(\frac{1}{2}) - G(y - \frac{1+r}{2}) - G(1 - (y - \frac{1-r}{2}))\right].
\end{aligned} \tag{28}
$$

Since both $G(x)$ and $g(x) > 0$, and noting from equation (24) the ranges of y in which each function is active, this establishes that U is decreasing in y and in t

(b) Using equation (27), it is easily verified that

$$U_1^1(\tfrac{1+r}{2},t) = U_1^2(\tfrac{1+r}{2},t) = -tg(r), \tag{29}$$

and that

$$U_1^2(\tfrac{2-r}{2},t) = U_1^3(\tfrac{2-r}{2},t) = -t[g(\tfrac{1}{2}) - g(\tfrac{1}{2} - r)]. \tag{30}$$

This establishes that $U_1(y,t)$ is continuous. Inspection of (27) establishes that it is piece-wise differentiable with respect to both its arguments.

(c) Differentiating both sides of equation (27) with respect to y yields:

$$U_{11}^1(y,t) = -t\left[g_1(y-\frac{1-r}{2})+g_1(\frac{1+r}{2}-y)\right], \tag{31}$$

$$U_{11}^2(y,t) = -t\left[g_1(y-\frac{1-r}{2})-g_1(y-\frac{1+r}{2})\right], \tag{32}$$

$$U_{11}^3(y,t) = t\left[g_1(1-(y-\frac{1-r}{2}))+g_1(y-\frac{1+r}{2})\right]. \tag{33}$$

Since $g(x)$ is strictly increasing, $g_1(x) > 0$, which, in conjuction with equations (31) and (33), establishes that $U_{11}^1(y,t) < 0$ and that $U_{11}^3(y,t) > 0$. Finally, if $g_{11}(x) > 0$, then

$$g_1(y-\frac{1-r}{2}) > g_1(y-\frac{1+r}{2}), \tag{34}$$

which along with equations (32) implies that $U_{11}^2(y,t) < 0$. Conversely, if $g_{11}(x) < 0$, then

$$g_1(y-\frac{1-r}{2}) < g_1(y-\frac{1+r}{2}), \tag{35}$$

and equations (34) and (35) imply that $U_{11}^2(y,t) > 0$.

(d) Differentiating both sides of equation (28) with respect to y yields:

$$\begin{aligned} U_{12}^1(y,t) &= -\left[g(y-\frac{1-r}{2})-g(\frac{1+r}{2}-y)\right], \\ U_{12}^2(y,t) &= -\left[g(y-\frac{1-r}{2})-g(y-\frac{1+r}{2})\right], \\ U_{12}^3(y,t) &= -\left[g(1-(y-\frac{1-r}{2}))-g(y-\frac{1+r}{2})\right]. \end{aligned} \tag{36}$$

Now, $g(x) > 0$, and noting from equation (24) the ranges of y in which each of the U functions is active, this establishes that $U_2(y,t)$ is decreasing in y. This completes the proof.

Proof of Lemma 2

Recall that:

$$P(q,t) = U(q+\tfrac{1}{2},t), \tag{37}$$

and

$$\pi(q,t) = nq(P(q,t)-c). \tag{38}$$

(a) Differentiating both sides of equation (37) with respect to q yields:

$$P_1(q,t) = U_1(q+\tfrac{1}{2},t) \tag{39}$$

which in conjuction with equation (27) establishes that if $r > 0$:

$$P_1(q,t) < 0 \text{ for } 0 \le q \le \tfrac{1-r}{2}. \tag{40}$$

Furthermore, differentiating both sides of equation (39) with respect to q yields:

$$P_{11}(q,t) = U_{11}(q+\tfrac{1}{2},t). \tag{41}$$

From Lemma 1(c), we know that if $g(y)$ is linear, $U_{11}(y,t) \leq 0$ for all $\frac{1}{2} \leq y \leq \frac{2-r}{2}$, which implies that:

$$P_{11}(q,t) \leq 0 \text{ for } 0 \leq q \leq \tfrac{1-r}{2}. \tag{42}$$

Now, differentiating both sides of equation (38) twice with respect to q yields

$$\pi_{11}(q,t) = 2nP_1(q,t) + nqP_{11}(q,t) \tag{43}$$

Equations (40), (42) and (43) establish that

$$\pi_{11}(q,t) < 0 \text{ for } 0 \leq q \leq \tfrac{1-r}{2}. \tag{44}$$

(b) Recall that for $\frac{1-r}{2} \leq y \leq \frac{1}{2}$:

$$\pi(q,t) = n(q(r-c) - tq(q(1-q) - \frac{(1-r)^2}{4}), \tag{45}$$

and therefore:

$$\pi(\tfrac{1-r}{2},t) = n(\frac{(r-c)(1-r)}{2} - \frac{tr(1-r)^2}{4}); \tag{46}$$

$$\pi(\tfrac{1}{2},t) = n(\frac{(r-c)}{2} - \frac{tr(2-r)}{8}). \tag{47}$$

By direct comparison of the expressions on the RHS on equations (46) and (47), it can be established that:

$$\pi(\tfrac{1-r}{2},t) \geq \pi(\tfrac{1}{2},t) \text{ for } t \geq t_1, \tag{48}$$

