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TESTING FOR LONG MEMORY IN VOLATILITY

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We consider the asymptotic behavior of log-periodogram regression estimators of the memory parameter in long-memory stochastic volatility models, under the null hypothesis of short memory in volatility. We show that in this situation, if the periodogram is computed from the log squared returns, then the estimator is asymptotically normal, with the same asymptotic mean and variance that would hold if the series were Gaussian. In particular, for the widely used GPH estimator $\hat{d}_{GPH}$ under the null hypothesis, the asymptotic mean of $m^{1/2}\hat{d}_{GPH}$ is zero and the asymptotic variance is $\pi^2/24$ where m is the number of Fourier frequencies used in the regression. This justifies an ordinary Wald test for long memory in volatility based on the log periodogram of the log squared returns.

1. INTRODUCTION

Many recent works have discussed the phenomenon of long memory in the volatility of financial and economic time series. Early empirical observations on persistence in volatility were given by Ding, Granger, and Engle (1993) and de Lima and Crato (1993). Two models that capture this phenomenon are the fractionally integrated GARCH (FIGARCH) model of Baillie, Bollerslev, and Mikkelsen (1996) and the long memory stochastic volatility (LMSV) model proposed independently by Breidt, Crato, and de Lima (1998) and Harvey (1998). Semiparametric estimation of the memory parameter in LMSV models is justified theoretically by Deo and Hurvich (2001), who consider the widely used log periodogram (GPH) estimator of Geweke and Porter-Hudak (1983), computed from the logarithms of the squared returns of the series. Deo and Hurvich (2001) establish the consistency and asymptotic normality of this estimator under conditions that require the assumption that long memory is in fact present in the volatility. Although this justifies the use of the estimator under certain circumstances, it does not justify the widespread practice of using the GPH estimator to construct a test for long memory in volatility. (Using this method on squared returns, Andersen and Bollerslev (1997a, 1997b) and Andersen, Bol-

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lerslev, Diebold, and Labys (2001) find evidence of long memory in volatility.) The difficulty is that to construct a test for long memory in volatility, it is necessary to know the asymptotic distribution of the test statistic under the null hypothesis of short memory in the volatility, a case that is not covered in the semiparametric theory of Deo and Hurvich (2001) or in the theory for the fully parametric case presented by Hosoya (1997). Here, we derive the asymptotic distribution of the GPH estimator based on log squared return data under an LMSV model in the short-memory case. This serves to justify the corresponding test for long memory in volatility. In practice, it is important to have such a test, as the long-range forecasts of volatility are crucially altered by the presence of long memory in volatility.

Giraitis, Kokoszka, and Leipus (1999) have constructed a test for long memory in volatility, but the model generating the stochastic volatility, developed in Giraitis, Robinson, and Surgailis (2000), is quite different from either the LMSV or FIGARCH frameworks, and the test does not yield a corresponding estimator of the memory parameter. Furthermore, Lobato and Robinson (1998) provide a test for long memory of a linear process, and this test is used by Lobato and Savin (1998) on squared stock returns to test for long memory in volatility. P.M. Robinson, in his discussion of the paper of Lobato and Savin (1998), conjectures that their test statistic, applied to squared returns, has the appropriate χ_1^2 limit distribution under the $I(0)$ null hypothesis, under suitable strong mixing conditions.

The LMSV model for returns $\{r_t\}$ takes the form $r_t = \eta \exp(Y_t/2) e_t$ where $\eta > 0$ is a scale parameter, $\{e_t\}$ are independent and identically distributed (i.i.d.) shocks with zero mean, and $\{Y_t\}$ is a stationary Gaussian process, independent of $\{e_t\}$, with spectral density $f_Y(x) \sim C x^{-2d}$ as $x \to 0^+$ $(C > 0)$ and memory parameter d such that $0 \le d < \frac{1}{2}$. Deo and Hurvich (2001) assume that

$$f_Y(x) = |2\sin(x/2)|^{-2d} g^*(x),$$

where $g^*(\cdot)$ is continuous on $[-\pi, \pi]$, bounded above and bounded away from zero. In this paper we focus on the case $d = 0$. Under the LMSV model, the logarithms of the squared returns, $X_t = \log(r_t^2)$, may be expressed as

$$X_t = \mu + Y_t + Z_t, \tag{1}$$

where $\mu = \log \eta^2 + E[\log e_t^2]$ and $\{Z_t\} = \{\log e_t^2 - E[\log e_t^2]\}$ is i.i.d. with mean zero and variance σ^2.

Define the periodogram of the observations $X_1, \dots, X_n$ at the kth Fourier frequency $x_k = 2\pi k/n$ by

$$I_{n,k}^X = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} X_t e^{itx_k} \right|^2 .$$

The GPH estimator of d using the first m Fourier frequencies may be written as

$$\hat{d}_{GPH} = -\frac{1}{2S_{ww}} \sum_{k=1}^{m} a_k \log I_{n,k}^X,$$

where $a_k = W_k - \overline{W}$, $W_k = \log|2\sin(x_k/2)|$ $(k = 1, \dots, m)$, $\overline{W} = m^{-1}\sum_{k=1}^{m} W_k$, and $S_{ww} = \sum_{k=1}^{m} a_k^2$.

Deo and Hurvich (2001) have established that for the LMSV model, $m^{1/2}(\hat{d}_{GPH} - d)$ is asymptotically normal with mean zero and variance $\pi^2/24$ assuming that $0 < d < \frac{1}{2}$, subject to restrictions on m that become more stringent as d approaches zero. The results in Deo and Hurvich (2001) are based on the fact that when $d > 0$ the spectral density of Y_t dominates that of Z_t at low frequencies. Therefore, it does not seem likely that the methodology used in Deo and Hurvich (2001) can be easily generalized to treat the case $d = 0$.

Theorem 1 establishes that $m^{1/2}\hat{d}_{GPH}$ is asymptotically normal with mean zero and variance $\pi^2/24$ when $d = 0$, thereby justifying the usual Wald test of $d = 0$ versus $d > 0$ based on $\hat{d}_{GPH}$ in the LMSV model. Note that the asymptotic distribution of $\hat{d}_{GPH}$ in this case is the same as that derived earlier for Gaussian processes by Robinson (1995) and Hurvich, Deo, and Brodsky (1998). Theorem 1 was conjectured by Deo and Hurvich (2002) on the basis of simulation results for $\hat{d}_{GPH}$ in the case of $d = 0$. Combining Theorem 1 with the results of Deo and Hurvich (2001), it is clear that the Wald test of $d = 0$ would be consistent against any alternative $d = d_1 > 0$ but that its local power would be low. On the other hand, it is not clear that any other test would have higher local power.

As in Hurvich et al. (1998) and Deo and Hurvich (2001), we avoid the need for trimming of low frequencies in $\hat{d}_{GPH}$. The low frequencies present no special problems here, because our theory is derived for the case $d = 0$. Because the noise term Z_t does not affect the regularity of the spectral density of X_t when $d = 0$, we are also able to avoid the restrictive conditions on m required in Deo and Hurvich (2001).

Theorem 2, which includes Theorem 1 as a special case, establishes the asymptotic normality of a general linear combination of $\log I_{n,k}^X$ when $d = 0$. Theorem 2 can be easily generalized to include the FEXP estimator proposed by Janacek (1982), and studied in Robinson (1994), Moulines and Soulier (1999), and Hurvich and Brodsky (2001), although the properties of the fractional exponential (FEXP) estimator when $d > 0$ in the LMSV model have not yet been established.

