ISTITUTO DI ANALISI DEI SISTEMI ED INFORMATICA
CONSIGLIO NAZIONALE DELLE RICERCHE

M. Padberg

ALMOST PERFECT MATRICES AND GRAPHS

R. 478 Ottobre 1998

Manfred Padberg – Stern School of Business, New York University, 44 West 4th Street - New York, New York 10012, USA.

Supported in part by a grant from the Office of Naval Research (N00014-96-0327). Work done in part while visiting IASI-CNR in Rome, Italy, and the Forschungsinstitut für Diskrete Mathematik, Universität Bonn, Germany.

Istituto di Analisi dei Sistemi ed Informatica, CNR
viale Manzoni 30
00185 ROMA, Italy

tel. ++39-06-77161
fax ++39-06-7716461
email: iasi@iasi.rm.cnr.it
URL: http://www.iasi.rm.cnr.it

Abstract

We introduce the notions of ω-projection and κ-projection that map almost integral polytopes associated with almost perfect graphs G with n nodes from $\mathbb{R}^n$ into $\mathbb{R}^{n-\omega}$ where ω is the maximum clique size in G. We show that C. Berge's strong perfect graph conjecture is correct if and only if the projection (of either kind) of such polytopes is again almost integral in $\mathbb{R}^{n-\omega}$. Several important properties of ω-projections and κ-projections are established. We prove that the strong perfect graph conjecture is wrong if an ω-projection and a related κ-projection of an almost integral polytope with $2 \leq \omega \leq (n-1)/2$ produce different polytopes in $\mathbb{R}^{n-\omega}$.

Introduction

A graph $G = (V, E)$ is *perfect* if $\alpha(G') = \theta(G')$ for all (node-induced) subgraphs $G' \subseteq G$. $\alpha(G)$ is the stability (or independence) number of G, i.e., the maximum number of pairwise nonadjacent nodes of G, and $\theta(G)$ is the clique-covering number of G, i.e., the minimum number of cliques (or maximal complete subgraphs of G) that are necessary to cover all nodes of G.

A graph $G = (V, E)$ is *almost perfect* (or minimally imperfect) if G is imperfect, i.e., $\alpha(G) < \theta(G)$, but $\alpha(G') = \theta(G')$ for every proper subgraph G' of G. Clearly, every imperfect graph contains an almost perfect graph. Claude Berge formulated around 1960 several conjectures regarding perfect and almost perfect graphs. One of these conjectures (the so-called weak perfect graph conjecture) was proven by Lovàsz (1972) and is known as the perfect graph theorem. It states that a graph $G = (V, E)$ is perfect if and only if its complement graph $\overline{G} = (V, \overline{E})$ is perfect, where $\overline{E} = \{(u, v) \in V \times V : u \neq v \text{ and } (u, v) \notin E\}$.

Berge's strong perfect graph conjecture (SPGC) asserts that the only almost perfect graphs are the chordless cycles C_n having an odd number n of nodes and their complement graphs $\overline{C}_n$. In other words, Claude Berge conjectured (and still does so) that a graph is perfect if and only if it does not contain a chordless odd cycle or its complement as an induced subgraph.

The apparent elegance of the SPGC and its relevance to the problem of characterizing the *integrality* of certain polytopes in $\mathbb{R}^n$ have prompted a good deal of work on perfect graphs, see e.g. the book edited by Berge and Chvátal (1984), but the status of the conjecture is still open today.

From among the special graphs for which the SPGC has been proven to be correct the most remarkable result is a theorem of Tucker (1977) which states that the SPGC is correct for all graphs with $\alpha(G) \leq 3$ and thus by Lovàsz's perfect graph theorem for all graphs with $\omega(G) = \alpha(\overline{G}) \leq 3$ as well. $\omega(G)$ denotes the clique-number of G, i.e., the maximum number of pairwise adjacent nodes of G.

In this paper we connect to our earlier work on perfect matrices and almost integral polytopes, see Padberg (1973, 1974, 1976), and give geometric reformulations of the SPGC in terms of almost integral polytopes. We assume familiarity of the reader with polyhedral theory and recommend e.g. Padberg (1995, Chapter 7) for a review.

1. Almost Integral Polytopes

Let $\mathbf{A}$ be any $m \times n$ matrix of zeros and ones having no zero row or column and let $\mathbf{e_m}$ be the vector having m components equal to one. Define two polytopes $P(\mathbf{A})$ and $P_I(\mathbf{A})$ as follows

$$P(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^\mathbf{n} : \mathbf{Ax} \leq \mathbf{e_m} \ , \ \mathbf{x} \geq \mathbf{0}\} \ , \ P_I(\mathbf{A}) = conv(\mathbf{P}(\mathbf{A}) \cap \mathbb{Z}^\mathbf{n}) \ . \tag{1}$$

From $\mathbf{x} \geq \mathbf{0}$ and the assumption that $\mathbf{A}$ contains no zero column it follows that $\mathbf{x} \leq \mathbf{e_n}$ for all $\mathbf{x} \in P(\mathbf{A})$ and by definition $P_I(\mathbf{A}) \subseteq P(\mathbf{A})$. The containment is in general proper, i.e., $P(\mathbf{A})$ typically has (fractional) extreme points $\mathbf{x}$ satisfying $0 < x_j < 1$ for some $j \in V = \{1, \ldots, n\}$.

If $P(\mathbf{A}) = P_I(\mathbf{A})$, i.e., if all extreme points of $P(\mathbf{A})$ are zero-one valued, then $P(\mathbf{A})$ is integral and $\mathbf{A}$ is called a *perfect zero-one matrix*, see Padberg (1974). If the inequalities $\mathbf{Ax} \leq \mathbf{e_m}$ in the definition of $P(\mathbf{A})$ are reversed, a zero-one matrix with the integrality property for the corresponding polyhedron is called an *ideal zero-one matrix*, see e.g. Padberg (1993) for more detail. Here we concern ourself solely with the first case.

With any zero one matrix $\mathbf{A}$ we associate the *intersection graph* $G_\mathbf{A} = (V, E)$ as follows. The node set $V = \{1, \ldots, n\}$ corresponds to the column set of $\mathbf{A}$ and we define $(u, v) \in E$ if

$u \neq v \in V$ and the columns u and v of $\mathbf{A}$ are nonorthogonal, i.e., if they have an entry equal to one in common in some row of $\mathbf{A}$.

Every zero-one extreme point of $P(\mathbf{A})$ thus corresponds to some stable set in $G_{\mathbf{A}}$, i.e., to some subset of pairwise nonadjacent nodes of $G_{\mathbf{A}}$, and vice versa.

On the other hand, let $K \subseteq V$ be a clique in $G_{\mathbf{A}}$. Since every stable set S in $G_{\mathbf{A}}$ satisfies $|K \cap S| \leq 1$, every $\mathbf{x} \in P_I(\mathbf{A})$ satisfies the inequality $\sum_{v \in K} x_v \leq 1$ and it is not difficult to prove that this inequality defines a facet of $P_I(\mathbf{A})$, see Padberg (1973). Let $a_v^K = 1$ for $v \in K$, $a_v^K = 0$ for $v \in V - K$ and $\mathbf{a^K} \in \mathbb{R}^{\mathbf{n}}$ be the row vector with components a_v^K for $v \in V$. If $\mathbf{a^K}$ is missing from the rows of $\mathbf{A}$ then $|K \cap R_i| \leq \omega - 1$ where $\omega = |K|$ and $R_i = \{v \in V : a_{iv} = 1\}$ for all $i = 1, \dots, m$. Consequently, $\mathbf{x} \in P(\mathbf{A})$ where $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$ is given by $x_v = 1/(\omega - 1)$ for $v \in K$, $x_v = 0$ for all $v \in V - K$ and hence $P(\mathbf{A}) \neq P_I(\mathbf{A})$ since $\mathbf{a^K x} = \frac{\omega}{\omega - \mathbf{1}} > \mathbf{1}$.

Consequently, for $\mathbf{A}$ to be perfect to every clique K of $G_{\mathbf{A}}$ there must correspond some row of $\mathbf{A}$ and we call a zero-one matrix with this property a *clique-matrix*.

With and without using the perfect graph theorem it has been shown that if $\mathbf{A}$ is a clique-matrix and $G_{\mathbf{A}}$ a perfect graph, then the matrix $\mathbf{A}$ is perfect and vice versa.

In other words, every perfect zero-one matrix is the clique-matrix of some perfect graph.

Further references and different proofs of this remarkable theorem, which is equivalent to the perfect graph theorem, can be found e.g. in Berge and Chvátal (1984), Golumbic (1980) or Padberg (1976).

Definition 1. *(i) An $m \times n$ matrix $\mathbf{A}$ is* almost perfect *if $\mathbf{A}$ is a zero-one matrix, $P(\mathbf{A}) \neq P_I(\mathbf{A})$ and $P^j(\mathbf{A}) = \mathbf{P_I^j}(\mathbf{A})$ for $j = 1, \dots, n$ where*

$$P^j(\mathbf{A}) = P(\mathbf{A}) \cap \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}} : \mathbf{x_j} = \mathbf{0}\} \tag{2}$$

and $P_I^j(\mathbf{A})$ is defined likewise.
(ii) A polyhedron $P \subseteq \mathbb{R}^n$ is an almost integral polytope *if there exists an almost perfect matrix $\mathbf{A}$ such that $P = P(\mathbf{A})$.*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

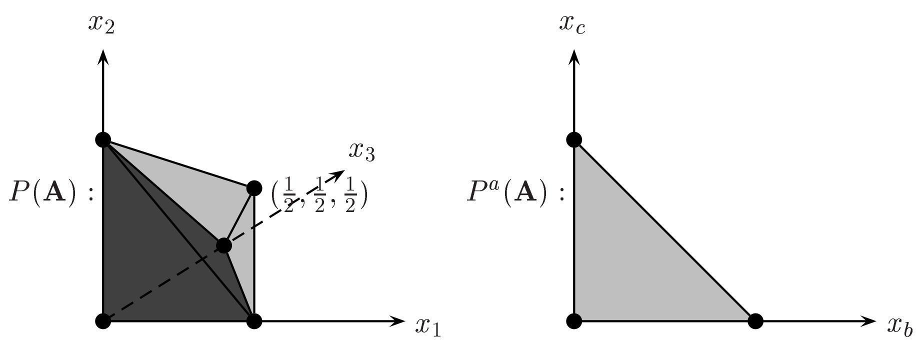

Figure 1: An almost integral polytope in $\mathbb{R}^3$ and its projection on $\mathbb{R}^2$

If $P(\mathbf{A})$ is almost integral, then since $P(\mathbf{A}) \neq P_I(\mathbf{A})$ the matrix $\mathbf{A}$ is imperfect and since $P^j(\mathbf{A}) = \mathbf{P_I^j}(\mathbf{A})$ for $j = 1, \dots, n$ every $m \times (n-1)$ submatrix of $\mathbf{A}$ is perfect.

By the above it follows that clique-matrices of almost perfect graphs are almost perfect and give rise to almost integral polytopes.

Different from the case of perfect matrices here the reverse statement is not correct.

The matrix $\mathbf{A} = \mathbf{E_n} - \mathbf{I_n}$ where $\mathbf{E_n}$ is the $n \times n$ matrix consisting of ones only and $\mathbf{I_n}$ the $n \times n$ identity matrix is clearly almost perfect, but its intersection graph $G_{\mathbf{A}}$ is the complete graph K_n and hence perfect.

Padberg (1976) has shown that *modulo* identical rows $\mathbf{A} = \mathbf{E_n} - \mathbf{I_n}$ is the only exception, i.e., every almost perfect matrix $\mathbf{A}$ with $G_{\mathbf{A}} \neq K_n$ is the clique-matrix of an almost perfect graph; see also Shepherd (1990).

To summarize more precisely what is known about almost perfect matrices and their polytopes we denote by $\mathbf{a^1}, \ldots, \mathbf{a^m}$ the rows of $\mathbf{A}$. Let $\mathbf{b^1}, \ldots, \mathbf{b^r}$ be the (nonzero) extreme points of $P_I(\mathbf{A})$ and denote by $\mathbf{B}$ the $r \times n$ matrix having rows $\mathbf{b^1}, \ldots, \mathbf{b^r}$. We define

$$\omega = \max\{\mathbf{a^i e_n} : \mathbf{1} \leq \mathbf{i} \leq \mathbf{m}\} \ , \ \alpha = \max\{\mathbf{b^i e_n} : \mathbf{1} \leq \mathbf{i} \leq \mathbf{r}\} \ , \tag{3}$$

$$Q(\mathbf{B}) = \{\mathbf{y} \in \mathbb{R}^{\mathbf{n}} : \mathbf{B y} \leq \mathbf{e_r}, \ \mathbf{y} \geq \mathbf{0}\} \ , \ Q_I(\mathbf{B}) = conv(Q(\mathbf{B}) \cap \mathbb{Z}^{\mathbf{n}}) \ . \tag{4}$$

In the following det denotes the absolute value of the respective determinants and r-unique is to be read as "unique modulo identical rows."