$$\pi(\tfrac{1}{2},t) \geq \pi(\tfrac{1-r}{2},t) \text{for } t \leq t_1. \tag{49}$$

where

$$t_1 = \frac{4(r-c)}{r(3-2r)}. \tag{50}$$

Next, for $q \geq \frac{1-r}{2}$, direct comparison of the expressions on the RHS on equations (45) and (46) establishes that

$$\pi(\tfrac{1-r}{2},t) \geq \pi(q,t) \text{ for } t \geq t_2, \tag{51}$$

where

$$t_2 = \frac{2(r-c)}{r(1-r+q) + q(1-2q)}. \tag{52}$$

Comparing the expressions on the RHS of (50) and (52) establishes that:

$$t_2 \leq t_1 \text{ for } q \geq \frac{r}{2}. \tag{53}$$

Equations (48), (51) and (53) establish that

$$\pi(\tfrac{1-r}{2},t) \geq \pi(\tfrac{1}{2},t) \Rightarrow \pi(\tfrac{1-r}{2},t) \geq \pi(q,t) \text{ for all } q \geq \tfrac{1-r}{2}.$$

Similarly, direct comparison of the expressions on the RHS on equations (45) and (47) establishes that

$$\pi(\tfrac{1}{2},t) \geq \pi(q,t) \text{ for } t \geq t_3, \tag{54}$$

where

$$t_3 = \frac{4(r-c)}{r(2-r) + 2q(1-2q)}. \tag{55}$$

Comparing the expressions on the RHS of (50) and (55) establishes that:

$$t_3 \geq t_1 \text{ for } q \geq \frac{1-r}{2}. \tag{56}$$

Equations (49), (54) and (56) establish that

$$\pi(\tfrac{1}{2}, t) \geq \pi(\tfrac{1-r}{2}, t) \Rightarrow \pi(\tfrac{1}{2}, t) \geq \pi(q, t) \text{ for all } q \geq \tfrac{1-r}{2},$$

which completes the proof.

Proof of Proposition 1

Recall that

$$\pi(q,t) = \begin{array}{ll} nq(r-c) - ntq(q^2 + \frac{r^2}{4}) & \text{for } 0 \leq q \leq \frac{r}{2}; \\ nq(r-c) - ntrq^2 & \text{for } \frac{r}{2} \leq q \leq \frac{1-r}{2}; \\ nq(r-c) - ntq(q(1-q) - \frac{(1-r)^2}{4}) & \text{for } \frac{1-r}{2} \leq q \leq \frac{1}{2}. \end{array} \tag{57}$$

Lemma 2 has established that over the range $q \in [0, \frac{1-r}{2}]$, the function $\pi(q,t)$ is strictly concave and has at most one interior maximum, and that over the range $q \in [\frac{1-r}{2}, \frac{1}{2}]$, it is maximized at one of its end-points. Since $\frac{1-r}{2}$ is contained in $[0, \frac{1-r}{2}]$, to find the global maximum of $\pi(q,t)$, all one needs to do is to compare the value of the maximum of $\pi(q,t)$ over $q \in [0, \frac{1-r}{2}]$ with the end-point value $\pi(\frac{1}{2}, t)$.

The interior maximum in $q \in [0, \frac{1-r}{2}]$ could occur in either $[0, \frac{r}{2}]$, or in $[\frac{r}{2}, \frac{1-r}{2}]$ – the functional form of $\pi(q,t)$ is different in each of these intervals. There are therefore three candidate maxima.

(1) Interior maximum in $[0, \frac{r}{2}]$: this value q_a^* solves $\pi_1(q_a^*, t) = 0$, which, based on (57), reduces to:

$$q_a^* = \sqrt{\frac{r - c - \frac{r^2 t}{4}}{3t}}, \tag{58}$$

and is relevant only if $q_a^* \leq \frac{r}{2}$. Using (58), this condition simplifies to:

$$t \geq \frac{r-c}{r^2}. \tag{59}$$

(2) Interior maximum in $[\frac{r}{2}, \frac{1-r}{2}]$: this value q_b^* solves $\pi_1(q_b^*, t) = 0$, which, based on (57), solves to:

$$q_b^* = \frac{r-c}{2rt}, \tag{60}$$

and is relevant only if $\frac{r}{2} \leq q_b^* \leq \frac{1-r}{2}$. Using (60), this simplifies to:

$$\frac{r-c}{r^2} \geq t \geq \frac{r-c}{r(1-r)}. \tag{61}$$

(3) End-point maximum at $\frac{1}{2}$: This is relevant only if $\pi(\frac{1}{2}, t) \geq \pi(\frac{1-r}{2}, t)$, which we know from (50) occurs only when

$$t \leq \frac{4(r-c)}{r(3-2r)}. \tag{62}$$

Now, for $r \leq \frac{1}{2}$, it is easily verified that $\frac{4(r-c)}{r(3-2r)} \leq \frac{r-c}{r^2}$. In conjunction with (61) and (62), (59) establishes that q_a^* is the global maximizing value for $t \geq \frac{r-c}{r^2}$.