Theorem 2 only requires that Z_t have finite moments up to the fourth order. This is a less stringent assumption than was made in Deo and Hurvich (2001) for $d > 0$. Those authors assumed that Z_t has finite moments up to the eighth order. Theorem 2 is first proved under the provisional assumption that Z_t has finite moments of all orders. The proof is by the method of moments, using Edgeworth expansions for discrete Fourier transforms (DFTs) of an i.i.d. series developed in Fay and Soulier (2001). Lemma 2 then shows that the moment assumption on Z_t can be weakened. Theorem 2 does not require conditions on the characteristic function such as those assumed by Velasco (2000) on the innovations in his work on log-periodogram regression for linear, non-Gaussian processes. We are able to avoid such assumptions by conditioning first on the DFTs of Z_t, so that the Edgeworth expansion is for the density of a smooth function of the DFTs of Z_t. This point, and also the overall validity of our Edgeworth expansions, is explained more fully in Section 3.3.

2. ASSUMPTIONS AND MAIN RESULT

We now introduce a precise assumption on the process Y. Because we only consider functions of the periodogram at nonzero Fourier frequencies, we set $\mu = 0$ in (1) without loss of generality.

(A1)

(1) Y is a centered stationary Gaussian process with spectral density f_Y that is bounded above and away from zero and

$$\sum_{p=1}^{\infty} p|\operatorname{cov}(Y_0, Y_p)| < \infty. \tag{2}$$

(2) Z is a sequence of i.i.d. centered random variables with variance σ^2 and finite moments up to the fourth order.

(3) The processes Y and Z are independent.

Assumption (2) implies that f is continuously differentiable over the whole frequency range.

Define $\tilde{n} = [(n-1)/2]$.

THEOREM 1. *Suppose that Assumption (A1) holds. Let β be the largest real number in $[1,2]$ for which there exist positive reals x^* and c such that for all $x \in [-x^*, x^*]$,*

$$|f_Y(x) - f_Y(0)| \le c|x|^\beta. \tag{3}$$

Let $m := m(n)$ be a nondecreasing sequence of integers such that $\lim_{n\to\infty} m^{-1} + m^{2\beta+1}n^{-2\beta} = 0$. Then $m^{1/2}\hat{d}_{GPH}$ is asymptotically normal with zero mean and variance $\pi^2/24$.

It is frequently assumed in the literature that for a short-memory process Y, f_Y is C^2 on $[-\pi,\pi]$, which implies that (2) holds and (3) holds with $\beta = 2$ (because a spectral density is even, hence its first derivative vanishes at 0). Under this assumption, the GPH estimator is asymptotically normal for any choice of m_n such that $\lim_{n\to\infty} m_n^5/n^4 = 0$.

3. A THEOREM FOR GENERAL LINEAR COMBINATIONS OF LOG-PERIODOGRAM ORDINATES

Theorem 1 is a consequence of a more general result for linear combinations of log-periodogram ordinates. We will require the following conditions on the weights in the linear combinations.

(A2) $(\beta_{n,k})_{1\le k\le \tilde{n}}$ is a triangular array of real numbers such that

$$\sum_{k=1}^{\tilde{n}} \beta_{n,k}^2 = 1, \tag{4}$$

$$b_n := \max_{1\le k\le \tilde{n}} |\beta_{n,k}| = o(1), \tag{5}$$

$$\mu_n := \#\{k : \beta_{n,k} \ne 0\} = o(n), \tag{6}$$

$$\forall\, \epsilon > 0, \exists C(\epsilon), \quad \mu_n b_n^2 \le C(\epsilon)\mu_n^{\epsilon}. \tag{7}$$

Remark. Assumptions (4) and (5) are the classical Lindeberg–Liapounov conditions that ensure asymptotic normality of a weighted sum $\sum_{k=1}^{\tilde{n}} \beta_{n,k} Y_{n,k}$, for i.i.d. summands $Y_{n,k}$. Assumption (6) is not necessary; it is assumed here only to simplify the proof of Theorem 2. Assumption (7) is a technical restriction that is easily checked. Note that Assumptions (4) and (5) imply that μ_n tends to infinity. Define $X = Y + Z$ and let $f = f_Y + (\sigma^2/2\pi)$ be the spectral density of the process X. Let $\gamma = 0.577216\ \ldots$ be Euler's constant.

THEOREM 2. *If Assumptions (A1) and (A2) hold, then* $\sum_{k=1}^{\tilde{n}} \beta_{n,k} \times [\log(I_{n,k}^X/f(x_k)) + \gamma]$ *tends weakly to the Gaussian distribution with zero mean and variance* $\pi^2/6$.

Proof of Theorem 2. We first introduce more notation. Throughout the paper, a standard complex Gaussian variable means a complex random variable with i.i.d. $\mathcal{N}(0,\frac{1}{2})$ components. A function h of v complex variables will be identified with a function of $2v$ real variables and will be denoted indifferently $h(z)$, $h(z_1,\ldots,z_v)$, $h(u)$, or $h(u_1,\ldots,u_{2v})$ or using any other convenient symbol. For any process U, denote $d_{n,k}^U = (2\pi n)^{-1/2}\sum_{t=1}^n U_t e^{itx_k}$ and $I_{n,k}^U = |d_{n,k}^U|^2$. Let ϵ be a zero mean Gaussian white noise with variance σ^2 and define the process $\xi = Y + \epsilon$ so that ξ is a Gaussian process with spectral density f. For $z \in \mathbb{C}$, denote $\phi(z) = \log(|z|^2) + \gamma$ and $\phi_{n,k}(z) = \log(|z|^2/f(x_k)) + \gamma$. It is well known that if ζ is standard complex Gaussian, $\mathbb{E}[\phi(\zeta)] = 0$ and $\mathbb{E}[\phi^2(\zeta)] = \pi^2/6$.

The main tools used to prove Theorem 2 are Lemma 5 (applied with $\alpha = 1$), which is stated and proved in Section 3.2, and Edgeworth expansions of the joint density of DFTs of a white noise, based on the results of Fay and Soulier (2001). The theory of Edgeworth expansions for DFTs is reviewed and shown to be valid in the present context in Section 3.3.

The proof of Theorem 2 is based on the method of moments. Thus we first assume that all moments of the noise Z are finite. Under that assumption, we must first prove that the moments of $\phi_{n,k}(d_{n,k}^X)$ are bounded uniformly with respect to n and k.

LEMMA 1. *If Assumptions (A1) and (A2) hold and if* $\mathbb{E}[|Z_0|^q] < \infty$ *for some integer* $q \ge 2$, *then for all sufficiently large* n, *there exists a constant* C_q *such that*

$$\mathbb{E}[|\phi_{n,k}(d_{n,k}^X)|^q] \le C_q.$$

Proof of Lemma 1. As will be shown in Section 3.3, a first-order Edgeworth expansion is valid and yields

$$\mathbb{E}[|\phi_{n,k}(d_{n,k}^X)|^q] = \mathbb{E}[|\phi_{n,k}(d_{n,k}^\xi)|^q] + o(n^{-1/2}),$$

where the term $o(n^{-1/2})$ is uniform with respect to k (but not necessarily q). Applying Lemma 5 (with $\alpha = 1$) we also have

$$\mathbb{E}[|\phi_{n,k}(d_{n,k}^\xi)|^q] = \mathbb{E}[|\phi(\zeta)|^q] + O(n^{-1/2}),$$

where ζ is standard complex Gaussian and the term $O(n^{-1/2})$ is uniform with respect to k (but not necessarily q). Because $\phi(\zeta)$ is distributed as the (centered) logarithm of an exponential random variable, it follows that $\mathbb{E}[|\phi(\zeta)|^q]$ is finite for all q, and the proof of Lemma 1 is complete. ■

Define $S_n = \sum_{k=1}^{\tilde{n}} \beta_{n,k}\phi_{n,k}(d_{n,k}^X)$. We now prove that if all moments of Z are finite, the the moments of S_n tend to those of a Gaussian variable with zero mean and variance $\pi^2/6$; i.e., for all even positive integers q,

$$\lim_{n\to\infty} \mathbb{E}[S_n^q] = \frac{q!(\pi^2/6)^{q/2}}{(q/2)!\,2^{q/2}} \tag{8}$$

and $\lim_{n\to\infty} \mathbb{E}[S_n^q] = 0$ for all odd integers q.