Theorem 1. (Padberg (1974, 1976)) *(i) Every almost integral $P(\mathbf{A}) \subseteq \mathbb{R}^n$ has a unique non-integer extreme point given by $\mathbf{x^0} = \frac{\mathbf{1}}{\omega}\mathbf{e_n}$ and*

$$P_I(\mathbf{A}) = P(\mathbf{A}) \cap \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}} : \mathbf{e_n^T x} \leq \alpha\} \ . \tag{5}$$

(ii) The submatrix $\mathbf{A_1}$ of $\mathbf{A}$ defining $\mathbf{x^0}$ is r-unique and there exists an r-unique submatrix $\mathbf{B_1}$ of $\mathbf{B}$ satisfying the matrix equation

$$\mathbf{A_1 B_1^T} = \mathbf{E_n} - \mathbf{I_n} \ . \tag{6}$$

$\mathbf{x^0}$ *has precisely n adjacent extreme points given by the rows of $\mathbf{B_1}$. Moreover, $\alpha\omega = n - 1$ and*

$$\det \mathbf{A_1} = \omega, \ \det \mathbf{B_1} = \alpha, \ \mathbf{e_n^T A_1} = (\mathbf{A_1 e_n})^{\mathbf{T}} = \omega \mathbf{e_n^T} \ , \ \mathbf{e_n^T B_1} = (\mathbf{B_1 e_n})^{\mathbf{T}} = \alpha \mathbf{e_n^T} \ . \tag{7}$$

(iii) If $\alpha = 1$, $Q(\mathbf{B}) = Q_I(\mathbf{B})$ is the unit cube in $\mathbb{R}^n$. Otherwise, $Q(\mathbf{B})$ is almost integral and

$$Q_I(\mathbf{B}) = Q(\mathbf{B}) \cap \{\mathbf{y} \in \mathbb{R}^{\mathbf{n}} : \mathbf{e_n^T y} \leq \omega\} \ . \tag{8}$$

The properties of almost perfect matrices stated in the theorem, except the case where $\alpha = 1$, were originally derived for almost perfect graphs by Padberg (1974).

They show that almost perfect graphs G with $\alpha = \alpha(G)$, $\omega = \omega(G)$ and n nodes have among others the following properties:

(i) $n = \alpha\omega + 1$,

(ii) G has precisely n cliques of size ω and every node of G is in exactly ω such cliques,

(iii) G has precisely n stable sets of size α and every node of G is in exactly α such stable sets,

(iv) the n stable sets S_i of size α and the n cliques K^j of size ω can be arranged such that $S_i \cap K^j = \emptyset$ if and only if $i = j$ where $1 \leq i, j \leq n$.

More properties of almost perfect graphs can be derived by observing that (6) implies $\mathbf{A_1 B_1^T} = \mathbf{B_1^T A_1}$, i.e. the commutativity of $\mathbf{A_1}$ and $\mathbf{B_1^T}$.

These properties are, however, not sufficient to characterize such graphs. There exist many graphs (the so-called partitionable or (α, ω)-graphs) having the properties (i),...,(iv), see Bland *et al* (1979) and Chvátal *et al* (1979) for examples none of which, however, contradicts the SPGC; see Boros and Gurvich (1993), Maffray and Preissmann (1993) and Sebö (1996).

The fact that the above properties of almost perfect graphs do not characterize such graphs completely is hardly surprising. Inspection of the proof of Theorem 1 shows that besides the nonintegrality of $P(\mathbf{A})$ only the integrality of $P^j(\mathbf{A})$ is exploited. We can characterize at present the integrality of $P^j(\mathbf{A})$, however, only by *forbidding* the occurrence of polytopes of the very same kind as the one that we are studying, namely $P(\mathbf{A})$.

We are thus – so to speak – caught in a circle, because we will evidently need other, lower dimensional faces of $P(\mathbf{A})$ to fully characterize almost integral polytopes and almost perfect graphs.

Theorem 1 has, however, a corollary that is worth noting.

Corollary 1. *If the SPGC is correct for graphs with n nodes, then every almost integral $P(\mathbf{A}) \subseteq \mathbb{R}^n$ satisfies $\omega = 2$ or $\omega = \lfloor \frac{n-1}{2} \rfloor$ or $\omega = n-1$. Moreover, if $\omega \neq n-1$ then n is odd.*

2. A Reformulation of the SPGC

Let $P(\mathbf{A}) \subseteq \mathbb{R}^n$ be an almost integral polytope. We shall assume without loss of generality that $2 \leq \omega \leq (n-1)/2$. For any $v \in V$ let $K_v^1, \ldots, K_v^\omega$ be the cliques of size ω containing v and define

$$P^{=} = P(\mathbf{A}) \cap \{\mathbf{x} \in \mathbb{R}^\mathbf{n} : \sum_{\mathbf{u} \in \mathbf{K_v^i}} \mathbf{x_u} = \mathbf{1} \text{ for } \mathbf{i} = \mathbf{1}, \ldots, \omega\} \ . \tag{9}$$

We select one of the cliques K_v^j with $1 \leq j \leq \omega$ and denote by π_ρ the orthoprojection from $\mathbb{R}^n$ onto $\mathbb{R}^{n-\omega}$ that projects out all x_u with $u \in K_v^j$. There are ω different choices for the variables to be projected out, there are n choices for the special column $v \in V$ and thus $n\omega$ different ways of selecting the projection π_ρ.

We call π_ρ an *ω-projection* and denote by Ω the index set of all possible ω-projections (in some arbitrary order). For $\rho \in \Omega$ we denote by P^ρ the orthogonal projection of $P^{=}$, i.e.,

$$P^\rho = \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \exists \mathbf{x} \in P^{=} \text{ such that } \mathbf{z} = \pi_\rho \mathbf{x}\} \ , \tag{10}$$

and call P^ρ the ω-projection of $P(\mathbf{A})$ for short.

If $\mathbf{A}$ is the incidence matrix of a chordless odd cycle on $n = 2\alpha+1$ nodes then a straightforward calculation shows that P^ρ is the polytope corresponding to a chordless odd cycle on $2(\alpha-1)+1 = n-2$ nodes for all $\rho \in \Omega$.

Moreover, if $\mathbf{A}$ is the incidence matrix of all cliques of the complement of a chordless odd cycle on n nodes then as shown in Appendix A

$$P^\rho = \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : (\mathbf{E}_{\mathbf{n}-\omega} - \mathbf{I}_{\mathbf{n}-\omega})\mathbf{z} \leq \mathbf{e}_{\mathbf{n}-\omega} \ , \ \mathbf{z} \geq \mathbf{0}\} \tag{11}$$

for all $\rho \in \Omega$ where $\omega = (n-1)/2$.

Thus if the SPGC is true for graphs with n nodes then the ω-projection P^ρ of any almost integral $P(\mathbf{A}) \subseteq \mathbb{R}^n$ with $2 \leq \omega \leq (n-1)/2$ is almost integral and moreover, $P^\rho = P_{=}$ for all $\rho \in \Omega$. We are thus led to the following *almost integral polytope conjecture* (AIPC).

Conjecture A. *The ω-projection P^ρ of an almost integral $P(\mathbf{A}) \subseteq \mathbb{R}^n$ with $2 \leq \omega \leq (n-1)/2$ is almost integral for some $\rho \in \Omega$.*

Theorem 2. *The SPGC is true if and only if the AIPC is true.*

Proof. As we have seen, the truth of the SPGC implies the truth of the AIPC. Suppose that the opposite is not correct. Then there exists a smallest $n \geq 10$ such that the AIPC applies, but the SPGC is incorrect. Hence there exists an almost perfect graph having n nodes that violates the SPGC. Let $P(\mathbf{A}) \subseteq \mathbb{R}^n$ be the associated almost integral polytope. Since the AIPC applies there exists some $\rho \in \Omega$ such that $P^\rho \subseteq \mathbb{R}^{n-\omega}$ is almost integral. Since the SPGC is true for graphs with $n-\omega$ nodes (by the minimality of n) it follows from Corollary 1 that $\omega = 2$ or $\omega = (n-\omega-1)/2$ or $\omega = n-\omega-1$. Since the SPGC fails, $\omega \neq 2$ and $\alpha \neq 2$ and thus $n-\omega$ is odd and $n = 3\omega+1$. But then by Tucker's theorem the SPGC is correct, which is a contradiction. ∎

As we have seen the truth of the SPGC implies more than we need to establish the equivalence of the SPGC and the AIPC. Indeed the symmetries that must be present in almost integral polytopes suggest the following conjecture.

Conjecture B. *The ω-projection P^ρ of an almost integral $P(\mathbf{A}) \subseteq \mathbb{R}^n$ with $2 \leq \omega \leq (n-1)/2$ is some almost integral polytope $P_= \subseteq \mathbb{R}^{n-\omega}$ for all $\rho \in \Omega$.*

Clearly, Conjecture B is also equivalent to the SPGC since it is implied by the SPGC and it implies Conjecture A.

Example: ω-projection for $\overline{C}_7$ with $v = 1$ and $K = \{1,2,3\}$.

$$
\begin{array}{llllllll}
x_1 + x_2 + x_3 & & & & & & = 1 \\
x_1 + x_2 & & & & & + x_7 & = 1 \\
x_1 & & & & + x_6 & + x_7 & = 1 \\
& x_2 + x_3 + x_4 & & & & & \leq 1 \\
& & x_3 + x_4 + x_5 & & & & \leq 1 \\
& & & x_4 + x_5 + x_6 & & & \leq 1 \\
& & & & x_5 + x_6 + x_7 & & \leq 1 \\
& & & x_i \geq 0, & & i = 1, \ldots, 7 &
\end{array}
\qquad \xrightarrow{\pi_\rho}
$$

$$
\begin{array}{lllll}
z_1 & & + z_3 + z_4 & \leq 1 \\
z_1 + z_2 & & + z_4 & \leq 1 \\
z_1 + z_2 + z_3 & & & \leq 1 \\
& z_2 + z_3 + z_4 & & \leq 1 \\
z_i \geq 0, & & i = 1, \ldots, 4 &
\end{array}
$$

Note: $z_i = x_{3+i}$, for $i = 1, \ldots, 4$.

Indeed, if the SPGC is true, then almost integral polytopes $P(\mathbf{A})$ with $2 \leq \omega \leq (n-1)/2$ exist only in odd-dimensional spaces because they are precisely those corresponding to odd cycles without chords and their complements. ω-projections apply to all polytopes $P(\mathbf{A})$ where $\mathbf{A}$ is the clique-matrix of some graph and thus in particular to partitionable or (α,ω)-graphs. If e.g. in the first example of Bland *et al* (1979), see Figure 2, the clique $\{2,3,4\}$ with $v=3$ as the special node is projected out this way, then the almost integral polytope associated with $\overline{C}_7$ results. [This (α,ω)-graph has $n=10$ nodes, exactly ten cliques, all of size $\omega=3$, given by $\{1,2,3\}$, $\{1,6,7\}$, $\{1,2,10\}$, $\{2,3,4\}$, $\{3,4,5\}$, $\{4,5,6\}$, $\{5,9,10\}$, $\{6,7,8\}$, $\{7,8,9\}$, $\{8,9,10\}$ and ten stable sets of size $\alpha=3$ given by $\{1,4,8\}$, $\{1,4,9\}$, $\{1,5,8\}$, $\{2,5,7\}$, $\{2,5,8\}$, $\{2,6,9\}$, $\{3,6,9\}$, $\{3,6,10\}$, $\{3,7,10\}$, $\{4,7,10\}$.] It would thus be wrong to believe that the almost integrality of some ω-projection of a polytope $P(\mathbf{A})$ implies the almost integrality of the "mother" polytope $P(\mathbf{A})$. Rather, to prove the SPGC, one has to show that if the ω-projection of $P(\mathbf{A})$ is not almost integral, then $P(\mathbf{A})$ cannot be almost integral either.

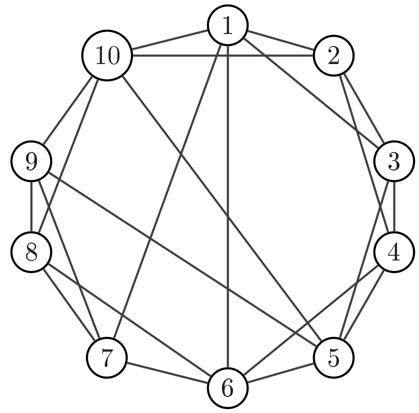

Figure 2: An (α,ω)-graph

3. Some Properties of ω-Projections of $P(\mathbf{A})$

We assume throughout this section that $P(\mathbf{A})$ is almost integral with $2 \le \omega \le (n-1)/2$ and that the rows of $\mathbf{A}$ and $\mathbf{B}$ are indexed such that the rows of the submatrices $\mathbf{A_1}$ and $\mathbf{B_1}$ of Theorem 1 correspond to the rows $1,\dots,n$. Since $\mathbf{A_1B_1^T} = \mathbf{E_n} - \mathbf{I_n} = \mathbf{B_1^T A_1}$, $(\mathbf{A_1P})(\mathbf{B_1P})^{\mathbf{T}} = \mathbf{A_1B_1^T}$ and $(\mathbf{PA_1})(\mathbf{PB_1})^{\mathbf{T}} = \mathbf{E_n} - \mathbf{I_n}$ for all $n \times n$ permutation matrices $\mathbf{P}$, we can arrange the rows and the columns of $\mathbf{A_1}$ and $\mathbf{B_1}$ such that

$$\mathbf{A_1} = \begin{pmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{pmatrix} \;,\; \mathbf{B_1} = \begin{pmatrix} \mathbf{B_{11}} & \mathbf{B_{12}} \\ \mathbf{B_{12}^T} & \mathbf{B_{22}} \end{pmatrix} \tag{12}$$

and the $\omega \times \omega$ submatrices $\mathbf{A_{11}}$ and $\mathbf{B_{11}}$ are given by

$$\mathbf{A_{11}} = \begin{pmatrix} 1 & \mathbf{e^T_{\omega-1}} \\ \mathbf{e_{\omega-1}} & \mathbf{G} \end{pmatrix} \;,\; \mathbf{B_{11}} = \begin{pmatrix} 0 & \mathbf{0}^T_{\omega-1} \\ \mathbf{0}_{\omega-1} & \mathbf{I_{\omega-1}} \end{pmatrix} \;, \tag{13}$$

where $\mathbf{G}$ is a zero-one matrix of size $(\omega-1)\times(\omega-1)$ having zeros on its diagonal. The submatrices $\mathbf{B_{12}}$ and $\mathbf{B_{22}}$ are of size $\omega \times (n-\omega)$ and $(n-\omega)\times(n-\omega)$, respectively, and satisfy

$$\mathbf{B_{12}} = \begin{pmatrix} \mathbf{e^T_\alpha} & \mathbf{0}^T & \cdots & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{e^T_{\alpha-1}} & \cdots & \mathbf{0}^T \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{e^T_{\alpha-1}} \end{pmatrix} \;,\; \mathbf{e^T_{n-\omega}B_{22}} = (\mathbf{B_{22}e_{n-\omega}})^{\mathbf{T}} = (\alpha-\mathbf{1})\mathbf{e^T_{n-\omega}} \;. \tag{14}$$

This follows more or less immediately from the properties of almost perfect graphs listed in the previous section and the matrix equation (6) remains correct after such a rearrangement.