Comparing $\pi(\frac{1}{2}, t)$ to $\pi(q_b^*, t)$ yields:

$$\pi(\frac{1}{2}, t) \geq \pi(q_b^*, t) \text{ if } t \leq \frac{2(r-c)}{r(2-\sqrt{2r})}. \tag{63}$$

Again, it is straightforward to verify that $\frac{2(r-c)}{r(2-\sqrt{2r})} \leq \frac{4(r-c)}{r(3-2r)}$ for all $r \leq 1$. Therefore, $\pi(\frac{1}{2},t) \geq \pi(q_b^*,t)$ only in the region where $\pi(\frac{1}{2},t) \geq \pi(\frac{1-r}{2},t)$. Finally, for $r \leq \frac{1}{2}$:

$$\frac{r-c}{r^2} \geq \frac{2(r-c)}{r(2-\sqrt{2r})} \geq \frac{r-c}{r(1-r)}. \tag{64}$$

Therefore, q_b^* is the global maximizer for $\frac{r-c}{r^2} \geq t \geq \frac{2(r-c)}{r(2-\sqrt{2r})}$, after which $\pi(q,t)$ is maximized at its end-point $\frac{1}{2}$. Substituting these q values into the inverse demand function and into the profit function completes the proof.

Proof of Proposition 2

The slope of the gross total surplus function $s^M(q^*(t),t)$ with respect to t, can be computed to be:

$$s_2^M(q^*(t),t) = \begin{array}{l} -\frac{n(3r-1)}{24} \text{ for } 0 \leq t \leq \frac{2(r-c)}{r(2-\sqrt{2r})} \\ -\frac{n(r^4t^2+9(r-c)^2)}{24rt^2} \text{ for } \frac{2(r-c)}{r(2-\sqrt{2r})} \leq t \leq \frac{(r-c)}{r^2} \\ -\frac{n(2(r-c)+r^2t)}{9t}\sqrt{\frac{4(r-c)-r^2t}{3t}} \text{ for } \frac{(r-c)}{r^2} \leq t \leq 2 \end{array} \tag{65}$$

Similarly, the slope of the gross profit function with respect to t, can be shown to be:

$$\pi_2(q^*(t),t) = \begin{array}{l} -\frac{nr(2-r)}{8} \text{ for } 0 \leq t \leq \frac{2(r-c)}{r(2-\sqrt{2r})} \\ -\frac{n(r-c)^2}{4rt^2} \text{ for } \frac{2(r-c)}{r(2-\sqrt{2r})} \leq t \leq \frac{(r-c)}{r^2} \\ -\frac{n(7(r-c)^2+(r(rt-1)+c)^2)}{12\sqrt{3}t^2}\sqrt{\frac{t}{4(r-c)-r^2t}} \text{ for } \frac{(r-c)}{r^2} \leq t \leq 2 \end{array} \tag{66}$$

Using $r < \frac{1}{2}, 0 \leq t \leq 2$ and $c \leq P(\frac{1}{2},t)$ establishes the following:

$$\pi_2(q^*(t),t) \quad < \quad s_2^M(\frac{1}{2},t) \text{ for } 0 \leq t \leq \frac{2(r-c)}{r(2-\sqrt{2r})} \tag{67}$$

$$\pi_2(q^*(t),t) \quad > \quad s_2^M(\frac{1}{2},t) \text{ for } \frac{2(r-c)}{r(2-\sqrt{2r})} \leq t \leq \frac{(r-c)}{r^2} \tag{68}$$

$$\pi_2(q^*(t),t) \quad > \quad s_2^M(\frac{1}{2},t) \text{ for } \frac{(r-c)}{r^2} \leq t \leq 2 \tag{69}$$

Note from Proposition 1 that for $0 \leq t \leq \frac{2(r-c)}{r(2-\sqrt{2r})}$, full market coverage is optimal and hence the corresponding optimal level of product scope will be t_f^*. For $\frac{2(r-c)}{r(2-\sqrt{2r})} \leq t \leq 2$, partial market coverage is optimal and hence the optimal level of scope is given by t_p^*.

Now, the socially optimal level t^* satisfies:

$$s_2^M(q^*(t),t) = F_1(t). \tag{70}$$

while the profit-maximizing levels t_f^*, t_p^* satisfy

$$\pi_2(q^*(t),t) = F_1(t). \tag{71}$$

Since $F_1(t)$ is strictly increasing (because F is strictly convex), (67) - (71) imply that $t_f^* < t^*$ and $t_p^* > t^*$, thus establishing the result.

Proof of Lemma 3

Define:

$$R^M(q,t) = q(P^M(q,t)-c), \tag{72}$$

and

$$R^C(q_i, t_i, t_j, p_j) = q_i(P^C(q_i, t_i, t_j, p_j) - c). \tag{73}$$

The function $\pi^i(q_i, t_i, t_j, p_j)$, $i = A, B$, therefore takes either the form $R^M(q_i, t_i)$ or the form $R^C(q_i, t_i, t_j, p_j)$. The former is simply the monopoly profit function, which has already been shown to have at most one interior maximum, and from (7) the latter can be expanded to:

$$R^C(q_i, t_i, t_j, p_j) = R^M(q_i, t_i) - [q_i(U(1 - q_i, t_j) - p_j)]. \tag{74}$$

Since $-U_1(1 - q_i, t_j) > 0$, the function $[q_i(U(1 - q_i, t_j) - p_j)]$ is increasing and convex in q_i for all q_i. Therefore, the function $R^C(q_i, t_i, t_j, p_j)$ also has no more than one interior maximum, and it is also strictly concave for $q_i \leq \frac{1-r}{2}$.

