IPOs and The Growth of Firms*

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Abstract

Recent years have witnessed a rapid accumulation of empirical evidence documenting firm dynamics around the IPO date. A particularly striking finding is that operating performance, as measured by Returns on Assets for example, peaks in the fiscal year preceding the offering, worsens on impact at the IPO date, and keeps on declining for a few more years. In this paper, I provide a novel rationalization of this evidence. To this end, I construct a simple dynamic stochastic model of firm behavior in which the decision to go public is modelled explicitly. The model predicts that the operating performance reaches its peak in the period before the offering and experiences a sudden decline at the IPO date. The comparative advantage of my approach is that it produces further implications that are in line with the data. Most importantly, the model predicts that the IPO coincides with an increase in sales and capital expenditures. Consistently with evidence pointed out by the Industrial Organization literature, the firm growth rate is shown to be decreasing in age and size.

Key words. Firm Dynamics, IPO, Operating Performance.

JEL Codes: D21, D92, G32.

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1 Introduction.

The Initial Public Offering of equity is a special moment in the life cycle of a firm. In most cases, it results in radical changes in ownership structure, capital structure, and size of operations. Empirical studies have also found that the IPO moment is characterized by a series of puzzling regularities. The most studied of them is the well-known phenomenon of underpricing. Another one is that IPO stocks consistently underperform with respect to the stocks of non-issuing firms.[1] In this paper I focus on a further stylized fact, documented by Degeorge and Zeckhauser [12], Jain and Kini [20], Mikkelson, Partch and Shah [26], Pagano, Panetta and Zingales [27], and Fama and French [15]. These studies present evidence that, for IPO firms, measures of operating performance such as ratios of Operating Earnings and Cash Flows over Book Value of Assets exhibit a sudden decline in the fiscal year in which the offering takes place, and keep on worsening for a few more years. If, as I argue in Section 2, the observed dynamics of such measures is not the result of mismeasurement or accounting manipulation, then we need to acknowledge that profitability does actually decrease in the aftermath of the IPO. Understanding why this is the case is an interesting research question.[2]

In this paper I build on the existing literature on firm dynamics, embedding the IPO decision in a simple dynamic stochastic model of firm behavior. I show that for variables such as profitability, capital expenditures, and sales, the post-IPO dynamics predicted by the model is consistent with the available empirical evidence. Operating performance peaks in the fiscal year preceding the IPO and declines thereafter. Such decline is accompanied by an increase in capital expenditures and sales. The model's predictions are also consistent with a series of empirical regularities on the process of firm growth, pointed out by the Industrial Organization literature. Most importantly, the firm growth rate decreases with age and size.

I consider the intertemporal decision problem of a risk-averse agent (entrepreneur), with preferences defined over consumption and effort. The entrepreneur is endowed with a talent for managing, a production technology (firm), and an initial amount

[1] See Loughran and Ritter [23].

[2] A number of answers to this question have been conjectured in the past, none of which has won universal acclaim. In Section 2 I survey all of them in some detail.

of wealth. Production is assumed to depend on physical capital and on managerial effort. Factors' productivity is stochastic. The needs of physical capital can be financed through issue of equity and debt. However, at every date, the debt outstanding must be lower than a certain level, which is an increasing function of the equity. In other words, the entrepreneur is borrowing constrained.

Within every period, the timing is as follows. First, the entrepreneur has to decide whether to go public or stay private. Going public implies selling stock to outside investors, assumed to be risk neutral. Then, before the productivity level becomes known, the entrepreneur decides how much to borrow. After the shock to the technology is realized, the effort choice is made, production is undertaken, and consumption, dividend distribution, and portfolio decisions are taken.

Going public is costly. First, there are fixed and proportional listing costs. Second, and more importantly, rational and forward-looking investors expect that, everything else equal, managerial effort will drop when the entrepreneur's equity stake falls. Thus the price at which subscribers will be willing to buy the stock is positively related to the stake retained by the initial owner. There are two benefits from going public. First, the IPO allows the firm to overcome the borrowing constraints that keep production at a sub-optimal level. Second, it gives the entrepreneur the chance to unload part of the risk she bears to risk-neutral investors.[3]

The decision to go public is triggered by a sudden and persistent change in total factor productivity. Such event has the effect of widening the gap between efficient and actual capital level in such a way that the marginal benefit deriving from the expansion of the operations outweighs the marginal cost due to issuing costs and absence of commitment.

[3]There is a considerable literature on the costs and benefits of going public. See Roell [29] for a comprehensive list and Pagano, Panetta, and Zingales [27] for an assessment of their relative empirical importance. My own modelling choices are motivated by the following considerations. First of all, there is large evidence (see Section 2) that at least in the United States, firms use the IPO proceeds in order to finance capital expenditures. I argue that this might be due to the existence of financing constraints that preclude firms from expanding through the issue of debt. The benefits of diversification, instead, arise naturally in the model once I assume that the entrepreneur is risk-averse and there exists a large number of diversified potential subscribers for the issue. Finally, the cost of going public that is caused by the decrease in the entrepreneur's equity stake also emerges endogenously, once I make the natural assumption that firm performance depends on managerial effort.

To my knowledge, this is the first paper that attempts to provide a rationalization for the observed post-IPO dynamics of operating performance. However, there is an extensive theoretical literature that focuses on other aspects of the IPO decision. Allen and Faulhaber [2], Benveniste and Spindt [4], and Welch [33] provide explanations for the underpricing phenomenon. Alti [3] and Cao and Shi [5] present models that produce clustering as an equilibrium outcome. Zingales [35] and Mello and Parsons [25] focus on the role of the IPO in maximizing the proceeds an initial owner obtains when selling her company. Pagano and Roell [28], Chemmanur and Fulghieri [6], and Subrahmanyam and Titman [31] consider the problem of an entrepreneur that wishes to raise new equity while retaining control and faces the choice between going public and selling stock to private financiers. Maksimovic and Pichler [24] analyze how the determinants of the IPO decision, its timing, and the magnitude of the underpricing, are affected by technological risks and by the information revealed by rival firms' offerings.

The remainder of this paper is organized as follows. In Section 2 I analyze in greater detail the empirical evidence on the post-IPO dynamics of operating performance indicators. I argue that such dynamics is not the result of either mismeasurement or accounting manipulation. Next, I review the explanations that have been conjectured so far, underlining their shortcomings. Finally, I provide a simple example that illustrates the mechanism at work in my model, to be introduced in Section 3. In Section 4 I characterize the dynamics implied by a version of the model, in which I abstract from the IPO. In Section 5 I analyze the determinants of the IPO decision. Section 6 describes the implications of the model for the dynamics of equity, capital expenditure, and profitability around the IPO date. Finally, Section 7 concludes and provides tentative directions for future research.

2 Operating Performance of IPO Firms: Empirical Evidence and Competing Explanations.

2.1 The Empirical Evidence.

At the beginning of Section 1, I recalled a number of recent papers that document the operating performance of IPO firms. As I have already stated, the finding common to all of these studies is that the post-issue operating performance of IPO firms, as measured by their Operating Return on Assets,[4] deteriorates in the aftermath of the IPO and keeps on doing so for a few more years. Here I focus on the evidence presented by Jain and Kini [20] (JK, from now on) and Mikkelson, Partch and Shah [26] (MPS).[5] The figures reported in Table 1 are borrowed from JK and represent median percentage changes with respect to the fiscal year preceding the IPO.[6] The data in the first row tells us that, at the median, the Operating Return on Assets falls in the IPO year and keeps on declining for two more years. It recovers a little in the following year, remaining however well below its pre-IPO level. Obviously these results could be driven by a generalized decline in profitability that hit the industries to which a large part of the IPO firms belong. This seems not to be the case. In fact, the values reported by JK remain statistically and economically significant also when adjusted for industry effect.[7]

One could also argue that in the aftermath of the IPO the denominator of the ROA (i.e. the book value of assets) increases because of the cash injection generated by the offering, while the numerator (i.e. the earnings) remains constant, since new assets become productive only with a lag. A first objection to this explanation is

[4]The Operating Return on Assets is defined as the ratio between operating income (before depreciation and taxes) and total assets.

[5]Degeorge and Zeckhauser [12] study a peculiar subset of IPO firms, namely reverse leveraged buyouts. Pagano, Panetta, and Zingales [27] consider a sample of firms that went public on the Milan Stock Exchange. Both sets of firms differ in many respects from the average US firm at its first IPO. In the United States IPO firms tend to be small and young firms that use the IPO proceeds in order to finance growth. The reverse LBO firms considered by Degeorge and Zeckhauser and the Italian IPO firms studied by Pagano, Panetta, and Zingales, are much larger and older entities, and tend to use the cash raised at the IPO in order to reduce their leverage.

[6]Year 0 is with the fiscal year in which the IPO takes place.

[7]JK compute the industry-adjusted change in operating performance as the difference between the change in operating performance as reported in Table 1 and the median change of all firms in the industry.

	Periods following the IPO			
	0	+1	+2	+3
Operating ROA	-3.58	-7.6	-10.53	-9.09
Operating Cash Flow/ Total Assets	-3.92	-7.92	-7.4	-6.44
Sales	37.22	69.92	108.09	143.15
Capital Expenditures	90.45	141.61	144.66	167.33

Table 1: Post-IPO Dynamics

Source: Jain and Kini (1994)

that, if it held, it would imply that the Operating ROA reaches its lowest value in year 0. According to Table 1 instead, it keeps on declining for a few more years. Furthermore, JK show that if the numerator is augmented by the interest proceeds on the cash raised at the IPO, the decrease in profitability turns out to be less important, but still significant.[8]

It has also been suggested that the decline in Operating ROA could be due to the discretion that managers enjoy when preparing income statements. The argument is that firms preparing to go public artificially inflate their earnings in order to influence investors' valuation. Teoh, Welch and Wong [32] provide evidence that supports this conjecture. They also argue that immediate and radical accounting reversals in the aftermath of the IPO are not feasible, because they would render earnings management activities transparent enough to trigger lawsuits against the management. For this reason, I conclude, the *creative accounting* explanation predicts that operating returns, after having peaked in the pre-IPO year, decrease smoothly over a few years, rather than display such a dramatic drop in the IPO year. JK test for the robustness of their finding by computing a measure of operating performance, cash-flows over book value of assets, that is much less vulnerable to accounting manipulation. The second row in Table 1 reports their results. While the magnitudes change, the finding seems to withstand the test. In light of the above discussion, from now on I will assume that the decline in the measures of operating performance is not an artifact, but actually attests a decrease in profitability.

[8]As an alternative robustness test, MPS scale the operating returns by sales, instead of assets. The dynamics of this ratio is very similar to the one displayed by the operating ROA.

