# Optimal Auctions with General Distributions

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**Abstract**

This note characterizes revenue maximizing auctions in a single unit independent private value environment when buyers' distributions of valuations can be discrete, continuous, or any mixture of the two possibilities. The procedure described is applicable to many other single- or multi-agent mechanism design problems with transferable utility and single-dimensional types. *Keywords: mechanism design, optimal auctions, ironing, Dirac's delta functions. JEL Classification Codes: C72, D44, D82.*

## 1. Introduction

The literature on optimal auctions[1] is one of the most important parts of auction theory, and of the theory of mechanism design, more generally. A maintained assumption in those works is that buyers' valuations spaces are either finite or they are continuous. From these earlier works it follows that there are important qualitative differences between the continuous and finite cases. For instance, in the continuous case the revenue equivalence theorem, allows us to conclude that the allocation rule determines the payment rule up to a constant. However, with discrete distributions there can be *different* payment rules implemented with the same allocation rule. For an illustration of this point see Fudenberg and Tirole (1991). For a more recent and thorough exposition of the differences between the finite and the continuous type model see Lovejoy (2006).

This paper derives revenue maximizing auctions without imposing any structure on the distributions of buyers' valuations. They can be continuous, discrete or a mixture of these possibilities. The initial difficulty is to find a way to deal with distributions that have convex supports, and those that do not, in an unified way. This difficulty is addressed in Skreta (2006). That paper established that the mechanism designer can obtain an optimum by

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[1] The seminal contributions are Myerson (1981) and Riley and Samuelson (1981).

solving an artificial problem obtained by extending each agent's type space to its convex hull. With the help of that result we obtain an expression of the seller's revenue as a function of the allocation rule and the sum of the payoffs that accrue to buyers at their lowest possible valuations. In other words we obtain a "revenue equivalence" result. Unfortunately, that expression of the revenue is not very operational because distributions in general do not have densities. The existence of densities is important because it allows one to combine in a common term the cost and benefits of assigning the good to buyer $i$ with valuation $v_i$. This regrouping is important because it facilitates comparisons across buyers in order to decide the most profitable assignment of the object. In Myerson (1981) where densities are strictly positive, the term that combines the cost and benefits of assigning the good to buyer $i$ with valuation $v_i$ is the virtual valuation.

The present paper shows how to sidestep this difficulty and obtains a characterization of optimal auctions both in the regular and the general case. We use Dirac's delta function to obtain expressions of densities which allows us to treat discrete, mixed and continuous distributions in a unified way. Still the resulting problem is non-standard because virtual valuations are not well defined for all vectors of valuations. We show that one can obtain a solution of such a problem, by solving an artificial problem where virtual valuations are appropriately extended on all vectors of valuations. In the regular case, that is the case where pointwise optimization of the seller's revenue leads to a feasible mechanism, this problem can be solved exactly as in Myerson (1981). However, with distributions that can be discrete, and/or mixed, an optimal auction problem fails to be regular much more often than in the standard continuous case. If regularity fails, new complications arise since Myerson's (1981) ironing technique is not applicable because it requires distributions to be strictly increasing. This condition is violated when distributions are discrete or mixed and it can be also violated when distributions are continuous, but do not have strictly positive densities. In this paper we construct an "ironing procedure" that is applicable in environments where distributions are not invertible. This procedure is intuitive and it does rely on variational methods that impose unnecessary differentiability assumptions on the mechanisms.

Our analysis is completed with an illustration of the procedure in a two-buyer example where we allow for atoms and gaps in the supports.

## 2. Analysis of The Problem

We characterize revenue maximizing auctions in the standard independent private value environment, where buyers' valuations are drawn from distributions that can be continuous, discrete or even mixed.

A risk neutral seller owns a unit of an indivisible object, and faces $I$ risk neutral buyers. The seller's valuation for the object is common knowledge and is normalized to zero, whereas that of buyer $i$ is denoted by $v_i$, and it is distributed on $V_i$ according to $F_i$. The convex hull of $V_i$ is denoted by $\bar{V}_i$ and it is an interval $[a_i, b_i]$, where $-\infty < a_i \leq b_i < \infty$. A buyer's valuation $v_i$ is private and is independently distributed across buyers. All elements of the game except the realization of the buyers' valuations are common knowledge. The seller's goal is to design a mechanism that maximizes expected revenue. The buyers aim to maximize expected surplus. We use $V = \times_{i\in I} V_i,$ to denote the set of all possible vectors of valuations of all the buyers; $V_{-i} = \times_{j\in I, j\neq i} V_i$, stands for the set of all possible vectors of valuations of $I\backslash\{i\}$; $v = (v_1, v_2, ..., v_I)$ denotes a vector of valuations of all the buyers, and $v_{-i} = (v_1, v_2, ..., v_{i-1}, v_{i+1}, .., v_I)$ denotes a vector of valuations of $I\backslash\{i\}$. The joint distribution of $v$ on $V$ is denoted by $F = F_1 \times F_2 \times ... \times F_I$ and $F_{-i} = F_1 \times .. \times F_{i-1} \times F_{i+1} ... \times F_I$, denotes the joint distribution of $v_{-i}$ on $V_{-i}$. This completes the description of the environment under consideration. We move on to introduce *mechanisms*.

From the revelation principle one can restrict attention to direct revelation mechanisms, "$DRM$", where truth-telling is an equilibrium. A $DRM$ consists of an allocation rule $p : V \rightarrow [0,1]^I$ $p(v) = (p_1(v), p_2(v), ..., p_I(v))$, and a payment rule $x : V \rightarrow \mathbb{R}^I$, $x(v) = (x_1(v), x_2(v), ..., x_I(v))$.

Given a $DRM$ $(p, x)$, buyer $i's$ expected payoff at a truth telling equilibrium when his valuation is $v_i$, is given by

$$U_i(p, x, v_i) = P_i(v_i)v_i - X_i(v_i), \tag{1}$$

where $P_i(v_i) = \int_{V_{-i}} p_i(v_i, v_{-i})dF_{-i}(v_{-i})$ and $X_i(v_i) = \int_{V_{-i}} x_i(v_i, v_{-i})dF_{-i}(v_{-i})$.