Now, suppose $\pi^i(q_i, t_i, t_j, p_j)$ has an interior maximum q^* in its monopoly region, implying that $R_1^M(q^*, t_i) = 0$. Based on (74),

$$R_1^C(q_i, t_i, t_j, p_j) = R_1^M(q_i, t_i) - [U(1 - q_i, t_j) - p_j] + q_i U_1(1 - q_i, t_j). \tag{75}$$

Since $R_1^M(q_i, t_i) < 0$ for $q_i > q^*$, and $U_1(1 - q_i, t_j) < 0$ for all q_i, and by definition, $U(1 - q_i, t_j) - p_j > 0$ for any q_i in the competitive region of the duopolist's profit function, the RHS of (75) is strictly negative, and hence if there is an interior maximum in the monopoly region, there cannot be one in the competitive region.

Similarly, if $\pi^i(q_i, t_i, t_j, p_j)$ has an interior maximum q^* in its competitive region, based on (75) this means that

$$\pi_1^M(q^*, t_i) = [U(1 - q^*, t_j) - p_j] - q^* U_1(1 - q^*, t_j) > 0, \tag{76}$$

which means that $\pi_1^M(q_i, t_i) > 0$ for any $q_i < q^*$, which in turn implies that $\pi^i(q_i, t_i, t_j, p_j)$ cannot have an interior maximum in its monopoly region. The result follows.

Proof of Proposition 3

Let the demand for firm A and firm B under a candidate pure strategy Nash equilibrium be nq_A^* and nq_B^* respectively. Let the corresponding prices be p_A^* and p_B^*. We continue to use the functions defined in (72) and (73).

(a) Under a local monopoly equilibrium configuration, we know that $q_A + q_B < \frac{1}{2}$. In a feasible Nash equilibrium, q_i^* has to be a local maximizer of $\pi^i(q_i, t_i, t_j, p_j)$. Since q_i^* is in the monopoly region of $\pi^i(q_i, t_i, t_j, p_j)$, and we know that the unique local maximizer in this region, if it exists, is $Q^M(t_i)$,the only possible value of q_i^* is $Q^M(t_i)$. Consequently, the local monopoly equilibrium configuration is feasible only if $Q^M(t_A) + Q^M(t_B) < \frac{1}{2}$.

(b) Under a kinked equilibrium configuration, we know that $q_A^* + q_B^* = \frac{1}{2}$, and that q_i^* is at the kink of the duopoly inverse demand curve of firm i. Given firm j's strategy, there should be no incentive for firm i to deviate from its choice of q_i. Locally, that means that in any kinked equilibrium, either a small decrease or a small increase should not increase firm i's payoff, or that

$$R_1^M(q_i^*, t_i) \geq 0, \tag{77}$$

and that

$$R_1^C(q_i^*, t_i, t_j, p_j^*) \leq 0. \tag{78}$$

Since $R_1^M(q, t_i) < 0$ for $q > Q^M(t_i)$, it follows from (77) that

$$q_i^* \leq Q^M(t_i),\ i = A, B. \tag{79}$$

Define $q^K(p_j)$ as the value of q at the kink in the inverse demand function, for any opponent price p_j. From this definition of $q^K(p_j)$, we know that

$$U(1 - q^K(p_j), t_j) = p_j. \tag{80}$$

Differentiating both sides of (80) with respect to p_j and rearranging yields:

$$q_1^K(p_j) = \frac{1}{-U_1(1-q^K(p_j),t_j)} > 0. \tag{81}$$

Substituting in the expression for $R_1^M(q_i,t_i)$ into (75) yields:

$$R_1^C(q_i,t_i,t_j,p_j) = [U(q_i,t_i) - c - U(1-q_i,t_j) + p_j] + q_i[U_1(q_i,t_i) + U_1(1-q_i,t_j)]. \tag{82}$$

Also define $s^K(p_j)$ as the value of $R_1^C(q_i,t_i,t_j,p_j)$, evaluated at the kink $q^K(p_j)$, as a function of opponent price p_j. (80) and (82) yield:

$$s^K(p_j) = [U(q^K(p_j),t_i) - c] + q^K(p_j)[U_1(q^K(p_j),t_i) + U_1(1-q^K(p_j),t_j)]. \tag{83}$$

Differentiating both sides of (83) with respect to p_j yields:

$$\begin{aligned} s_1^K(p_j) &= q_1^K(p_j)[2U_1(q^K(p_j),t) + U_1(1-q^K(p_j),t_j)] \\ &\quad + q^K(p_j)[U_{11}(q^K(p_j),t_i) - U_{11}(1-q^K(p_j),t_j)]. \end{aligned} \tag{84}$$

From (81), we know that $q_1^K(p_j) > 0$. Since $U_1(q,t) < 0$ for all q, and based on Lemma 1, $U_{11}(q^K(p_j),t_i) < 0$, and $U_{11}(1-q^K(p_j),t_j) > 0$ so long as $q^K(p_j) \le \frac{1-r}{2}$, it follows that

$$s_1^K(p_j) < 0. \tag{85}$$

In other words, the slope of the competitive profit function, evaluated at the kink, is decreasing in p_j.