2.2 Existing Explanations.

The empirical papers I am relying upon have conjectured other potential explanations for the phenomenon under study. Such explanations rely either on adverse selection or on moral hazard considerations. According to the adverse selection argument, the decline in operating performance would occur simply because insiders have private information with regard to their firms' prospects, and decide to take them public only when profitability is about to decline permanently. Leland and Pyle [22] suggest that, in presence of adverse selection, firms with good projects could separate themselves from firms with worse prospects, simply by retaining a significant ownership stake in the firm. Under some conditions, the latter would not find it profitable to imitate the strategy of the former. Thus, everything else equal, the adverse selection argument predicts a positive correlation between the stake retained by pre-IPO owners and post-IPO performance. According to the moral hazard hypothesis, first outlined by Jensen and Meckling [21], higher ownership retention by managers reduces their incentives to undertake value-reducing projects. Since IPOs are associated with a sudden and significant decline in managerial ownership, such hypothesis could actually explain the drop in profitability. The same argument also predicts that profitability is positively related to the equity stake retained by managers. JK and MPS attempt at testing the relevance of either theory for the problem at hand, reaching opposite results. My opinion is that the shortcomings of both testing strategies limit severely the scope of their conclusions. The main problem is that both papers fail to recognize that the equity stake retained by managers is endogenous and thus is correlated with a number of firm- and possibly industry- specific characteristics. I argue that the only way to control for such endogeneity is to build a model that links explicitly managerial actions to managerial ownership and other firm characteristics.

2.3 A New Explanation.

We have just seen that the moral hazard explanation relates the change in profitability to a structural change that characterizes firms at the IPO stage, namely the decrease in the managers' equity stake. In the same fashion, it seems sensible to investigate whether the IPO triggers other relevant changes in firms' fundamental characteristics and conduct. It is to such analysis that I turn now.

In the case of the United States, IPO firms tend to be smaller and younger than public firms,[9] and most of them use the IPO proceeds to finance novel capital expenditures. Both JK and MPS find that in their samples median capital expenditures and sales increase dramatically in the years following the IPO. MPS contribute further evidence by reporting that the median liabilities/assets ratio declines on impact but then rises. Given that in the years following the offering firms exhibit high growth in assets, I deduce that also liabilities increase in the aftermath of the IPO. Several scholars have found it hard to reconcile the post-IPO decrease in profitability with the concurrent increase in capital expenditures and sales. In this paper I argue that 1) the two phenomena are perfectly compatible, 2) they can be rationalized as implications of a standard dynamic model of firm growth, and 3) the same model is able to replicate further, non-IPO related, features of the life-cycle of a firm.

In recent years, the empirical research in Industrial Organization has produced a body of (by now) well-established findings on firm dynamics, that naturally complement the evidence reported above. Contributions by Hall [19], Evans [14], and Dunne, Roberts and Samuelson [13] suggest that large firms grow at a slower rate than smaller firms and young firms grow faster than older ones. Further, small firms pay fewer dividends, invest more (relative to size) and are more leveraged. Finally, small firms have higher values of Tobin's q.

Among the possible mechanisms that could be responsible for the observed relation between size, age, and growth, one that finds favor among scholars relies on the role of financing constraints, and in particular on the finding that such constraints seem to affect young and small firms more than older and larger ones.[10] In the last

[9]See, for example, Fama and French [15] and Gompers [17] for detailed evidences supporting this statement.

[10]Up to date, the most cited study in the area is by Fazzari, Hubbard and Petersen [16], who estimate reduced-form investment equations across groups of firms classified by their investment behavior. Their results indicate a substantially greater sensitivity of investment to cash flow for those firms that plow back most of their earnings. Whited [34] applies a methodology already used in consumption studies in order to study the effects of borrowing constraints on household

few years, a number of theoretical papers have shown that by incorporating borrowing constraints in an optimizing model of firm dynamics with decreasing returns to scale, it is possible to replicate the cited empirical evidence on the growth process. In brief, the mechanism is as follows. Consider a borrowing constrained, risk-neutral agent operating a deterministic technology that exhibits decreasing returns to physical capital, the only input. Her optimal strategy is to postpone consumption and reinvest all earnings at least until the marginal return equals the opportunity cost of capital. As the capital grows, its average return falls, and so does its growth rate.[11]

The basic insight of this paper is that, enriching this simple structure in order to accommodate the IPO decision, one obtains a model that, while retaining the properties just outlined, is able to replicate the evidence on firm dynamics around the IPO date. Here is an heuristic rendering of why this is the case. First, notice that the hypothesis of decreasing returns to scale is equivalent to assuming that business projects available to entrepreneurs are heterogenous with respect to their profitability, and that rational managers implement them in decreasing order of profitability. Think of a small, private firm endowed with a list of projects. Each of them requires the same level of capital expenditures and has the same level of operating costs. However, they differ in the cash-flows they generate. Figure 1 illustrates the example. Every bin represents a project. The surface of the bin measures the gross cash-flow. The level of total per-period costs is given by c_h. As a result, the net cash-flow of a given project is given by the area of the bin that lies above the line of height c_h. With costs c_h, only projects 1 through 4 produce strictly positive cash-flows. The firm will find it worth investing in all of them and no one else. It will do so, conditional on the availability of the necessary resources. Now

expenditures. This methodology exploits the observation that the investment Euler equation of the standard neoclassical model should be violated for firms that face financing constraints. Whited partitions the firms in her sample in two sets depending on several measures of financial distress and finds that the unconstrained Euler equation fails to hold for the a priori constrained firms, while it performs much better for the remaining firms.

[11] Among the models that exploit this mechanism are Clementi and Hopenhayn [7] and Cooley and Quadrini [9]. Clementi and Hopenhayn [7] show that the borrowing constraint arises as a feature of the optimal lending contract in an environment characterized by asymmetric information with respect to the firm's cash flow. Cooley and Quadrini [9] consider a general equilibrium model with heterogenous firms, whose operations are constrained by an exogenous financing constraint. The industry dynamics implied by their model displays many of the features I have described above.

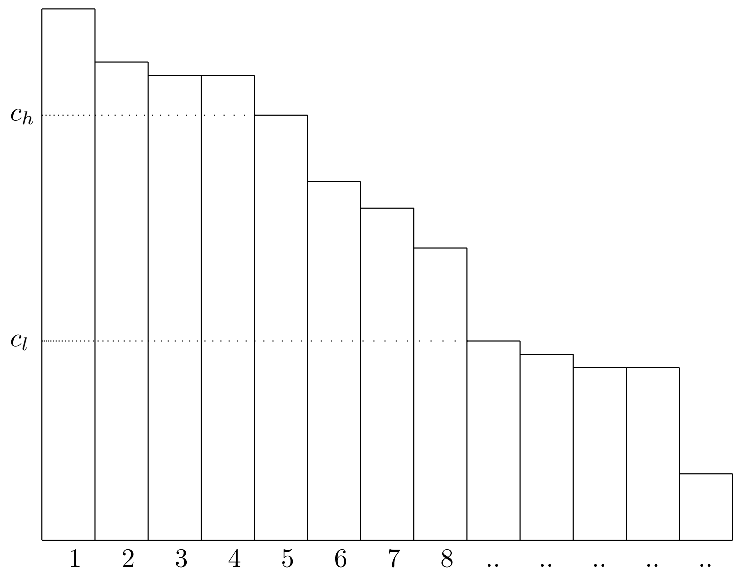

Figure 1: An Illustrative Example.

assume that a sudden technological improvement lowers the operating costs, so that total costs fall to the level c_l. The immediate effect is an increase in the measures of operating performance, because the decrease in costs translates one to one in an increase in net cash flows. A further effect is that projects 5 through 8 become viable. In case the management decides to raise the cash needed to finance all the projects through a public offering of equity, we would observe an increase in capital expenditures, sales, and total earnings, but also a deterioration in the measures of performance (i.e. average productivity) with respect to the pre-IPO levels. Further, if the program of capital expenditures takes time to implement, the deterioration of such measures will not occur instantaneously, but rather over a time.

3 The Model.

Time is discrete and the time horizon is infinite. I consider the decision problem as of time zero of a single agent (entrepreneur), who is endowed with an amount of cash M and a production technology.

In every period t, the technology produces one homogeneous good with inputs of physical capital and managerial effort (e_t), according to the function $F(k_t, e_t, z_t) = z_t k_t^{\gamma} e_t^{\varphi}$. The random variable $z_t \in Z$ indicates the level of total factor productivity and is distributed according to $G(z_t \mid z_{t-1})$. I assume decreasing returns to scale, i.e. $\gamma, \varphi > 0$, $\gamma + \varphi < 1$.

Preferences are defined by the utility function[12] $u(c_t, e_t) = \frac{1}{1-\xi}\left[\left(c_t - \frac{e_t^{1+\eta}}{1+\eta}\right)^{1-\xi} - 1\right]$, where $0 < \xi < 1$, $\eta > 0$, and where c_t denotes consumption at time t.

The entrepreneur can smooth consumption over time by investing in two assets, namely her company's equity and a riskless asset, yielding a constant per-period return r. I denote the book value of the firm's equity as k_t and the entrepreneur's stake in it as $\alpha_t \in [0, 1]$. I will say that the company is privately held when $\alpha_t = 1$, and publicly held when $\alpha < 1$. The entrepreneur's holding of the risk-free asset is denoted by a_t, $a_t \geq 0$. The restriction $a_t \geq 0$ implies that the entrepreneur cannot sell the riskless asset short (alternatively, she cannot borrow). Finally, I assume that the entrepreneur discounts future utility at the constant rate $\beta < \frac{1}{1+r}$.[13]

While the entrepreneur is not allowed to borrow to finance consumption, the firm can borrow (at the rate r) in order to increase the level of physical capital employed in production beyond the value of equity k_t. I assume that the firm can borrow up to a level that is a linear increasing function of the equity. Letting b_t denote debt, I impose that $b_t \leq sk_t$, $s > 0$.[14]

A further assumption is that in every period there is a positive and constant probability ρ_L that the company's assets become worthless. Since in such an event the firm would have to file for bankruptcy, from now on I will refer to it as the *bankruptcy* event (state).[15,16] For simplicity, I assume that in case the bankruptcy state real-

[12]This particular utility function is usually referred to as GHH because it was first introduced by Greenwood, Herkowitz and Huffman [18]. In the remainder of the paper it will be sometimes useful to rewrite it as the composition of two functions. That is, I will write $u(c, e) = v(x(c, e))$, where $v(x) = \frac{1}{1-\xi}\left[x^{1-\xi} - 1\right]$ and $x(c, e) = c - \frac{e^{1+\eta}}{1+\eta}$.

[13]The restriction $\beta < \frac{1}{1+r}$ is introduced to insure that the support of the distribution of asset holdings be bounded.

[14]The choice of imposing an exogenous borrowing constraint is motivated by the need to keep a rather complicated structure as tractable as possible. Recent papers by Albuquerque and Hopenhayn [1] and Clementi and Hopenhayn [7] show how the borrowing constraint can emerge as an endogenous feature of the optimal lending contract in environments characterized by enforcement problems and asymmetric information, respectively.