Moreover, we are free to choose any column of $\mathbf{A}$ as the "first" column in this rearrangement and thus it will suffice to study the ω-projection with column 1 being the special column and $x_1,\dots,x_\omega$ the variables to be projected out.

We will write P, P_I and P_j to mean $P(\mathbf{A})$, $P_I(\mathbf{A})$ and $P_j(\mathbf{A})$ in the following for notational simplicity. In this notation the polytope $P^=$ defined in (9) thus becomes

$$P^= = \{\mathbf{x} \in \mathbb{R}^\mathbf{n} : (\mathbf{A_{11}}\ \mathbf{A_{12}})\mathbf{x} = \mathbf{e}_\omega\ ,\ (\mathbf{A_{21}}\ \mathbf{A_{22}})\mathbf{x} \le \mathbf{e_{n-\omega}}\ ,\ \mathbf{A_3x} \le \mathbf{e_t}\ ,\ \mathbf{x} \ge \mathbf{0}\}\ , \tag{15}$$

where $\mathbf{A_3}$ are all $t = m - n \ge 0$ rows of $\mathbf{A}$ having less than ω ones. We assume that $\mathbf{A}$ has no identical rows and define for $j = 1, \dots, n$

$$P_j^= = P^= \cap \{\mathbf{x} \in \mathbb{R}^\mathbf{n} : \mathbf{x_j} = \mathbf{0}\}\ ,\ \mathbf{P_I^=} = conv(P^= \cap \mathbb{Z}^\mathbf{n})\ .$$

Proposition 1. *(i) $P^=$ has exactly one fractional extreme point $\mathbf{x^0} = \frac{\mathbf{1}}{\omega}\mathbf{e_n}$.*
(ii) The extreme point $\mathbf{x^0}$ of $P^=$ has precisely $n-\omega$ linearly independent adjacent extreme points given by the rows $\mathbf{b}^{\omega+1}, \dots, \mathbf{b^n}$ of $\mathbf{B_1}$.
(iii) $\dim P^= = \dim P_I^= = n - \omega$.
(iv) $\sum_{j=\omega+1}^n x_j \le \alpha - 1$ defines a facet of $P_I^=$ and

$$P_I^= = P^= \cap \{\mathbf{x} \in \mathbb{R}^\mathbf{n} : \sum_{\mathbf{j}=\omega+\mathbf{1}}^{\mathbf{n}} \mathbf{x_j} \le \alpha - \mathbf{1}\}\ . \tag{16}$$

(v) $P_j^=$ is integral for $j = 1, \dots, n$.

Proof. (i) $P^=$ is a face of P, $\mathbf{x^0} \in P^=$ and thus the assertion follows.
(ii) By Theorem 1, $\mathbf{b^1}, \dots, \mathbf{b^n}$ are precisely the adjacent extreme points of $\mathbf{x^0}$ in P. Since $\mathbf{b^i} \notin P^=$ for $i = 1, \dots, \omega$, $\mathbf{b^i} \in P^=$ for $i = \omega + 1, \dots, n$ and since $P^=$ is a face of P, the statement follows from the nonsingularity of $\mathbf{B_1}$.
(iii) Since $r(\mathbf{A_{11}}\ \mathbf{A_{12}}) = \omega$, $\dim P^= \le n - \omega$. Consider the $n - \omega + 1$ points $\mathbf{b^0}, \mathbf{b}^{\omega+1}, \dots, \mathbf{b^n}$ where $(\mathbf{b^0})^\mathbf{T} = \mathbf{u_n^1} \in P^=$ is the first unit vector in $\mathbb{R}^n$. Suppose that they are linearly dependent. Because $\mathbf{b}^{\omega+1}, \dots, \mathbf{b^n}$ are linearly independent, there exist $\lambda_{\omega+1}, \dots, \lambda_n$, not all zero, such that $\mathbf{b_0} = \sum_{\mathbf{j}=\omega+\mathbf{1}}^{\mathbf{n}} \lambda_\mathbf{j} \mathbf{b^j}$, i.e., there exists $\boldsymbol{\lambda} \in \mathbb{R}^n$, $\boldsymbol{\lambda} \ne \mathbf{0}$, $\lambda_i = 0$ for $i = 1, \dots, \omega$ such that $\mathbf{b_0} = \boldsymbol{\lambda}\mathbf{B_1}$. Consequently, from $\mathbf{A_1B_1^T} = \mathbf{E_n} - \mathbf{I_n} = \mathbf{B_1A_1^T}$ it follows that $\mathbf{b_0A_1^T} = \boldsymbol{\lambda}(\mathbf{E_n} - \mathbf{I_n})$ or $\mathbf{a^1} = \boldsymbol{\lambda}(\mathbf{E_n} - \mathbf{I_n})$, where $\mathbf{a^1} = (\mathbf{e}_\omega^\mathbf{T}, \mathbf{0_{n-\omega}})$ is the first row of $\mathbf{A_1}$. Solving for $\boldsymbol{\lambda}$ we get $\boldsymbol{\lambda} = \frac{1}{\alpha}\mathbf{e_n^T} - \mathbf{a^1}$ and thus $\lambda_i = -(\alpha - 1)/\alpha$ for $i = 1, \dots, \omega$. This is a contradiction and thus $\mathbf{b^0}, \mathbf{b}^{\omega+1}, \dots, \mathbf{b^n}$ are linearly independent. Since $\mathbf{b^0} \in P_I^= \subseteq P^=$ and $\mathbf{b^j} \in P_I^= \subseteq P^=$ for $j = \omega + 1, \dots, n$ it follows that $\dim P^= = \dim P_I^= = n - \omega$.
(iv) Since $\sum_{j=1}^\omega x_j = 1$ for all $\mathbf{x} \in P^=$ the claim follows from $P_I = P \cap \{\mathbf{x} \in \mathbb{R}^\mathbf{n} : \mathbf{e_n^T x} \le \alpha\}$ because $P_I^=$ is a face of P_I.
(v) $P_j^=$ is a face of P_j which is integral and thus $P_j^=$ is integral for $j = 1, \dots, n$. ∎

Consider now the orthogonal projection P^ρ of $P^=$ into $\mathbb{R}^{n-\omega}$, i.e.,

$$P^\rho = \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \exists \mathbf{x} \in P^= \text{ such that } \mathbf{z} = \boldsymbol{\pi}\mathbf{x}\}\ , \tag{17}$$

where the ω-projection $\mathbf{z} = \boldsymbol{\pi}\mathbf{x}$ is in matrix form given by $\mathbf{z} = (\mathbf{O}\ \mathbf{I_{n-\omega}})\mathbf{x}$. Clearly

$$P^\rho \subseteq \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \mathbf{0} \le \mathbf{z_j} \le \mathbf{1} \text{ for } \mathbf{j} = \mathbf{1}, \dots, \mathbf{n} - \omega\} \tag{18}$$

and thus every $\mathbf{z} \in P^\rho$ with $\mathbf{z} \in \{\mathbf{0}, \mathbf{1}\}^{\mathbf{n}-\omega}$ is an extreme point of P^ρ. Moreover since $\mathbf{u_n^1} \in P^=$ it follows that $\mathbf{0}_{n-\omega} = \boldsymbol{\pi}\mathbf{u_n^1} \in P^\rho$, i.e., P^ρ contains the origin of $\mathbb{R}^{n-\omega}$. Like we did above we define correspondingly P_I^ρ and P_j^ρ for $j = 1, \dots, n - \omega$. The following proposition is stated in terms of P^ρ and $P^=$, but its proof shows that it remains true for orthoprojections of polytopes in general.

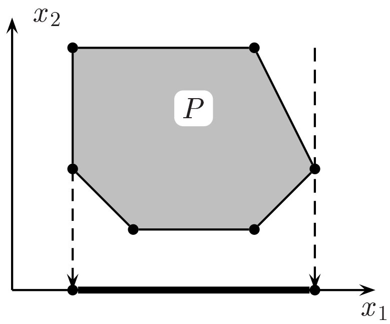

Figure 3: Orthoprojection

Proposition 2. *If F is a face of P^ρ, then there exists a face F^* of $P^=$ such that $F = \boldsymbol{\pi}F^*$. In particular, if $\mathbf{z}$ is an extreme point of P^ρ, then there is an extreme point $\mathbf{x} \in P^=$ such that $\mathbf{z} = \boldsymbol{\pi}\mathbf{x}$.*

Proof. Since F is a face of P^ρ there exists $(\mathbf{f}, \mathbf{f_0}) \in \mathbb{R}^{\mathbf{n}-\omega+\mathbf{1}}$ such that $F = P^\rho \cap \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \mathbf{fz} = \mathbf{f_0}\}$ and $\mathbf{fz} < \mathbf{f_0}$ for all $\mathbf{z} \in P^\rho - \mathbf{F}$. Let $\mathbf{g} = (\mathbf{0}_\omega, \mathbf{f})$. By construction $\mathbf{gx} \le \mathbf{f_0}$ for all $\mathbf{x} \in P^=$, $\mathbf{gx} = \mathbf{f_0}$ for $\mathbf{x} \in P^=$ if and only if $\boldsymbol{\pi}\mathbf{x} \in \mathbf{F}$ and $\mathbf{gx} < \mathbf{f_0}$ for $\mathbf{x} \in P^=$ if and only if $\mathbf{f}(\boldsymbol{\pi}\mathbf{x}) < \mathbf{f_0}$. It follows that $F^* = P^= \cap \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}} : \mathbf{gx} = \mathbf{f_0}\}$ is a face of $P^=$ that satisfies $F = \boldsymbol{\pi}F^*$. If F consists of a single extreme point $\mathbf{z}$ of P^ρ, then by the first part there is some nonempty face F^* of $P^=$ that is mapped into $\mathbf{z}$. Because $P^=$ is a polytope F^* has extreme points which are all mapped into $\mathbf{z}$. ∎

Proposition 3. *(i) $\mathbf{z^0} = (\mathbf{1}/\omega)\mathbf{e_{n-\omega}}$ is an extreme point of P^ρ. All other extreme points $\mathbf{z}$ of P^ρ are integral and satisfy $\mathbf{e^T_{n-\omega}z} \le \alpha - \mathbf{1}$.*
(ii) Every extreme point $\mathbf{z}$ of P^ρ that is adjacent to $\mathbf{z^0}$ satisfies $\mathbf{e^T_{n-\omega}z} = \alpha - \mathbf{1}$ and there are at most $n - \omega$ such extreme points which are given by the rows of $\mathbf{B_{22}}$.
(iii) $\mathbf{e^T_{n-\omega}z} \le \alpha - \mathbf{1}$ defines a facet of P^ρ_I and

$$P^\rho_I = P^\rho \cap \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \mathbf{e^T_{n-\omega}z} \le \alpha - \mathbf{1}\}\ . \tag{19}$$

(iv) P^ρ_j is integral for $j = 1, \ldots, n - \omega$.

