OPTIMAL SCHEDULING OF PURCHASING ORDERS FOR LARGE PROJECTS

Boaz Ronen
New York University

and

Dan Trietsch
Streamline Operations Products, Inc.

July 1986

Center for Research on Information Systems
Information Systems Area
Graduate School of Business Administration
New York University

Working Paper Series

CRIS #127

GBA #86-64

Table of Contents

OPTIMAL SCHEDULING OF PURCHASING ORDERS FOR LARGE PROJECTS

by

Boaz Ronen[1] and Dan Trietsch[2]

Abstract

The completion of a typical project hinges upon receiving all the purchased components by the time they are scheduled to be used. Some of these components may have long stochastic lead times, so the project manager is tempted to order them ASAP, to avoid the (usual) high penalties associated with delays. (The penalties may be tangible or intangible, but we assume that they can be measured by monetary units.) This in turn may bring about excessive inventory holding costs. Clearly an optimization is called for to minimize the total expected cost of the project. This is achieved by timing the orders optimally, for the one component case as well as for the n component general case.

1. Computer Applications and Information Systems, Graduate School of Business Administration, New York University, New York, NY 10006.

2. Streamline Operations Products, Inc. P.O. Box 7368 Berkeley, CA 94707

1. Introduction:

By way of introduction, let us consider the following special case: A project requires one purchased component, which must be on hand at a specified time, t*. If the item is received earlier, the project will be completed in time --i.e., without penalties--but an inventory holding of cost C will be incurred for each time unit the item is held in inventory after arrival and until t*. On the other hand, if the component is late, a penalty P is incurred for each time unit of delay-- since the whole project is consequently delayed. Assume now that the lead time of the component has a given stochastic distribution, and the project manager has to decide when to place the order in such a manner that the total expected cost of the inventory holding cost and the delay penalty will be minimized.

This problem can now be generalized for n independently distributed components , each of which has its own holding cost, where it is enough that one of them will not be in time, to incur the penalty in full, while inventory holding cost applies to all the components which did arrive.

Section 2 describes the single component model. We will show an analytic solution provided the lead times has an analytically invertible cumulative distribution function (CDF)--otherwise, a numerical procedure will be required to find the exact optimal order point.

In section 3 we discuss another special case, namely, the two components model. We include this case as an introduction to the general case.

Indeed, Section 4 presents a set of equations, which solve the general case. However, this solution requires a numerical search procedure, even if the CDFs of the lead times are analytically invertible.

Finally in Section 5, we discuss some other applications, and suggestions for future extensions.

2. The Single Component Model

Our basic premise is that the project manager is responsible for all the costs associated with the purchasing decision. Therefore, it's in her or his interest and power to minimize the expected total cost of holding the inventory, in case of early delivery, and of the penalty, in case of late delivery (e.g., see Taha [2], Chap. 13). We also assume that the component's lead time is a stationary stochastic variable with a given distribution. In order to minimize the expected cost, the manager has to optimize the scheduling of the order placement--which is the decision variable under her or his control.

The objective function is:

$$(2.1) \quad \underset{T}{MIN}\{E(Penalty\ Cost)+E(Holding\ Cost)\}$$

where T is the time in which the order is placed. Figure 2.1 illustrates the relationship between t*, T and the lead time distribution. Note that the distribution "starts" at T (the item cannot arrive before be ordered), and consequently the area to the right of t*, i.e. the penalty probability, increases with T, as expected.

Place figure 2.1 about here

Expanding the target function (2.1), we may write:

$$(2.2) \quad \underset{T}{MIN} \quad \{ C\int_{T}^{t^*} F(t-T)dt + P\int_{t^*}^{\infty} [1-F(t-T)]dt \}$$

where:

- F() is the CDF of the lead time
- C is the holding cost per period
- P is the penalty cost per period

Note that these costs are assumed to be linear, as mentioned above.

By taking the derivatives of (2.2), and using the Leibnitz method to differentiate under the intergral, it can be shown that the optimal order point, T* , satisfies

$$(2.3) \quad F(t^* - T^*) = \frac{P}{P+C}$$

In other words, T^* satisfies (2.4)

$$(2.4) \quad T^* = arg\{F(t^* - T^*) = P \ / \ (P + C)\}$$

Assuming that F() has an analytic inverse, we may now obtain T^* directly from it, i.e.

$$(2.5) \quad T^* = t^* - F^{-1}\,(1/(1 + C \ / \ P)).$$

If F() does not have an explicit inverse, this part of the solution has to be carried out numerically.