[15]A further restriction on the parameter space is needed to insure that in all states of nature in which the company is viable, total resources are enough to pay back principal and interest. Such restriction is $s < \frac{1}{r}$. It is easy to check that it implies $k_t < sk_t r$, where sk_t is the maximum level of debt when the book value of equity is k_t.

[16]Notice that my assumptions imply that firm's debt is risky. However, the rate the firm pays on

izes, the entrepreneur is not allowed to start a new company. After bankruptcy, her risk-free asset holdings will be her only source of income.

In every period, the timing is the same for public and private firms, with one important exception. If the firm is private, the entrepreneur has to decide whether or not to take it public. I assume that the IPO decision is taken at the very beginning of the period. If she decides to go public, the entrepreneur chooses the equity stake she is going to retain after the IPO. The remaining of the equity of the new public company is sold to outside risk-neutral investors. Going public is costly. I assume a fixed cost N and proportional costs in measure of a fraction d of the IPO proceeds. I also assume that the entrepreneur's equity stake α is irrevocably set at the IPO. This implies that the entrepreneur will not be able to either sell part of her holdings in the open market, conduct a seasoned offering, or take the company private. These assumptions are made in order to insure tractability of an already very complex problem.[17] In later sections, as I present the results, I will discuss at length the role of these hypothesis and I will argue that the most relevant achievements of my analysis do not depend on them.

The timing of the remaining events does not depend on the status of the firm (public of private). I describe it with the help of Figure 2. In case bankruptcy hits, the entrepreneur faces a simple deterministic consumption/saving problem. She has to choose the optimal allocation of her wealth between consumption and savings. The decision problem upon survival is more interesting. Before the state of productivity z_t is observed, the entrepreneur decides how much debt to issue (b_t). Once z_t becomes known, the entrepreneur takes the effort decision e_t. Then production occurs. Finally, the agent consumes and chooses next period's book equity level k_{t+1} and risk-free asset holdings a_{t+1}.

The decision problem of the entrepreneur can be written as a sequence of two dynamic programming problems, which I am going to describe in the remainder of

it is set exogenously at the level r. From the discussion that follows, it will be clear that allowing the borrowing rate to depend on the risk of cash flows would considerably complicate the analysis, without having any impact on the most relevant results of the paper.

[17]In Appendix C I outline the technical issues that prevent the characterization of the solution, once these simplifying assumptions are dispensed with.

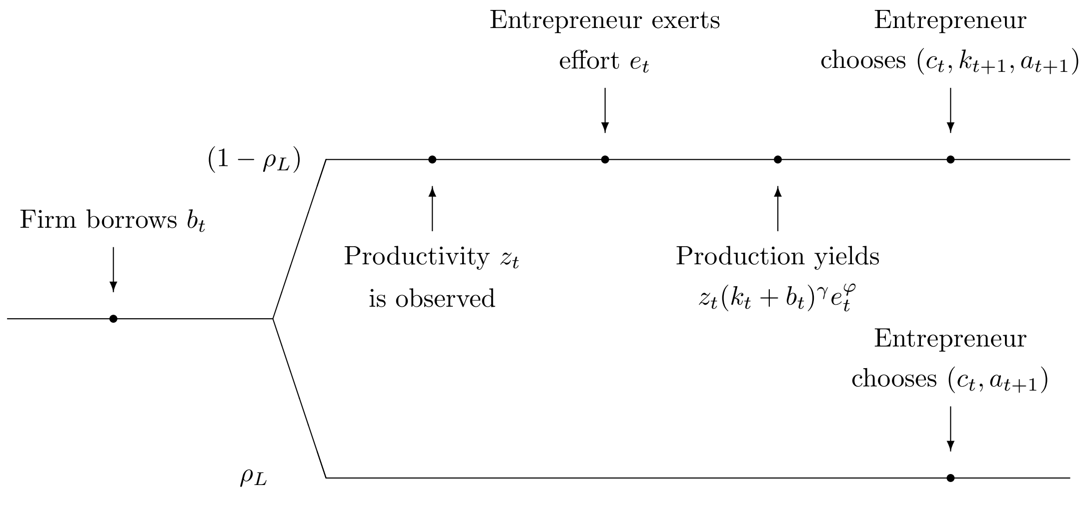

Figure 2: The Timing.

this section. In doing so, I proceed backwards. First, I describe the problem when the company is public. Then I move on to discuss the decision problem when the firm is private.

3.1 The Post-IPO Decision Problem.

Let $W(\alpha, a, k, z_{-1})$ represent the expected discounted utility of the entrepreneur when she holds the riskless asset in quantity a, total book equity is k, her stake in it is α, and the productivity state in the previous period was z_{-1}. Then $W(\alpha, a, k, z_{-1})$ solves the functional equation (P1):

$$
\begin{aligned}
W(\alpha,&a, k, z_{-1}) = \qquad\qquad \text{(P1)}\\
&\max_{b<sk}\left\{(1-\rho_L)\int\left[\max_{\substack{c_{(z)},e_{(z)}\\ a'(z),k'_{(z)}}} u(c_{(z)},e_{(z)})+\beta W(\alpha,a'_{(z)},k'_{(z)},z)\right]dG(z\mid z_{-1})+\right.\\
&\qquad\left.+\rho_L\max_{c_L,a'_L} u(c_L,0)+\beta W_L(a'_L)\right\},\\
&\qquad s.t.\ c_{(z)}=\alpha\left[k+z(k+b)^\gamma e^\varphi_{(z)}-rb-k'_{(z)}\right]+a(1+r)-a'_{(z)}\ \ \forall\, z, \qquad (1)\\
&\qquad\quad k+z(k+b)^\gamma e^\varphi_{(z)}-rb-k'_{(z)}\geq 0\ \ \forall\, z, \qquad (2)\\
&\qquad\quad c_L=a(1+r)-a'_L, \qquad (3)\\
&\qquad\quad a'_{(z)},a'_L\geq 0\ \ \forall\, z. \qquad (4)
\end{aligned}
$$

Conditional on the choice of b, the value of the entrepreneur is given by the average of two terms. The first one represents the expected discounted utility, conditional on survival of the firm, while the second one is the expected discounted utility, conditional on bankruptcy. In the latter case the physical capital is unproductive, thus the entrepreneur will exert no effort. From next period on, she will consume out of her risk-free asset holdings only. The function $W_L(a'_L)$ yields the continuation value of the entrepreneur, conditional on the occurrence of the bankruptcy state.[18]

Condition (1) is the budget constraint in all states in which the company is viable. The expression in square brackets corresponds to dividends. In fact, dividends equal the difference between total resources (i.e. initial equity plus revenues minus interest payment) and tomorrow's equity. The entrepreneur holds the claim to a fraction α of the distributed dividends. Consumption equals dividends plus the gross proceeds from the investment in the asset a, minus next period holding a'. Condition (2) makes sure that dividends are always non-negative.[19]

Equation (3) is the budget constraint in case bankruptcy occurs. Since the firm's assets are not productive, dividends are identically zero.

The solution to the functional equation (P1) yields policy functions (optimal decision rules) for debt issue (b^*), managerial effort ($e^*_{(z)}$), book equity ($k'^*_{(z)}$), and risk-free asset holdings ($a'^*_{(z)}$). Such functions can be used in order to compute the expected discounted sum of dividends distributed by the firm, which coincides with the market value of equity in the case in which shareholders are risk neutral. Given a set of state variables (α, a, k, z_{-1}), the equity's market value is given by the *price function* $P(\alpha, a, k, z_{-1})$ that solves the functional equation

$$P(\alpha, a, k, z_{-1}) =$$
$$(1-\rho_L) \int \left[(k + z(k+b^*)^\gamma e^{*\varphi}_{(z)} - rb^* - k'^*_{(z)}) + \frac{1}{1+r} P(\alpha, a'^*_{(z)}, k'^*_{(z)}, z) \right] dG(z \mid z_{-1}). \tag{5}$$

[18] The function $W_L(a'_L)$ is computed analytically in Appendix A.

[19] In other words, this conditions insures that equity increases only through the reinvestment of earnings.

3.2 The Pre-IPO Decision Problem.

When the firm is private, the functional equation describing the entrepreneur's problem is slightly more involved. The IPO decision process can be decomposed into three steps. First, the entrepreneur needs to figure out the value of staying private at least one more period. Then she will gauge the value of going public. Finally, she will decide for the opportunity that yields the highest expected discounted utility. The first step is the simplest. Let $V(a, k, z_{-1})$ represent the continuation utility, conditional on keeping the company private at least one more period.[20] Then, if the entrepreneur decides not to go public, her value is given by $V_{pr}(a, k, z_{-1})$, where

$$\begin{aligned} V_{pr}(a,k,z_{-1}) = \max_{b<sk} \Bigg\{ (1-\rho_L) \int & \left[\max_{\substack{c_{(z)}, e_{(z)} \\ a'_{(z)}, k'_{(z)}}} u(c_{(z)}, e_{(z)}) + \beta V(a'_{(z)}, k'_{(z)}, z) \right] dG\left(z \mid z_{-1}\right) + \\ & + \rho_L \max_{c_L, a'_L} u\left(c_L, 0\right) + \beta W_L(a'_L) \Bigg\}, \\ & s.t.\ c_{(z)} = \left[k + z(k+b)^{\gamma} e^{\varphi}_{(z)} - rb - k'_{(z)} \right] + a(1+r) - a'_{(z)} \ \ \forall\ z, \\ & k + z(k+b)^{\gamma} e^{\varphi}_{(z)} - rb - k'_{(z)} \geq 0 \ \ \forall\ z, \\ & c_L = a(1+r) - a'_L, \\ & a'_{(z)}, a'_L \geq 0 \ \ \forall z. \end{aligned} \tag{P2}$$

The second step is modelled as follows. The entrepreneur enters the new period with a portfolio allocation (a, k). Let I denote her contribution to the equity of the new public company. Then, if a_{ipo} denotes her holdings of risk-free asset in the aftermath of the IPO, her budget constraint reads as follows:[21]

$$a_{ipo} = a + (k - I). \tag{6}$$

[20] Recall that I have assumed that at time zero the entrepreneur owns all of the equity and does not have other means to sell shares other than the IPO. This implies that before the IPO takes place, $\alpha = 1$.

[21] In the case in which $I > k$, the entrepreneur subscribes a fraction of the newly issued shares. Instead, in the case in which $I < k$, the entrepreneur is actually selling part of her stock holding to new investors. That is, she is conducting a secondary offering.