Proof. (i) Since by Proposition 2 every extreme point of P^ρ is the image of some extreme point of $P^=$ it follows that P^ρ has at most one fractional extreme point and that all other extreme points of P^ρ are zero-one valued. Moreover, every integer extreme point $\mathbf{z}$ of P^ρ satisfies $\mathbf{e^T_{n-\omega}z} \le \alpha - \mathbf{1}$ because $\mathbf{z}$ is the image of some integer extreme point $\mathbf{x}$ of $P^=$ with $\mathbf{a^1x} = \mathbf{1}$ and $\mathbf{e^T_n x} \le \alpha$. Since $\mathbf{z^0} = \boldsymbol{\pi}\mathbf{x^0}$ where $\mathbf{x^0} = (\mathbf{1}/\omega)\mathbf{e_n}$ is the unique fractional extreme point of $P^=$ we have $\mathbf{z^0} \in P^\rho$. But $\mathbf{e^T_{n-\omega}z^0} = (\mathbf{n} - \omega)/\omega > \alpha - \mathbf{1}$ and thus P^ρ has at least one extreme point $\mathbf{z}$ with $\mathbf{e^T_{n-\omega}z} > \alpha - \mathbf{1}$. Since $\mathbf{z}$ is the image of some extreme point of $P^=$ it follows that $\mathbf{z} = \mathbf{z^0}$ is the only fractional extreme point of P^ρ.
(ii) Let $\mathbf{z} \in P^\rho$ be any integer extreme point of P^ρ that is adjacent to $\mathbf{z^0}$ and suppose that $\mathbf{e^T_{n-\omega}z} < \alpha - \mathbf{1}$. Then the point $\mathbf{z}^* = \mu\mathbf{z} + (\mathbf{1} - \mu)\mathbf{z^0}$ for $\mu = [n - \omega - \omega\mathbf{e^T_{n-\omega}z}]^{-\mathbf{1}}$ is in P^ρ because $0 < \mu < 1$ and satisfies $\mathbf{e^T_{n-\omega}z^*} = \alpha - \mathbf{1}$. Moreover, since $\mathbf{z}$ and $\mathbf{z^0}$ are adjacent on P^ρ the convex combination for $\mathbf{z}^*$ is unique. Let $\mathbf{x}^* \in P^=$ be such that $\mathbf{z}^* = \boldsymbol{\pi}\mathbf{x}^*$. Since $\mathbf{a^1x}^* = \mathbf{1}$ it follows that $\mathbf{e^T_n x}^* = \alpha$. Consequently, by Proposition 1, there exist $\mu_i \ge 0$ with $\sum_{\omega+1}^n \mu_i = 1$

and $\mathbf{x}^* = \sum_{\mathbf{i}=\omega+\mathbf{1}}^{\mathbf{n}} \mu_{\mathbf{i}}(\mathbf{b^i})^{\mathbf{T}}$. But then

$$\mathbf{z}^* = \boldsymbol{\pi}\mathbf{x}^* = \sum_{\mathbf{i}=\omega+\mathbf{1}}^{\mathbf{n}} \mu_{\mathbf{i}}\boldsymbol{\pi}(\mathbf{b^i})^{\mathbf{T}}$$

contradicts the uniqueness of the convex combination for $\mathbf{z}^*$ and thus every integer extreme point $\mathbf{z}$ of P that is adjacent to $\mathbf{z^0}$ satisfies $\mathbf{e}^{\mathbf{T}}_{\mathbf{n}-\omega}\mathbf{z} = \alpha - \mathbf{1}$. Since the only extreme points $\mathbf{z}$ of P^ρ with $\mathbf{e}^{\mathbf{T}}_{\mathbf{n}-\omega}\mathbf{z} = \alpha - \mathbf{1}$ are given by $\boldsymbol{\pi}(\mathbf{b^i})^{\mathbf{T}}$ for $i = \omega+1, \dots, n$ there are at most $n-\omega$ distinct extreme points adjacent to $\mathbf{z^0}$ on P^ρ and they correspond to the rows of $\mathbf{B_{22}}$.
(iii) By part (i) of the proposition we have

$$P_I^\rho \subseteq P^\rho \cap \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \mathbf{e}^{\mathbf{T}}_{\mathbf{n}-\omega}\mathbf{z} \le \alpha - \mathbf{1}\} \subset P^\rho \ ,$$

were the last inclusion is proper because $\mathbf{z^0} \in P^\rho$ and $\mathbf{e}^{\mathbf{T}}_{\mathbf{n}-\omega}\mathbf{z^0} > \alpha - \mathbf{1}$. Suppose that the first inclusion is proper as well. Then there exists an extreme point $\mathbf{z}^*$ of $P^\rho \cap \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \mathbf{e}^{\mathbf{T}}_{\mathbf{n}-\omega}\mathbf{z} \le \alpha - \mathbf{1}\}$, $\mathbf{z}^* \notin P_I^\rho$. As we are intersecting P^ρ with a single inequality it follows that $\mathbf{z}^*$ lies on a 1-dimensional face of P^ρ and satisfies $\mathbf{e}_{\mathbf{n}-\omega}\mathbf{z}^* = \alpha - \mathbf{1}$. Consequently there exists some extreme point $\mathbf{z}$ of P^ρ with $\mathbf{e}^{\mathbf{T}}_{\mathbf{n}-\omega}\mathbf{z} < \alpha - \mathbf{1}$ that is adjacent to $\mathbf{z^0}$. By part (ii) of the proposition this is impossible and thus the claim follows.
(iv) By definition we have

$$\begin{aligned} P_j^\rho &= \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \exists \mathbf{x} \in P^= \text{ such that } \mathbf{z} = \boldsymbol{\pi}\mathbf{x} \text{ and } z_j = 0\} \\ &= \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \exists \mathbf{x} \in P^=_{\omega+\mathbf{j}} \text{ such that } \mathbf{z} = \boldsymbol{\pi}\mathbf{x}\} \ . \end{aligned}$$

But $P^=_{\omega+j}$ for $j = 1, \dots, n-\omega$ is an integral polytope by Proposition 1(v) and thus by Proposition 2 the result follows. ∎

Proposition 3 shows that ω-projections of almost integral $P(\mathbf{A}) \subseteq \mathbb{R}^n$ have many of the properties of almost integral polytopes in $\mathbb{R}^{n-\omega}$, but it leaves open two important aspects of such polytopes: (i) the dimension of the ω-projections and (ii) the linear description of them by way of zero-one matrices. We partition the $t = m - n \ge 0$ nonmaximal rows $\mathbf{A_3} = (\mathbf{A_{31}}\ \mathbf{A_{32}})$ of $\mathbf{A}$ like the rest of $\mathbf{A}$.

Proposition 4. *(i)* $\dim P^\rho = \dim P_I^\rho = r(\mathbf{B_{22}}) = \mathbf{n} - \mathbf{2}\omega + \mathbf{r}(\mathbf{A_{11}}) \le \mathbf{n} - \omega$.
(ii) $\det \mathbf{B_{22}} = (\alpha - \mathbf{1}) \det \mathbf{A_{11}}$ *and* $\det \mathbf{A_{22}} = \mathbf{1}$.
(iii) If $\det \mathbf{A_{11}} \ne \mathbf{0}$ *then the linear description of* P^ρ *is*

$$P^\rho = \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}} : \mathbf{H_1}\mathbf{z} \le \mathbf{e}_{\mathbf{n}-\omega} \ , \ \mathbf{H_2}\mathbf{z} \le \mathbf{h_2} \ , \ \mathbf{H_3}\mathbf{z} \le \mathbf{u}^{\mathbf{1}}_\omega \ , \ \mathbf{z} \ge \mathbf{0}\} \ , \tag{20}$$

where $\mathbf{H_1} = \mathbf{A_{22}} - \mathbf{A_{21}}\mathbf{A_{11}^{-1}}\mathbf{A_{12}}$, $\mathbf{H_2} = \mathbf{A_{32}} - \mathbf{A_{31}}\mathbf{A_{11}^{-1}}\mathbf{A_{12}}$, $\mathbf{H_3} = \mathbf{A_{11}^{-1}}\mathbf{A_{12}}$ *and* $\mathbf{h_2} = \mathbf{e_t} - \mathbf{A_{31}}\mathbf{u}^{\mathbf{1}}_\omega$. *Moreover,* $\mathbf{e}^{\mathbf{T}}_{\mathbf{n}-\omega}\mathbf{H_1} = (\mathbf{H_1}\mathbf{e}_{\mathbf{n}-\omega})^{\mathbf{T}} = \omega\mathbf{e}^{\mathbf{T}}_{\mathbf{n}-\omega}$, $\det \mathbf{H_1} \det \mathbf{A_{11}} = \omega$,

$$\mathbf{H_1}\mathbf{B}^{\mathbf{T}}_{\mathbf{22}} = \mathbf{B}^{\mathbf{T}}_{\mathbf{22}}\mathbf{H_1} = \mathbf{E}_{\mathbf{n}-\omega} - \mathbf{I}_{\mathbf{n}-\omega} \ , \tag{21}$$

and P^ρ *has precisely* $n - \omega$ *integer extreme points* $\mathbf{z}$ *with* $\mathbf{e}^{\mathbf{T}}_{\mathbf{n}-\omega}\mathbf{z} = \alpha - \mathbf{1}$ *given by the rows of* $\mathbf{B_{22}}$.

Proof. (i) Because $\mathbf{0}_{n-\omega} \in P^\rho$ every valid equation for P^ρ is of the form $\mathbf{f}\mathbf{z} = \mathbf{0}$. It follows that $(\mathbf{0}_\omega\ \mathbf{f})\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in P^=$. By Proposition 1(iii) the minimal system of valid equations for $P^=$ is $(\mathbf{A_{11}}\ \mathbf{A_{12}})\mathbf{x} = \mathbf{e}_\omega$. Consequently, there exists $\boldsymbol{\eta} \in \mathbb{R}^\omega$ such that $\boldsymbol{\eta}\mathbf{A_{11}} = \mathbf{0}_\omega$, $\boldsymbol{\eta}\mathbf{A_{12}} = \mathbf{f}$ and

$\boldsymbol{\eta}\mathbf{e}_\omega = \mathbf{0}$. The last equation is redundant because the first column of $\mathbf{A_{11}}$ has all entries equal to one. There are precisely $s = \omega - r(\mathbf{A_{11}}) \geq \mathbf{0}$ linearly independent solutions $\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^s$ to $\boldsymbol{\eta}\mathbf{A_{11}} = \mathbf{0}_\omega$. Let $\mathbf{f^i} = \boldsymbol{\eta}^\mathbf{i}\mathbf{A_{12}}$ for $i = 1, \ldots, s$. Since $r(\mathbf{A_{11}}\ \mathbf{A_{12}}) = \omega$ it follows that $\mathbf{f^i} \neq \mathbf{0}_{\mathbf{n}-\omega}$ for $i = 1, \ldots, s$. We claim that $\mathbf{f^1}, \ldots, \mathbf{f^s}$ are linearly independent. Assume the contrary and let $\mathbf{H}$ be the $s \times \omega$ matrix with rows $\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^s$. Then there exists $\boldsymbol{\lambda} \in \mathbb{R}^s, \boldsymbol{\lambda} \neq \mathbf{0}_s$, such that $\boldsymbol{\lambda}\mathbf{H}(\mathbf{A_{11}}\ \mathbf{A_{12}}) = \mathbf{0_n}$. But from $\mathbf{A_1B_1^T} = \mathbf{E_n} - \mathbf{I_n}$ it follows that $\mathbf{H}(\mathbf{A_{11}}\ \mathbf{A_{12}})(\mathbf{b^i})^\mathbf{T} = \mathbf{H}(\mathbf{e}_\omega - \mathbf{u}^\mathbf{i}_\omega) = -\mathbf{H}\mathbf{u}^\mathbf{i}_\omega$ for $i = 1, \ldots, \omega$ where $\mathbf{u}^\mathbf{i}_\omega \in \mathbb{R}^\omega$ is the i-th unit vector. Hence $\boldsymbol{\lambda}\mathbf{H} = \mathbf{0}_\omega$ which contradicts the linear independence of $\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^s$. Consequently P^ρ has precisely $\omega - r(\mathbf{A_{11}})$ linearly independent equations and thus $\dim P^\rho = n - \omega - (\omega - r(\mathbf{A_{11}})) = \mathbf{n} - \mathbf{2}\omega + \mathbf{r}(\mathbf{A_{11}})$. By Proposition 1(iv) the same reasoning applies to P^ρ_I and hence $\dim P^\rho = \dim P^\rho_I$. By Proposition 3(ii) the rows of $\mathbf{B_{22}}$ are precisely the extreme points of P^ρ satisfying $\mathbf{e}^\mathbf{T}_{\mathbf{n}-\omega}\mathbf{z} = \alpha - \mathbf{1}$. By Proposition 3(iii) $\mathbf{e}^\mathbf{T}_{\mathbf{n}-\omega}\mathbf{z} = \alpha - \mathbf{1}$ is a facet of P^ρ_I and thus $r(\mathbf{B_{22}}) = (\dim P^\rho_I - \mathbf{1}) + \mathbf{1} = \dim P^\rho_I$.
(ii) From $\mathbf{A_1B_1^T} = \mathbf{E_n} - \mathbf{I_n}$ it follows that $\mathbf{A_1^{-1}} = \frac{\mathbf{1}}{\omega}\mathbf{E_n} - \mathbf{B_1^T}$. We partition $\mathbf{A_1^{-1}}$ according to the partitioning of $\mathbf{A_1}$ as

$$\mathbf{A_1^{-1}} = \begin{pmatrix} \mathbf{X} & \mathbf{W} \\ \mathbf{Y} & \mathbf{Z} \end{pmatrix} .$$

Multiplying the respective matrices we find the identities

$$\begin{pmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{pmatrix} \begin{pmatrix} \mathbf{X} & \mathbf{O} \\ \mathbf{Y} & \mathbf{I}_{\mathbf{n}-\omega} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_\omega & \mathbf{A_{12}} \\ \mathbf{O} & \mathbf{A_{22}} \end{pmatrix} , \begin{pmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{pmatrix} \begin{pmatrix} \mathbf{I}_\omega & \mathbf{W} \\ \mathbf{O} & \mathbf{Z} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}} & \mathbf{O} \\ \mathbf{A_{21}} & \mathbf{I}_{\mathbf{n}-\omega} \end{pmatrix}$$

and thus we get from our formula for $\mathbf{A_1^{-1}}$ the *Jacobi* identities

$$\det \mathbf{A_1} \det(\frac{\mathbf{1}}{\omega}\mathbf{E}_\omega - \mathbf{B_{11}^T}) = \det \mathbf{A_{22}}\ ,\ \det \mathbf{A_1} \det(\frac{\mathbf{1}}{\omega}\mathbf{E}_{\mathbf{n}-\omega} - \mathbf{B_{22}^T}) = \det \mathbf{A_{11}}\ .$$