Denote $\eta_{n,k} = \phi_{n,k}(d_{n,k}^X)$.

$$\mathbb{E}(S_n^q) = \sum_{v=1}^{q} {\sum}'_{v,q} \frac{q!}{q_1!\dots q_v!}\,\frac{1}{v!}\,A_n(q_1,\dots,q_v),$$

$$A_n(q_1,\dots,q_v) = {\sum}''_{v,n} \prod_{j=1}^{v} \beta_{n,k_j}^{q_j}\,\mathbb{E}\left[\prod_{i=1}^{v} \eta_{n,k_i}^{q_i}\right].$$

The term ${\sum}'_{v,q}$ extends on all v-tuples of positive integers $(q_1,\dots,q_v)$ such that $q_1 + \cdots + q_v = q$ and ${\sum}''_{v,n}$ extends on all v-tuples $(k_1,\dots,k_v)$ of pairwise distinct integers in the range $\{1,\dots,\tilde{n}\}$. For a v-tuple $(q_1,\dots,q_v)$ such that $q_1 + \cdots + q_v = q$, let s be the number of indices i such that $q_i = 1$ and let u be the number of indices i such that $q_i = 2$. We will consider three cases:

- $s = 0$ and $2u < q$ (or equivalently $2v < q$): the corresponding sums are easily proved to be $o(1)$;
- $s = 0$ and $2u = q$ (or equivalently $2v = q$): these terms are the leading term; a first-order Edgeworth expansion proves that Z can be replaced by ϵ;
- $s > 0$: for these terms we will use a higher order Edgeworth expansion and Lemma 3 in Fay and Soulier (2001) to prove that they do not contribute to the limit.

Case 1 ($s = 0$, $2u < q$). Because $s = 0$ and $2u < q$, it follows that $2v < q$. Because $q_1 + \cdots + q_v = q$, we moreover find that $\sum_{i=1}^{v}(q_i - 2) = q - 2v > 0$. Applying Lemma 1 and Hölder's inequality, it always holds that $\mathbb{E}[\prod_{i=1}^{v} \eta_{n,k_i}^{q_i}]$ is uniformly bounded by C_q. Recall that $b_n = \max_{1\le k\le \tilde{n}}|\beta_{n,k}|$. Thus,

$$|A_n(q_1,\dots,q_v)| \le C_q b_n^{q-2v} \sum\nolimits''_{v,n} \prod_{j=1}^{v} \beta_{n,k_j}^2 \le C_q b_n^{q-2v} \left(\sum_{k=1}^{\tilde{n}} \beta_{n,k}^2\right)^{v} = C_q b_n^{q-2v}.$$

By assumption, $b_n = o(1)$, and thus $A_n(q_1,\dots,q_v) = o(1)$.

Case 2 ($s = 0$, $2u = q$). In this case, $u = v = q/2$ and $q_1 = \cdots = q_v = 2$. Denote $\mathbf{k} = (k_1,\dots,k_{q/2})$ and let $\psi_{\mathbf{k}}$ be defined as

$$\begin{aligned}\psi_{\mathbf{k}}(u_1,\dots,u_{q/2}) &= \mathbb{E}\left[\prod_{i=1}^{q/2} \phi_{n,k_i}^2(d_{n,k_i}^X) | d_{n,k_1}^Z = u_1,\dots,d_{n,k_{q/2}}^Z = u_{q/2}\right]\\ &= \mathbb{E}\left[\prod_{i=1}^{v} \phi_{n,k_i}^2(d_{n,k_i}^Y + u_i)\right].\end{aligned}$$

With this notation, we get

$$\mathbb{E}\left[\prod_{i=1}^{q/2} \eta_{n,k_i}^2\right] = \mathbb{E}[\psi_{\mathbf{k}}(d_{n,k_1}^Z,\dots,d_{n,k_{q/2}}^Z)].$$

A first-order Edgeworth expansion yields

$$\mathbb{E}[\psi_{\mathbf{k}}(d_{n,k_1}^Z,\dots,d_{n,k_{q/2}}^Z)] = \mathbb{E}[\psi_{\mathbf{k}}(d_{n,k_1}^\epsilon,\dots,d_{n,k_{q/2}}^\epsilon)] + o(n^{-1/2}),$$

where the term $o(n^{-1/2})$ is uniform with respect to $\mathbf{k}$. By definition of the process ξ and the functions $\phi_{n,k}$,

$$\mathbb{E}[\psi_{\mathbf{k}}(d_{n,k_1}^\epsilon,\dots,d_{n,k_{q/2}}^\epsilon)] = \mathbb{E}\left[\prod_{i=1}^{q/2} \phi_{n,k_i}^2(d_{n,k_i}^\xi)\right].$$

Applying Lemma 5, we now get that

$$\mathbb{E}\left[\prod_{i=1}^{q/2} \phi_{n,k_i}^2(d_{n,k_i}^\xi)\right] = (\pi^2/6)^{q/2} + O(n^{-1/2}),$$

where the term $O(n^{-1/2})$ is uniform with respect to $k_1,\dots,k_{q/2}$. Recall that $\sum_{k=1}^{\tilde{n}} \beta_{n,k}^2 = 1$ and $b_n = \max_{1\le k\le\tilde{n}}|\beta_{n,k}|$. Thus it is easily seen that

$$\sum\nolimits''_{q/2,n} \prod_{i=1}^{q/2} \beta_{n,k_i}^2 \le 1, \quad \sum\nolimits''_{q/2,n} \prod_{i=1}^{q/2} \beta_{n,k_i}^2 = 1 + O(b_n^2).$$

Hence $A_n(2,\dots,2) = (\pi^2/6)^{q/2}(1 + O(b_n^2)) + O(n^{-1/2})$. Because it is assumed that $b_n = o(1)$, we conclude that $\lim_{n\to\infty} A_n(2,\dots,2) = (\pi^2/6)^{q/2}$.