Obviously, when P >> C (which is very often the case in practice), the model will push T^* as far to the left as possible (i.e., ASAP). It may even happen that (2.5) cannot be satisfied for any non-negative T, which implies an immediate order. On the other hand, if the probability that the item will be delivered immediately upon order is high enough, we may have to order "just in time." In both cases above, the expected total cost will be larger than "optimal."

It is interesting to note that our result is not dependent upon the form of the distribution, or any of its moments, except for the cumulated probability itself. However, the optimal value of the target function which results, is very dependent upon the distribution, and especially its tails.

3. Purchasing Decisions for Two Independent Orders

In this section we assume that the project requires two independent orders. It is enough that one order will be delayed, to delay the whole project, and thus incur the penalty cost P. However, in case one of the orders arrives in time, and the other is delayed, we also have to carry the holding cost for the order which arrived.

For simplicity, we assume that the orders are required at the same time t^*. However, the readers can verify later, that this assumption may be dropped very easily by a simple adjustment.

Let C_1, C_2 denote the inventory holding costs per period of item 1 and 2 respectively. Let $F_1 = F_1(t-T_1)$ and $F_2 = (t-T_2)$ denote the probabilities that the respective orders will arrive before t given that they were ordered at T (regardless of what happens to the other item). Let P be the penalty cost per period in case the project is delayed.

For tractability reasons, it seems convenient to calculate the expected costs till t^* and from t^* separately. It is easy to see then, that the expected holding cost till t^*, $h = h(T_1,T_2)$, is

$$(3.1)\quad h(T_1,T_2) = C_1\int_{T_1}^{t^*} F_1\,(t - T_1)dt \;+ C_2 \int_{T_2}^{t^*} F_2\,(t - T_2)dt$$

From t* onwards, at any given time t, there are four mutually exclusive and exhaustive possible random events:

(i) both items are on hand, and the implied cost is zero;
(ii) item 1 is on hand, but item 2 is missing, resulting at a cost of $P + C_1$;
(iii) similarly, where item 1 is the missing one, costing $P + C_2$; and finally,
(iv) both are missing at the same time, costing P.

Combining these costs with h, we obtain the target function as follows:

$$(3.2)\quad z = z\ (T_1,T_2) = h + (P + C_1)\int_{t^*}^{\infty} F_1(1-F_2)dt +$$

$$(P + C_2)\int_{T^*}^{\infty} (1 - F_1)\,F_2 dt + P\int_{t^*}^{\infty} (1-F_1)\,(1-F_2)dt$$

Our objective is to minimize z, by optimizing the order points of item 1 and item 2, i.e. T_1 and T_2 respectively. To that end, we take the partial derivatives (again, using the Leibnitz method where required), and set them to zero. This yields the following equation:

$$(3.3)\quad C_1 = (P + C_1 + C_2)\int_{T^*}^{\infty} \frac{\delta F_1}{\delta(t-T_1)}\;F_2(t-T_2)dt$$

and a symmetrical results holds for the second item.

We should observe, that due to the interaction between the two components, it is no longer enough to have the fractiles of the distributions -- as was the case for a single item. These equations can be solved numerically.

4. Purchasing Decisions for n Independent Orders

As in the former case, it is enough that one order will be delayed, to delay the whole project, and thus incur the penalty cost P, and, if at least one of the orders arrives in time while at least one of the others is delayed, we also have to carry the holding cost for the orders which have arrived.

Let C_i (i = 1, . . . , n) be the holding cost of item i, analog to the notation used in the previous section, and similarly let F_i, F_i^*, T_i and T_i^* be defined analog to F_1 (or F_2), F_1^*, T_1 and T_1^* for item i. Clearly, these can be written in vector form, e.g $\underset{\sim}{T} = (T_1, \ldots, T_n)$, etc.

Again we calculate the costs till and after t* separately, and if we use h ($\underset{\sim}{T}$) to denote the expected holding costs till t* (as before), we have

$$(4.1) \quad h(\underset{\sim}{T}) = \sum_i C_i \int_{T_i}^{t^*} F_i\,(t-T_i)dt$$

From t* onwards, at any given time t, there are now 2^n mutually exclusive and exhaustive possible random events, which can be categorized as follows:

(i) all items on hand, and the implied cost is zero;

(ii) all items except item i (where i can be chosen arbitrarily) are on hand, yielding a cost of

$$P + \sum_j C_j - C_i;$$

(iii) item i is on hand, but at least one other item is missing, resulting in a cost of P + possibly some other C_j's;

(iv) similarly, where item i is missing and at least one other item is missing, costing P + possibly some other C_j's.