An optimal mechanism solves

$$\max_{p,x\in DRM} \int_V \Sigma_{i\in I} x_i(v)dF(v),$$

subject to:

$IC$, **incentive compatibility**, $P_i(v_i)v_i - X_i(v_i) \geq P_i(v_i')v_i - X_i(v_i')$, for all $v_i, v_i' \in V_i$ and $i \in I$

$PC$, **participation constraints**, $P_i(v_i)v_i - X_i(v_i) \geq 0$, for all $v_i \in V_i$ and $i \in I$,

$RES$, **resource constraints**, $\Sigma_{i\in I} p_i(v_i, v_{-i}) \leq 1$ and $0 \leq p_i(v_i, v_{-i}) \leq 1$, for all $v \in V$ and $i \in I$.

We will refer to this program as Program $A$. Even though this problem is isomorphic to the problem in the classical work of Myerson (1981), the solution approach does not go through because it requires that the type spaces be intervals. The key step there is to rewrite revenue as a function solely of the allocation rule, which is the gist of the famous revenue equivalence theorem. That step relies on the type space being an interval. When supports are not necessarily convex, one can proceed as follows. Extend[2] the definitions of

[2] The Appendix describes how the extension can be performed.

$p$ and $x$ on the convex hull of $V$, which we call $\bar{V}$, and solve the resulting artificial problem, which we call Program $B$. The following result is established in Skreta (2006).

**Proposition 1** *A solution of Program $B$ restricted on $V$ solves Program $A$.*

For the artificial Program $B$ we can obtain properties of feasible mechanisms using standard techniques, from which we obtain the famous revenue equivalence results. More importantly, this extension shows how one can bridge the gap between the continuous and the discrete case. Then, the set of feasible mechanisms satisfies all standard properties and from Lemma 2 in Myerson (1981) we have:

**Lemma 1** *A mechanism $p, x$ satisfies IC, PC and RES constraints on $\times_{i\in I}[a_i, b_i]$, if and only if for all $v_i \in [a_i, b_i]$ (a) $P_i(v_i)$ is increasing in $v_i$ (b) $U_i(p, x, v_i) = \int_{a_i}^{v_i} \int_{\bar{V}_{-i}} p_i(s_i, v_{-i})dF_{-i}(v_{-i})ds_i + U_i(p, x, a_i)$ (c) $U_i(p, x, a_i) \geq 0$ and (d) $0 \leq p_i(v_i, v_{-i}) \leq 1$, $\Sigma_{i\in I} p_i(v_i, v_{-i}) \leq 1$ for all $i \in I$ and $v \in \times_{i\in I}[a_i, b_i]$.*

Given this Lemma, and using fairly standard procedures, the seller's problem becomes:

$$\max_p \int_{\bar{V}} \Sigma_{i\in I} p_i(v_i, v_{-i})v_i dF_i(v_i)dF_{-i}(v_{-i}) \tag{2}$$
$$- \int_{\bar{V}} \Sigma_{i\in I} p_i(v_i, v_{-i})[1 - F_i(v_i)]dv_i dF_{-i}(v_{-i}) - \Sigma_{i\in I} U_i(p, x, a_i),$$

subject to (a) and (d) of Lemma 1.

**Review: Optimal Auctions with Strictly Positive & Continuous Densities**

When densities are strictly positive that is $f_i(v_i) > 0$, for all $v_i$, as is the case in Myerson (1981), we usually factor $f_i(.)$ out, by dividing by it and (2) can be rewritten as:

$$\int_{\bar{V}} \Sigma_{i\in I} p_i(v_i, v_{-i}) \left( v_i - \frac{(1 - F_i(v_i))}{f_i(v_i)} \right) f(v)dv - \Sigma_{i\in I} U_i(p, x, a_i). \tag{3}$$

Then, as we can see from (3) each buyer's virtual valuation $J_i(v_i) = v_i - \frac{(1-F_i(v_i))}{f_i(v_i)}$ is weighted with the same number, namely $f(v)$.

As it is well known, when all buyers' virtual valuations are increasing, an optimal auction assigns the object to the buyer with the highest virtual valuation, provided that it is above the seller's valuation. When this condition fails, Myerson (1981) shows that virtual valuations can be replaced by their "ironed" versions without essentially changing the objective function. The ironed virtual valuation is constructed as follows. Integrate the virtual valuation

$$H_i(v_i) = \int_{a_i}^{v_i} \left[ F_i^{-1}(k) - \frac{1 - k}{f_i(F^{-i}(k))} \right] dk, \text{ where } k = F_i(v_i), \tag{4}$$

and convexify the resulting integral

$$\begin{aligned} G_i(v_i) &= convH_i(v_i) \quad (5)\\ &= \min\{\omega H_i(r_1) + (1-\omega)H_i(r_2)\,|\omega \in [0,1],\ r_1, r_2 \in [a_i, b_i] \text{ and } \omega r_1 + (1-\omega)r_2 = v_i\}. \end{aligned}$$

Then, the ironed virtual valuation is $\bar{J}_i$ is obtained by differentiating $G_i$,

$$\bar{J}_i(v_i) \equiv \frac{dG_i(v_i)}{dv_i},$$

which is increasing because it is the derivative of a convex function.