Now, by the definition of q^K and of $Q^C(t_i,t_j)$,

$$q^K(U(1-Q^C(t_i,t_j),t_j)) = Q^C(t_i,t_j), \tag{86}$$

and

$$R_1^C(Q^C(t_i,t_j),t_i,t_j,U(1-Q^C(t_i,t_j),t_j)) = 0, \tag{87}$$

which in conjunction with (78) and the fact that $s_1^K(p_j) < 0$, establishes that

$$U(1-Q^C(t_i,t_j),t_j) \le p_j^* \tag{88}$$

(81), (86) and (88) together imply that

$$Q^C(t_i,t_j) \le q^K(p_j^*). \tag{89}$$

Since the candidate equilibrium is at the kink, it follows that $q_i^* = q^K(p_j^*)$. Therefore,

$$Q^C(t_A,t_B) + Q^C(t_B,t_A) \le q_A^* + q_B^*, \tag{90}$$

and the result follows.

(c) If the candidate equilibrium is in the competitive region, it follows that the corresponding q values occur after the kink, or that

$$q_i^* \ge q^K(p_j^*). \tag{91}$$

Also, since q_i^* is part of a candidate equilibrium in this region, it must be the case that it occurs at a local maximum of $R^C(q_i,t_i,t_j,p_j^*)$, or that:

$$R_1^C(q_i^*,t_i,t_i,p_;^*) = 0, \tag{92}$$

which in conjunction with (91) implies that

$$R_1^C(q^K(p_j^*), t_i, t_j, p_j^*) \geq 0. \tag{93}$$

Based on (85) and the definition of $Q^C(t_i, t_j)$, (93) implies that

$$q^K(p_j^*) \geq Q^C(t_i, t_j). \tag{94}$$

From (94), it follows that

$$Q^C(t_A, t_B) + Q^C(t_B, t_A) \geq q_A^* + q_B^*. \tag{95}$$

Since $q_A^* + q_B^* = \frac{1}{2}$, the result follows.

Proof of Proposition 4

Based on Proposition 3(a), a necessary condition for a local monopoly equilibrium to exist is that

$$2Q^M(t) \leq \frac{1}{2}. \tag{96}$$

Based on the bounds on $Q^M(t)$ in Table 1, it is easily seen that when $r \leq \frac{1}{2}$, this is always true in the region $2 \geq t \geq \frac{r-c}{r^2}$. Also, based on the derived values of $Q^M(t)$ in Table 1, this condition holds in $\frac{r-c}{r^2} \geq t \geq \frac{r-c}{r(1-r)}$ so long as

$$2\left(\frac{r-c}{2rt}\right) \leq \frac{1}{2}, \tag{97}$$

which reduces to

$$t \geq \frac{2(r-c)}{r}. \tag{98}$$

Therefore, a local monopoly equilibrium exists if $2 \geq t \geq \frac{2(r-c)}{r}$. Substituting in the appropriate values of q and prices from Tables 3 and 4, and computing the corresponding profits completes the proof.

Proof of Proposition 5

Based on Proposition 3(b), necessary conditions for a kinked equilibrium to exist are

$$2Q^M(t) \geq \frac{1}{2}, \tag{99}$$

and

$$2Q^C(t,t) \leq \frac{1}{2}. \tag{100}$$

Following the proof of Proposition 4, (99) reduces to:

$$t \leq \frac{2(r-c)}{r}. \tag{101}$$

For (100) to be true, it must be the case that $Q^C(t,t) < \frac{1-r}{2}$. Therefore, it can easily be seen that for symmetric t, given (101), and the fact that $r \leq \frac{1}{2}$, the relevant range of values is the set in the second row, corresponding to

$$Q^C(t_i, t_j) = \frac{r-c}{r(2t_i + t_j)} \tag{102}$$

Substituting (102) into (100) for symmetric t and rearranging yields:

$$4\left(\frac{r-c}{3rt}\right) \leq t. \tag{103}$$

(101) and (103) establish that the range of values in which kinked equilibria exist is $\frac{2(r-c)}{r} \geq t \geq \frac{4(r-c)}{3rt}$. For $q_A^* = q_B^* = \frac{1}{4}$, substituting into the price function from Tables 3 and 4, and computing profits at this value completes the proof.

Proof of Proposition 6

Based on Proposition 3(c), the necessary condition for a kinked equilibrium to exist is that:

$$2Q^C(t,t) > \frac{1}{2}, \tag{104}$$

which we know from (103) is true if:

$$4\left(\frac{r-c}{3rt}\right) > t \tag{105}$$

for $Q^C(t,t) < \frac{1-r}{2}$, and is always true if $Q^C(t,t) \geq \frac{1-r}{2}$, since $r \leq \frac{1}{2}$. Substituting in the relevant q and price values from Tables 3 and 4, for symmetric t, and computing profits at these values completes the proof.