In addition, the entrepreneur has to determine k_{ipo}, i.e. the magnitude of the firm's equity after the offering, and the stake she is going to retain (α). These two values are jointly determined by the following condition:

$$k_{ipo} - I = (1-d)(1-\alpha)P(\alpha, a_{ipo}, k_{ipo}, z_{-1}) - N. \tag{7}$$

Equation (7) is an accounting relation. The left-hand side represents the contribution of outside shareholders to the equity of the public company. The right-hand side consists of the net proceedings of the IPO. Substituting (6) in (7), I obtain the restriction

$$a_{ipo} = [a + k - N] + (1-d)(1-\alpha)P(\alpha, a_{ipo}, k_{ipo}, z_{-1}) - k_{ipo}. \tag{8}$$

For given choices of α and k_{ipo}, condition (8) yields the volume of post-IPO risk-free asset holding a_{ipo}. Finally, the value of going public is given by $V_{ipo}(a, k, z_{-1})$ as determined by

$$\begin{aligned} V_{ipo}(a, k, z_{-1}) = \max_{\alpha, k_{ipo}} \; & W(\alpha, a_{ipo}, k_{ipo}, z_{-1}), \\ & s.t. \; (8), \\ & \quad \alpha \in [0,1], \\ & \quad a_{ipo} \geq 0. \end{aligned} \tag{P3}$$

The final step consists of comparing the value of staying private with the value of going public and opting for the solution that yields the highest reward. This means that the expected discounted utility for the entrepreneur when the company is private equals the maximum of the two. Thus the value function $V(\alpha, a, k, z_{-1})$ must solve the functional equation given by (P2), (P3), and (P4) below, with $P(\alpha, a_{ipo}, k_{ipo}, z_{-1})$ satisfying (5).

$$V(a, k, z_{-1}) = \max\left\{V_{pr}(a, k, z_{-1}), V_{ipo}(a, k, z_{-1})\right\}. \tag{P4}$$

4 Post-IPO Dynamics.

In this section I study the dynamics of the firm after the IPO has taken place. In other words, I characterize the solution to the decision problem described by the program (P1). Such solution consists of a set of policy functions (optimal decisions rules) for managerial effort, dividend distribution, capital expenditures, and debt. It will also be of interest to understand how the evolution of these variables differ from that implied by existing dynamic optimizing models of entrepreneurial behavior.[22] It is worth reminding which innovations I introduce with respect to this literature. First of all, in my model production depends on managerial effort. Second, the entrepreneur is risk-averse. Finally, the entrepreneur has access to a further (risk-free) financial asset, besides her firm's equity. All of these hypothesis were introduced in order to accommodate the IPO decision. However, it is easy to argue that they make for a much closer approximation of reality.

I begin by describing the determinants of the effort level e. Given the particular choice of preferences I have made, there is no wealth effect on the effort choice. This implies that when (2) does not bind (i.e. when dividends strictly positive), managerial effort depends on contemporary variables only. Namely, the productivity level z, the capital employed in production $(k+b)$, and the stake α. It is easy to show that

$$e = (\alpha z \varphi)^{\frac{1}{1+\eta-\varphi}} (k+b)^{\frac{\gamma}{1+\eta-\varphi}} . \tag{9}$$

As common sense dictates, conditional on capital and productivity level, managerial effort is increasing in the stake α. However, as we will see in Section 6, this does not imply that the IPO, by bringing about a decrease in α, causes a decline in the effort supplied by the entrepreneur. The reason is that, given the complementarity of the factors in production, effort is an increasing function of the capital in place and the productivity parameter z. Therefore, the change in effort at the IPO stage will also depend on the evolution of the random variable z and of the endogenous variables k and b.

[22] See Cooley and Quadrini [8] for example.

In Section 4.1 I describe the optimal dividend distribution and portfolio allocations policies of the entrepreneur, in a version of the model without bankruptcy state. This exercise will be helpful in building intuition for the predictions of the general model with bankruptcy, to be discussed in Section 4.2.

4.1 The Model Without Bankruptcy.

An important observation is that there exists a level of equity beyond which the entrepreneur becomes indifferent between reinvesting the earnings in the business (i.e. increasing k) and distributing them to shareholders (i.e. increasing a). The key to understanding why this is so is to consider that, because of decreasing returns to scale, there exists a scale of production $(k+b)$ such that any further unit of physical capital would be rented out at the rate r rather than being used in production. Hence, once that scale is reached, equity and the asset a yield the same marginal (riskless) return r. The two assets become perfect substitutes at the margin. The analysis that follows gives formal support to this reasoning.

Let $x_{(z)} = c_{(z)} - \frac{e_{(z)}^{1+\eta}}{1+\eta}$. This quantity is the aggregate of consumption and effort that constitutes the argument of the utility function when the productivity shock takes the value z. Using (9) along with the budget constraint (1), I can write $x(z)$ as

$$x_{(z)} = \alpha \left\{ f\left[\alpha, (k+b), z\right] - r(k+b) \right\} + (a + \alpha k)(1+r) - (a'_{(z)} + \alpha k'_{(z)}),$$

where

$$f\left[\alpha, (k+b), z\right] = (\alpha z)^{\frac{\varphi}{1+\eta-\varphi}} \left[\varphi^{\frac{\varphi}{1+\eta-\varphi}} - \frac{\varphi^{\frac{1+\eta}{1+\eta-\varphi}}}{1+\eta} \right] (k+b)^{\frac{\gamma(1+\eta)}{1+\eta-\varphi}}.$$

Given the assumption of decreasing returns, the expression $f\left[\alpha, (k+b), z\right] - r(k+b)$ is strictly concave in the scale of production $(k+b)$ and admits a unique maximum. Thus, for every α and for every realization of z, there will be a scale such that any unit of capital beyond it, if employed in production, will yield a return strictly lower than r. In turn, this implies that, as long as the set Z is bounded, for every pair (α, z) there exists a threshold $k(\alpha, z)$ such that any unit of equity beyond it will be rented out (earning the riskless return r), rather than being employed in production. In other words, any increase of equity beyond the threshold will correspond to a decrease in b of the same magnitude. Notice that, once the threshold $k(\alpha, z)$ is reached, the value achieved keeping on accumulating equity (i.e. plowing back earnings) while decreasing b, can also be obtained by holding k constant and distributing earnings so to increase a. This occurs because the marginal returns of the assets a and k are the same. In subsection A.2 I illustrate the argument in yet another way, by showing that for levels of equity greater than $k(\alpha, z)$, the value function W exhibits linear indifference sets (with slope $-\alpha$) in the space (a, k).

In light of this discussion, I decide to introduce the following transformation on the state space. I define $h \equiv a + \alpha k$ as the book value of the entrepreneur's accumulated wealth (total wealth, for simplicity), so that the state variables now are (α, h, k, z_{-1}). For every triple (α, h, z) there will be a level of equity $k^*(\alpha, h, z)$ such that when $k > k^*(\alpha, h, z)$ the entrepreneur is indifferent between couples (a, k) that satisfy the restriction $a+\alpha k = h$, provided that $a \geq 0$. The value function of the problem, redefined on the new state space, will be linear in the k dimension for every level of equity k such that $k^*(\alpha, h, z) \leq k \leq \frac{h}{\alpha}$. Once the level of capital $k^*(\alpha, h, z)$ is reached, the solution of the problem is not unique anymore. This occurs for the reason outlined earlier in this section. For $k \geq k^*(\alpha, h, z)$ the dividend distribution policy is indeterminate.

4.2 The Complete Model.

Following the introduction of the new state variable h, the functional equation (P1) is recast in the following terms:[23]

[23] In order to avoid introducing further notation, I denote the new value function with the letter W.

$$
\begin{aligned}
W(\alpha, & h, k, z_{-1}) = && \text{(P5)}\\
& \max_{b<sk}\left\{(1-\rho_L)\int\left[\max_{\substack{c_{(z)},e_{(z)}\\ h'_{(z)},k'_{(z)}}} u(c_{(z)},e_{(z)})+\beta W(\alpha,h'_{(z)},k'_{(z)},z)\right]dG(z\mid z_{-1})+\right.\\
& \quad\left.+\rho_L\max_{c_L,h'_L}\left[u(c_L,0)+\beta W_L(h'_L)\right]\right\},\\
& \quad s.t.\ c_{(z)}=\alpha\left[z(k+b)^{\gamma}e^{\varphi}_{(z)}-r(k+b)\right]+h(1+r)-h'_{(z)}\ \ \forall z, && (10)\\
& \quad\quad z(k+b)^{\gamma}e^{\varphi}_{(z)}-r(k+b)\geq k'_{(z)}-k(1+r)\ \ \forall z, && (11)\\
& \quad\quad k'_{(z)}\leq\frac{h'_{(z)}}{\alpha}\ \ \forall z, && (12)\\
& \quad\quad c_L=(h-\alpha k)+h(1+r)-h'_L, && (13)\\
& \quad\quad h'_{(z)},h'_L\geq 0\ \ \forall z.
\end{aligned}
$$

The new budget constraints (10) and (13) are obtained by rearranging the original constraints (1) and (3), respectively. The same holds for condition (11), which is a re-elaboration of (2). Thus (11) imposes that dividends are non-negative. Finally, condition (12) is implied by the constraint $a' > 0$.[24] To see that the program (P5) is equivalent to the original problem (P1), notice that any choice of $(h'_{(z)}, k'_{(z)})$ such that (10)-(13) are satisfied corresponds to a feasible couple $(a'_{(z)}, k'_{(z)})$ in the original problem.

The transformation I have introduced has an economic interpretation. It is equivalent to a redefinition of the assets available to the entrepreneur. The asset h is a risk-free asset that yields a constant return r, while the asset k yields a per-period return equal to the surplus of production $\frac{z(k+b)^{\gamma}e^{\varphi}-r(k+b)}{k}$. The isomorphism is not exact because the constraint $a'_{(z)} \geq 0$ implies that condition (12) must hold.

The introduction of the bankruptcy event eliminates the indeterminacy of the dividend distribution policy that I have outlined in the previous section. I recall here that if the bankruptcy state occurs, the assets (k) invested in the firm become worthless. The entrepreneur is left with her holdings of the risk-free asset a. This

[24] The function $W_L(h)$, that yields the continuation value for the entrepreneur following bankruptcy, is unchanged. In fact after bankruptcy the equity is identically zero and thus $h \equiv a$.

implies that investing in such asset is the only way for the entrepreneur to insure against the adverse event. Thus, even when k and a yield the same marginal (riskless) return r, their marginal values to the entrepreneur will be different. She will always prefer to distribute earnings in terms of dividends, rather than plowing them back and increase the firm's equity.

I am now ready to introduce three propositions that state basic properties of the value function W.[25]

Proposition 1 *There exists one and only one function W that solves the functional equation (P5).*

Proposition 2 *The function W is jointly strictly concave in (h, k) for every (α, z).*

Proposition 3 *For every (α, h, z) there exist a value $k^*(\alpha, k, z) < \frac{h}{\alpha}$ such that $W(\alpha, h, k, z) < W(\alpha, h, k^*, z)$ for all $k \in \left(k^*, \frac{h}{\alpha}\right]$.*

Strict concavity of the value function is again a consequence of the introduction of the bankruptcy state. It implies that the policy correspondence $k'^*_{(z)}$ is always single-valued and therefore the dividend distribution policy is never indeterminate. Proposition 3 simply states that since the utility function satisfies the Inada condition at the origin, it will always be the case that $\alpha k < h$ or, alternatively, that the entrepreneur holds the riskless asset a in strictly positive quantity. Since for $c = 0$ the marginal utility of consumption is infinite, the entrepreneur will never run the risk of being surprised by the bankruptcy event with no wealth invested in the riskless asset.