**Optimal Auctions with General Distributions**

When the distributions do not have densities, it is not clear how one can obtain the revenue maximizing auction from (2). This is because we cannot combine the cost and benefits of assigning the good to buyer $i$ with valuation $v_i$ in a common term, as is done in the expression of virtual valuation[3] in the standard case. This regrouping is important because it allows easy comparisons across buyers. We bypass this difficulty using Dirac's delta functions, usually denoted by $\delta$.[4] These generalized functions allow us to obtain expressions for densities even for discrete and/or mixed distributions as follows. Let $\pi_i^1$ denote the probability that buyer $i's$ valuation is equal to $v_i^k$, where $k \in \{1, ..., K\}$. For such a distribution the density can be written as

$$f_i(v_i) = \Sigma_{k=1}^K \pi_i^k \delta(v_i - v_i^k). \quad (6)$$

Then, irrespective of whether the distributions of buyers' valuations are discrete, mixed or continuous, (2) can be rewritten as:[5]

$$\int_{\bar{V}} \Sigma_{i\in I} p_i(v_i, v_{-i}) \cdot (v_i f_i(v_i) - (1 - F_i(v_i)))\, f_{-i}(v_{-i}) dv_{-i} dv_i - \Sigma_{i\in I} U_i(p, x, a_i). \quad (7)$$

When $f_i's$ can be zero we cannot divide by them and obtain expressions of the virtual valuations for all $v_i's$. One way to proceed is to try to mimic the standard procedure,

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[3] $$J_i(v_i) = \underbrace{v_i}_{\text{benefit}} - \underbrace{\frac{1-F_i(v_i)}{f_i(v_i)}}_{\text{cost}}$$

[4] Delta functions can be defined as: $\delta(x) = \lim_{\varepsilon\to 0} d_\varepsilon(x)$, where

$$d_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon} \text{ for } -\frac{\varepsilon}{2} \le x \le \frac{\varepsilon}{2} \\ 0 \text{ otherwise} \end{cases}.$$

It follows that $\delta(x) = 0$ for all $x \in \mathbb{R}\backslash\{0\}$, and that at $x = 0$, $\delta(0) \to \infty$ . Moreover it is always true that $\delta(x)x = 0$.

[5] Strictly, speaking when the densities are expressed with the help of delta functions, we have to integrate over $\mathbb{R}^I$ instead of $\bar{V}$.

but instead of "prioritizing" the buyers with their virtual valuations, to do so using the quasi-virtual valuations $[v_i f_i(v_i) - (1 - F_i(v_i))]$, for $i \in I$. However, this is problematic because each buyer is assigned a different weight in the objective function. For buyer $i$ this weight is given by $f_{-i}(v_{-i})$. A more promising way, is to prioritize buyers according to their "*weighted*" virtual valuations, defined by

$$\mathcal{J}_i(v_i, v_{-i}) \equiv [v_i f_i(v_i) - (1 - F_i(v_i))] \, f_{-i}(v_{-i}). \tag{8}$$

Then each buyer is assigned the same weight in the objective function, namely one. Then buyer $i$ should be assigned the object if

$$\begin{aligned} \mathcal{J}_i(v_i, v_{-i}) &\geq \mathcal{J}_j(v_j, v_{-j}) \text{ and} \\ \mathcal{J}_i(v_i, v_{-i}) &\geq v_0 f(v) \text{ for all } i, j \in I \text{ with } j \neq i. \end{aligned} \tag{9}$$

Recalling (8), (9) reduces to

$$\begin{aligned} [v_i f_i(v_i) - (1 - F_i(v_i))] \, f_j(v_j) &\geq [v_j f_j(v_j) - (1 - F_j(v_j))] \, f_i(v_i) \text{ and} \\ [v_i f_i(v_i) - (1 - F_i(v_i))] &\geq v_0 f_i(v_i) \text{ for all } i, j \in I \text{ with } j \neq i. \end{aligned} \tag{10}$$

Notice, however, that (10) is trivially satisfied for all $v_i$ and $v_j$ when $f_i(v_i) = f_j(v_j) = 0$. This essentially says that "anything goes" when $f_i(v_i) = f_j(v_j) = 0$. Now when $f_i(v_i) = 0$, and $f_j(v_j) \neq 0$, (10) says that $i$ can be getting the good only when $v_i = b_i$, whereas in the case where $f_j(v_j) = 0$, but $f_i(v_i) > 0$, $i$ must be getting the good for all $v_i \in [a_i, b_i]$. This suggests that the allocation rule obtained using the weighted virtual valuations in order to prioritize buyers,[6] will fail to be incentive compatible, because when $f_i(v_i) = 0$, pointwise optimization dictates $p_i(v_i, v_{-i}) = 0$ for all $v'_{-i}s$, which implies that $P_i(v_i) = 0$. Another reason why (10) is not that useful, is because it is not clear how one could proceed in the cases where "ironing" may be required. If we were to "iron," that is replace the expression $[v_i f_i(v_i) - (1 - F_i(v_i))]$ by some increasing version, this would not work, because the RHS of (10) also depends on $v_i$, and hence the allocation obtained could still violate incentive compatibility even though $[v_i f_i(v_i) - (1 - F_i(v_i))]$ and $[v_j f_j(v_j) - (1 - F_j(v_j))]$ are replaced by increasing versions.

From all these considerations it follows that the inequalities in (10) are not that useful for the vectors of valuations where $f(v) = 0$. However, when they are useful when $f(v) > 0$, and they in fact reduce to the standard set of inequalities, namely buyer $i$ obtains the good if

$$\begin{aligned} v_i - \frac{(1 - F_i(v_i))}{f_i(v_i)} &\geq v_j - \frac{(1 - F_j(v_j))}{f_j(v_j)} \text{ for all } j \in I \text{ with } j \neq i \text{ and} \\ v_i - \frac{(1 - F_i(v_i))}{f_i(v_i)} &\geq v_0. \end{aligned} \tag{11}$$

[6] This essentially amounts to pointwise optimization of (7).

We proceed as follows. We rely on the standard expressions of virtual valuations in order to determine a pointwise optimal assignment when $f(v) > 0$. Then we extend this allocation rule for the vectors of valuations that occur with probability zero in a way that the resulting extended allocation is (i) incentive compatible and (ii) it leaves unaffected the rents enjoyed by valuations occurring with strictly positive probability.[7]

**Regular Case**

Following Myerson (1981) we call *regular* the case where for all $i$, the virtual valuation $J_i(v_i) = v_i - \frac{(1-F_i(v_i))}{f_i(v_i)}$, is increasing in $v_i$, for all $v_i$ with $f_i(v_i) > 0$. Then, pointwise optimization on the regions where $f(v) > 0$ gives us

$$p_i^*(v_i, v_{-i}) = \begin{cases} \frac{1}{\#\mathcal{I}(v)} \text{ if } i \in \mathcal{I}(v) \\ 0 \text{ otherwise} \end{cases}, \tag{12}$$

where $\mathcal{I}(v) \equiv \{i \in I, \text{ s.t. } i \in \arg\max_{i \in I} J_i(v_i), \text{ and } J_i(v_i) \geq 0\}$, and it denotes the set of buyers that have maximal virtual valuations when the vector of valuations is equal to $v$.[8]