We calculate using elementary row operations

$$\det(\frac{1}{\omega}\mathbf{E}_\omega - \mathbf{B_{11}^T}) = \det \begin{pmatrix} \frac{1}{\omega} & \frac{1}{\omega} \cdots \frac{1}{\omega} \\ \mathbf{0} & -\mathbf{I}_{\omega-\mathbf{1}} \end{pmatrix} = \frac{\mathbf{1}}{\omega}\ ,$$

$$\det(\frac{1}{\omega}\mathbf{E}_{\mathbf{n}-\omega} - \mathbf{B_{22}^T}) = \det(\mathbf{B_{22}}(\frac{\mathbf{1}}{\omega(\alpha - \mathbf{1})}\mathbf{E}_{\mathbf{n}-\omega} - \mathbf{I}_{\mathbf{n}-\omega})) = \frac{\mathbf{1}}{(\alpha - \mathbf{1})\omega} \det \mathbf{B_{22}}\ ,$$

for the absolute values of the respective determinants where we have used $\mathbf{B_{22}e}_{\mathbf{n}-\omega} = (\alpha - \mathbf{1})\mathbf{e}_{\mathbf{n}-\omega}$ in the second calculation. By Theorem 1(ii) $\det \mathbf{A_1} = \omega$ and thus the formulas follow.
(iii) From $\mathbf{A_{11}x^1} + \mathbf{A_{12}x^2} = \mathbf{e}_\omega$ we have $\mathbf{x^1} = \mathbf{u}^\mathbf{1}_\omega - \mathbf{A_{11}^{-1}A_{12}x^2}$ where $\mathbf{x^1}$, $\mathbf{x^2}$ correspond to the ω first and the $n - \omega$ last components of $\mathbf{x}$, respectively. The linear description of P^ρ_I follows by eliminating $\mathbf{x^1}$ and the observation that $\mathbf{A_{21}u}^\mathbf{1}_\omega = \mathbf{0}_{\mathbf{n}-\omega}$. From $\mathbf{e}^\mathbf{T}_{\mathbf{n}-\omega}\mathbf{A_{22}} = \omega\mathbf{e}^\mathbf{T}_{\mathbf{n}-\omega} - \mathbf{e}^\mathbf{T}_\omega\mathbf{A_{12}}$ and $\mathbf{e}^\mathbf{T}_{\mathbf{n}-\omega}\mathbf{A_{21}} = \omega\mathbf{e}^\mathbf{T}_\omega - \mathbf{e}^\mathbf{T}_\omega\mathbf{A_{11}}$ we get

$$\mathbf{e}^\mathbf{T}_{\mathbf{n}-\omega}\mathbf{H_1} = \mathbf{e}^\mathbf{T}_{\mathbf{n}-\omega}\mathbf{A_{22}} - \mathbf{e}^\mathbf{T}_{\mathbf{n}-\omega}\mathbf{A_{21}A_{11}^{-1}A_{12}} = \omega\mathbf{e}^\mathbf{T}_{\mathbf{n}-\omega} - \mathbf{e}^\mathbf{T}_\omega\mathbf{A_{12}} - (\omega(\mathbf{u}^\mathbf{1}_\omega)^\mathbf{T} - \mathbf{e}^\mathbf{T}_\omega)\mathbf{A_{12}} = \omega\mathbf{e}^\mathbf{T}_{\mathbf{n}-\omega}\ ,$$

because the first row of $\mathbf{A_{12}}$ is zero. $\mathbf{H_1e}_{\mathbf{n}-\omega} = \omega\mathbf{e}_{\mathbf{n}-\omega}$ is verified likewise. From $\mathbf{A_1B_1^T} = \mathbf{E_n} - \mathbf{I_n}$ we find that $\mathbf{A_{11}B_{12}} + \mathbf{A_{12}B_{22}^T} = \mathbf{E}^\omega_{\mathbf{n}-\omega}$ and thus $\mathbf{B_{12}} + \mathbf{A_{11}^{-1}A_{12}B_{22}^T} = \mathbf{u}^\mathbf{1}_\omega\mathbf{e}^\mathbf{T}_{\mathbf{n}-\omega}$, where $\mathbf{E}^\omega_{\mathbf{n}-\omega} = \mathbf{e}_\omega\mathbf{e}^\mathbf{T}_{\mathbf{n}-\omega}$. Moreover, $\mathbf{A_{21}B_{12}} + \mathbf{A_{22}B_{22}^T} = \mathbf{E}_{\mathbf{n}-\omega} - \mathbf{I}_{\mathbf{n}-\omega}$. Consequently,

$$\mathbf{H_1B_{22}^T} = \mathbf{A_{22}B_{22}^T} - \mathbf{A_{21}A_{11}^{-1}A_{12}B_{22}^T} = \mathbf{E}_{\mathbf{n}-\omega} - \mathbf{I}_{\mathbf{n}-\omega} - \mathbf{A_{21}}(\mathbf{B_{21}} + \mathbf{A_{11}^{-1}A_{12}B_{22}^T}) = \mathbf{E}_{\mathbf{n}-\omega} - \mathbf{I}_{\mathbf{n}-\omega}\ ,$$

because the first column of $\mathbf{A_{21}}$ is zero. Thus

$$\mathbf{H_1} = \mathbf{E}_{\mathbf{n}-\omega}(\mathbf{B_{22}^T})^{-\mathbf{1}} - (\mathbf{B_{22}^T})^{-\mathbf{1}} = (\mathbf{B_{22}^T})^{-\mathbf{1}}(\mathbf{E}_{\mathbf{n}-\omega} - \mathbf{I}_{\mathbf{n}-\omega})$$

shows that $\mathbf{B_{22}^T H_1} = \mathbf{H_1 B_{22}^T} = \mathbf{E_{n-\omega}} - \mathbf{I_{n-\omega}}$ as asserted. Consequently, $\det \mathbf{H_1} \det \mathbf{B_{22}} = \mathbf{n} - \omega - \mathbf{1}$ and thus from (ii) $\det \mathbf{H_1} \det \mathbf{A_{11}} = \omega$, since $n - \omega - 1 = (\alpha - 1)\omega$. The last assertion follows from Proposition 3(ii) and the fact that $\dim P^\rho = n - \omega$. ∎

Proposition 4 shows that an ω-projection P^ρ of $P(\mathbf{A})$ cannot possibly be almost integral if $\mathbf{A_{11}}$ is singular, because almost integral polytopes are in particular full-dimensional. But if $\det \mathbf{A_{11}} \neq \mathbf{0}$ then P^ρ has several properties that we know from $P(\mathbf{A})$, see Theorem 1(ii). Recall that a full-dimensional polytope $P \subseteq \mathbb{R}^n_+$ is an *independence system* (or down-monotone in $\mathbb{R}^n_+$) if $\mathbf{x} \in \mathbf{P}$ and $\mathbf{0} \leq \mathbf{x}' \leq \mathbf{x}$ imply that $\mathbf{x}' \in \mathbf{P}$. It is easy to prove that all nontrivial facet defining inequalities $\mathbf{fx} \leq \mathbf{f_0}$ (i.e., those with $f_0 \neq 0$) of such polytopes satisfy $\mathbf{f} \geq \mathbf{0}$ and $f_0 > 0$, while all other (trivial) facets are given by the nonnegativity constraints.

Theorem 3. *P^ρ is almost integral if and only if* $\det \mathbf{A_{11}} = \mathbf{1}$ *and P^ρ is an independence system.*

Proof. If P^ρ is almost integral, then there exists an almost perfect matrix $\widetilde{\mathbf{A}}$ of size $\widetilde{m} \times (n-\omega)$ such that $P^\rho = P(\widetilde{\mathbf{A}})$. Consequently, P^ρ is a full-dimensional independence system and thus by Proposition 4(i) $\det \mathbf{A_{11}} \neq \mathbf{0}$. By Theorem 1(ii) there exists a $(n-\omega) \times (n-\omega)$ submatrix $\widetilde{\mathbf{A}}_1$ of $\widetilde{\mathbf{A}}$ such that $\widetilde{\mathbf{A}}_1 \mathbf{B_{22}} = \mathbf{E_{n-\omega}} - \mathbf{I_{n-\omega}}$ with $\widetilde{\mathbf{A}}_1 \mathbf{e_{n-\omega}} = \omega' \mathbf{e_{n-\omega}}$, say, satisfying $\omega'(\alpha - 1) = n - \omega - 1$. Thus $\omega' = \omega$, $\det \widetilde{\mathbf{A}}_1 = \omega$ and from $\det \widetilde{\mathbf{A}}_1 \det \mathbf{B_{22}} = \mathbf{n} - \omega - \mathbf{1}$ we have $\det \mathbf{B_{22}} = \alpha - \mathbf{1}$. Consequently by Proposition 4(ii) $\det \mathbf{A_{11}} = \mathbf{1}$. Suppose on the other hand that $\det \mathbf{A_{11}} = \mathbf{1}$ and that P^ρ is an independence system. Thus $\dim P^\rho = n - \omega$ and $\mathbf{u^j} \in P^\rho$ for all unit vectors $j = 1, \ldots, n - \omega$. Let $\mathbf{hz} \leq \mathbf{1}$ be any nontrivial facet defining inequality of P^ρ in the linear description of P^ρ given in Proposition 4(iii). It follows that $\mathbf{0} \leq \mathbf{h} \leq \mathbf{e^T_{n-\omega}}$ and since $\det \mathbf{A_{11}} = \mathbf{1}$ the vector $\mathbf{h}$ is integer and thus zero-one. Consequently, P^ρ is defined by a zero-one matrix which by Proposition 3 is almost perfect. ∎

From Theorem 2 it thus follows that the strong perfect graph conjecture is correct if for some ω-projection of an almost integral polytope $P(\mathbf{A})$ with $2 \leq \omega \leq (n-1)/2$ the corresponding matrix $\mathbf{A_{11}}$ has a determinant of 1 and the corresponding P^ρ forms an independence system.

It is interesting to note (and not difficult to prove) that the case distinction made by Tucker (1977) in the proof of his theorem is along the two cases where $\mathbf{A_{11}}$ is singular for some $v \in V$ and some ω-clique K containing v or not.

4. A Different Reformulation of the SPGC

Here we look at the orthoprojection of a facet of an almost integral polytope $P(\mathbf{A})$ satisfying $2 \leq \omega \leq (n-1)/2$. Let $K \subset V$ be any ω-clique in $G_\mathbf{A}$ and define

$$\mathcal{P}^{\#} = P(\mathbf{A}) \cap \{\mathbf{x} \in \mathbb{R}^\mathbf{n} : \sum_{\mathbf{u} \in \mathbf{K}} \mathbf{x_u} = \mathbf{1}\} \ , \ \mathcal{P}^{\#}_{\mathcal{I}} = conv(\mathcal{P}^{\#} \cap \mathbb{Z}^\mathbf{n}) \ . \tag{22}$$

Denote by π_κ the orthoprojection from $\mathbb{R}^n$ into $\mathbb{R}^{n-\omega}$ that projects out all variables x_u with $u \in K$. There are exactly n different choices for K and we denote by Φ the index set (in an arbitrary order) of all such *κ-projections.*

For $\kappa \in \Phi$ we denote by $\mathcal{P}^\kappa$ the orthogonal projection of $\mathcal{P}^{\#}$, i.e.,

$$\mathcal{P}^\kappa = \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \exists \mathbf{x} \in \mathcal{P}^{\#} \text{ such that } \mathbf{z} = \pi_\kappa \mathbf{x}\} \ , \ \mathcal{P}^\kappa_{\mathcal{I}} = conv(\mathcal{P}^\kappa \cap \mathbb{Z}^\mathbf{n}) \ , \tag{23}$$

and call $\mathcal{P}^\kappa$ the κ-projection of $P(\mathbf{A})$ for short.

The polytopes $\mathcal{P}^\kappa$ are contained in the unit cube of $\mathbb{R}^{n-\omega}$ and thus every 0-1 point in $\mathcal{P}^\kappa$ is an extreme point of $\mathcal{P}^\kappa$. ω-projections and κ-projections are related to one another as follows. Since the κ-projections are in one-to-one correspondence with the ω-cliques in $G_{\mathbf{A}}$, we denote

$$\Omega_\kappa = \{\rho \in \Omega : x_u \text{ with } u \in K^\kappa \text{ are projected out}\} \ , \tag{24}$$

where K^κ is the clique in $G_{\mathbf{A}}$ given by row κ of $\mathbf{A_1}$. For every choice of $K = K^\kappa$ in (22) there are exactly ω possible ways of choosing a special variable x_v with $v \in K$ in the ω-projection of $P(\mathbf{A})$ and the corresponding polytopes $P^=$ are all contained in $\mathcal{P}^\#$. Thus

$$P^\rho \subseteq \mathcal{P}^\kappa \text{ for all } \rho \in \Omega_\kappa \text{ and } \kappa \in \Phi \ . \tag{25}$$

Like in the case of ω-projections, a straightforward calculation (using Fourier-Motzkin elimination) shows that $\mathcal{P}^\kappa$ is the polytope corresponding to the chordless odd cycle on $n-2$ nodes for all $\kappa \in \Phi$ if $\mathbf{A}$ is the clique matrix of a chordless odd cycle on n nodes. In Appendix A we show that the κ-projection of the polytope $P(\mathbf{A})$ is for all $\kappa \in \Phi$ the almost integral polytope (11), when $\mathbf{A}$ is the clique-matrix of the complement of a chordless odd cycle on n nodes. We are thus led to another *almost integral polytope conjecture* ($AIPC^\#$) which differs from (AIPC) because a different orthoprojection is used.

Conjecture C. *The κ-projection $\mathcal{P}^\kappa$ of an almost integral $P(\mathbf{A}) \subseteq \mathbb{R}^n$ with $2 \le \omega \le (n-1)/2$ is almost integral for some $\kappa \in \Phi$.*

The proof of Theorem 2 applies unchanged to conjecture ($AIPC^\#$) and thus we have:

Theorem 4. *The SPGC is true if and only if the $AIPC^\#$ is true.*

Conjecture B has its equivalent for κ-projections which is equivalent to SPGC as well.