Proof of Lemma 4

The proof uses the following properties of the functions $R^M(q,t)$ and $R^C(q,t_i,t_j,p_j)$ defined in (72) and (73):

$$R_2^M(q,t) = qU_2(q+\frac{1}{2},t) < 0; \tag{106}$$

$$R_{12}^M(q,t) = qU_{12}(q+\frac{1}{2},t) + U_2(q+\frac{1}{2},t) < 0; \tag{107}$$

$$R_2^C(q,t_i,t_j,p_j) = qU_2(q+\frac{1}{2},t_i) < 0, \text{ and} \tag{108}$$

$$R_{12}^C(q,t_i,t_j,p_j) = qU_{12}(q+\frac{1}{2},t_i) + U_2(q+\frac{1}{2},t_i) < 0. \tag{109}$$

(a) The form of the profit function for local monopoly equilibria is the same as the one given by equation (6). A straightforward application of the envelope theorem yields

$$\frac{d}{dt_i}\pi^i(q_i^*(t_i,t_j),t_i,t_j,P_j^*(t_j,t_i)) = R_2^M(q_i^*(t_i,t_j),t_i) < 0.$$

Note that an application of the envelope theorem is valid here as the equilibrium values are local maxima in q_i – moreover, q_i^* is a function of t_i alone, and $P_j^*(t_j,t_i)$ is independent of t_i.

(b) The argument is based on analyzing the impact of a small increase in t_i, to $t_i+\varepsilon$. To ease exposition, let (q_A^*,q_B^*) be the equilibrium q pair under the original value of t_i.

If the pair (q_A^*,q_B^*) continues to be a feasible kinked equilibrium pair after t_i increases to $t_i+\varepsilon$, we assume that the firms stay at this q pair. In a kinked equilibrium, firms still price on their own monopoly inverse demand functions, and hence a change in the value of t_i while holding t_j constant does not change p_j. Therefore, firm i's new profits are $R^M(q_i^*,t_i+\varepsilon)$, which based on (106) is strictly less than $R^M(q_i^*,t_i)$.

If the pair (q_A^*,q_B^*) is no longer a feasible kinked equilibrium, we assume that the firms move to the closest pair $(q_A^\varepsilon,q_B^\varepsilon)$. The proof then consists of the following steps:

(i) $q_i^\varepsilon < q_i^*$: Assume the converse, i.e., that $q_i^\varepsilon > q_i^*$. Since q_i^* is part of a kinked equilibrium at the original scope value t_i, we know that:

$$R_1^M(q_i^*,t_i) \geq 0. \tag{110}$$

and

$$R_1^C(q_i^*, t_i, t_j, p_j) \leq 0. \tag{111}$$

Similarly, since q_i^ε is part of a kinked equilibrium at the new scope value $t_i + \varepsilon$, we know that:

$$R_1^M(q_i^\varepsilon, t_i + \varepsilon) \geq 0. \tag{112}$$

and

$$R_1^C(q_i^\varepsilon, t_i + \varepsilon, t_j, p_j) \leq 0. \tag{113}$$

(107) and (112) imply that if $q_i^\varepsilon > q_i^*$:

$$R_1^M(q_i^*, t_i + \varepsilon) > 0. \tag{114}$$

Since q_i^* is not a kinked equilibrium for $t_i + \varepsilon$, (114) must mean that

$$R_1^C(q_i^*, t_i + \varepsilon, t_j, p_j) > 0, \tag{115}$$

which in conjuction with (111) contradicts (109). Therefore, we have established that in any new candidate q pair,

$$q_i^\varepsilon < q_i^*. \tag{116}$$

(ii) Profits reduce: Given (112) and (114), (116) implies that:

$$R^M(q_i^*, t_i + \varepsilon) > R^M(q_i^\varepsilon, t_i + \varepsilon). \tag{117}$$

Also, at the fixed level of quantity q_i^*, (107) implies that

$$R^M(q_i^*, t_i) > R^M(q_i^*, t_i + \varepsilon), \tag{118}$$

since the increase in t_i only reduces firm i's own price, without changing firm j's price or quantity. (117) and (118) imply that:

$$R^M(q_i^*, t_i) > R^M(q_i^\varepsilon, t_i + \varepsilon), \tag{119}$$

which establishes that even when a small increase in t_i necessitates a change in the equilibrium q values, it still reduces profits. As a consequence, we can now conclude that for any kinked equilibrium,

$$\frac{d}{dt_i}\pi^i(q_i^*(t_i, t_j), t_i, t_j, P_j^*(t_j, t_i)) < 0. \tag{120}$$

(c) The simplicity of the price and q values makes direct computation of the equilibrium profit function and its total derivative straightforward in this case. Substituting in these values from Tables 3 and 4 , we get:

$$\pi^i(q_i^*(t_i, t_j), t_i, t_j, P_j^*(t_j, t_i)) = \frac{nr(t_i + 2t_j)^2}{36(t_i + t_j)}. \tag{121}$$

Differentiating both sides with respect to t_i yields:

$$\frac{d}{dt_i}\pi^i(q_i^*(t_i, t_j), t_i, t_j, P_j^*(t_j, t_i)) = \frac{nrt_i(t_i + 2t_j)}{36(t_i + t_j)^2} > 0, \tag{122}$$

which establishes the result.