Because we are in the regular case, $p^*$ it is incentive compatible on $V$, which consists of the vectors $v$, with $f(v) > 0$. An extension of $p^*$ on all $\bar{V}$ that maintains incentive compatibility and does not affect the the rents enjoyed by valuations occurring with strictly positive probability, is as follows:

$$p_i^E(v_i, v_{-i}) = \begin{cases} p_i^*(v_i, \hat{v}_{-i}(v_{-i})) \text{ for each } v_i \in [a_i, b_i] \text{ s.t. } f_i(v_i) > 0 \\ p_i^*(\hat{v}_i(v_i), \hat{v}_{-i}(v_{-i})) \text{ where } \hat{v}_i(v_i) = \sup\{t_i \in [a_i, b_i] \text{ s.t. } t_i \leq v_i \text{ and } f_i(t_i) > 0 \end{cases}, \tag{13}$$

and where $\hat{v}_{-i}(v_{-i}) = (\hat{v}_1(v_1), ..., \hat{v}_{i-1}(v_{i-1}), \hat{v}_{i+1}(v_{i+1}), ..., \hat{v}_i(v_i))$.

The allocation rule $p^E$ is incentive compatible because it is a flat extension of $p^*$ over $v_i's$ that occur with probability zero. It is also easy to see that this flat extension does not alter the information rents of types that occur with strictly positive probability. For each $v_{-i}$ all that matters for the rents of buyer $i$ is $\underline{v}_i(v_{-i})$, which stands for the smallest valuation of $i$ where $i$ wins the object, when the valuations of all other bidders are equal to $v_{-i}$.[9] Indeed the extension (13) leaves $\underline{v}_i(v_{-i})$ unaffected.

Finally, using standard arguments it is easy to see that the $DRM$ $p^E, x^E$, where $x_i^E(v_i, v_{-i}) = p_i^E(v_i, v_{-i})v_i - \int_{v_i}^{b_i} p_i^E(t_i, v_{-i})dt_i$, satisfies the resource and participation constraints. This

[7] From condition (b) of Lemma 1 it follows that at an incentive compatible mechanism, both the expected payment, and the expected utility of buyer $i$ with valuation $v_i$ depend on the allocation rule $p_i$ for all valuations in $[a_i, v_i)$, even those occurring with probability zero.

[8] Observe that for this more general problem, ties can occur for regions of valuations that have strictly positive measure, and for this reason ties have to be broken in a consistent way to avoid obtaining an allocation rule that violates incentive compatibility. Here we use the tie-breaking rule employed by Myerson (1981) for the general case.

[9] For a valuation $v_i > \underline{v}_i(v_{-i})$, the rent is $v_i - \underline{v}_i(v_{-i})$, and it is zero otherwise.

completes the derivation of a solution in our regular case. Before we move to the general case, we will offer a practical reinterpretation of the solution we have obtained.

**A Practical Reinterpretation:**

It is immediate to see that the solution we have obtained solves the following problem[10]

$$\max_p \int_{\bar{V}} \Sigma_{i\in I} p_i(v_i, v_{-i}) J_i^E(v_i) f(v) dv - \Sigma_{i\in I} U_i(p, x, a_i). \tag{14}$$

subject (a) and (d) of Lemma 1.

This problem is constructed by extending the virtual valuations of the problem of interest, (which are only defined for $v_i$, with $f_i(v_i) > 0$), on all $v_i' s$, as follows:

$$J_i^E(v_i) = \begin{cases} v_i - \frac{(1-F_i(v_i))}{f_i(v_i)} \text{ for each } v_i \in [a_i, b_i] \text{ s.t. } f_i(v_i) > 0 \\ \hat{v}_i(v_i) - \frac{(1-F_i(\hat{v}_i(v_i)))}{f_i(\hat{v}_i(v_i))} \text{ where } \hat{v}_i(v_i) = \sup\{t_i \in [a_i, b_i] \text{ s.t. } t_i \leq v_i \text{ and } f_i(t_i) > 0\} \end{cases}. \tag{15}$$

The extended virtual valuation of buyer $i$ with valuation $v_i$ is equal to his actual virtual valuation, if $f_i(v_i) > 0$, otherwise is it equal to the actual virtual valuation of the highest valuation below $v_i$ that occurs with strictly positive probability. From the definition of the extended virtual valuations it follows that if the problem is regular, so will the problem with the extended virtual valuations.

After this reinterpretation of our findings for the regular case, we move on to examine the general case.

**General Case (Ironing)**

When the $J_i' s$ fail to be increasing, we cannot "iron" using Myerson's (1981) ironing technique because, as it can be seen from (4), is requires $F_i' s$ to be strictly increasing in order to be invertible. This condition is violated when distributions are discrete or mixed and it can be also violated when distributions are continuous, but do not have strictly positive densities. We move on to show how one can go about "ironing" a virtual valuation in cases where $F_i$ need not be invertible.

[10] Observe that the objective function of (14) is different from the objective function we are interested in. To see this, notice that it is possible that

$$[v_i f_i(v_i) - (1 - F_i(v_i))] f_{-i}(v_{-i}) \neq J_i^E(v_i) f(v) \text{ for a } v \text{ with } f(v) = 0,$$

in particular think about the case where $f_i(v_i) = 0$ and $f_j(v_j) \neq 0$ for all $j \in I$, then $J^E(v_i)f(v) = 0$, whereas $[v_i f_i(v_i) - (1 - F_i(v_i))] f_{-i}(v_{-i}) = -(1 - F_i(v_i)) f_{-i}(v_{-i})$ which imply that it is possible that

$$\Sigma_{i\in I} \int_V p_i(v_i, v_{-i}) [v_i f_i(v_i) - (1 - F_i(v_i))] f_{-i}(v_{-i}) dv_{-i} dv_i \neq \Sigma_{i\in I} \int_V p_i(v_i, v_{-i}) J_i^E(v_i) f(v) dv.$$

Still, as it turns out, a solution of our problem can be obtained by pointwise optimization of this "extended problem."