Case 3 ($s > 0$). Denote $\mathbf{k} = (k_1, \dots, k_v)$ and $\mathbf{q} = (q_1, \dots, q_v)$, and let $\psi_{\mathbf{k},\mathbf{q}}$ be defined as

$$\psi_{\mathbf{k},\mathbf{q}}(u_1, \dots, u_v) = \mathbb{E}\left[\prod_{i=1}^{v} \phi_{n,k_i}^{q_i}(d_{n,k_i}^Y + d_{n,k_i}^Z) | d_{n,k_1}^Z = u_1, \dots, d_{n,k_v}^Z = u_v\right]$$
$$= \mathbb{E}\left[\prod_{i=1}^{v} \phi^{q_i}(d_{n,k_i}^Y + u_i)\right].$$

With this notation, we get

$$\mathbb{E}\left[\prod_{i=1}^{v} \eta_{n,k}^{q_i}\right] = \mathbb{E}[\psi_{\mathbf{k},\mathbf{q}}(d_{n,k_1}^Z, \dots, d_{n,k_v}^Z)].$$

Using the notation of Section 3.3, an sth-order Edgeworth expansion can be written as

$$\mathbb{E}[\psi_{\mathbf{k},\mathbf{q}}(d_{n,k_1}^Z, \dots, d_{n,k_v}^Z)] = \sum_{r=0}^{s} n^{-r/2} \mathbb{E}_r[\psi_{\mathbf{k},\mathbf{q}}] + o(n^{-s/2}),$$

where the term $o(n^{-s/2})$ is uniform with respect to $\mathbf{k}$, $\mathbb{E}_0[\psi_{\mathbf{k},\mathbf{q}}] = \mathbb{E}[\psi_{\mathbf{k},\mathbf{q}}(d_{n,k_1}^\epsilon, \dots, d_{n,k_v}^\epsilon)]$ and, for $r \geq 1$,

$$\mathbb{E}_r[\psi_{\mathbf{k},\mathbf{q}}] = \sum_{t=1}^{r} \frac{1}{t!} \sum\nolimits_{r,t}^{*} \mathbb{E}_{r,t,k}(\nu_1, \dots, \nu_t, \psi_{\mathbf{k},\mathbf{q}}),$$

where $\sum_{r,t}^{*}$ and the quantities $\mathbb{E}_{r,t,k}$ are defined in (19) and (20), which follow. Define

$$S_{n,r,t}(\nu_1, \dots, \nu_t) = n^{-r/2} \sum\nolimits_{v,n}'' \prod_{i=1}^{v} \beta_{n,k_i}^{q_i} \mathbb{E}_{r,t,k}(\nu_1, \dots, \nu_t, \psi_{\mathbf{k},\mathbf{q}}),$$
$$S_{n,r} = \sum_{t=1}^{r} \frac{1}{t!} \sum\nolimits_{r,t}^{*} S_{n,r,t}(\nu_1, \dots, \nu_t).$$

Two kinds of arguments will be used to prove that the terms $S_{n,r,t}$ are asymptotically negligible. The orthogonality properties of the sine and cosine functions computed at Fourier frequencies will restrict the number of multi-indices $(k_1, \dots, k_v)$ such that $\mathbb{E}_{r,t,k}(\nu_1, \dots, \nu_t, \psi_{\mathbf{k},\mathbf{q}}) \neq 0$ and the expectations appearing in $\mathbb{E}_{r,t,k}$ will be bounded by Lemma 5. Let U be a $2v$-dimensional Gaussian vector whose components are the real and imaginary parts of $d_{n,k_i}^\epsilon / \sqrt{(\sigma^2/4\pi)}$, $i = 1, \dots, v$. Because ϵ is a Gaussian white noise, the components of U are i.i.d. $\mathcal{N}(0,1)$. Following Section 3.3, $\mathbb{E}_{r,t,k}(\nu_1, \dots, \nu_t, \psi_{\mathbf{k},\mathbf{q}})$ is expressed as

$$\mathbb{E}_{r,t,k}(\nu_1, \dots, \nu_t, \psi_{\mathbf{k},\mathbf{q}}) = \frac{\chi_{\nu_1}(k) \dots \chi_{\nu_t}(k)}{\nu_1! \dots \nu_t!} \mathbb{E}[H_{\nu_1 + \dots + \nu_t}(U) \psi_{\mathbf{k},\mathbf{q}}(d_{n,k_1}^\epsilon, \dots, d_{n,k_v}^\epsilon)],$$

where ν_j, $j = 1,\dots,t$ are multi-indices in $\mathbb{N}^{2v}$ such that (21) (which follows) holds, and the multidimensional Hermite polynomial H_ν is defined in (22). Recall that s among the indices $q_1,\dots,q_v$ are equal to 1. Assume for convenience and without loss of generality that $q_1 = \cdots = q_s = 1$. Let a, b, and c be the number of indices $j \le s$ such that $\nu_1(2j-1) + \nu_1(2j) + \cdots + \nu_t(2j-1) + \nu_t(2j) = 0, = 1$, and ≥ 2, respectively. By definition, $a + b + c = s$. Assume also for simplicity that for $j \le a$, $\nu_1(2j-1) + \nu_1(2j) + \cdots + \nu_t(2j-1) + \nu_t(2j) = 2$. Then H_ν actually does not depend on its first $2a$ arguments. The following arguments are the key tools to conclude the evaluation of $A_n(q_1,\dots,q_v)$.

- Let $\tilde{\phi}_{n,k}$ be a function defined on $\mathbb{C}^2$ by $\tilde{\phi}_{n,k}(z_1,z_2) = \phi_{n,k}(\sqrt{\sigma^2/2\pi}z_1 + \sqrt{f_Y(x_k)}z_2)$. Then $\tilde{\phi}_{n,k}$, considered as a function of four real variables, has Hermite rank 2. Indeed, it is easily checked that if ζ_1 and ζ_2 are i.i.d. standard complex Gaussian, then $\mathbb{E}[\zeta_i\tilde{\phi}_{n,k}(\zeta_1,\zeta_2)] = 0$ $(i = 1,2)$. Now define

$$\tilde{\Phi}_{\mathbf{k},\mathbf{q}}(z_1,\dots,z_{2v}) = H_\nu(\sqrt{2}(z_1,\dots,z_v))\prod_{i=1}^{v}\tilde{\phi}_{n,k_i}^{q_i}(z_{2i-1},z_{2i}).$$

As was noted previously, H_ν, considered as a function of v complex Gaussian variables, actually does not depend on $z_1,\dots,z_a$. Hence, $\tilde{\Phi}_{\mathbf{k},\mathbf{q}}$ obviously has Hermite rank at least $2s - b - 2c$, because it can be written as

$$\tilde{\Phi}_{\mathbf{k},\mathbf{q}}(z_1,\dots,z_{2v}) = \prod_{i=1}^{a}\tilde{\phi}_{n,k_i}^{q_i}(z_{2i-1},z_{2i})\hat{\Phi}_{\mathbf{k},\mathbf{q}}(z_{a+1},\dots,z_{2v}),$$

where $\hat{\Phi}$ is implicitly defined. Applying Lemma 5 yields

$$\begin{aligned}
&\mathbb{E}[H_{\nu_1+\cdots+\nu_t}(U)\psi_{\mathbf{k},\mathbf{q}}(d^\epsilon_{n,k_1},\dots,d^\epsilon_{n,k_v})]\\
&\quad= \mathbb{E}\left[H_{\nu_1+\cdots+\nu_t}(U)\prod_{i=1}^{v}\phi_{n,k_i}^{q_i}(d^\epsilon_{n,k_i} + d^Y_{n,k_i})\right] && \textbf{(9)}\\
&\quad= \mathbb{E}[\tilde{\Phi}_{\mathbf{k},\mathbf{q}}(d^\epsilon_{n,k_1}/\sqrt{\sigma^2/2\pi}, d^Y_{n,k_1}/\sqrt{f_Y(x_k)},\dots,d^\epsilon_{n,k_1}/\\
&\qquad\qquad \sqrt{\sigma^2/2\pi}, d^Y_{n,k_1}/\sqrt{f_Y(x_k)})] && \textbf{(10)}\\
&\quad= \mathbb{E}[\tilde{\Phi}_{\mathbf{k},\mathbf{q}}(\zeta_1,\dots,\zeta_{2v})] + O(n^{-s+c+b/2}) && \textbf{(11)}
\end{aligned}$$

uniformly with respect to $\mathbf{k}$. Moreover, if $c < s$ the last expectation vanishes because the Hermite rank of $\tilde{\Phi}_{\mathbf{k},\mathbf{q}}$ is then positive.