Combining these costs with h, we obtain the target function. In order to do that, we can choose any i. For convenience we actually show the case for T1 only, but the generalization is immediate, due to symmetry. (Note that case (i) can be paired with case (ii), and all the cases of (iii) form similar pairs with cases of (iv). Eq. (4.2) is arranged according to these pairs. For clarity we also show the entry for the case where all items arrived--namely (i) --even though its contribution is zero.)

$$(4.2) \quad z = z(\underset{\sim}{T}) = \sum_i C_i \int_{T_i}^{t^*} F_i dt +$$

$$0 \int_{t^*}^{\infty} F_1 F_2 \cdots F_n \, dt + [P + \sum_2^n C_j] \int_{t^*}^{\infty} (1-F_1) F_2 \cdots F_n \, dt +$$

$$[P + C_1 + \sum_{j \neq 1, 2} C_j] \int_{t^*}^{\infty} F_1 (1-F_2) F_3 \cdots F_n \, dt +$$

$$[P + \sum_{j \neq 1, 2} C_j] \int_{t^*}^{\infty} (1-F_1)(1-F_2) F_3 \cdots F_n \, dt + \cdots$$

$$(P + C_1) \int_{t^*}^{\infty} F_1 (1-F_1)(1-F_2) \cdots (1-F_n) \, dt +$$

$$P \int_{t^*}^{\infty} (1-F_1)(1-F_2) \cdots (1-F_n) \, dt$$

Note that if we would write Eq. (4.2) in full, it would have 2^{n-1} pairs of integrals, in addition to h (the first line).

In order to minimize z, we have to set its gradient by $\underset{\sim}{T}$ to $\underset{\sim}{O}$. Due to symmetry, if we find the partial derivative by T_1, we practically have all the components of the gradient. Looking at h first, only its first component is a function of T_1, and thus we may disregard the others. As for the pairs which are not functions of T_1, each pair contributes

$$(4.3) \quad C1 \int_{t^*}^{\infty} F_1 \prod \, (some\ combination\ of\ F_i\ (1-F_j)\ dt$$

But, if we sum all the values for the various combinations of F_j and 1-F_j in 4.3, obviously all the possible combinations except the case where all the items arrived are represented, and therefore the sum must be:

$$(4.4) \quad C_1 \int_{t^*}^{\infty} F1 \, (1 - \prod_2^n F_j) \, dt$$

By regrouping and some algebraic manipulations we finally get the following expression for the partial derivative

$$(4.5)\quad \frac{\partial}{\partial T_1}[C_1 \int_{T_1}^{\infty} F_1\, dt - S \int_{t*}^{\infty} \prod_1^n F_j\, dt]$$

Where S *is* $P + \sum_i C_i$

And after setting this derivative to zero:

$$(4.6)\quad C_1 = S \int_{t*}^{\infty} \frac{\partial F_1}{\partial (t-T_i)} \prod_{j \neq 1} F_j (t-T_j)\, dt$$

(4.6) is a set of n nonlinear equations, which can be handled numerically.

We can also evaluate a lower bound for the Ti's for this problem. Since $\prod_j F_j < 1$,

$$(4.7)\quad \frac{C_i}{S} < \int_{t*}^{\infty} \frac{\partial F_i}{\partial\,(t-T_i)}\, dt$$

$$(4.8)\quad \int_{T_i}^{t*} \frac{\partial F_i}{\partial\,(t-T_i)}\, dt < 1 - \frac{C_i}{S}$$

In other words

$$(4.9)\quad F_i = optimal\ Pr\ (arrival\ in\ time) < 1 - \frac{C_i}{S}$$

thus, the lower bound may lead to assess Fi as follows:

$$(4.10)\quad F_i = 1 - \frac{C_i}{S}$$

and can be calculated for each item independently.

This means that a project manager using this policy would never have a greater expected penalty than the expected penalty derived by this limit. This might be considered as a "conservative limit" and might be perceived by managers as a "conservative policy," because the manager takes less penalty risks than at the optimal policy.

5. Conclusion

We have shown that this seemingly complex problem can be solved very easily, indeed. Hi-tech projects, and modern production in general, are characterized by an ever growing need for timeliness. Management, and competitors, push for shorter cycle times. Thus a model such as ours may fill an important gap.