Instead of using the somewhat indirect, but elegant way, of integrating the actual virtual valuation and then convexifying the integral, we will be working directly with the virtual valuation. Our goal is to replace it with an increasing function that is as close as possible to the true virtual valuation. The best we can hope for, is for an "ironed" virtual valuation that (i) lies on the true virtual valuation as much as possible and (ii) whenever the two are not equal pointwise, they are equal in expectation. These two properties are guaranteed by Myerson's construction. To see this, note that when $J_i$ is increasing its integral is convex and it obviously coincides with its convex hull which implies that $J_i = g_i$. When the integral is not convex, it is replaced by its convex hull. By the construction of the convex hull in (5) we have that $G_i(v_i) = H_i(v_i)$ at $v_i = r_1$ and at $v_i = r_2$. These equalities are equivalent to

$$\int_{a_i}^{r_1} \left[ t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \right] dF_i(t_i) = G_i(r_1) = \int_{a_i}^{r_1} g_i(t_i) dF_i(t_i) \tag{16}$$

and

$$\int_{a_i}^{r_2} \left[ t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \right] dF_i(t_i) = G_i(r_2) = \int_{a_i}^{r_2} g_i(t_i) dF_i(t_i). \tag{17}$$

Because the integral for a $v_i \in [r_1, r_2]$ is a straight line, the "ironed" virtual valuation will be constant in this region. Then, from (16) and (17) it follows that

$$\int_{r_1}^{r_2} g_i dF_i(t_i) = \int_{r_1}^{r_2} \left[ t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \right] dF_i(t_i). \tag{18}$$

Equality (18) guarantees that when the virtual valuation differs from the "ironed" virtual valuation, they are equal in expectation.

We now proceed to show how one can obtain an "ironed" virtual valuation with exactly these properties, when $F_i's$ are not invertible and/or the virtual valuations are not defined on all the region $[a_i, b_i]$. The "ironing" can proceed as follows. Suppose that the virtual valuation of some buyer $i$ is increasing up to some valuation $v_i^*$ and it drops at $v_i^*$. For simplicity we assume for the moment that there is only one such $v_i^*$ where $J_i$ turns from increasing to decreasing. Then the region where $\bar{J}_i$ and $J_i$ differ, namely $[r_1, r_2]$, must contain $v_i^*$, otherwise obviously $\bar{J}_i$ fails to be increasing as well. Hence $r_1$ and $r_2$ must satisfy:

$$a_i \le r_1 \le v_i^* \le r_2 \le b_i. \tag{19}$$

Along the region $[r_1, r_2]$ the virtual valuation of $i$ must be flat. With some abuse of notation, we let $\bar{J}_i$ denote the value of the virtual valuation along this region. From the requirement that $J_i$ and $\bar{J}_i$ have the same expected value over the range where they differ, the following equality must hold:

$$\int_{r_1}^{r_2} \bar{J}_i dF_i(v_i) = \int_{r_1}^{r_2} J_i(v_i) dF_i(v_i). \tag{20}$$

Then the ironed virtual valuation is derived as follows:

$$\begin{aligned} \bar{J}_i(v_i) &= J_i(v_i) \text{ for } v_i \in [a_i, r_1] \cup [r_2, b_i] \\ \bar{J}_i(v_i) &= \bar{J}_i \text{ for } v_i \in (r_1, r_2) \end{aligned}$$

**Remark 1** *Observe that is it without any loss to view the $J_i'$s as their **extended** versions obtained in (15). The reason is that the extension does not add any new values of $J_i$, meaning that the range of $J_i$ and that of $J_i^E$ is the same. This says that no new possible "levels" of $\bar{J}_i$ are added. The level of $\bar{J}_i$ is, roughly, the relevant variable for the ironing. Also, since the extension is over valuations that occur with probability zero, we immediately have that $\int_{r_1}^{r_2} J_i(v_i)dF_i(v_i) = \int_{r_1}^{r_2} J_i^E(v_i)dF_i(v_i)$, for any $r_1, r_2 \in [a_i, b_i]$.*

In order to find the "ironed" virtual valuation we simply need to find three numbers $\bar{J}_i$, $r_1$ and $r_2$.

First note that if $r_2 < b_i$ and $J_i$ is continuous and strictly increasing at $r_2$, then it will hold that

$$\bar{J}_i = J(r_2) \text{ and } r_2 = \min\{J_i^{-1}(\bar{J}_i), b_i\}, \tag{21}$$

whereas if $J_i$ is continuous, but it fails to be *strictly* increasing at $r_2$, then

$$r_2 = \inf\{v_i \geq r_1 : \ J_i(v_i) = \bar{J}_i\}. \tag{22}$$

If $r_2 < b_i$ falls at a point of a discontinuity of $J_i$ the following must be true

$$\bar{J}_i \in [J_i(r_2 - \varepsilon), J_i(r_2 + \varepsilon)], \text{ for some } \varepsilon > 0 \text{ very small.}$$

Now if $r_1 > a_i$, and $J_i$ is continuous and strictly increasing at $r_1$, then it will hold that

$$J_i(r_1) = \bar{J}_i \text{ and } r_1 = \max\{a_i, J_i^{-1}(\bar{J}_i)\}, \tag{23}$$

whereas if $J_i$ is continuous but it fails to be *strictly* increasing at $r_1$, then

$$r_1 = \sup\{v_i \leq r_2 : \ J_i(v_i) = \bar{J}_i\}. \tag{24}$$

It follows that once we have $\bar{J}_i$, we can pin down $r_2$: it will either satisfy (21) or (22), or it will be a corner solution and it will be equal to $b_i$. From $\bar{J}_i$ we can also pin down $r_1$: it will either satisfy (23) or (24), or it will be a corner solution and it will be equal to $a_i$.