- If $\nu_1,\dots,\nu_t$ satisfy (21), with c defined as before, then it is shown in the proof of Lemma 3 in Fay and Soulier (2001) the number of $\mathbf{k} \in \mathbb{N}^v$ such that $\chi_{\nu_1}(k)\dots\chi_{\nu_t}(k) \ne 0$ is of order $\mu_n^{v+(r-c)/2-1}$ at most. This is a consequence of the orthogonality properties of the sine and cosine functions evaluated at the Fourier frequencies.

Thus, for $r < s$,

$$S_{n,r} = O(\mu_n^{v+(r-c)/2-1} b_n^q n^{-s+b/2+c-r/2}) = O((\mu_n/n)^{r/2} \mu_n^{v-s/2-1} b_n^q) = o(1).$$

The last bound is a consequence of the fact that by definition of s, v, and q, $v - s/2 \le q/2$ and Assumption (A2) (7).

For $r = s$, because by assumption $\mu_n = o(n)$, and because $\mathbb{E}_s(\psi_{\mathbf{k},\mathbf{q}})$ is uniformly bounded with respect to $\mathbf{k}$, we get that $n^{-s/2}\mathbb{E}_s(\psi_{\mathbf{k},\mathbf{q}}) = O((\mu_n/n)^s) = o(1)$. Finally, we conclude that

$$\sum_{r=0}^{s} n^{-s/2} \mathbb{E}_r[\psi_{\mathbf{k},\mathbf{q}}] = o(1).$$

Hence $\lim_{n\to\infty} A_n(q_1, \ldots, q_v) = 0$ in the case $s > 0$.

There now only remains to prove that we can get rid of the assumption that all moments of Z are finite. For any integer M, define $Z_t^{(M)} = Z_t \mathbf{1}_{\{|Z_t|\le M\}}$ and $X^{(M)} = Y + Z^{(M)}$. For each M, $S_n(M) := \sum_{k=1}^{\tilde{n}} \beta_{n,k} \phi(d_{n,k}^{X^{(M)}})$ converges weakly to $\mathcal{N}(0, \pi^2/6)$. Lemma 2 implies that

$$\lim_{M\to\infty} \limsup_n \mathbb{E}\left[\left(\sum_{k=1}^{\tilde{n}} \beta_{n,k}\{\phi(d_{n,k}^X) - \phi(d_{n,k}^{X^{(M)}})\}\right)^2\right] = 0.$$

Hence we can apply Theorem 4.2 in Billingsley (1968) to conclude that S_n converges weakly to $\mathcal{N}(0, \pi^2/6)$. ∎

3.1. Proof of Theorem 1

In the case of the GPH estimator, we apply Theorem 2 with $\beta_{n,k} = -a_k \big/ \sqrt{S_{ww}}$, using the convention that $a_k = 0$ for $k > m$. By construction, $\sum_{k=1}^{\tilde{n}} \beta_{n,k} = 0$. Thus,

$$\begin{aligned}\sum_{k=1}^{\tilde{n}} \beta_{n,k} \log(I_n(x_k)) &= \sum_{k=1}^{\tilde{n}} \beta_{n,k}[\log(I_n(x_k)) + \gamma] \\ &= \sum_{k=1}^{\tilde{n}} \beta_{n,k}[\log(I_n(x_k)/f(x_k)) + \gamma] \\ &\quad + \sum_{k=1}^{\tilde{n}} \beta_{n,k} \log(f(x_k)/f(0)) =: S_n + R_n.\end{aligned}$$

It has been shown that $\max_{1\le k\le m} |\beta_{n,k}| = O(\log(m)/m)$ (see, e.g., Hurvich et al., 1998). Thus Assumption (A2) holds as long as $m_n = o(n)$. Hence Theorem 2 implies that S_n is asymptotically normal with zero mean and variance $\pi^2/6$.

We must now prove that $\lim_{n\to\infty} R_n = 0$. By applying Hölder's inequality, we get

$$|R_n| \le \left\{ \sum_{k=1}^{\tilde n} \beta_{n,k}^2 \right\}^{1/2} \left\{ \sum_{k=1}^{m_n} \log^2(f(x_k)/f(0)) \right\}^{1/2} = \left\{ \sum_{k=1}^{m_n} \log^2(f(x_k)/f(0)) \right\}^{1/2}.$$

Because f is bounded away from zero, $\log(f)$ has the same regularity as f. Because $m/n \to 0$, for large enough n, $x_k < x^*$; hence

$$|R_n| \le C \left\{ \sum_{k=1}^{m_n} (k/n)^{2\beta} \right\}^{1/2} \le C m_n^{\beta+1/2} n^{-\beta} = o(1),$$

where the constant C depends only on the function f.

To conclude the proof of Theorem 1, note that $\lim_{n\to\infty} m^{-1} S_{ww} = \frac{1}{4}$. ∎

3.2. Lemmas

LEMMA 2. *If Assumptions (A1) and (A2) hold, then*

$$\lim_{M\to\infty} \limsup_n \mathbb{E}\left[\left(\sum_{k=1}^{\tilde n} \beta_{n,k} \{ \phi(d_{n,k}^X) - \phi(d_{n,k}^{X^{(M)}}) \} \right)^2 \right] = 0.$$

Proof of Lemma 2. Define $\sigma_M^2 = \mathbb{E}[Z_t^2 \mathbf{1}_{\{|Z_t|\le M\}}]$, $\tilde\sigma_M^2 = \mathbb{E}[Z_t^2 \mathbf{1}_{\{|Z_t|>M\}}]$, and $\tilde Z_t^{(M)} = Z_t \mathbf{1}_{\{|Z_t|>M\}}$. Recall that $\eta_{n,k} = \phi_{n,k}(d_{n,k}^X)$ and denote similarly $\eta_{n,k}^{(M)} = \phi(d_{n,k}^{X^{(M)}})$. Then

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{k=1}^{\tilde n} \beta_{n,k} \{ \phi(d_{n,k}^X) - \phi(d_{n,k}^{X^{(M)}}) \} \right)^2 \right] \\
&\qquad = \sum_{k=1}^{\tilde n} \beta_{n,k}^2 \mathbb{E}[(\eta_{n,k} - \eta_{n,k}^{(M)})^2] \\
&\qquad\quad + \sum_{1\le j\ne k\le \tilde n} \beta_{n,j}\beta_{n,k} \mathbb{E}[(\eta_{n,j} - \eta_{n,j}^{(M)})(\eta_{n,k} - \eta_{n,k}^{(M)})] =: A_{n,M} + B_{n,M}.
\end{aligned}$$

The term $A_{n,M}$ would be easily dealt with if the function $\phi(x) = \log(|x|^2) + \gamma$ was replaced by a bounded function with polynomially bounded derivatives. To that purpose, we must use a tightness argument.