The model may be used in many other management areas. For instance, many insurance related problems, involving risk management may lead to similar results. In actual P.E.R.T. problems, (see, for instance Buffa [1], ch.10) each activity has its own t* so to speak, so the model should be adjusted accordingly.

Further research is being done now in order to build an effective Decision Support System that will take advantage of the concepts and methods derived in this work. For instance, modifications of purchasing policies as well as cost considerations could be done. Another direction is to develop a Negotiation Support System for better cost estimation and scheduling of large projects. This problem is being investigated as well.

Bibliography

1. Buffa, E. S., MODERN PRODUCTION/OPERATIONS MANAGEMENT, 7th edition, John Wiley and Sons, New York, 1983

2. Taha, H. A., OPERATIONS RESEARCH, Third edition, MacMilan Publishing Co., Inc. New York, 1982

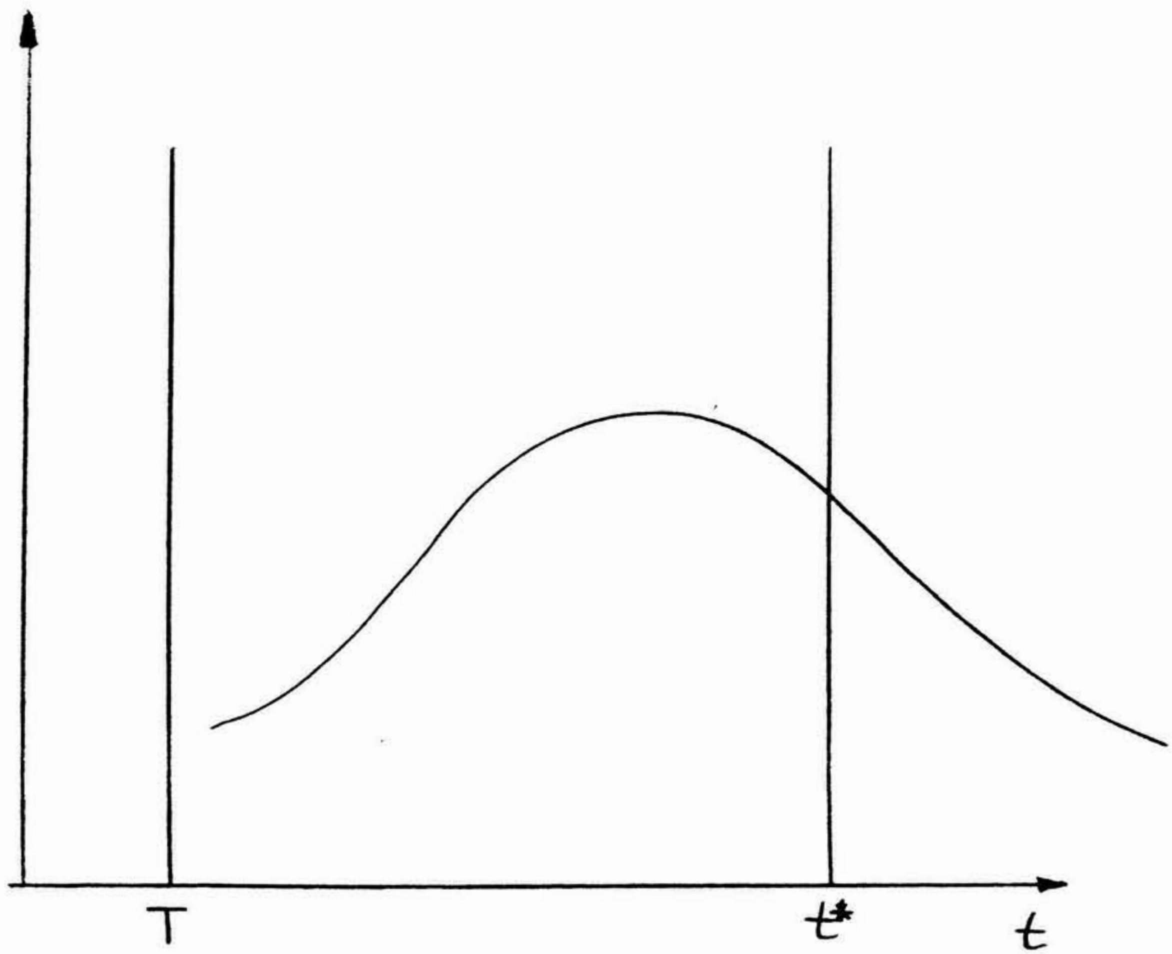

Figure 21: The relationship between
t*, T and the lead time distribution.