From all these considerations it follows that the problem reduces to essentially finding $\bar{J}_i$. One can start assuming that $r_1$ and $r_2$ do not fall at a point of discontinuity and where $J_i$ is strictly increasing. Then substituting (21) and (23) in (20) we get an equation in terms of $\bar{J}_i$ :

$$\int_{\max\{a_i, J_i^{-1}(\bar{J}_i)\}}^{\min\{J_i^{-1}(\bar{J}_i), b_i\}} \bar{J}_i dF_i(v_i) - \int_{\max\{a_i, J_i^{-1}(\bar{J}_i)\}}^{\min\{J_i^{-1}(\bar{J}_i), b_i\}} J_i(v_i) dF_i(v_i) = 0.$$

Notice that the LHS of this equality is continuous in $\bar{J}_i$. Also there exists a $\bar{J}_i$ that is always above $J_i$ and the LHS is strictly positive. It is also easy to find a $\bar{J}_i$ that is always below $J_i$ and then the difference is negative. For an illustration of these possibilities see Figure 1. Then by the continuity wrt to $\bar{J}_i$, we get that this equality has a solution.

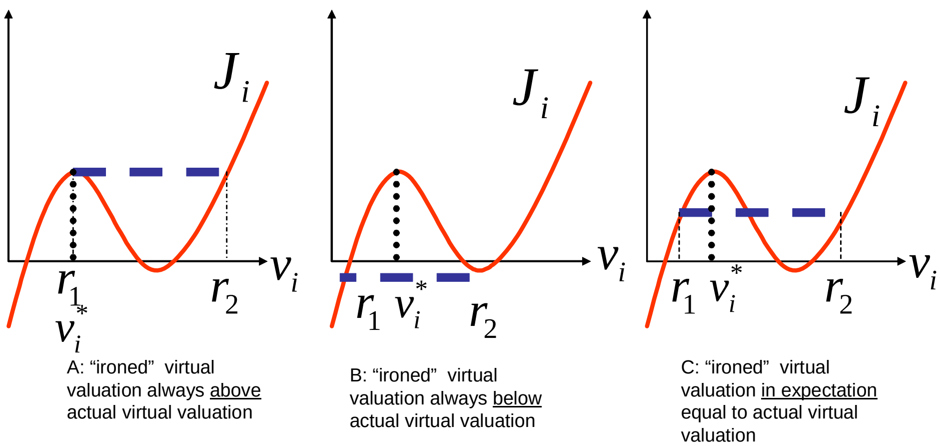


Figure 1

In the case that a solution $\bar{J}_i$ is likely to be such that $r_1$ and $r_2$ fall on points where $J_i$ is discontinuous, and/or where it is not invertible, the analysis can get a bit tedious, but it is straightforward. Typically inspection of the virtual valuation, will give one a good idea where the $r_1$ and $r_2$ could lie, as one can see from the example we solve below.

In the event where there are many points where $J_i(v_i)$ drops, the ironing procedure can be straightforwardly modified. One should start the ironing from the largest type $v^{*(\max)}$ where $J_i$ drops and follow the procedure we just described. If $r_1^{(\max)}$ is at a valuation below the smallest valuation where $J_i$ drops, let us call it $v^{*(\min)}$, then we are done. If not, we continue with the ironing procedure around the largest valuation below $r_1^{(\max)}$ where $J_i$ drops, and so forth.

**Example 1** ***Finding the Optimal Auction when Distributions have Atoms, and "Gaps":*** *Suppose that there are two buyers and that the seller's valuation is 0. Buyer 1's valuation is uniformly distributed on* $[0,1]$, *that is*

$$F_1(v_1) = \begin{cases} v_1 \text{ for } v_1 \in [0,1] \\ 0 \text{ otherwise} \end{cases}, \text{ with } f_1(v_1) = \begin{cases} 1 \text{ for } v_1 \in [0,1] \\ 0 \text{ otherwise} \end{cases}, \tag{25}$$

*whereas buyer's 2 valuation is $\frac{1}{6}$ with probability $\frac{1}{3}$, and with probability two thirds his* *valuation is uniformly distributed on* $[\frac{1}{3}, 1]$, *that is*

$$F_2(v_2) = \begin{cases} 0 \text{ for } v_2 < \frac{1}{6} \\ \frac{1}{3} \text{ for } v_2 \in [\frac{1}{6}, \frac{1}{3}) \\ v_2 \text{ for } v_2 \in [\frac{1}{3}, 1] \\ 0 \text{ otherwise} \end{cases}, \text{ with } f_2(v_2) = \begin{cases} \frac{\delta(v_2 - \frac{1}{6})}{3} \text{ for } v_2 \in [\frac{1}{6}, \frac{1}{3}) \\ \frac{\delta(v_2 - \frac{1}{6})}{3} + 1 \text{ for } v_2 \in [\frac{1}{3}, 1] \\ 0 \text{ otherwise} \end{cases}. \tag{26}$$

*Now with the help of (25) and (26) we can write the virtual valuations of buyer 1 and buyer 2 as follows:*

$$J_1(v_1) = 2v_1 - 1 \text{ for all } v_1 \in [0, 1) \text{ and} \tag{27}$$

$$J_2(v_2) = \begin{cases} v_2 - \frac{\frac{2}{3}}{\frac{\delta(v_2 - \frac{1}{6})}{3}} \text{ for } v_2 = \frac{1}{6} \\ v_1 - \frac{1 - v_1}{\frac{\delta(v_2 - \frac{1}{6})}{3} + 1} \text{ for } v_2 \in [\frac{1}{3}, 1] \end{cases}$$

*note that* $J_2$ *is not defined in the region* $(\frac{1}{6}, \frac{1}{3})$. *From properties of delta functions,*[11] *we know that* $\delta(v_2 - \frac{1}{6}) = 0$ *for all* $v_2 \neq \frac{1}{6}$, *and that* $\delta(v_2 - \frac{1}{6}) = \delta(0) = \infty$ *and with the help of this, buyer's 2 extended virtual valuation becomes:*

$$J_2^E(v_2) = \begin{cases} \frac{1}{6} \text{ for } v_2 \in [\frac{1}{6}, \frac{1}{3}) \\ 2v_2 - 1 \text{ for } v_2 \in [\frac{1}{3}, 1] \end{cases}. \tag{28}$$