Figure 3 shows the evolution over time of the most relevant firm-related variables, as predicted by the model. The time paths constitute the outcome of a simple simulation exercise carried out using a parameterized version of the model.[26] In order to keep the computational burden low, I employ a very simple stochastic structure. I assume that the random variable z is *i.i.d.* and can take either one of two values. I set the entrepreneur's initial endowment M and I run a large number

[25] Proofs of the propositions are reported in Appendix B.

[26] The algorithm that solves for the functions W and P is described in subsection C.1.

of 40-period long simulations. In Figure 3 I plot first moments against time. The remainder of this section is dedicated to the analysis of such dynamics.

4.2.1 Growth and Dividend Distribution Policy.

Consider the entrepreneur at time zero, endowed with a relatively low initial wealth M. If there was no chance of failure (i.e. if there was no bankruptcy state) and she could sell the riskless asset short, the entrepreneur would start by borrowing heavily (i.e. by setting $a < 0$) to finance consumption and by plowing back all earnings in order to increase equity and the scale of production. Consumption would be set in such a way to equate marginal utility over time and across states. However, when the probability of failure ρ_L is strictly positive, the entrepreneur will hold the asset a in positive amount from the outset. Once again, this occurs because upon bankruptcy the risk-free asset will be her only source of income. Since the marginal value of reinvesting earnings decreases as the scale of production increases, the entrepreneur will choose an increasing path for consumption. As the marginal value of increasing the scale decreases, so will the marginal utility of current consumption. Given the complementarity in production between effort and physical capital, managerial effort will also increase over time. The dynamics of debt tracks closely the one of equity, because as long as marginal returns on capital are high, the firm will borrow the maximum allowed by the constraint, i.e. $b_t = sk_t$.[27] Dividends will be also increasing over time, since they have to finance an increasing level of consumption and the growth in the holding of riskless asset. Such holding must increase, because the entrepreneur equates marginal utility across states of nature.

Notice that the model predicts that older and larger companies grow at a slower rate and pay more dividends. These findings are qualitatively consistent with the empirical evidence mentioned in Section 2.[28]

[27] In the example represented in Figure 3, I have set $s = 1$. This is the reason why the paths for equity and debt coincide.

[28] One might argue that the prediction that firms pay dividend from the outset is at odd with the empirical evidence, in that the typical start-up company does not pay dividends at all. However, it must be considered that in my setup dividends include also the entrepreneur's compensation.

4.2.2 Scale of Production and Production Efficiency.

A further prediction of the model is that the company is borrowing constrained at every point in time. The reason why this is the case is that if the entrepreneur reached a level of equity such that the firm was unconstrained, she could increase her life-time utility by distributing part of that equity in forms of dividends. In fact the only effect of doing so would be to enlarge the consumption possibility set in the bankruptcy state. Analytically, this can be seen by writing the optimality condition for k' as follows:

$$(1-\rho_L)\left(1+\frac{\partial b'}{\partial k'}\right)\int \frac{dv(x(z))}{dx(z)}\frac{\partial x\left(z\right)}{\partial (k'+b')}\,dG(z\mid z_{-1})=\rho_L\frac{dv(c_L)}{dc_L}. \tag{14}$$

Necessary condition for (14) to hold is that $\frac{\partial b'}{\partial k'}>-1$. In turn, this implies that the optimality condition for b' must hold with strict inequality. That is:

$$\int \frac{dv(x(z))}{dx(z)}\frac{\partial x\left(z\right)}{\partial (k'+b')}\,dG(z\mid z_{-1})>0.$$

Finally, this means that the entrepreneur is borrowing constrained, or $\frac{\partial b'}{\partial k'}=s$.

Thus, under my assumptions it turns out to be optimal for the entrepreneur to keep the scale of production at an inefficient level. This result would not hold if the entrepreneur had the opportunity to transfer her risk to the firm's outside financiers. Obviously this is not a new finding. It just restates that when the access of entrepreneurs to risk-sharing opportunities is limited, the penalty inflicted by the bankruptcy code to defaulting entrepreneurs becomes an important determinant of firm size. The greater the punishment inflicted to the entrepreneurs, the higher will be their need to insure against their businesses' failure, and thus the lower the scale of production.

4.2.3 The Market Value of Equity.

Given the transformation I have operated on the state space, I also redefine the price function. Now the function $P(\alpha,h,k,z_{-1})$ solves the functional equation

$$
P(\alpha, h, k, z_{-1}) =
$$

$$
(1-\rho_L) \int \left[z\,(k+b^*)^\gamma \, e^{*\varphi}_{(z)} - r(k+b^*) + \frac{1}{1+r} P(\alpha, h'^*_{(z)}, k'^*_{(z)}, z) \right] dG\,(z \mid z_{-1})\,. \tag{15}
$$

Notice that, in the new definition, $P(\alpha, h, k, z_{-1})$ is equal to the expected discounted value of the surplus generated in production. Here I define the surplus as the difference between the value of production and the opportunity cost of the physical capital employed as input. Having obtained the value P, it is easy to recover the market value of equity, given by the expected discounted sum of dividends and defined by equation (5). Figure 4 shows an example of the price function as computed by my computer code, on the space (α, k). Notice in particular that the surplus increases with the equity stake held by the entrepreneur. This occurs because, by (9), the effort level is strictly increasing in α.

5 The IPO Decision.

This section is dedicated to the analysis of the entrepreneur's decision problem when the company is private. In particular, I illustrate the determinants of the IPO decision and I provide economic intuition that will help the reader appreciate the results reported in Section 6.

The properties that I am going to characterize hold for the general model, but they are more transparent in the case of a simplified environment that allows for closed form expressions for the value functions. It is to such environment that I turn now. Assume that the entrepreneur lives for two period only, indexed by $t = 1, 2$. Further, let $z \equiv 1$ and $b \equiv 0$ (i.e. assume that total factor productivity is constant and that the firm cannot borrow). In period 2, conditional on survival, an entrepreneur holding a stake $\alpha \in [0, 1]$ faces the following simple problem:

$$
\max_{c,e} u(c, e),
$$

$$
s.t \quad c = \alpha \left[k + z k^\gamma e^\varphi \right] + a(1+r).
$$

The value of this program is given by $v[y + a(1 + r)]$, where $y \equiv \alpha k + \left(\varphi^{\frac{\varphi}{1+\eta-\varphi}} - \frac{\varphi^{\frac{1+\eta}{1+\eta-\varphi}}}{1+\eta}\right) \alpha^{\frac{1+\eta}{1+\eta-\varphi}} k^{\frac{\gamma(1+\eta)}{1+\eta-\varphi}}$. Instead, conditional on bankruptcy, the entrepreneur's value is just $v[a(1 + r)]$. As a consequence, the value of going public at the beginning of period 2 is given by

$$
\begin{aligned}
V_{ipo}(a, k) = \max_{\alpha, k_{ipo}} \; & (1 - \rho_L) v[y_{ipo} + a_{ipo}(1 + r)] + \rho_L v[a_{ipo}(1 + r)], \\
s.t. \quad & y_{ipo} = \alpha k_{ipo} + \left(\varphi^{\frac{\varphi}{1+\eta-\varphi}} - \frac{\varphi^{\frac{1+\eta}{1+\eta-\varphi}}}{1+\eta}\right) \alpha^{\frac{1+\eta}{1+\eta-\varphi}} k_{ipo}^{\frac{\gamma(1+\eta)}{1+\eta-\varphi}}, \\
& a_{ipo} = [a + k - N] + (1 - d)(1 - \alpha) P(\alpha, k_{ipo}) - k_{ipo}, \\
& P(\alpha, k_{ipo}) = k_{ipo} + (\alpha\varphi)^{\frac{\varphi}{1+\eta-\varphi}} k_{ipo}^{\frac{\gamma(1+\eta)}{1+\eta-\varphi}}.
\end{aligned}
$$

It is interesting to notice that the value of going public does not depend on the equity of the private company (k), but rather on $h \equiv a + k$, the book value of the entrepreneur's accumulated wealth. The reason is that at the IPO stage the entrepreneur is free to change the composition of her portfolio, and the market value of the firm depends on the post-IPO level of book equity.

A first conclusion is that, for every $\alpha \in [0, 1]$, the optimal choice of equity $k_{ipo}(\alpha)$ is bounded. To see that this is the case, notice that because of the complementarity of effort and capital in production, the term y_{ipo} is clearly monotone in k_{ipo}. For low levels of k_{ipo}, the same holds for a_{ipo}. Eventually, however, for high enough levels of k_{ipo}, the term a_{ipo} becomes decreasing in k_{ipo}. This occurs because for given α, the function P is strictly concave in the capital input. As a result, for every α there will be a value for k_{ipo} such that $a_{ipo} = 0$. Since the utility function satisfies Inada conditions at the origin, it will never be optimal for the entrepreneur to increase the equity up to such level.

A second, important conclusion, is that the entrepreneur will never choose to dispose of all of her equity stake. That is, she will never set $\alpha = 0$. In fact, for $\alpha = 0$, the value of the company is given by the book value of assets (i.e. $P(0, k_{ipo}) = k_{ipo}$). If the entrepreneur chose to liquidate completely her position, it would follow that $y_{ipo} = 0$ and $a_{ipo} = [a + k] - N - dk_{ipo}$. In turn, this implies that the choice of going public and setting $\alpha = 0$ is strictly dominated by the decision to keep the company private.

Depending on the parameterization, the optimal choice of k_{ipo} may or may not be monotone in the equity stake α. Again because of complementarity in production between effort and physical capital, the cross derivative $\frac{\partial^2 y_{ipo}}{\partial k_{ipo} \partial \alpha}$ is always strictly positive. This implies that y_{ipo} is supermodular and thus maximization of its value would require k_{ipo} to be increasing in α. On the other hand, simple inspection reveals that the cross derivative $\frac{\partial^2 a_{ipo}}{\partial k_{ipo} \partial \alpha}$ is positive for low values of α_{ipo}, and negative for high values. The intuition behind this result is simple. Notice that $\frac{\partial a_{ipo}}{\partial \alpha} = -(1-d)P + (1-d)(1-\alpha)\frac{\partial P}{\partial \alpha}$. For given market value P, an increase in the stake α raises the resources that the entrepreneurs commits to the company. On the other hand, it also raises the value of the firm P and thus IPO proceeds. In absolute value, both of these marginal effects are increasing in k_{ipo}. It turns out that the first prevails for low levels of α, while the second prevails for high levels. I conclude that the schedule $k_{ipo}(\alpha)$ is definitely increasing for low level of the stake. At higher levels, the slope will still be positive if the term $[a + k - N]$ is large enough. Otherwise, there will be a level for α above which the schedule is decreasing.