Conjecture D. *The κ-projection $\mathcal{P}^\kappa$ of an almost integral $P(\mathbf{A}) \subseteq \mathbb{R}^n$ with $2 \le \omega \le (n-1)/2$ is some almost integral polytope $\mathcal{P}_\# \subseteq \mathbb{R}^{\setminus-\omega}$ for all $\kappa \in \Phi$.*

In case the SPGC is true, it follows that the polytope $P_=$ of Conjecture B and the polytope $\mathcal{P}_\#$ of Conjecture D coincide –which evidently gives rise to a yet another conjecture, since in general $P_=$ and $\mathcal{P}_\#$ must be expected to be different.

Like in the case of ω-projections, the almost integrality of the κ-projection of some polytope $P(\mathbf{A})$ does not imply the almost integrality of the "mother" polytope $P(\mathbf{A})$. An example to this effect is the graph given at the end of Section 2, where the κ-projection using the clique $K = \{2,3,4\}$ also yields the almost integral polytope associated with $\overline{C}_7$, while $P(\mathbf{A})$ in this case is known not to be almost integral.

5. Some Properties of κ-Projections of $P(\mathbf{A})$

We make the same assumptions as in Section 3, in particular as regards the arrangement of the rows and columns of $\mathbf{A_1}$ and $\mathbf{B_1}$. We use the same notation as done there with the necessary changes and choose without restriction of generality the ω-clique $K = \{1, \dots, \omega\}$ as the set of variables to be projected out.

We start by listing some properties of the facet $\mathcal{P}^\#$ of $P(\mathbf{A})$.

Proposition 5. *(i)* $\mathcal{P}^\#$ *has exactly one fractional extreme point* $\mathbf{x^0} = \frac{\mathbf{1}}{\omega}\mathbf{e_n}$.
(ii) The extreme point $\mathbf{x^0}$ *of* $\mathcal{P}^\#$ *has precisely* $n-1$ *linearly independent adjacent extreme points given by the rows* $\mathbf{b^2}, \ldots, \mathbf{b^n}$ *of* $\mathbf{B_1}$.
(iii) $\dim \mathcal{P}^\# = \dim \mathcal{P}^\#_{\mathcal{I}} = n-1$.
(iv) $\sum_{j=\omega+1}^{n} x_j \le \alpha - 1$ *defines a facet of* $\mathcal{P}^\#_{\mathcal{I}}$ *and*

$$\mathcal{P}^\#_{\mathcal{I}} = \mathcal{P}^\# \cap \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}} : \sum_{\mathbf{j}=\omega+\mathbf{1}}^{\mathbf{n}} \mathbf{x_j} \le \alpha - \mathbf{1}\} \ . \tag{26}$$

(v) $\mathcal{P}^\#_{|}$ *is integral for* $j = 1, \ldots, n$.

The proof of Proposition 5 goes like the proof of Proposition 1 and is omitted. Let now

$$\mathcal{P}^\kappa = \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \exists \mathbf{x} \in \mathcal{P}^\# \text{ such that } \mathbf{z} = \boldsymbol{\pi}\mathbf{x}\} \ , \tag{27}$$

where $\mathbf{z} = \boldsymbol{\pi}\mathbf{x}$ is in matrix form $\mathbf{z} = (\mathbf{O} \ \mathbf{I_{n-\omega}})\mathbf{x}$ the orthoprojection from $\mathbb{R}^n$ to $\mathbb{R}^{n-\omega}$ that we consider. $\mathcal{P}^\kappa$ is a polytope that lies in the unit cube of $\mathbb{R}^{n-\omega}$ and thus every 0-1 point in $\mathcal{P}^\kappa$ is an extreme point of $\mathcal{P}^\kappa$.

Denote by $\mathbf{B^\#_{12}}$ the $(\omega-1) \times (n-\omega)$ matrix that is obtained by deleting the first row of $\mathbf{B_{12}}$ and by $\mathbf{B^\#_1}$ the $(n-1) \times (n-1)$ matrix that is obtained by deleting the first row and column of $\mathbf{B_1}$.

Proposition 6. *(i)* $\mathbf{z^0} = (\mathbf{1}/\omega)\mathbf{e_{n-\omega}}$ *is an extreme point of* $\mathcal{P}^\kappa$. *All other extreme points* $\mathbf{z}$ *of* $\mathcal{P}^\kappa$ *are integral and satisfy* $\mathbf{e^T_{n-\omega}}\mathbf{z} \le \alpha - \mathbf{1}$.
(ii) Every extreme point $\mathbf{z}$ *of* $\mathcal{P}^\kappa$ *that is adjacent to* $\mathbf{z^0}$ *satisfies* $\mathbf{e^T_{n-\omega}}\mathbf{z} = \alpha - \mathbf{1}$ *and there are at most* $n-1$ *such extreme points which are given by the rows of* $\mathbf{B^\#_{12}}$ *and* $\mathbf{B_{22}}$.
(iii) $\mathbf{e^T_{n-\omega}}\mathbf{z} \le \alpha - \mathbf{1}$ *defines a facet of* $\mathcal{P}^\kappa_{\mathcal{I}}$ *and*

$$\mathcal{P}^\kappa_{\mathcal{I}} = \mathcal{P}^\kappa \cap \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \mathbf{e^T_{n-\omega}}\mathbf{z} \le \alpha - \mathbf{1}\} \ . \tag{28}$$

(iv) $\mathcal{P}^\kappa_{|}$ *is integral for* $j = 1, \ldots, n-\omega$.
(v) $\det \mathbf{B^\#_1} = \alpha - \mathbf{1}$, $\dim \mathcal{P}^\kappa = \dim \mathcal{P}^\kappa_{\mathcal{I}} = n - \omega$ *and* $\mathcal{P}^\kappa$ *and* $\mathcal{P}^\kappa_{\mathcal{I}}$ *are independence systems.*

Proof. (i) The proof is *mutatis mutandis* the proof of Proposition 3(*i*).
(ii) The proof goes exactly like the one of Proposition 3(*ii*) by observing that the points $\mathbf{b^2}, \ldots, \mathbf{b^n}$ must be used in lieu of the points $\mathbf{b^{\omega+1}}, \ldots, \mathbf{b^n}$.
(iii) The proof goes like the proof of Proposition 3(*iii*).
(iv) The proof goes like the proof of Proposition 3(*iv*).
(v) $\mathbf{B^\#_1}$ is the matrix obtained by deleting the first row and column of $\mathbf{B_1}$. From (6) we have $\mathbf{B_1^{-1}} = \frac{1}{\alpha}\mathbf{E_n} - \mathbf{A^T_1}$ and thus from Cramer's rule, applied to the element with index $\{1,1\}$ of $\mathbf{B_1^{-1}}$, and $\det \mathbf{B_1} = \alpha$ we get $\det \mathbf{B^\#_1} = \alpha - \mathbf{1}$. From the nonsingularity of $\mathbf{B^\#_1}$ it follows that

$$r(\begin{pmatrix} \mathbf{B^\#_{12}} \\ \mathbf{B_{22}} \end{pmatrix}) = n - \omega \ . \tag{29}$$

Since $\mathbf{u^1_n} \in \mathcal{P}^\#$ we have $\mathbf{0}_{n-\omega} \in \mathcal{P}^\kappa$ and thus by (*ii*) $\dim \mathcal{P}^\kappa = \dim \mathcal{P}^\kappa_{\mathcal{I}} = n - \omega$, because $\mathcal{P}^\kappa$ and $\mathcal{P}^\kappa_{\mathcal{I}}$ contain $n-\omega+1$ affinely independent points. Let $\mathbf{z} \in \mathcal{P}^\kappa$. So there exists $\mathbf{x} \in \mathcal{P}^\#$ such that $\mathbf{z} = \boldsymbol{\pi}\mathbf{x}$. Let $\mathbf{x} = (\mathbf{x^1}, \mathbf{x^2})$ where $\mathbf{x^1} \in \mathbb{R}^\omega$ and $\mathbf{x^2} \in \mathbb{R}^{\mathbf{n}-\omega}$. Since $\mathbf{a^1}\mathbf{x} = \mathbf{e^T_\omega}\mathbf{x^1} = \mathbf{1}$ it follows that $(\mathbf{x^1}, \mathbf{z'}) \in \mathcal{P}^\#$ for all $0 \leq \mathbf{z'} \leq \mathbf{z} = \mathbf{x^2}$ because $\mathbf{A}$ is a nonnegative matrix and thus $\mathbf{z'} \in \mathcal{P}^\kappa$. If $\mathbf{z} \in \mathcal{P}^\kappa_\mathcal{I}$ and $0 \leq \mathbf{z'} \leq \mathbf{z}$ then by (28) $\mathbf{z'} \in \mathcal{P}^\kappa_\mathcal{I}$, because $\mathbf{z'} \in \mathcal{P}^\kappa$ and $\mathbf{e^T_{n-\omega}}\mathbf{z'} \leq \mathbf{e^T_{n-\omega}}\mathbf{z} \leq \alpha - \mathbf{1}$ from the nonnegativity of $\mathbf{z'}$. ∎

Different from the ω-projection, in the case of κ-projection of an almost integral $P(\mathbf{A}) \subseteq \mathbb{R}^n$ we thus have always a (full-dimensional) independence system, but no explicit linear description like in Proposition 4(*iii*).

Theorem 5. *$\mathcal{P}^\kappa$ is almost integral if and only if $\mathcal{P}^\kappa = P^\rho$ for some $\rho \in \Omega_\kappa$.*

Proof. If $\mathcal{P}^\kappa$ is almost integral, then by Theorem 4 the SPGC is correct, $\mathbf{A}$ is the clique-matrix of a chordless odd cycle or its complement and thus $\mathcal{P}^\kappa = P^\rho$ for all $\rho \in \Omega_\kappa$ and $\kappa \in \Phi$. So suppose that $\mathcal{P}^\kappa = P^\rho$ for some $\rho \in \Omega_\kappa$. By Proposition 6(v) thus $\dim P^\rho = n - \omega$ and P^ρ is an independence system. By Proposition 4(*ii*) we thus have $\det \mathbf{A_{11}} \neq \mathbf{0}$ and by Proposition 4(*iii*) P^ρ has exactly $n - \omega$ linearly independent 0-1 extreme points $\mathbf{z}$ with $\mathbf{e^T_{n-\omega}}\mathbf{z} = \alpha - \mathbf{1}$ given by the rows of $\mathbf{B_{22}}$. Consequently by Proposition 6(*ii*) for every row of $\mathbf{B^\#_{12}}$ there is some identical row of $\mathbf{B_{22}}$, because otherwise P^ρ has *more* than $n - \omega$ distinct 0-1 extreme points satisfying $\mathbf{e^T_{n-\omega}}\mathbf{z} = \alpha - \mathbf{1}$. From the factorization

$$\mathbf{B^\#_1} = \begin{pmatrix} \mathbf{I_{\omega-1}} & \mathbf{B^\#_{12}} \\ (\mathbf{B^\#_{12}})^\mathbf{T} & \mathbf{B_{22}} \end{pmatrix} = \begin{pmatrix} \mathbf{I_{\omega-1}} & \mathbf{B^\#_{12}} \\ \mathbf{O} & \mathbf{B_{22}} \end{pmatrix} \begin{pmatrix} \mathbf{I_{\omega-1}} - \mathbf{B^\#_{12}}\mathbf{B^{-1}_{22}}(\mathbf{B^\#_{12}})^\mathbf{T} & \mathbf{O} \\ \mathbf{B^{-1}_{22}}(\mathbf{B^\#_{12}})^\mathbf{T} & \mathbf{I_{n-\omega}} \end{pmatrix} \tag{30}$$

we get the determinantal identity in absolute values

$$\det \mathbf{B^\#_1} = \det \mathbf{B_{22}}\ \det(\mathbf{I_{\omega-1}} - \mathbf{B^\#_{12}}\mathbf{B^{-1}_{22}}(\mathbf{B^\#_{12}})^\mathbf{T})\ .$$

But $\mathbf{B^{-1}_{22}}(\mathbf{B^\#_{12}})^\mathbf{T}$ is a $(n-\omega) \times (\omega - 1)$ matrix of unit vectors in $\mathbb{R}^{n-\omega}$ (because the rows of $\mathbf{B^\#_{12}}$ are duplicates of some rows of $\mathbf{B_{22}}$) and thus the second determinant on the right is some integer number $\delta \geq 0$ in absolute value. By Proposition 4(*ii*) and Proposition 6(v) we thus get $\alpha - 1 = (\alpha - 1)\det \mathbf{A_{11}}\delta$ and hence $\det \mathbf{A_{11}} = \delta = \mathbf{1}$ since $\alpha \geq 2$ and $\det \mathbf{A_{11}}$ is some integer number as well. Consequently, by Theorem 3, P^ρ is almost integral and so is $\mathcal{P}^\kappa$. ∎

Theorem 5 shows that the strong perfect graph conjecture is wrong if for almost integral $P(\mathbf{A}) \subseteq \mathbb{R}^n$ with $2 \leq \omega \leq (n-1)/2$ the ω-projection and the κ-projection differ for some $\omega \in \Omega_\kappa$ and $\kappa \in \Phi$.

Denote by $\mathbf{B_3} = (\mathbf{B_{31}}\ \mathbf{B_{32}})$ the matrix formed by the $r - n \geq 0$ nonmaximal rows of $\mathbf{B}$ and by $\mathbf{B^\#_3} = (\mathbf{B^\#_{31}}\ \mathbf{B^\#_{32}})$ the subset of all $s \geq 0$ rows of $\mathbf{B_3}$ that correspond to the 0-1 extreme points of $P(\mathbf{A})$ that belong to $\mathcal{P}^\#$ where $\mathbf{B_3}$ and $\mathbf{B^\#_3}$ are partitioned like the rest of $\mathbf{B}$.