Let $\tilde\phi$ denote either ϕ or a C^∞ function with compact support or a linear combination of these. If Z has three finite moments, we get, by a first-order Edgeworth expansion (which is shown to be valid in Section 3.3) and following the same line of reasoning as in the proof of (8),

$$\mathbb{E}[\tilde\phi^2(d_{n,k}^X \big/ \sqrt{f(x_k)})] = \mathbb{E}[\tilde\phi^2(d_{n,k}^\xi \big/ \sqrt{f(x_k)})] + o(n^{-1/2}),$$

where again ξ denotes a Gaussian process with the same spectral density as X. Applying Lemma 5 then yields

$$\mathbb{E}[\tilde{\phi}^2(d_{n,k}^{\xi}/\sqrt{f(x_k)})] = \mathbb{E}[\tilde{\phi}^2(\zeta)] + O(n^{-1/2}),$$

where ζ is a standard complex Gaussian. In the last two equations, the terms $o(n^{-1/2})$ and $O(n^{-1/2})$ are uniform with respect to k. Hence, given that $\sum_{k=1}^{\tilde{n}} \beta_{n,k}^2 = 1$,

$$\lim_{n\to\infty} \sum_{k=1}^{\tilde{n}} \beta_{n,k}^2 \mathbb{E}[\tilde{\phi}^2(d_{n,k}^X/f(x_k))] = \mathbb{E}[\tilde{\phi}^2(\zeta)]. \tag{12}$$

Let ϕ_M be a sequence of C^∞ functions with compact support such that

$$\lim_{M\to\infty} \mathbb{E}[\{\phi(\zeta) - \phi_M(\zeta)\}^2] = 0.$$

The sequence ϕ_M can be chosen such that $\lim_{M\to\infty} \|\phi_M'\|_\infty \tilde{\sigma}_M = 0$, where ϕ_M' is the first derivative of the function ϕ_M and $\|.\|_\infty$ is the supremum norm. Now $A_{n,M}$ is split into three terms:

$$\begin{aligned} A_{n,M} \le 3 \sum_{k=1}^{\tilde{n}} \beta_{n,k}^2 \mathbb{E}[\{\phi(d_{n,k}^X) - \phi_M(d_{n,k}^X)\}^2] \\ + 3 \sum_{k=1}^{\tilde{n}} \beta_{n,k}^2 \mathbb{E}[\{\phi(d_{n,k}^{X^{(M)}}) - \phi_M(d_{n,k}^{X^{(M)}})\}^2] \\ + 3 \sum_{k=1}^{\tilde{n}} \beta_{n,k}^2 \mathbb{E}[\{\phi_M(d_{n,k}^X) - \phi_M(d_{n,k}^{X^{(M)}})\}^2]. \end{aligned}$$

Applying (12) with $\tilde{\phi} = \phi - \phi_M$, we get

$$\begin{aligned} \limsup_n \sum_{k=1}^{\tilde{n}} \beta_{n,k}^2 \mathbb{E}[\{\phi(d_{n,k}^X) - \phi_M(d_{n,k}^X)\}^2] \\ = \limsup_n \sum_{k=1}^{\tilde{n}} \beta_{n,k}^2 \mathbb{E}[\{\phi(d_{n,k}^{X^{(M)}}) - \phi_M(d_{n,k}^{X^{(M)}})\}^2] \\ = \mathbb{E}[\{\phi(\zeta) - \phi_M(\zeta)\}^2]. \end{aligned}$$

Applying the mean value theorem, it is trivially seen that

$$\sum_{k=1}^{\tilde{n}} \beta_{n,k}^2 \mathbb{E}[\{\phi_M(d_{n,k}^X) - \phi_M(d_{n,k}^{X^{(M)}})\}^2] \le \|\phi_M'\|_\infty^2 \tilde{\sigma}_M^2.$$

Altogether, we get that

$$\lim_{M\to\infty} \limsup_n A_{n,M} = 0.$$

Consider now the term $B_{n,M}$. It can be expanded as

$$B_{n,M} = \sum_{1\le j\ne k\le\tilde{n}} \beta_{n,j}\beta_{n,k}\mathbb{E}[\eta_{n,j}\eta_{n,k}] + \sum_{1\le j\ne k\le\tilde{n}} \beta_{n,j}\beta_{n,k}\mathbb{E}[\eta_{n,j}^{(M)}\eta_{n,k}^{(M)}] \quad \textbf{(13)}$$

$$- 2\sum_{1\le j\ne k\le\tilde{n}} \beta_{n,j}\beta_{n,k}\mathbb{E}[\eta_{n,j}\eta_{n,k}^{(M)}]. \quad \textbf{(14)}$$

The terms that involve only one noise can be dealt with easily using the same arguments as in the proof of (8). A second-order Edgeworth expansion, which is valid as soon as Z has four finite moments (see Section 3.3), and an application of Lemma 5 yield

$$\lim_{n\to\infty} \sum_{1\le j\ne k\le\tilde{n}} \beta_{n,j}\beta_{n,k}\mathbb{E}[\eta_{n,j}\eta_{n,k}] = 0. \quad \textbf{(15)}$$

Because all that is needed for (15) to hold is that Z has four finite moments, the preceding limit obviously holds with $\eta_{n,k}^{(M)}$ instead of $\eta_{n,k}$. Thus we need only consider the terms $c_M(j,k) = \mathbb{E}[\eta_{n,k}\eta_{n,j}^{(M)}]$. For short, define $a_M^2 = \sigma_M^2/2\pi$ and $\tilde{a}_M^2 = \tilde{\sigma}_M^2/2\pi$. Define

$$\psi_{j,k}(u,v) = \mathbb{E}[\phi(d_{n,k}^X + a_M u + \tilde{a}_M v)\phi(d_{n,j}^X + a_M u)].$$

With this notation,

$$c_M(j,k) = \mathbb{E}[\psi_{j,k}(a_M^{-1} d_{n,k}^{Z^{(M)}}, \tilde{a}_M^{-1} d_{n,k}^{\tilde{Z}^{(M)}})].$$

A second-order Edgeworth expansion of $c_M(j,k)$ can be shown valid as in Section 8.1 in Fay and Soulier (2001) and can be written as

$$c_M(j,k) = \mathbb{E}[\psi_{j,k}(\zeta_1,\zeta_2)] + n^{-1/2}\mathbb{E}_1[\psi_{j,k}] + O(n^{-1}),$$

where ζ_1 and ζ_2 are i.i.d. standard complex Gaussian and the term $O(n^{-1})$ is uniform with respect to k and j. If the process Y were Gaussian white noise, the terms $\mathbb{E}[\psi_{j,k}(\zeta_1,\zeta_2)]$ and $\mathbb{E}_1[\psi_{j,k}]$ would vanish identically. Here, using Lemma 5, it is seen $\mathbb{E}[\psi_{j,k}(\zeta_1,\zeta_2)] = O(n^{-1})$ and $\mathbb{E}_1[\psi_{j,k}] = O(n^{-1/2})$, uniformly with respect to k and j. Hence,

$$\sum_{1\le k<j\le\tilde{n}} |\beta_{n,k}\beta_{n,j}c_M(j,k)| = O(\mu_n/n).$$

Note that all the previous bounds depend on M, but for any fixed M, we have proved that

$$\lim_{n\to\infty} B_{n,M} = 0. \qquad \blacksquare$$

LEMMA 3. *Let Φ be a function such that $\|\Phi\|^2 := (2\pi)^{-a/2} \times \int_{\mathbb{R}^a} \Phi^2(x)e^{-x^T x/2}\,dx < \infty$ and with Hermite rank at least τ. Let X be an a-dimensional centered Gaussian vector with covariance matrix Γ such that the* *spectral radius ρ of $I_a - \Gamma$ satisfies $\rho < \frac{1}{3} - \epsilon$ for some $0 < \epsilon < \frac{1}{3}$. Then there exists a constant $c(\epsilon, \tau, a)$ that depends only on ϵ, τ, and a such that*

$$|\mathbb{E}[\Phi(X)]| \leq c(\epsilon, \tau, a)\|\Phi\| \rho^{\tau/2}.$$

Proof of Lemma 3. Denote $\Delta = \Gamma^{-1} - I_a$ and $\nu = [(\tau + 1)/2]$.