Let's now go back to the general version of the model. Once I introduce the transformation $h = a + \alpha k$, the value of going public is determined by

$$
\begin{aligned}
V_{ipo}(h, k, z_{-1}) = \max_{\alpha, h_{ipo}, k_{ipo}} & W(\alpha, h_{ipo}, k_{ipo}, z_{-1}), \\
s.t.\ & h_{ipo} + (1-\alpha)k_{ipo} - h + N = \\
& = (1-d)(1-\alpha)\left[k_{ipo} + P(\alpha, h_{ipo}, k_{ipo}, z_{-1})\right], \\
& 0 < k_{ipo} \leq \frac{h_{ipo}}{\alpha}, \\
& \alpha \in [0, 1].
\end{aligned}
\tag{P6}
$$

Given the transformation on the state space I have introduced in the last section, the value of staying private $V_{pr}(h, k, z_{-1})$ is now given by

$$
V_{pr}(h,k,z_{-1}) = \max_{b \le sk} \left\{ (1-\rho_L) \int \left[\max_{\substack{c_{(z)}, e_{(z)} \\ h'_{(z)}, k'_{(z)}}} u(c_{(z)}, e_{(z)}) + \beta V(h'_{(z)}, k'_{(z)}, z) \right] dG(z \mid z_{-1}) + \right.
$$

$$
\left. + \rho_L \max_{c_L, h'_L} \left[u(c_L, 0) + \beta W_L(h'_L) \right] \right\}, \tag{P7}
$$

$$
\begin{aligned}
& s.t.\ c_{(z)} = \left[z(k+b)^{\gamma} e^{\varphi}_{(z)} - r(k+b) \right] + h(1+r) - h'_{(z)} \ \ \forall\, z, \\
& z(k+b)^{\gamma} e^{\varphi}_{(z)} - r(k+b) \ge k'_{(z)} \ \ \forall\, z, \\
& k'_{(z)} \le h'_{(z)} \ \ \forall\, z, \\
& h'_{(z)}, k'_{(z)} > 0 \ \ \forall\, z, \\
& c_L = [h - \alpha k](1+r) - h'_L.
\end{aligned}
$$

Then, the value of the entrepreneur when the company is private solves the functional equation given by programs (P6), (P7), and (P8):

$$
V(h,k,z_{-1}) = \max \left\{ V_{pr}(h,k,z_{-1}), V_{ipo}(h,k,z_{-1}) \right\}. \tag{P8}
$$

While useful in order to easily understand results that apply to the general model, the simplified scenario considered earlier in this section does not help to understand what actually triggers the IPO or, alternatively, why the incentives to go public change over time. As it will become clear in the next section, the timing of the IPO depends crucially on the stochastic process governing the dynamics of the variable z. The opportunity of going public can be thought of as an option. The problem of the entrepreneur is to figure out the best time to exercise it. As long as the firm is privately held, the entrepreneur compares the value of going public with the value of waiting at least one more period. both of these values depend crucially on the evolution of z. Assume, for example, that z exhibits persistence. This implies that when total factor productivity is low, it will be expected to remain relatively low in the following periods. In such a scenario, the entrepreneur will be able to raise only a limited amount of cash by going public and, importantly, will not have access to further offerings in the future. Therefore the option to exercise is likely to be out of the money. However, when z is relatively high, prospects for the near future will look good and the firm will be able to raise much more, for the same fixed cost. In the latter scenario, the value of exercising the option would be much greater.

6 IPO and Firm Dynamics.

In this section I study the effects of the IPO event on the dynamics of firm equity, sales, capital expenditures, productivity, and managerial effort. In particular, I show that the model is able to generate predictions that are qualitatively in line with the empirical evidence discussed in Section 2. Given the complexity of the environment, it is not possible to provide an analytical characterization. Therefore, I resort to numerical methods. I parameterize the model and compute approximations to the optimal decision rules. Then, I conduct a simulation exercise, as explained in detail below.[29]

I assume that the random variable z takes value in the finite set $\{z_i\}_{i=1}^{5}$, with $z_j > z_i$ for $j > i$. The probability distribution for z is given by the transition matrix represented in Table 2.[30] In the first period, z can take one of three values (z_i, $i = 1, 2, 3$). That is, the firm starts out with a low-mean distribution for the productivity shock. In case either z_1 or z_2 occur, the distribution of z will be unchanged in the following period. Instead, if z_3 occurs, the firm enjoys what I call a sudden technological breakthrough, as a consequence of which, starting from the next period, it will face a distribution with an higher mean. In every period of this new stage of the firm's existence, z will assume one of two values (z_4, z_5) with constant probability. In the discussion that follows, I will refer to the company as *private* or *public*, according to whether the IPO has already occurred or not, and as *start-up* or *mature*, according to the distribution of the variable z. At any time t, the firm is a start-up if $z_t \in \{z_1, z_2\}$, while it is mature if $z_t \in \{z_3, z_4, z_5\}$.

I conduct the following simple exercise. I set a value for M, the initial entrepreneur's endowment, and I run a very large number of 50-period long simula-

[29] It turns out that approximating numerically the solution to the functional equation given by (P6), (P7), and (P8) is a very challenging exercise. In Appendix C I discuss the technical issues in detail.

[30] The number in the cell (i, j) represents the probability (conditional on survival) of moving from state i to state j.

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=5$				.5	.5
$i=4$				.5	.5
$i=3$				.5	.5
$i=2$	.4525	.4525	.095		
$i=1$	.4525	.4525	.095		

Table 2: Transition Probabilities

tions. Figure 5 shows a sample of the paths for equity generated by the program. For every single path, the occurrence of the IPO is signalled by the sudden increase in the equity level. It is possible to appreciate that the model delivers heterogeneity in the date of the IPO and in the size at the moment of the IPO.

Beginning at time $t = 0$, in every period the entrepreneur increases the firm's equity by reinvesting part of the earnings. In the particular example, her target is a stationary level around 10. However, when the technological breakthrough hits (i.e. when the shock to the technology assumes the value z_3), the entrepreneur decides to go public. In my example, the equity rises by at least 4 times and the stake held by the entrepreneur falls from 1 to 0.5. The economic intuition behind this dynamics is fairly simple. Early in time, when the firm is a start-up, the borrowing constraint precludes the entrepreneur from reaching the efficient size. An infusion of capital such as the one made possible by an IPO would definitely allow the firm to at least partially bridge the gap between actual and efficient size. However, the IPO is not the optimal move at that stage. The reason is that by going public the entrepreneur would forgo the opportunity to do so in the future, should her firm's prospects improve (i.e. should the firm become mature). The optimal policy for the entrepreneur is to wait for the technological breakthrough. In fact, consider the implications of going public when $z \in \{z_1, z_2\}$. Should the transition to maturity occur, the only way to increase the scale of operations would be through a lengthy process of earnings re-investment.

Let's now consider the consequences of the IPO for sales, capital expenditures, operating performance, and managerial effort. Table 3 reports the average levels of such variables around the IPO date, as computed by my computer code. Period 0 is

	Periods					
	-2	-1	0	+1	+2	+3
ROA	.315	.373	.204	.183	.179	.189
Sales	.871	1.241	6.266	6.702	6.704	7.078
Capital Expenditures	.434	.456	26.68	5.88	.769	.002
Effort	.704	.772	.822	.836	.836	.847

Table 3: Dynamics around the IPO date

the one in which the IPO takes place. Period -1 is the one in which the technological breakthrough hits. In period -1 the level of the shock to the technology (z_3) is higher than in the previous periods, thus production (sales) expand. Since the level of physical capital employed in production is set in advance at a level that is lower than the ex-post efficient one, the return on asset (which equals average productivity in this setting) increases too. This situation has a real world counterpart in the case of a firm that enjoys a sudden productivity enhancement but cannot upgrade its organization instantaneously. Since the entrepreneur knows that the level of total factor productivity z will be permanently higher in the future, she decides to go public. At the date of the IPO, equity increases and the entrepreneur's stake falls. While the increase in equity has a positive effect on effort, the decrease in the stake tends to depress it. In the case of this simulation exercise, the net effect is positive. Thus the IPO brings about a sudden increase in both output and effort. Given decreasing returns to scale, the return on assets declines sharply in period 0.

Comparing the results of this simple exercise with the evidence considered in Section 2, one realizes that the model is able to generate predictions that are (qualitatively) in line with the data. However, my model lacks persistence. The data shows that performance worsens further in the periods following the IPO and sales and capital expenditures keep on growing. The version I have simulated fails to replicate this pattern because I have made the simplifying assumption that the proceedings of the IPO translate immediately into physical capital in place. If I assumed that it takes time to put physical capital in the condition to be productive, my model would predict that scale of production and effort would keep on increasing for a few periods after the IPO, and the measures of performance would decrease further (because of decreasing returns to scale).

Several scholars have argued that the decrease in the management's stake, which results in a weakening of the incentive system, might be responsible for the post-IPO decline in performance. Consistently with this argument, my assumptions imply that, *ceteris paribus*, managerial effort falls as the entrepreneur's stake decreases. However, the simple exercise just considered shows that effort can actually increase in the aftermath of the IPO. This is due to the fact that the effort choice depends not only on the manager's equity stake, but also on the size of equity and on the level of total factor productivity z. In the case of my model, the IPO is triggered by a permanent increase in z and brings about an increase in the scale of operations. These factors imply changes in effort that go in opposite directions. In the example considered, the net effect is an increase in managerial effort.

7 Conclusion.

This paper operates a synthesis between the I&O literature on firm dynamics and the Corporate Finance literature on IPOs. It does so, by embedding the IPO decision in an otherwise standard dynamic optimizing model of firm behavior. My analysis suggests that the post-IPO firm dynamics as documented by Jain and Kini [20] and Mikkelson, Partch, and Shah [26], might not constitute a puzzle, after all. For variables such as profitability, capital expenditures, and sales, the post-IPO dynamics predicted by the model is consistent with the available empirical evidence. Operating performance peaks in the fiscal year preceding the IPO and declines thereafter. Such decline is accompanied by an increase in capital expenditures and sales. The model's predictions are also consistent with a series of empirical regularities on the process of firm growth, pointed out by the Industrial Organization literature. In particular, firm growth decreases with age and size.

The robustness of my results can be assessed by enriching the model along several dimensions. In particular, I think of the opportunity of allowing the entrepreneur to modify her stake in the company's equity after the IPO and to conduct seasoned offerings. I conjecture that if the company was allowed to undergo multiple offerings, the solution to the problem would be of the (S,s) type. That is, the company would refrain from a further offering until the expected productivity growth, as reflected in the post-offering market value, was large enough to offset the fixed cost of the offering.

A Computations.