By (14) $\mathbf{B^\#_{12}}\mathbf{e_{n-\omega}} = (\alpha - \mathbf{1})\mathbf{e_{\omega-1}}$, $\mathbf{B_{22}}$ has row and column sums equal to $\alpha - 1$ and $\mathbf{B^\#_{32}}\mathbf{e_{n-\omega}} \leq (\alpha - \mathbf{2})\mathbf{e_s}$. Let $\mathbf{B}^\kappa$ be the $q \times (n - \omega)$ matrix consisting of $\mathbf{B^\#_{12}}$, $\mathbf{B_{22}}$ and $\mathbf{B^\#_{32}}$ where $q = n - 1 + s$ denotes the number of rows of $\mathbf{B}^\kappa$. The polytope

$$Q^\kappa = \{\mathbf{y} \in \mathbb{R}^{\mathbf{n}-\omega}\ :\ \mathbf{B}^\kappa\mathbf{y} \leq \mathbf{e_q}\ ,\ \mathbf{y} \geq \mathbf{0}\} \tag{31}$$

is by definition the *antiblocker* of $\mathcal{P}^\kappa_\mathcal{I}$, since the rows of $\mathbf{B}^\kappa$ is a list of all (nonzero) extreme points of $\mathcal{P}^\kappa_\mathcal{I}$. Because $\mathcal{P}^\kappa_\mathcal{I}$ is an independence system, Q^κ gives a (nonminimal) extremal characterization of the nontrivial facets of $\mathcal{P}^\kappa_\mathcal{I}$, see Padberg (1995, Chapter 10.3.1). We define

$$Q^\kappa_I = conv(Q^\kappa \cap \mathbb{Z}^{n-\omega})\ , \quad \widetilde{Q^\kappa} = Q^\kappa \cap \{\ \mathbf{y} \in \mathbb{R}^{\mathbf{n}-\omega}\ :\ \mathbf{e^T_{n-\omega}}\mathbf{y} \leq \lfloor(\mathbf{n} - \omega)/(\alpha - \mathbf{1})\rfloor\ \}\ , \tag{32}$$

where $\lfloor\delta\rfloor$ is the largest integer less-than-or-equal-to δ.

Proposition 7. *(i) If* $\mathbf{y} \in Q^\kappa$*, then* $\mathbf{y^T z} \leq \mathbf{1}$ *for all* $\mathbf{z} \in \mathcal{P}^\kappa_{\mathcal{I}}$*, i.e.,* $\mathbf{y^T z} \leq \mathbf{1}$ *is a* valid inequality *for* $\mathcal{P}^\kappa_{\mathcal{I}}$*.*
(ii) If $\mathbf{y^T z} \leq \mathbf{1}$ *defines a nontrivial facet of* $\mathcal{P}^\kappa_{\mathcal{I}}$*, then* $\mathbf{y}$ *is an extreme point of* Q^κ*.*
(iii) $Q^\kappa_I \subseteq \widetilde{\mathcal{Q}^\kappa} \subseteq Q^\kappa$ *,* $\dim Q^\kappa_I = n - \omega$ *and* $\mathbf{y^0} = \frac{\mathbf{1}}{\alpha - \mathbf{1}} \mathbf{e}_{\mathbf{n}-\omega}$ *is an extreme point of* Q^κ*.*
(iv) If $\alpha = 2$*, then* $Q^\kappa_I = \widetilde{\mathcal{Q}^\kappa} = Q^\kappa$ *is the unit cube in* $\mathbb{R}^{n-\omega}$*. If* $\alpha > 2$*, then* $\widetilde{\mathcal{Q}^\kappa}$ *is the antiblocker of* $\mathcal{P}^\kappa$*.*
(v) If $\mathbf{y} \in \widetilde{\mathcal{Q}^\kappa}$*, then* $\mathbf{y^T z} \leq \mathbf{1}$ *for all* $\mathbf{z} \in \mathcal{P}^\kappa$ *and if* $\mathbf{y^T z} \leq \mathbf{1}$ *defines a nontrivial facet of* $\mathcal{P}^\kappa$*, then* $\mathbf{y}$ *is an extreme point of* $\widetilde{\mathcal{Q}^\kappa}$*.*
(vi) Let $\widetilde{\mathcal{Q}^\kappa_{\mathcal{I}}} = \mathrm{conv}(\widetilde{\mathcal{Q}^\kappa} \cap \mathbb{Z}^{n-\omega})$*. Then* $Q^\kappa_I = \widetilde{\mathcal{Q}^\kappa_{\mathcal{I}}}$*.*

Proof. (i) Since $\mathbf{B}^\kappa$ is a list of all extreme points of $\mathcal{P}^\kappa_{\mathcal{I}}$ and $\mathbf{B}^\kappa \mathbf{y} \leq \mathbf{e_q}$ by assumption, it follows that $\mathbf{z^T y} \leq \mathbf{1}$ for all $\mathbf{z} \in \mathcal{P}^\kappa_{\mathcal{I}}$ because $\mathbf{z^T} = \boldsymbol{\mu} \mathbf{B}^\kappa$ for some $\boldsymbol{\mu} \geq \mathbf{0}$ with $\sum_{i=1}^q \mu_i = 1$, i.e., because $\mathbf{z}$ is a convex combination of the extreme points of $\mathcal{P}^\kappa_{\mathcal{I}}$.
(ii) If $\mathbf{y^T z} \leq \mathbf{1}$ defines a facet of $\mathcal{P}^\kappa_{\mathcal{I}}$, then there exist $n - \omega$ linearly independent 0-1 points in $\mathcal{P}^\kappa_{\mathcal{I}}$ satisfying the inequality at equality. Hence $\mathbf{y}$ is an extreme point of Q^κ; see (31).
(iii) By definition, $\widetilde{\mathcal{Q}^\kappa} \subseteq Q^\kappa$. Since $\mathbf{B_{22} y} \leq \mathbf{e}_{\mathbf{n}-\omega}$ for all $\mathbf{y} \in Q^\kappa_I$, we have

$$\mathbf{e}^{\mathbf{T}}_{\mathbf{n}-\omega} \mathbf{B_{22} y} = (\alpha - \mathbf{1}) \mathbf{e}^{\mathbf{T}}_{\mathbf{n}-\omega} \mathbf{y} \leq \mathbf{n} - \omega \ ,$$

hence $\mathbf{e}^{\mathbf{T}}_{\mathbf{n}-\omega} \mathbf{y} \leq \lfloor (\mathbf{n} - \omega)/(\alpha - \mathbf{1}) \rfloor$ for all $\mathbf{y} \in Q^\kappa_I$ and thus $Q^\kappa_I \subseteq \widetilde{\mathcal{Q}^\kappa}$. The origin and the $n - \omega$ unit vectors of $\mathbb{R}^{n-\omega}$ are in Q^κ_I and thus $\dim Q^\kappa_I = n - \omega$. Since every row of $\mathbf{B}^\kappa$ is zero-one and has at most $\alpha - 1$ entries equal to 1, $\mathbf{y^0} \in Q^\kappa$ and hence by (29) $\mathbf{y^0}$ is an extreme point of Q^κ.
(iv) If $\alpha = 2$, then the assertion follows from (29) and (31) by observing that the last inequality defining $\widetilde{\mathcal{Q}^\kappa}$ is redundant. If $\alpha > 2$, then $\lfloor (n - \omega)/(\alpha - 1) \rfloor = \omega$. By Proposition 6(*i*) the nontrivial inequalities defining $\widetilde{\mathcal{Q}^\kappa}$ correspond to the list of all extreme points of $\mathcal{P}^\kappa$ and thus $\widetilde{\mathcal{Q}^\kappa}$ is the antiblocker of $\mathcal{P}^\kappa$.
(v) The proof of this part is similar to that of parts (*i*) and (*ii*).
(vi) By part (*iii*) it follows that $Q^\kappa_I \subseteq \widetilde{\mathcal{Q}^\kappa_{\mathcal{I}}}$. Let $\mathbf{y} \in \widetilde{\mathcal{Q}^\kappa_{\mathcal{I}}}$ be integer. Then $\mathbf{y} \in Q^\kappa \cap \mathbb{Z}^{\mathbf{n}-\omega}$ and thus $\mathbf{y} \in Q^\kappa_I$, i.e., $Q^\kappa_I = \widetilde{\mathcal{Q}^\kappa_{\mathcal{I}}}$. ∎

Theorem 6. $\mathcal{P}^\kappa$ *is almost integral if and only if* $Q^\kappa_I = \widetilde{\mathcal{Q}^\kappa}$*.*

Proof. If $\mathcal{P}^\kappa$ is almost integral, then by Theorem 4 the SPGC is correct and $Q^\kappa_I = \widetilde{\mathcal{Q}^\kappa}$ follows. Suppose $Q^\kappa_I = \widetilde{\mathcal{Q}^\kappa}$. Since every nontrivial facet $\mathbf{y^T z} \leq \mathbf{1}$ of $\mathcal{P}^\kappa$ defines an extreme point $\mathbf{y}$ of $\widetilde{\mathcal{Q}^\kappa}$ and all extreme points of Q^κ_I are 0-1 valued, it follows that there exists a zero-one matrix $\mathbf{A}^\kappa$ with $n - \omega$ columns and some finite number of rows such that $\mathcal{P}^\kappa = P(\mathbf{A}^\kappa)$ and which by Proposition 6 is almost perfect. ∎

If $\mathcal{P}^\kappa$ is not almost integral, then by the preceding there exists an extreme point $\mathbf{y} \in \widetilde{\mathcal{Q}^\kappa}$ with $0 < y_j < 1$ for some $j \in \{1, \ldots, n - \omega\}$ that defines a facet $\mathbf{y^T x} \leq \mathbf{1}$ of both $\mathcal{P}^\kappa$ and $\mathcal{P}^\kappa_{\mathcal{I}}$.

Since $\mathcal{P}^\kappa$ "inherits" all of its facets and $\mathcal{P}^\kappa_{\mathcal{I}}$ all but one of its facets from $P(\mathbf{A})$, there should be a facet-defining inequality $\mathbf{fx} \leq \mathbf{1}$ of $P(\mathbf{A})$ or of $P_I(\mathbf{A})$ with $\mathbf{f} \neq \frac{1}{\alpha} \mathbf{e_n}$, $\mathbf{f} \notin \{\mathbf{0}, \mathbf{1}\}^{\mathbf{n}}$, thus making $P(\mathbf{A})$ not almost integral. If this is not the case, then the SPGC does not hold.

The SPGC is, of course, still open to proof or disproof, but the geometric reformulations presented here may help to resolve this long standing conjecture one way or the other.

Appendix A. ω-Projection and κ-Projection of Circulants

To prove that the ω-projection of almost integral polytopes associated with the complements of odd cycles without chords are given by (11) we prove first a more general proposition. We denote by

$$\mathbf{C_n^\omega} = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 1 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

the circulant matrix where ω consecutive ones are shifted cyclically over n components and $2 \le \omega \le (n-1)/2$. $\mathbf{C_n^2}$ for odd n is the clique-matrix of a chordless odd cycle. For odd n and $\omega = (n-1)/2$ $\mathbf{C_n^\omega}$ is the incidence matrix of the cliques of maximum cardinality of the complement of a chordless odd cycle. The clique-matrices of such graphs contain for $n \ge 9$ many additional cliques of cardinality $\omega - 1$ or less, but we will show below that these become redundant under ω-projections. It is well known that for $n \ge 9$ the polytopes $P(\mathbf{C_n^\omega})$ as defined by (1) are almost integral if and only if n is odd and $\omega = 2$. For $n \le 8$ the only exception from this statement is the polytope $P(\mathbf{C_7^3})$, which is almost integral.