Proof of Proposition 7

According to Lemma 4, at any first-stage candidate pair (t_A, t_B), under local monopoly and kinked equilibrium configurations, firms have an incentive to change their choice of t_i if it leaves them in the same equilibrium configuration. At the set of (t_A, t_B) pairs which border the local monopoly and kinked regions in the (t_A, t_B) space, the payoff functions are continuous and decreasing in t_i even at the point of transition. Therefore, these cannot be supported in an equilibrium.

However, at the set of feasible (t_A, t_B) pairs which border the kinked and competitive regions in the (t_A, t_B) space, an increase in t_i takes the firm into the kinked region and strictly reduces payoff, according to Lemma 4(b). Likewise, a decrease in t_i takes the firm into the competitive region and strictly reduces payoff, according to Lemma 4(c). As a consequence, for this set of pairs at which a second stage pure strategy price equilibrium exists, given firm j's choice of t_j, firm i has an incentive not to deviate from its choice t_i. This ensures that any pair (t_A, t_B) along the CK border for which a pure-strategy second stage equilibrium exists is part of a subgame perfect equilibrium, and proves part (b).

Note that for symmetric values of scope, a pure strategy price equilibrium always exists in the second stage. Thus, the symmetric value of t that forms the point of transition between the kinked and competitive equilibrium regions is at:

$$\frac{3(t_A + t_B)}{(2t_A + t_B)(t_A + 2t_B)} = \frac{r}{2(r-c)}, \tag{123}$$

and substituting $t_A = t_B = t$, we have $t = \frac{4}{3}(\frac{r-c}{r})$. From Table 3, any kinked equilibrium for values of product scope $t_A = t_B = \frac{4}{3}(\frac{r-c}{r})$ has to satisfy:

$$\frac{r-c}{3rt} \leq q_i^* \leq \frac{1}{2} - \frac{r-c}{3rt} \tag{124}$$

Substituting the value of t reduces this to $\frac{1}{4} \leq q_i^* \leq \frac{1}{2} - \frac{1}{4}$for $i = A, B$, or $q_A^* = q_B^* = \frac{1}{4}$. Since firms price in the monopoly region of their demand curves,

$$p_A^* = p_B^* = P^M(\frac{1}{4}, \frac{4}{3}(\frac{r-c}{r})) = \frac{c+2r}{3} \tag{125}$$

Substituting the values of t and q_i^* into the payoff functions yields the following gross profits:

$$\pi_A^* = \pi_B^* = \frac{1}{6}(r-c), \tag{126}$$

which completes the proof.

Proof of Proposition 8

The slope of the gross total surplus curve $s^D(q,t)$, evaluated at the duopolists' optimal quantity, is:

$$s_2^D(q^*(t),t) = \begin{array}{l} -\frac{nr(3+4r^2)}{48} \text{ for } \frac{4(r-c)}{3r} \leq t \leq \frac{2(r-c)}{r} \\ -n(\frac{r^3}{12} + \frac{3(r-c)^2}{4rt^2}) \text{ for } \frac{2(r-c)}{r} \leq t \leq \frac{r-c}{r^2} \\ -\frac{2n(2(r-c)+r^2t)}{9t}\sqrt{\frac{4(r-c)-r^2t}{3t}} \text{ for } \frac{r-c}{r^2} \leq t \leq 2. \end{array} \tag{127}$$

Correspondingly, the slope of the gross profit function is:

$$\pi_2(q^*(t),t) == \begin{array}{l} -\frac{nr}{16} \text{ for } \frac{4(r-c)}{3r} \leq t \leq \frac{2(r-c)}{r} \\ -\frac{n(r-c)^2}{4rt^2}) \text{ for } \frac{2(r-c)}{r} \leq t \leq \frac{r-c}{r^2} \\ -\frac{n(2(r-c)+r^2t)}{12t}\sqrt{\frac{4(r-c)-r^2t}{3t}} \text{ for } \frac{r-c}{r^2} \leq t \leq 2 \end{array} \tag{128}$$

The socially optimal level t^* satisfies:

$$t^* = \arg\max_t s^D(q^*(t),t) - 2F(t), \tag{129}$$

which reduces to:

$$F_1(t^*) = \frac{s_2^D(q^*(t^*), t^*)}{2}, \tag{130}$$

and the duopolists' optimal level t_d^* satisfies:

$$F_1(t_d^*) = \pi_2(q^*(t_d^*), t_d^*). \tag{131}$$

Straightforward algebraic manipulation establishes the following:

$$\pi_2(q^*(t), t) < \frac{s_2^D(q^*(t), t)}{2} \text{ for } \frac{4(r-c)}{3r} \leq t \leq \frac{2(r-c)}{r}; \tag{132}$$

$$\pi_2(q^*(t), t) > \frac{s_2^D(q^*(t), t)}{2} \text{ for } \frac{2(r-c)}{r} \leq t \leq \frac{r-c}{r^2}, \text{ and} \tag{133}$$

$$\pi_2(q^*(t), t) > \frac{s_2^D(q^*(t), t)}{2} \text{ for } \frac{r-c}{r^2} \leq t \leq 2. \tag{134}$$