*Notice that* $J_1$ *is strictly increasing so we do not need to iron it. However,* $J_2^E(v_2)$ *is not increasing, (it drops at* $\frac{1}{3}$*), and must be ironed. Our "generalized" ironing proceeds as follows. We are looking for an interval of valuations of the form* $[r_1, r_2]$ *where we replace* $J_2^E$ *with a constant* $\bar{J}_2$ *over this range and the resulting virtual valuation is increasing and it satisfies (20). Since* $J_2^E$ *violates monotonicity at* $v_2 = \frac{1}{3}$, *and* $J_2^E$ *is flat and equal to* $\frac{1}{6}$, *for all* $v_2 \leq \frac{1}{3}$, *it immediately follows that* $r_1 = \frac{1}{6}$, *because otherwise the resulting ironed virtual will be above* $J_2^E$ *for all* $v_2$ *and (20) cannot hold. We also know that* $r_2$ *must satisfy*

$$\frac{1}{3} \leq r_2 \leq 1, \tag{29}$$

*and because* $J_2$ *is continuous and strictly increasing for all* $v_2 > \frac{1}{3}$, *(21) reduces to:*

$$r_2 = \min\{1, \frac{\bar{J}_2}{2} + \frac{1}{2}\}. \tag{30}$$

*With the help of (30), and recalling the definition of* $J_2$, *and the requirement that* $r_2 \geq \frac{1}{3}$, *(20) for this example reduces to:*

$$\int_{-\infty}^{\infty} \bar{J}_2 \frac{\delta(v_2 - \frac{1}{6})}{3} dv_2 + \int_{\frac{1}{3}}^{\frac{\bar{J}_2}{2} + \frac{1}{2}} \bar{J}_2 dv_2 = \int_{-\infty}^{\infty} \frac{1}{6} \frac{\delta(v_2 - \frac{1}{6})}{3} dv_2 + \int_{\frac{1}{3}}^{\frac{\bar{J}_2}{2} + \frac{1}{2}} (2v_2 - 1) dv_2,$$

[11] See for instance Hoskins (1979), or Saichev and Woyczynski (1997).

*which from the properties of delta functions becomes:*

$$\frac{\bar{J}_2}{3} + \int_{\frac{1}{3}}^{\frac{\bar{J}_2}{2}+\frac{1}{2}} \bar{J}_2 dv_2 = \frac{1}{6} \cdot \frac{1}{3} + \int_{\frac{1}{3}}^{\frac{\bar{J}_2}{2}+\frac{1}{2}} (2v_2 - 1) dv_2.$$

*Solving for* $\bar{J}_2$, *we get* $\bar{J}_2 = 0.054$, *which implies from (30) that* $r_2 = 0.527$. *Then the ironed virtual valuation for buyer 2 becomes*

$$\bar{J}_2^E(v_2) = \begin{cases} 0.054 \ \textit{for} \ v_2 \in [\frac{1}{6}, 0.527) \\ 2v_2 - 1 \ \textit{for} \ v_2 \in [0.527, 1] \end{cases}, \tag{31}$$

*and it is depicted together with buyer's 2 actual virtual valuation in Figure 2.*

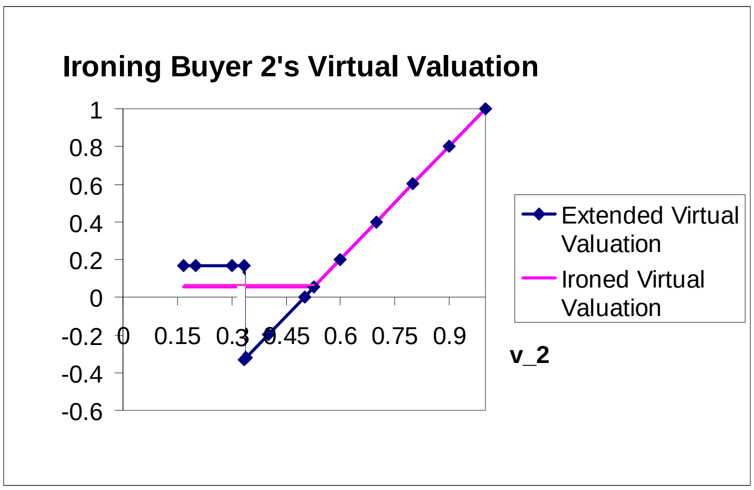


Figure 2

*Using (27) and (31) we obtain that a revenue maximizing allocation rule must satisfy:*

$$\begin{aligned} p_1(v_1, v_2) &= 1 \ \textit{if} \ v_1 > 0.527 \ \textit{and} \ v_1 > v_2 \\ p_2(v_1, v_2) &= 1 \ \textit{otherwise.} \end{aligned}$$

*The seller never gets to keep the object, because buyer 2's (ironed) virtual valuation is always greater than zero.*

## 3. Appendix

**A sketch of the proof of Proposition 1.** First note that since for $v \in \bar{V} \backslash V$ $F$ is constant, which implies that $dF$ is zero, we have that essentially the objective functions of Program $A$ and $B$ are identical. The programs differ only in the constraints sets.

Let $R(p^A, x^A)$, (respectively $R(p^B, x^B)$), denote the principal's expected maximized payoff at a solution of Program $A$, (respectively Program $B$). We first establish that $R(p^A, x^A) = R(p^B, x^B)$.