Proposition 8. *For all $\rho \in \Omega$ and $2 \le \omega \le (n-1)/2$ the ω-projection P^ρ of $P(\mathbf{C_n^\omega})$ is*

$$P^\rho = \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \mathbf{C_{n-\omega}^\omega} \mathbf{z} \le \mathbf{e_{n-\omega}} \ , \ \mathbf{z} \ge \mathbf{0}\} \ .$$

Proof. We index the rows and columns of $\mathbf{C_n^\omega}$ by $1, 2, \dots, n$ and because of symmetry we can choose x_ω to be the special variable and the variables with nonzero coefficients in any one of the first ω rows of $\mathbf{C_n^\omega}$ as the variables to be projected out. So let $1 \le k \le \omega$ be the selected row. We denote by $\mathbf{E_q^p}$ and $\mathbf{O}_q^p$ the $p \times q$ matrices consisting of all ones and of all zeros, respectively. $\mathbf{I_p}$ is the $p \times p$ identity matrix, $\mathbf{e_p^T} = (\mathbf{1}, \dots, \mathbf{1})$ a vector of p ones, $\mathbf{u_p^i}$ the i-th unit vector of $\mathbb{R}^p$ and for notational simplicity $\mathbf{r_p^i} = (\mathbf{u_p^i})^\mathbf{T}$ where $1 \le i \le p$. Moreover, we define $p \times p$ matrices

$$\mathbf{L_p} = \begin{pmatrix} 0 & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ 1 & \dots & 1 & 0 \end{pmatrix} \ , \ \mathbf{W_p} = \begin{pmatrix} -1 & 1 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & \dots & 0 & -1 \end{pmatrix} \ ,$$

$\mathbf{U_p} = \mathbf{L_p^T}$, $\mathbf{V_p} = \mathbf{W_p^T}$, $\mathbf{S_p} = \mathbf{I_p} + \mathbf{L_p}$ and $\mathbf{T_p} = \mathbf{I_p} + \mathbf{U_p}$. The matrix $\mathbf{A_{11}}$ of Proposition 4 corresponds to the submatrix of $\mathbf{C_n^\omega}$ given by the columns $k, \dots, \omega + k - 1$ and the rows $1, \dots, \omega$ where $1 \le k \le \omega$ is arbitrary. In our notation we find that $\mathbf{A_{11}}$ and its inverse $\mathbf{A_{11}^{-1}}$ are given by

$$\mathbf{A_{11}} = \begin{pmatrix} \mathbf{E_q^p} & \mathbf{e_p} & \mathbf{L_p} \\ \mathbf{e_q^T} & 1 & \mathbf{e_p^T} \\ \mathbf{U_q} & \mathbf{e_q} & \mathbf{E_p^q} \end{pmatrix} \ , \ \mathbf{A_{11}^{-1}} = \begin{pmatrix} \mathbf{O}_p^q & \mathbf{u_q^1} & \mathbf{V_q} \\ \mathbf{r_p^1} & \delta & \mathbf{r_q^q} \\ \mathbf{W_p} & \mathbf{u_p^p} & \mathbf{O}_q^p \end{pmatrix} \ ,$$

where $\delta = 1 - \mathbf{e_q^T u_q^1} - \mathbf{e_p^T u_p^p}$ and we have set $p = k-1$ and $q = \omega - k$ for notational convenience. If $p = 0$ or $q = 0$ then the corresponding matrices and vectors are empty. We leave the verification that $\mathbf{A_{11} A_{11}^{-1}} = \mathbf{I}_\omega$ to the reader. To form the matrix $\mathbf{A_{12}}$ of Proposition 4 we list the remaining

$$\mathbf{C_n^\omega} = \left(\begin{array}{c|ccc|cc} \mathbf{T_p} & \mathbf{E_q^p} & \mathbf{e_p} & \mathbf{L_p} & \mathbf{O}_q^p & \mathbf{O} \\ \mathbf{0}_p & \mathbf{e_q^T} & 1 & \mathbf{e_p^T} & \mathbf{0}_q & \mathbf{0} \\ \mathbf{O}_p^q & \mathbf{U_q} & \mathbf{e_q} & \mathbf{E_p^q} & \mathbf{S_q} & \mathbf{O} \\ \hline \mathbf{X_1} & \mathbf{O}_q^p & \mathbf{0}_p & \mathbf{T_p} & \mathbf{Y_1} & \mathbf{Z_1} \\ \mathbf{X_2} & \mathbf{O} & \mathbf{0} & \mathbf{O} & \mathbf{Y_2} & \mathbf{Z_2} \\ \mathbf{X_3} & \mathbf{S_q} & \mathbf{0}_q & \mathbf{O}_p^q & \mathbf{Y_3} & \mathbf{Z_3} \end{array}\right) \; , \; \mathbf{C_{n-\omega}^\omega} = \begin{pmatrix} \mathbf{X_1}+\mathbf{T_p} & \mathbf{Y_1} & \mathbf{Z_1} \\ \mathbf{X_2} & \mathbf{Y_2} & \mathbf{Z_2} \\ \mathbf{X_3} & \mathbf{S_q}+\mathbf{Y_3} & \mathbf{Z_3} \end{pmatrix}$$

Figure 4: The image of $\mathbf{C_n^\omega}$ under an ω-projection.

columns of $\mathbf{C_n^\omega}$ in the order $1,\dots,k-1,\omega+k,\dots,n$. Then we get in our notation

$$\mathbf{A_{12}} = \begin{pmatrix} \mathbf{T_p} & \mathbf{O}_q^p & \mathbf{O} \\ \mathbf{0}_p & \mathbf{0}_q & \mathbf{0} \\ \mathbf{O}_p^q & \mathbf{S_q} & \mathbf{O} \end{pmatrix} \; , \; \mathbf{A_{21}} = \begin{pmatrix} \mathbf{O}_q^p & \mathbf{0}_p & \mathbf{T_p} \\ \mathbf{O} & \mathbf{0} & \mathbf{O} \\ \mathbf{S_q} & \mathbf{0}_q & \mathbf{O}_p^q \end{pmatrix} \; ,$$

where $\mathbf{0}$ and $\mathbf{O}$ are zero vectors and matrices of the required size. The k-th row of $\mathbf{A_{12}}$ and the k-th column of $\mathbf{A_{21}}$, respectively, are vectors of zeros only. We compute

$$\mathbf{A_{11}^{-1}A_{12}} = \begin{pmatrix} \mathbf{O}_p^q & -\mathbf{I_q} & \mathbf{O} \\ \mathbf{e_p^T} & \mathbf{e_q^T} & \mathbf{0} \\ -\mathbf{I_p} & \mathbf{O}_q^p & \mathbf{O} \end{pmatrix} \; , \mathbf{A_{21}A_{11}^{-1}A_{12}} = \begin{pmatrix} -\mathbf{T_p} & \mathbf{O}_p^q & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O}_p^q & -\mathbf{S_q} & \mathbf{O} \end{pmatrix} \; .$$

We label the rows of $\mathbf{A_{22}}$ by $\omega+1,\dots,n$ as in the original matrix $\mathbf{C_n^\omega}$. Let $1 \le i \le k-1$. Then the first nonzero entry in column i of $\mathbf{A_{22}}$ occurs in row $n-\omega+1+i$, while the last nonzero entry in column i of $\mathbf{A_{21}A_{11}^{-1}A_{12}}$ occurs in row $\omega+i$. Since $n-\omega+1+i > \omega+i$ it follows that the two columns do not overlap. Let $k \le i \le \omega-1$ corresponding to the columns $\omega+k,\dots,2\omega-1$ of $\mathbf{A_{22}}$. Then the last nonzero entry in column i of $\mathbf{A_{22}}$ occurs in row $\omega+i$, while the first nonzero entry in $\mathbf{A_{21}A_{11}^{-1}A_{12}}$ occurs in row $n-\omega+1+i$. Thus the corresponding columns do not overlap either. But then it follows that

$$\mathbf{A_{22}} - \mathbf{A_{21}A_{11}^{-1}A_{12}} = \mathbf{C_{n-\omega}^\omega} \; ,$$

since the matrix on the left inherits the consecutive ones property from $\mathbf{C_n^\omega}$ and there are precisely ω ones per row, see also Figure 1 for an illustration. Consequently, by Proposition 4 the assertion follows since the matrix $\mathbf{A_3}$ is empty in our case and the constraints given by $\mathbf{A_{11}^{-1}A_{12}z} \le \mathbf{u}_\omega^\mathbf{k}$ are either nonnegativity conditions or dominated and thus redundant. ∎

It follows from the proposition that the ω-projections of almost integral polytopes associated with chordless odd cycles on n nodes are the corresponding polytopes for cycles on $n-2$ nodes. So let $\mathbf{A}$ be the clique-matrix of the complement of an odd cycle without chords and $P(\mathbf{A})$ be the associated polytope. Then without loss of generality $\mathbf{A_1} = \mathbf{C_n^\omega}$ with $\omega = (n-1)/2$ and thus by the proposition the matrix $\mathbf{H_1}$ of Proposition 4(iii) becomes $\mathbf{H_1} = \mathbf{E}_{\mathbf{n}-\omega} - \mathbf{I}_{\mathbf{n}-\omega}$ because $n-\omega \le n-(n-1)/2 = (n+1)/2$. The matrix $\mathbf{B_{22}}$ is the $(n-\omega)\times(n-\omega)$ identity matrix and thus by Proposition 3(ii) the ω-projection P^ρ of $P(\mathbf{A})$ contains the $n-\omega$ unit vectors of $\mathbb{R}^{n-\omega}$ as well as its origin. Since by Proposition 3(i) $\mathbf{z^0} = \frac{\mathbf{1}}{\mathbf{n}-\omega}\mathbf{e}_{\mathbf{n}-\omega}$ is the unique fractional extreme point of P^ρ it follows that all constraints $\mathbf{H_2z} \le \mathbf{h_2}$ and $\mathbf{H_3z} \le \mathbf{u}_\omega^\mathbf{1}$ of Proposition 4(iii) are redundant and thus (11) follows.

In the proof of the proposition we have made explicit use of the consecutiveness of the ones in the matrix $\mathbf{C_n^\omega}$. An open question is whether or not a similar result can be proven when general zero-one circulants (without the consecutive ones property) are considered. We do not know the answer, but conjecture that it is negative.

To carry out the κ-projection of $\mathcal{P}^{\#}$ for $\mathbf{A} = \mathbf{C}_{\mathbf{n}}^{\omega}$ we choose without restriction of generality the ω-clique $K = \{1, \ldots, \omega\}$ as the set of variables to be projected out. Here we will use Fourier-Motzkin elimination, see e.g. Ziegler (1995) for an excellent exposition of this method, which goes stepwise by projecting out a single variable at a time. Using this aspect of the method we can in our case detect redundant constraints for the next projection in the course of the computation and drop them from further consideration. We start by projecting out x_1. To do so we substitute $x_1 = 1 - \sum_{j=2}^{\omega} x_j$ everywhere. The nonnegativity condition $x_1 \geq 0$ gives rise to the constraint $\sum_{j=2}^{\omega} x_j \leq 1$ which is redundant because it is implied by $\sum_{j=2}^{\omega+1} x_j \leq 1$ and $x_{\omega+1} \geq 0$. We claim that after eliminating variable x_k for $1 \leq k \leq \omega$ the nonredundant linear inequalities for the current projected polytope are

$$
\begin{array}{llll}
x_{k+1} + \ldots + x_{\omega+k} & & & \leq 1 \\
\quad x_{k+2} + \ldots + x_{\omega+k+1} & & & \leq 1 \\
 & & & \vdots \\
 & & x_{n-\omega+1} + \ldots\ldots\ldots\ldots + x_n & \leq 1 \\
-x_{k+1} \ldots - x_{\omega} & & + x_{n-\omega+k+1} + \ldots + x_n & \leq 0 \\
 & & & \vdots \\
\quad -x_{\omega} & & + x_n & \leq 0 \\
 & x_{\omega+1} & + x_{n-\omega+2} + \ldots\ldots\ldots + x_n & \leq 1 \\
 & & & \vdots \\
 & x_{\omega+1} + \ldots + x_{\omega+k-1} & + x_{n-\omega+k} + \ldots + x_n & \leq 1
\end{array}
$$

and $x_i \geq 0$ for $i = k+1, \ldots, n$. The claim is correct for $k = 1$ with an empty set of "new" constraints. Suppose that it is correct for some $1 \leq k < \omega$. To project out variable x_{k+1} we write

$$
x_{k+1} \leq 1 - \sum_{j=k+2}^{\omega+k} x_j \ , \qquad x_{k+1} \geq - \sum_{j=k+2}^{\omega} x_j + \sum_{j=n-\omega+k+1}^{n} x_j \ , \qquad x_{k+1} \geq 0 \ .
$$

Combining the upper bound on x_{k+1} with $x_{k+1} \geq 0$ we get $\sum_{j=k+2}^{\omega+k} x_j \leq 1$ which is implied by $\sum_{j=k+2}^{\omega+k+1} x_j \leq 1$ and $x_{\omega+k+1} \geq 0$ and thus redundant. Combining the first two inequalities we find

$$
x_{\omega+1} + \ldots + x_{\omega+k} \qquad + x_{n-\omega+k+1} + \ldots + x_n \leq 1
$$

and thus the claim and the next proposition follow.

Proposition 9. *For all $\kappa \in \Phi$ and $2 \leq \omega \leq (n-1)/2$ the κ-projection $\mathcal{P}^{\kappa}$ of $P(\mathbf{C}_{\mathbf{n}}^{\omega})$ is*

$$
\mathcal{P}^{\kappa} = \{\mathbf{z} \in \mathbb{R}^{\mathbf{n}-\omega} : \mathbf{C}_{\mathbf{n}-\omega}^{\omega} \mathbf{z} \leq \mathbf{e}_{\mathbf{n}-\omega} \ , \ \mathbf{z} \geq \mathbf{0}\} \ .
$$

To prove that the κ-projection of $P(\mathbf{A})$ when $\mathbf{A}$ is the clique-matrix of $\overline{C}_n$ for odd n is given by (11) we note first that

$$
P(\mathbf{E_n} - \mathbf{I_n}) = conv\{\mathbf{0_n}, \mathbf{u_n^1}, \ldots, \mathbf{u_n^n}, \frac{\mathbf{1}}{\mathbf{n-1}} \mathbf{e_n}\} \ ,
$$

i.e., that the vectors on the right are a minimal pointwise generator of the polytope on the left for all $n \geq 2$. Let $\mathbf{A_1}$ be as usual denote the ω-cliques of $G_{\mathbf{A}}$ and note that $P(\mathbf{A}) \subseteq P(\mathbf{A_1})$. Since $\mathbf{A_1}$ is a circulant on $\omega = (n-1)/2$ ones it follows from Proposition 9 that the κ-projection of $P(\mathbf{A_1})$ is given by (11). Consequently, $\mathcal{P}^\kappa \subseteq P(\mathbf{E_{n-\omega}} - \mathbf{I_{n-\omega}})$. Since $n - \omega - 1 = \omega$ in our case, we have by Proposition 6 that the pointwise generator of $P(\mathbf{E_{n-\omega}} - \mathbf{I_{n-\omega}})$ is contained in the κ-projection $\mathcal{P}^\kappa$ of $P(\mathbf{A})$ and thus we have equality as claimed in Section 4.

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