Since $F_1(t)$ is strictly increasing, conditions (130) – (134) establish that:

$$\text{If } \frac{4(r-c)}{3r} \leq t_d^* \leq \frac{2(r-c)}{r}, \text{ then } t^* > t_d^*,$$

and that:

$$\text{If } \frac{2(r-c)}{r} \leq t_d^* \leq 2, \text{ then } t_d^* > t^*,$$

which completes the proof.

B Appendix B

Note the following about the bounds on $Q^M(t_i)$ and $Q^C(t_i, t_j)$ in Tables 1 and 2. Since $r \leq \frac{1}{2}$,

$$2(\frac{r}{2}) \leq \frac{1}{2} \text{ and } 2(\frac{1-r}{2}) \geq \frac{1}{2}. \tag{135}$$

From proposition 3, a local monopoly equilibrium is feasible only if

$$Q^M(t_A) + Q^M(t_B) < \frac{1}{2}. \tag{136}$$

From column 3 in Table 1 and (135), it is clear that condition (136) is always satisfied for $2 \geq t_A, t_B \geq \frac{r-c}{r^2}$ and never satisfied for $\frac{r-c}{r(1-r)} \geq t_A, t_B$. In the region $\frac{r-c}{r^2} \geq t_A, t_B \geq \frac{r-c}{r(1-r)}$, condition (136) is satisfied iff

$$\frac{r-c}{2rt_A} + \frac{r-c}{2rt_B} < \frac{1}{2} \text{ or } \frac{t_A + t_B}{t_A t_B} < \frac{r}{r-c}. \tag{137}$$

If $2 \geq t_i \geq \frac{r-c}{r^2}$, $\frac{r-c}{r(1-r)} \geq t_j \geq \frac{12(r-c)}{4-3(1-r)^2}$ for $i, j = A, B$, from column 2 in Table 1, we need the following for (136) to be satisfied:

$$\sqrt{\frac{4(r-c) - r^2 t_i}{12t_i}} + \frac{1}{3} - \sqrt{\frac{4-3(1-r)^2}{36} - \frac{r-c}{3t_j}} < \frac{1}{2}, \tag{138}$$

which simplifies to:

$$\sqrt{\frac{4(r-c) - r^2 t_i}{12t_i}} - \sqrt{\frac{4-3(1-r)^2}{36} - \frac{r-c}{3t_j}} < \frac{1}{6}. \tag{139}$$

The conditions (137) and (139) are each tighter than the other in their respective regions of the (t_A, t_B) parameter space and hence a conjunction of the two conditions defines the MK curve that partitions the local monopoly region and the kinked region.

Now consider the condition:

$$Q^C(t_A, t_B) + Q^C(t_B, t_A) \leq \frac{1}{2} \tag{140}$$

From column 3 of Table 2 and (135), it is clear that condition (140) will never be satisfied if $2t_i + t_j \leq \frac{(r-c)}{r(1-r)}$ $i, j = A, B$. From columns 1 and 2, if $2(\frac{r-c}{r^2} - t_i) \geq t_j \geq 2(\frac{r-c}{r(1-r)} - t_i)$, $i, j = A, B$, then condition (140) will only be satisfied if

$$\frac{r-c}{r(2t_A + t_B)} + \frac{r-c}{r(2t_B + t_A)} \leq \frac{1}{2} \quad \text{or} \quad \frac{3(t_A + t_B)}{(2t_A + t_B)(t_A + 2t_B)} \leq \frac{r}{2(r-c)} \tag{141}$$

Once again from columns 1 and 2 of Table 2, if for $i, j = A, B$, $t_j \geq 2(\frac{r-c}{r^2} - t_i)$ and $2(\frac{r-c}{r(1-r)} - t_i) \geq t_j \geq (A - 2t_i + \sqrt{A(A - t_i)})$ where $A = 4(r-c) + t_i(1-r)^2$, then condition (140) will only be satisfied if

$$\sqrt{\frac{4(r-c) - r^2 t_i}{4(3t_i + 2t_j)}} + \frac{2t_j + t_i - \sqrt{(2t_j + t_i)^2 - (4(r-c) + t_j(1-r)^2)(3t_j + 2t_i)}}{2(3t_j + 2t_i)} \leq \frac{1}{2} \tag{142}$$

It can be verified that each of conditions (141) and (142) have to be binding in their respective regions of the parameter space for condition (140) to be satisfied. Therefore a simple conjunction of conditions (141) and (142) yields the KC curve that separates the kinked equilibrium region from the competitive equilibrium region. It is straightforward to verify that every point (t_A, t_B) which satisfies $t_i \in [0, 2]$, $i = A, B$, and satisfies (137) and (139), also satisfies condition (141) and (142). Thus, the region between the MK and the KC curves corresponds to the kinked equilibrium region.