A.1 The Continuation Value in Case of Bankruptcy.

The value function $W_L(a)$, which represents the expected discounted lifetime utility when asset holdings are given by a, solves the following functional equation:

$$
\begin{aligned}
W_L(a) = \max_{c,a'} u(c,0) + \beta W_L\left(a'\right), \\
s.t.\ c = a(1+r) - a', \\
a' \geq 0.
\end{aligned}
$$

Rearranging the Euler equation for the problem, I obtain that the sequence of consumption levels has to satisfies the following first-order difference equation:

$$c_{t+1} = [\beta(1+r)]^{-\frac{1}{\xi}} c_t. \tag{16}$$

I now use the budget constraint in order to determine the initial condition for equation (16). The budget constraint at time t writes as follows:

$$c_t = a_t(1+r) - a_{t+1}.$$

Simply dividing by $(1+r)^t$ and summing over t, I obtain the following expression:

$$\sum_{t=1}^{T} \frac{c_t}{(1+r)^t} = a_0(1+r) - \frac{a_{T+1}}{(1+r)^T}$$

Finally, imposing that the sequence for h_{T+1} be bounded and taking the limit for $T \to +\infty$, I obtain

$$\sum_{t=1}^{\infty} \frac{c_t}{(1+r)^t} = a_0(1+r).$$

Using (16) I can express c_0 as a function of a_0 only:

$$c_0 = a_0(1+r)\left[1 - \beta^{\frac{1}{\xi}}(1+r)^{\frac{1-\xi}{\xi}}\right].$$

Then, the value function is given by

$$W^L(a) = \frac{1}{1-\xi}\left\{a(1+r)\left[1 - \beta^{\frac{1}{\xi}}(1+r)^{\frac{1-\xi}{\xi}}\right]\right\}^{1-\xi} \frac{1}{1-\beta^{\frac{1+\xi}{\xi}(1+r)}} - \frac{1}{(1-\beta)(1-\xi)}$$

A.2 The Model Without Bankruptcy State.

In the case in which the value function is differentiable, I can compute the envelope conditions as follows:

$$\frac{\partial W}{\partial k} = \alpha \int u^{'}(x) \left[(1+r) + \frac{\partial F(\cdot)}{\partial (k+b)} \left(1 + \frac{\partial b(k, z_{-1})}{\partial k} \right) \right] dG(z \mid z_{-1}),$$

$$\frac{\partial W}{\partial a} = (1+r) \int u^{'}(x) \frac{\partial x}{\partial (k+b)} dG(z \mid z_{-1}).$$

When the borrowing constraint does not bind, we have that $\frac{\partial b}{\partial k} = -1$ and thus the slope of the indifference sets is given by

$$\frac{\partial W / \partial k}{\partial W / \partial a} = \alpha.$$

B Proofs and Lemmas.

Before proving the results stated in the main body of the paper, I proceed to introduce further notation and state technical assumptions that I have decided to spare the reader so far. I assume that $e \in E$, $z \in Z$, and $b \in B$, where E, Z, and B are compact and convex subsets of $\Re_{+}$. Further, I impose that $h \leq \overline{h} < \infty$[31] and that $(h, k) \in S$, where $S = \left\{ (h,k) \in \Re^2_{+} : h < \overline{h}, k \leq \frac{h}{\alpha} \right\}$. Finally, I assume that the distribution function $G\left(z^{'} \mid z\right)$ has the Feller property.

Now let's consider the functional operator $\mathcal{T}$ as defined by

$$\begin{aligned} (\mathcal{T}W)(\alpha, h, k, z, b) \quad = \quad & \max_{e, h', k'} u(c, e) + \beta W\left(\alpha, h^{'}, k^{'}, z\right), \\ & c = \alpha \left[z (k+b)^{\gamma} e^{\varphi} - r(k+b) \right] + h(1+r) - h^{'}, \\ & z(k+b)^{\gamma} e^{\varphi} - r(k+b) \geq k^{'} - k(1+r), \\ & k' \leq \frac{h'}{\alpha}, \\ & h^{'}, k^{'} > 0. \end{aligned}$$

Lemma 1 *The operator $\mathcal{T}$ is a contraction mapping.*

[31] This assumption is without loss of generality, because my hypothesis $\beta < \frac{1}{1+r}$ implies that the distribution of h is bounded.

Proof. The feasibility correspondence for the problem is given by$\Gamma : Z{\times}S{\times}B \rightarrow E{\times}S$, where $\Gamma\left(h,k,z,b\right) = \left\{ (e,h,k) \in E \times S\text{: } z\left(k+b\right)^{\gamma} e^{\varphi} - r\left(k+b\right) \geq k^{'} - k\left(1+r\right), c \geq \frac{e^{1+\eta}}{1+\eta} \right\}$. It is easy (but tedious) to show that Γ is nonempty, compact valued and continuous. Let A denote the graph of Γ. Then the function $F : A \rightarrow \Re_{+}$ defined by $u(\alpha\left[z\left(k+b\right)^{\gamma} e^{\varphi} - r\left(k+b\right)\right] + h\left(1+r\right) - h^{'}, e)$ is bounded and continuous. Since $\beta < 1$, it is straightforward to verify that $\mathcal{T}$ satisfies Blackwell's necessary conditions for a contraction. ■

Lemma 2 *The operator $\mathcal{T}$ maps concave functions into concave functions.*

Proof. It is easy to verify that the concavity of the production function implies that the correspondence Γ is convex. Further, the function $F : A \rightarrow \Re_{+}$ defined by $u(\alpha\left[z\left(k+b\right)^{\gamma} e^{\varphi} - r\left(k+b\right)\right] + h\left(1+r\right) - h^{'}, e)$ is concave. In fact F is the composition of concave functions. Then the result follows from the application of the usual argument as in the proof of Theorem 4.8 in Stokey and Lucas [30]. ■

Notice however that the function F is not strictly concave. In fact there exist $\theta \in (0,1)$ and couples (k_1, b_1), (k_2, b_2) such that $u\left\{\alpha[z(k_\theta + b_\theta)^{\gamma} e^{\varphi} - r(k_\theta + b_\theta) + h(1+r) - h^{'}, e]\right\} = (1-\theta)\, u\left\{\alpha\left[z\left(k_1+b_1\right)^{\gamma} e^{\varphi} - r\left(k_1+b_1\right) + h\left(1+r\right) - h^{'}, e\right]\right\} + \theta u\left\{\alpha\left[z\left(k_2+b_2\right)^{\gamma} e^{\varphi} - r\left(k_2+b_2\right) + h\left(1+r\right) - h^{'}, e\right]\right\}$, with $k_\theta = (1-\theta)\, k_2 + \theta k_2$ and $b_\theta = (1-\theta)\, b_2 + \theta b_2$.

Now consider the operator $\widetilde{\mathcal{T}}$:

$$
\begin{aligned}
\left(\widetilde{\mathcal{T}}W\right)(\alpha,h,k) &= \max_{e,h^{'}} u\left(c,e\right) + \beta W^{L}\left(h^{'}\right), \\
&\qquad c = \left(h-\alpha k\right)\left(1+r\right) - h^{'}, \\
&\qquad h^{'} > 0.
\end{aligned}
$$

Lemma 3 *The operator $\widetilde{\mathcal{T}}$ is a contraction mapping.*

Proof. The feasibility correspondence for the problem is given by $\widetilde{\Gamma} : S \rightarrow S$, where $\widetilde{\Gamma}\left(\alpha,h,k\right) = \left\{\left(h^{'},k^{'}\right) \in S\text{: } \left(h-\alpha k\right)\left(1+r\right) - h^{'} > 0,\ h^{'} \geq 0\right\}$. Obviously $\widetilde{\Gamma}$ is nonempty, compact valued and continuous. Let $\widetilde{A}$ denote the graph of $\widetilde{\Gamma}$. Then the function $\widetilde{F} : \widetilde{A} \rightarrow \Re_{+}$ defined by $u\left(\left(h-\alpha k\right)\left(1+r\right) - h^{'}, 0\right)$ is bounded and continuous. It is straightforward to verify that $\widetilde{\mathcal{T}}$ satisfies Blackwell's necessary conditions for a contraction. ■

Lemma 4 *The operator* $\widetilde{\mathcal{T}}$ *maps concave functions into strictly concave functions.*

Proof. The concavity of the production function implies that the correspondence $\widetilde{\Gamma}$ is convex. Further, the function $\widetilde{F} : \widetilde{A} \rightarrow \Re_{+}$ defined by $u\left((h - \alpha k)(1 + r) - h^{'}, 0\right)$ is strictly concave. Then the result follows from the application of the usual argument as in the proof of Theorem 4.8 in Stokey and Lucas [30]. ■

Proposition 1.

Proof. The functional operator defined by the functional equation (P5) can be rewritten as follows:

$$\begin{aligned}(T(W, W_L))(\alpha, h, k, z_{-1}) = \max_{b<sk} \Big\{ &(1 - \rho_L) \int (\mathcal{T}W)(\alpha, h, k, z, b) dG(z \mid z_{-1}) \\ &+ \rho_L \left(\widetilde{\mathcal{T}} W_L\right)(\alpha, h, k) \Big\}.\end{aligned}$$

Since T satisfies monotonicity and $T(W + a, W_L + a) = T(W, W_L) + a \in \Re$, in force of Lemmata 1 and 3 I can apply Theorem 9.7 in Stokey and Lucas [30] to the composition of operators T, T, and $\widetilde{\mathcal{T}}$. ■

Proposition 2.

Proof. Since the operator T maps concave functions into concave functions, and given Lemmata 3 and 4, the composition of the operators $T, \mathcal{T}$, and $\widetilde{\mathcal{T}}$ satisfies the assumptions of Theorem 9.8 in Stokey and Lucas [30]. ■

Proposition 3.

Proof. By concavity, the value function W is differentiable almost everywhere. Strict concavity implies that the right (left) partial derivative with respect to k changes sign at most once. Thus it is sufficient to prove that the sequence of right (left) partial derivatives goes to $-\infty$ as k goes to $\frac{h}{\alpha}$. In turn, this is insured by the fact that the sequence is decreasing and that $\lim_{k \rightarrow \frac{h}{\alpha}} v^{'}(c_L) = -\infty$. ■

C The Algorithm.

The solution algorithm proceeds by backward induction. I begin by computing an approximation to the value function that solves the post-IPO decision problem (P5). Once obtained the policy functions for the post-IPO decision problem, it is easy to compute the an approximation to the price function by iterating on the contraction (15). I then proceed to compute the value of going public, by solving the problem (P5). Finally I turn to the pre-IPO decision problem (P8).[32]

C.1 The Post-IPO Decision Problem.