Since a solution of Program A has to satisfy $IC$, $PC$ and $RES$ on $V$, whereas a solution of Program $B$ has to satisfy all these constraints on the extended type space, that is on $\bar{V}$, Program $B$ has more constraints which gives us that

$$R(p^A, x^A) \geq R(p^B, x^B).$$

We will be done if we establish that

$$R(p^A, x^A) \leq R(p^B, x^B). \tag{32}$$

We argue by contradiction. Suppose not, then

$$R(p^A, x^A) > R(p^B, x^B). \tag{33}$$

Now we extend $p^A, x^A$ on the convex hull of $V_i$ in an appropriate way, and establish that this extension is feasible for Program $B$. In what follows we will take $V_i$ to be a closed set.[12]

Consider a $v_i \in \bar{V}_i \backslash V_i$ and define $v_i^L(v_i) = \max\{v_i' \in V_i : v_i^{'} \leq v_i\}$ and $v_i^H(v_i) = \min\{v_i' \in V_i : v_i^{'} \geq v_i\}$, (these maxima and minima exist because $V_i$ is closed). Now let $v_i^{Ind}(v_i) \in [v_i^L(v_i), v_i^H(v_i)]$ denote the type for which the following is true:

$$P_i^A(v_i^H(v_i))v_i^{Ind}(v_i) - X_i^A(v_i^H(v_i)) = P_i^A(v_i^L(v_i))v_i^{Ind}(v_i) - X_i^A(v_i^L(v_i)). \tag{34}$$

Such a $v_i$ exists by continuity, given that by the incentive compatibility of $p^A, x^A$ we have that

$$\begin{aligned} P_i^A(v_i^H(v_i))v_i^H(v_i) - X_i^A(v_i^H(v_i)) &\geq P_i^A(v_i^L(v_i))v_i^H(v_i) - X_i^A(v_i^L(v_i)) \text{ and} \\ P_i^A(v_i^H(v_i))v_i^L(v_i) - X_i^A(v_i^H(v_i)) &\leq P_i^A(v_i^L(v_i))v_i^L(v_i) - X_i^A(v_i^L(v_i)). \end{aligned}$$

[12] If $V_i's$ is not closed, it is very easy to show that $p^A, x^A$ can be extended on the closure of $V_i's$, for all $i \in I$.

Now consider the following extension of $p^A, x^A$, call it $\bar{p}^A, \bar{x}^A$ on $\bar{V}$[13]

$$\begin{aligned}\bar{p}_i^A(v_i, v_{-i}) &= p_i^A(\tilde{v}_i(v_i), \tilde{v}_{-i}(v_{-i})) \\ \bar{x}_i^A(v_i, v_{-i}) &= x_i^A(\tilde{v}_i(v_i), \tilde{v}_{-i}(v_{-i})),\end{aligned}$$

where

$$\tilde{v}_i(v_i) = \begin{cases} v_i \text{ if } v_i \in V_i(s_i) \\ v_i^L(v_i) \text{ if } v_i \in \bar{V}_i(s_i)\backslash V_i(s_i) \text{ and } v_i \leq v_i^{Ind}(v_i) \\ v_i^H(v_i) \text{ if } v_i \in \bar{V}_i(s_i)\backslash V_i(s_i) \text{ and } v_i > v_i^{Ind}(v_i) \end{cases}$$

and

$$\tilde{v}_{-i}(v_{-i}) = (\tilde{v}_1(v_1), ..., \tilde{v}_{i-1}(v_{i-1}), \tilde{v}_{i+1}(v_{i+1}), ..., \tilde{v}_i(v_i)). \tag{35}$$

Note that for $v_i \in V_i$ and $v_{-i} \in V_{-i}$ we have that

$$\bar{p}_i^A(v_i, v_{-i}) = p_i^A(v_i, v_{-i}) \text{ and } \bar{x}_i^A(v_i, v_{-i}) = x_i^A(v_i, v_{-i}).$$

Fix a $v_{-i} \in V_{-i}$, (so that this vector of types arises with strictly positive probability). It is easy to see that the "real options" that $i$ is choosing from are the same in both mechanisms, because for fixed $v_{-i}$, by the definition of $\bar{p}^A, \bar{x}^A$, the menus $\{p_i^A(v_i, .), x_i^A(v_i, .)\}_{v_i \in V_i}$ and $\{\bar{p}_i^A(v_i, .), \bar{x}_i^A(v_i, .)\}_{v_i \in \bar{V}_i}$ coincide. For a $v_{-i} \in \bar{V}_{-i}\backslash V_{-i}$, the menu $\{\bar{p}_i^A(v_i, .), \bar{x}_i^A(v_i, .)\}_{v_i \in \bar{V}_i}$ is actually equal to a menu for a $v_{-i} \in V_{-i}(s_{-i})$, namely $\tilde{v}_{-i}(v_{-i})$ defined in (35), so in this case too $\{\bar{p}_i^A(v_i, .), \bar{x}_i^A(v_i, .)\}_{v_i \in \bar{V}_i}$ is a menu that is identical to a menu $\{p_i^A(v_i, .), x_i^A(v_i, .)\}_{v_i \in V_i}$. Hence in extending $p^A, x^A$ on $\bar{V}$ no new "real options" options have been added for buyer $i$. Given the fact that there are no new options, the feasibility of $\bar{p}^A, \bar{x}^A$ on $V$ follows immediately from the feasibility of $p^A, x^A$ on $V$, since they coincide on those types. Now, the feasibility of $\bar{p}^A, \bar{x}^A$ on all of $\bar{V}$ can be easily verified by its definition with the help of (34).

Since $p^B, x^B$ is a solution for Program $B$ and $\bar{p}^A, \bar{x}^A$ is feasible for that problem, it must hold that

$$R(p^B, x^B) \geq R(\bar{p}^A, \bar{x}^A) = R(p^A, x^A),$$

which contradicts (33). Hence it must hold that

$$R(p^A, x^A) = R(p^B, x^B). \tag{36}$$

[13] Note, that this extension is slightly different than the one in Skreta (2006). The reason we choose to extend $p, x$ this way here, is that this extension works without further arguments when we extend allocation and payment rules that are subject to additional constraints, such that sequential rationality constraints, as is done in Skreta (2006b) in a single buyer environment. When we are looking at the classical auction problem, (that is the problem subject to incentive, participation and resource constraints), than one can show that

$$v_i^{Ind} = v_i^L \text{ for all } i \in I.$$

For more details see Skreta (2006).

Given that the values of Programs $A$ and $B$ are the same, it is now relatively straightforward to establish that a solution of $A$ can be obtained by solving $B$ and restricting its solution on $V$. Take a solution of $B$, $p^B, x^B$, and restrict it on $V$, then by (36) it immediately follows that

$$R(p^B, x^B) = R(p^A, x^A),$$

and hence we have a solution of Program $A$. ■

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