$$\begin{aligned}|\Gamma|^{1/2}\mathbb{E}[\Phi(x)] &= \int_{\mathbb{R}^a} \Phi(x) e^{-x^T\Delta x/2} e^{-x^T x/2} \frac{dx}{(2\pi)^{a/2}} \\ &= \sum_{k=0}^{\nu-1} \frac{(-1/2)^k}{k!} \int_{\mathbb{R}^a} \Phi(x)(x^T\Delta x)^k e^{-x^T x/2} \frac{dx}{(2\pi)^{a/2}} \\ &\quad + \int_{\mathbb{R}^a} \Phi(x) r_\nu(x) e^{-x^T x/2} \frac{dx}{(2\pi)^{a/2}},\end{aligned}$$

where r_ν is the remainder term in the νth order Taylor expansion of $e^{-x^T\Delta x/2}$. Because Φ has Hermite rank τ, the terms in the sum from 0 to $\nu - 1$ all vanish. Moreover, it is well known that

$$|r_\nu(x)| \leq \frac{|x^T\Delta x|^\nu}{2^\nu \nu!} e^{|x^T\Delta x|/2}.$$

Let δ be the spectral radius of Δ.

$$\begin{aligned}|\Gamma|^{1/2}|\mathbb{E}[\Phi(x)]| &\leq \frac{1}{2^\nu \nu!} \int_{\mathbb{R}^a} |\Phi(x)||x^T\Delta x|^\nu e^{|x^T\Delta x|/2} e^{-x^T x/2} \frac{dx}{(2\pi)^{a/2}} \\ &\leq \frac{\delta^\nu}{2^\nu \nu!} \|\Phi\| \left\{ \int_{\mathbb{R}^a} (x^T x)^{2\nu} e^{\delta x^T x} e^{-x^T x/2} \frac{dx}{(2\pi)^{a/2}} \right\}^{1/2}.\end{aligned}$$

If $\delta \leq \frac{1}{2} - \epsilon$ for some $\epsilon > 0$, then the last integral in the preceding expression is finite and depends only on ϵ, τ, and a. Moreover, by continuity of the function det, $|\Gamma|$ is bounded away from zero, and thus there exists a constant that depends only on ϵ, τ, and a such that

$$|\mathbb{E}[\Phi(X)]| \leq c(\epsilon, \tau, a)\|\Phi\|\delta^\nu \leq c(\epsilon, \tau, a)\|\Phi\|\delta^{\tau/2}.$$

Finally, it is easily seen that as soon as $\delta < 1$, Γ is invertible and $\Gamma^{-1} - I_a = \sum_{k=1}^\infty (I_a - \Gamma)^k$; thus $\rho \leq \delta/(1 - \delta)$, and $\delta < \frac{1}{3} - \epsilon$ implies $\rho < \frac{1}{2} - \epsilon'$ for some $\epsilon' > 0$. This concludes the proof of Lemma 3. ∎

LEMMA 4. *Let U be a stationary process with finite second moment. Let f_U be the spectral density and γ be the covariance function of U. If γ satisfies the following condition:*

$$\sum_{p=0}^\infty p^\alpha |\gamma(p)| < \infty, \tag{16}$$

for some $\alpha > 0$, *then for all* $1 \le k \ne j \le \tilde{n}$,

$$\mathbb{E}[I_{n,k}^U] = f_U(x_k) + O(n^{-(\alpha \wedge 1)}),$$

$$|\mathbb{E}[d_{n,k}^U d_{n,j}^U]| + |\mathbb{E}[d_{n,k}^U \overline{d_{n,j}^U}]| = O(n^{-(\alpha \wedge 1)}).$$

Proof of Lemma 4. For $1 \le k \le \tilde{n}$ and $-\tilde{n} \le j \le \tilde{n}$,

$$\mathbb{E}[d_{n,k}^U d_{n,j}^U] = (2\pi n)^{-1} \sum_{p=1-n}^{n-1} \gamma(p) e^{ipx_k} \sum_{s=1}^{n} \mathbf{1}_{\{1 \le s+p \le n\}} e^{is(x_k+x_j)}.$$

Note that $f_U(x) = (2\pi)^{-1} \sum_{p \in \mathbb{Z}} \gamma(p) e^{ipx}$. If $k + j = 0$, under assumption (16), denoting $\pi L := \sum_{p=1}^{\infty} p|\gamma(p)|$ we get

$$\begin{aligned} |\mathbb{E}[I_{n,k}^U] - f_U(x_k)| &\le (\pi n)^{-1} \sum_{p=1}^{n-1} |p||\gamma(p)| + \pi^{-1} \sum_{p \ge n} |\gamma(p)| \\ &\le \pi^{-1} n^{-(\alpha \wedge 1)} \sum_{p=1}^{\infty} p^{\alpha} |\gamma(p)| \le L n^{-(\alpha \wedge 1)}. \end{aligned}$$

If $1 \le |k + j| \le 2\tilde{n}$, then $|\sum_{1 \le s \le n} \mathbf{1}_{\{1 \le s+p \le n\}} e^{is(x_k+x_j)}| \le p$; thus

$$|\mathbb{E}[d_{n,k}^U d_{n,j}^U]| \le (\pi n)^{-1} \sum_{p=1}^{n-1} p|\gamma(p)| \le \pi^{-1} n^{-(\alpha \wedge 1)} \sum_{p=1}^{\infty} p^{\alpha} |\gamma(p)| \le L n^{-(\alpha \wedge 1)}. \quad \blacksquare$$

Let Γ_v denote the covariance matrix of the vector of DFTs $d_{n,k_1}^U / \sqrt{f_U(x_{k_1})}, \dots, d_{n,k_v}^U / \sqrt{f_U(x_{k_v})}$, considered as a $2v$-dimensional real Gaussian vector. Lemma 4 yields $\Gamma_v = \frac{1}{2} I_{2v} + \Delta_{n,v}$ where the spectral radius of $\Delta_{n,v}$ is of order $O(n^{-(\alpha \wedge 1)})$. Thus Lemmas 3 and 4 yield the following lemma.

LEMMA 5. *Let U be a stationary Gaussian process with spectral density f_U that satisfies condition (16) for some $\alpha \in (0,1]$. Let $\zeta_1, \dots, \zeta_u$ be i.i.d. standard complex Gaussian. Let Φ be a function defined on $\mathbb{C}^u$ such that $\|\Phi\|^2 := \mathbb{E}[\Phi^2(\zeta_1, \dots, \zeta_u)] < \infty$. If the Hermite rank of $\Phi - \mathbb{E}[\Phi(\zeta_1, \dots, \zeta_u)]$ is τ, then*

$$\begin{aligned} &|\mathbb{E}[\Phi(d_{n,k_1}^U / \sqrt{f_U(x_{k_1})}, \dots, d_{n,k_u}^U / \sqrt{f_U(x_{k_u})})] - \mathbb{E}[\Phi(\zeta_1, \dots, \zeta_u)]| \\ &\quad \le c(\tau, u) \|\Phi\| n^{-\tau\alpha/2} \end{aligned}$$

uniformly with respect to $k_1, \dots, k_u$.

3.3. Edgeworth Expansions

In this section, we check the validity of the Edgeworth expansions used in the proof of Theorem 2. It can be deduced from Theorem 3.17 in Götze and Hipp (1978) that if ψ is a C^{∞} function with polynomially bounded derivatives of all order, such that (i) $\sup_{\zeta \in \mathbb{C}^a} |\psi(\zeta)|/(1 + \|\zeta\|^s) < \infty$ and (ii) $\mathbb{E}[|Z_0|^{s+2}] < \infty$, then for any a-tuple of pairwise distinct integers $k = (k_1, \dots, k_a)$, an Edgeworth expansion of $\mathbb{E}[\psi(d^Z_{n,k_1},\dots,d^Z_{n,k_a})]$ up to the order s is valid, and the remainder term is uniform with respect to $k_1,\dots,k_a$ and depends only on bounds for the function ψ and its derivatives. (For more details, see Fay and Soulier, 2001, Sec. 8.1.) In the present context, all the functions ψ considered are C^∞ and their derivatives are polynomially bounded, because they can be written as convolutions of the Gaussian kernel. Moreover, they are uniformly bounded by a power of $\log(\|\zeta\|)$. Thus, the second-order expansions used in Lemma 2 are valid if Z has finite fourth moment. Let us illustrate this claim with the case of the function ψ defined on $\mathbb{C}$ as $\psi(u)=\mathbb{E}[\log|d^Y_n(x_k)+u|^2]$. Identifying $\mathbb{C}$ and $\mathbb{R}^2$, and denoting $\Gamma_{n,k}$ the covariance matrix of the real and imaginary parts of $d^Y_n(x_k)$, we have

$$\psi(u)=\int_{\mathbb{R}^2}\log(\|x+u\|^2)e^{-1/2(x-u)^T\Gamma^{-1}_{n,k}(x-u)}\,\frac{dx}{2\pi|\Gamma_{n,k}|^{1/2}}.$$

Under the assumptions on the spectral density of Y, $\Gamma_{n,k}/f_Y(x_k)$ converges uniformly with respect to k to $\frac{1}{2}I_2$, where I_2 is the two-dimensional identity matrix. Because f_Y is assumed bounded above and away from zero, there exist positive constants $c<C$ such that for all $x\in\mathbb{R}^2$ and for all sufficiently large n, $c\|x\|^2\le x^T\Gamma^{-1}_{n,k}x\le C\|x\|^2$. Thus, to prove (i), it is enough to check that $\int_{\mathbb{R}^2}\log^2(\|x\|)e^{-\|x-u\|^2}\,dx\le C\log^2(\|u\|)$, and to prove that ψ is C^∞ with uniformly (with respect to k and n) polynomially bounded derivatives, it suffices to prove that for all positive integer ν, $\int_{\mathbb{R}^2}\|x\|^\nu\log^2(\|x\|^2)e^{-\|x-u\|^2}\,dx$ is bounded by a power of $\|u\|$ on $\mathbb{R}^2$. Splitting the integral over the domains $\{\|x\|\le 1\}$ and $\{\|x\|\ge 1\}$, we get

$$\int_{\{\|x\|\le 1\}}\|x\|^\nu\log^2(\|x\|)e^{-\|x-u\|^2}\,dx\le e^{-\|u\|^2+2\|u\|}\int_{\{\|x\|\le 1\}}|\log(\|x\|)|\,dx\le C,$$

$$\int_{\{\|x\|>1\}}\|x\|^\nu\log^2(\|x\|)e^{-\|x-u\|^2}\,dx=\int_{\{\|x+u\|>1\}}\|x+u\|^\nu\log^2(\|x+u\|)e^{-\|x\|^2}\,dx.$$

If $\|x+u\|>1$, then $\log(\|x+u\|)\le\log(\|x\|)+\log(\|u\|)$ and $\log(\|x+u\|)\le\|x+u\|$. This yields

$$\begin{aligned}&\int_{\{\|x+u\|>1\}}\log^2(\|x+u\|)e^{-\|x\|^2}\,dx\\&\quad\le 2\int\log^2(\|x\|)e^{-\|x\|^2}\,dx+2\log^2(\|u\|)\int e^{-\|x\|^2}\,dx=A+B\log^2(\|u\|),\end{aligned}$$

$$\begin{aligned}&\int_{\{\|x+u\|>1\}}\|x+u\|^\nu\log^2(\|x+u\|)e^{-\|x\|^2}\,dx\\&\quad\le 2^\nu\int(\|x\|^{\nu+1}+\|u\|^{\nu+1})e^{-\|x\|^2}\,dx\le C+D\|u\|^{\nu+1},\end{aligned}$$

where A, B, C, and D are numerical constants.

We now give an explicit expression for this valid Edgeworth expansion. Let $U_1,\dots,U_{2a}$ be $2a$ i.i.d. $\mathcal{N}(0,1)$ random variables and denote $U = (U_1,\dots,U_{2a})^T$. Let ψ be a function such that $\mathbb{E}[\psi^2(\sqrt{\sigma^2/4\pi U})] < \infty$. A formal Edgeworth expansion of $\mathbb{E}[\psi(d^Z_{n,k_1},\dots,d^Z_{n,k_a})]$ up to the sth order can be written as

$$\mathbb{E}[\psi(d^Z_{n,k_1},\dots,d^Z_{n,k_a})] = \sum_{r=0}^{s} n^{-r/2}\mathbb{E}_{r,k}(\psi) + n^{-s/2}v_n R_n(\psi), \tag{17}$$

where the sequence v_n depends only on the distribution of Z_0 and s and satisfies $\lim_{n\to\infty} v_n = 0$; $R_n(\psi)$ is uniformly bounded with respect to n and $k_1,\dots,k_a$:

$$\mathbb{E}_{0,k}(\psi) = \mathbb{E}[\psi(\sqrt{\sigma^2/4\pi U})], \tag{18}$$

$$\mathbb{E}_{r,k}(\psi) = \sum_{t=1}^{r} \frac{1}{t!} \sum\nolimits^*_{r,t}\mathbb{E}_{r,t,k}(\psi), \tag{19}$$

$$\mathbb{E}_{r,t,k}(\psi) = \frac{\chi_{\nu_1}(k)\dots\chi_{\nu_t}(k)}{\nu_1!\dots\nu_t!}\,\mathbb{E}[H_{\nu_1+\dots+\nu_t}(U)\psi(\sqrt{\sigma^2/4\pi U})], \qquad (r>0); \tag{20}$$

$\sum^*_{r,t}$ extends over all t-tuples $\underline{\nu}$ of multi-indices $\nu_l := (\nu_l(1),\dots,\nu_l(2a)) \in \mathbb{N}^{2a}, l = 1,\dots,t$ such that

$$|\nu_l| := \nu_l(1) + \dots + \nu_l(2a) \geq 3, \quad l = 1,\dots,t \quad \text{and} \quad \sum_{l=1}^{t} |\nu_l| = r + 2t; \tag{21}$$

for $k \in \{1,\dots,K\}^a$ and $\nu \in \mathbb{N}^{2a}$, $\chi_\nu(k) = 2^{|\nu|/2}\kappa_{|\nu|}A_\nu(k)$ with

$$A_\nu(k) = n^{-1}\sum_{t=1}^{n}\prod_{j=1}^{a}\cos(tx_{k_j})^{\nu_{2j-1}}\sin(tx_{k_j})^{\nu_{2j}};$$

and $\kappa_{|\nu|}$ is the cumulant of order $|\nu|$ of Z_0; H_ν denotes a multidimensional Hermite polynomial:

$$H_\nu(U) = \prod_{j=1}^{2a} H_{\nu(j)}(U_j), \tag{22}$$

and for $k \in \mathbb{N}$, H_k is the usual Hermite polynomial of order k. For further details on multidimensional Hermite polynomials, see, e.g., Arcones (1994).

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