I compute an approximation of the value function $W(\alpha, h, k, z)$ by implementing the standard method known as Value Function Iteration. I begin by defining grids for the state variables $(\alpha_i, h_i, k_s, z_m)$. I then formulate a starting guess, defining the value of $W(\cdot)$ for every point $(\alpha_i, h_j, k_s, z_m)$. For every couple (α_i, z_m) I approximate the function over the space $S(\alpha_i) = \left\{(h, k) \in \Re^2_+ : \ h < \overline{h}, \, k \leq \frac{h}{\alpha_i}\right\}$ by computing the coefficients of the bi-dimensional shape-preserving spline developed by Costantini and Fontanella [10] [11]. For every (α_i, z_m) and $(h, k) \in S(\alpha_i)$ I denote the approximated value given by the spline as $\widetilde{W}_{im}(h, k)$. The next task is to compute the revised guess for $W(\alpha_i, h_j, k_s, z_m)$. In order to do so, for every point $(\alpha_i, h_i, k_s, z_m)$ I define a grid over the space for debt. The bounds of such grid are given by the optimal debt levels in the cases in which the productivity shock z is at the bounds. Then, for grid point every point $(\alpha_i, h_i, k_s, z_m)$ and debt level b_n I solve the following maximization problem:

[32] The programs that implement the algorithm are written in Fortran 90 and make use of routines from the IMSL Scientific Library.

$$
\begin{aligned}
W(\alpha_i, h_j, k_s, z_m, b_n) =& (1-\rho_L)\sum_q p_{mq}\left[\max_{e,h',k'} u(c,e) + \beta\widetilde{W}_{iq}(h',k')\right] + \\
&+ \rho_L\left[\max_{h'_L} u(c_L, 0) + \beta W_L(h'_L)\right], \\
& c = \alpha_i\left[z_q(k_s+b_{mn})^\gamma e^\varphi - r(k_s+b_{mn})\right] + h_i(1+r) - h', \\
& z_q(k_s+b_{mn})^\gamma e^\varphi - r(k_s+b_{mn}) \geq k', \\
& c_L = (h_j - \alpha_i k_s)(1+r) - h'_L.
\end{aligned}
$$

Then I proceed by feeding a one-dimensional shape-preserving spline over the debt dimension. In this way, for every b, I can obtain an approximation of the value of the above problem. Let $\widetilde{W}_{i,j,s,m}(b)$ denote such value. Finally, I am able to solve for $W(\alpha_i, h_j, k_s, z_m)$ by computing:

$$
W(\alpha_i, h_j, k_s, z_m) = \max_b \widetilde{W}_{i,j,s,m}(b).
$$

At this point I have all the elements to compute the revised value for $\widetilde{W}_{iq}(h,k)$ and repeat the procedure until convergence to the fixed point of the functional equation.

Using a spline to approximate the value function over the all space $S(\alpha_i)$ carries two great advantages. First, the resulting policy functions are continuous.[33]Second, I am able to obtain an accurate approximation to the solution by working with very few grid points over the dimensions (h_i, k_s). In turn, this notably increases the speed of my code. In general, one has to be very cautious in using polynomials to approximate discrete function outside the grid. A number of problems can emerge. In this particular case I am comforted by the facts that the maximization problem is concave and the spline that I use is shape-preserving.[34]As we will see shortly, my solution strategy will have to be different in approaching the pre-IPO decision problem.

[33]Here the term continuous is to be considered in a 'computational' sense. What I mean is that the grid I use is as fine as the numerical precision of the processor in use.

[34]The bi-dimensional spline introduced by Costantini and Fontanella [10] [11] is shape-preserving in the sense that the sign of first and second derivatives are preserved outside the grid. Please refer to the cited paper for an exact characterization of this property.

In order to compute an approximation to the price function $P(\cdot)$, I proceed in an analogous way. I begin by defining an initial guess for every grid point $(\alpha_i, h_j, k_s, z_m)$. Then, for every couple (α_i, z_m), I approximate the function over the space $S(\alpha_i)$ by computing the coefficients of the bi-dimensional spline. For every (α_i, z_m) and $(h, k) \in S(\alpha_i)$, I denote the approximated value given by the spline as $\widetilde{P}_{im}(h, k)$. Then I use the optimal decision rules of the entrepreneur's problem to compute the revised guess $P(\alpha_i, h_j, k_s, z_m)$ according to

$$P(\alpha_i, h_j, k_s, z_m) = \sum_q p_{mq} \left[z_q (k_s + b^*)^\gamma e^{*\varphi} - r(k_s + b^*) + \frac{1}{1+r} \widetilde{P}_{iq}(a'^*, k'^*) \right].$$

I keep on repeating the same steps until convergence of $P(\alpha_i, h_j, k_s, z_m)$.

C.2 The IPO Stage.

The value of going public by retaining a stake α_i, denoted by $V_{ipo}(\alpha_i, h_j, k_s, z_m)$, is given by the following problem:

$$\begin{aligned}
&\max_{k_{ipo}} \widetilde{W}_{im}(h_{ipo}, k_{ipo}), \\
&\qquad (1-\alpha_i) k_{ipo} + h_{ipo} - h + N = (1-d)(1-\alpha_i) \left[k_{ipo} + \widetilde{P}_{im}(h_{ipo}, k_{ipo}) \right], \\
&\qquad 0 \leq k_{ipo} \leq \frac{h_{ipo}}{\alpha_i}, \\
&\qquad h_{ipo} \geq 0.
\end{aligned}$$

Then the value of going public when the state variables are (h_i, k_s, z_m) is obtained by solving the following:

$$V_{ipo}(h_i, k_s, z_m) = \max_i \{ V(\alpha_i, h_j, k_s, z_m) \}.$$

C.3 The Pre-IPO Decision Problem.

In order to compute an approximation to the value function $V(h, k, z)$, I proceed once again by iterating on the Bellman equation. However, in this case I decide not to approximate the function outside of the grid. The Bellman equation is still a contraction, implying that the solution of the functional equation is unique. However, in general the solution will not be concave. Since I cannot control how the non-concavity will affect the procedure, I decide to solve the pre-IPO decision problem by discretizing the state space along all dimensions. I begin by formulating a starting guess, defining the value of $V(\cdot)$ at every point (h_j, k_s, z_m). The next task is to compute the revised guess for $W(\alpha_i, h_j, k_s, z_m)$. In order to do so, for every point $(\alpha_i, h_i, k_s, z_m)$ I define a grid over the space for debt.. The bounds of such grid are given by the optimal debt levels in the cases in which the productivity shock z is at the bounds of its domain.. Then, for every point $(\alpha_i, h_i, k_s, z_m)$ and debt level b_n I solve the following maximization problem:

$$
\begin{aligned}
V_{pr}(h_i, k_s, z_m, b_n) = (1-\rho_L) \sum p_{mq} \max_{u,v \in \Omega_{s,mn,q}} & \{u(c,e) + \beta V(h_v, k_u, z_q)\} + \\
& + \rho_L \left[\underset{w}{max}\; u(c_L, 0) + \beta W_L(h_w) \right], \\
& c = [z_q(k_s + b_{mn})^{\gamma} e^{\varphi} - r(k_s + b_{mn})] + h_i(1+r) - h_v, \\
& c_L = (h_i - k_s)(1+r) - h_w.
\end{aligned}
$$

where $\Omega_{s,n,q} = \{u, v : z_q (k_s + b_n)^{\gamma} e^{1-\gamma} - r(k_s + b_n) \geq k_u, h_v > k_u\}$.

As for the post-IPO decision problem, I proceed by feeding a one-dimensional shape-preserving spline over the debt dimension. In this way I can obtain the value of the above problem for every b. Let $\widetilde{V}^{pr}_{jsm}(b)$denote such value. Then I can solve for $V^{pr}(h_j, k_s, z_m)$by computing

$$V_{pr}(h_j, k_s, z_m) = \max_b \widetilde{V}^{pr}_{jsm}(b).$$

The revised guess for the value function is computed by applying the max operator:

$$V(h_j, k_s, z_m) = \max\left\{V^{IPO}(h_i, k_s, z_m), V^{pr}(h_j, k_s, z_m)\right\}.$$

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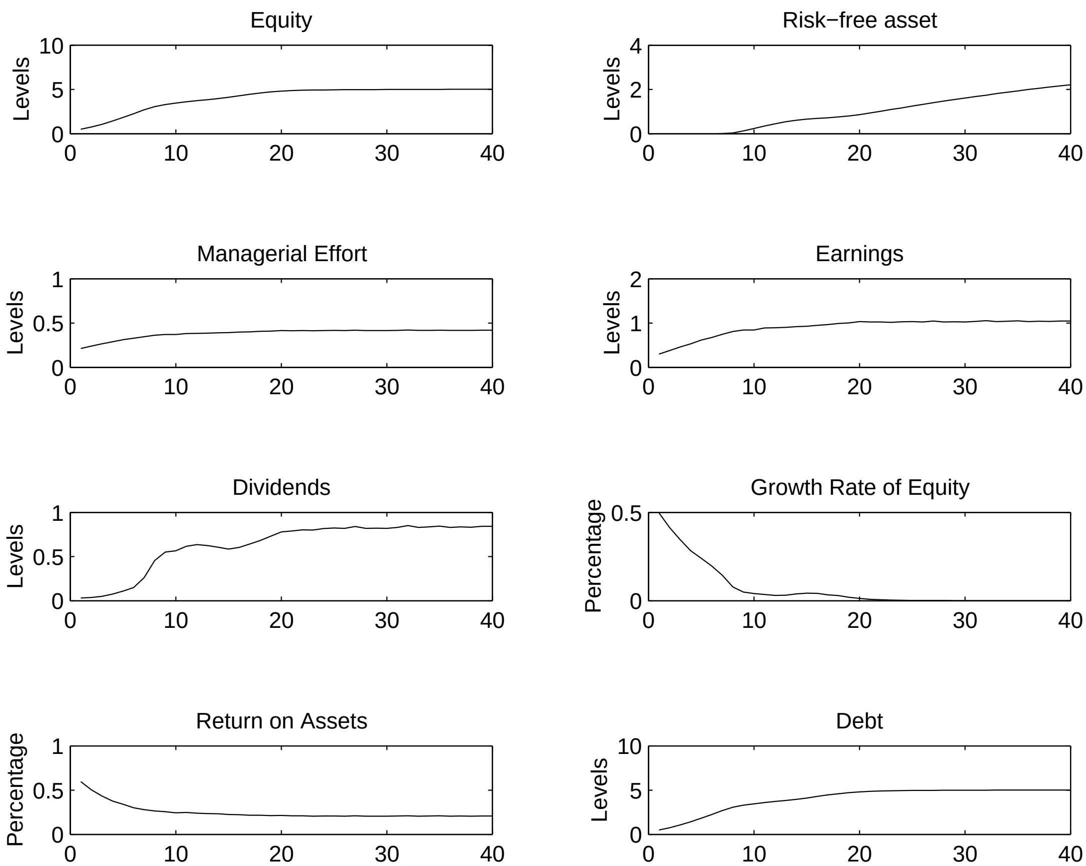

Figure 3: Dynamics of a Public Company.

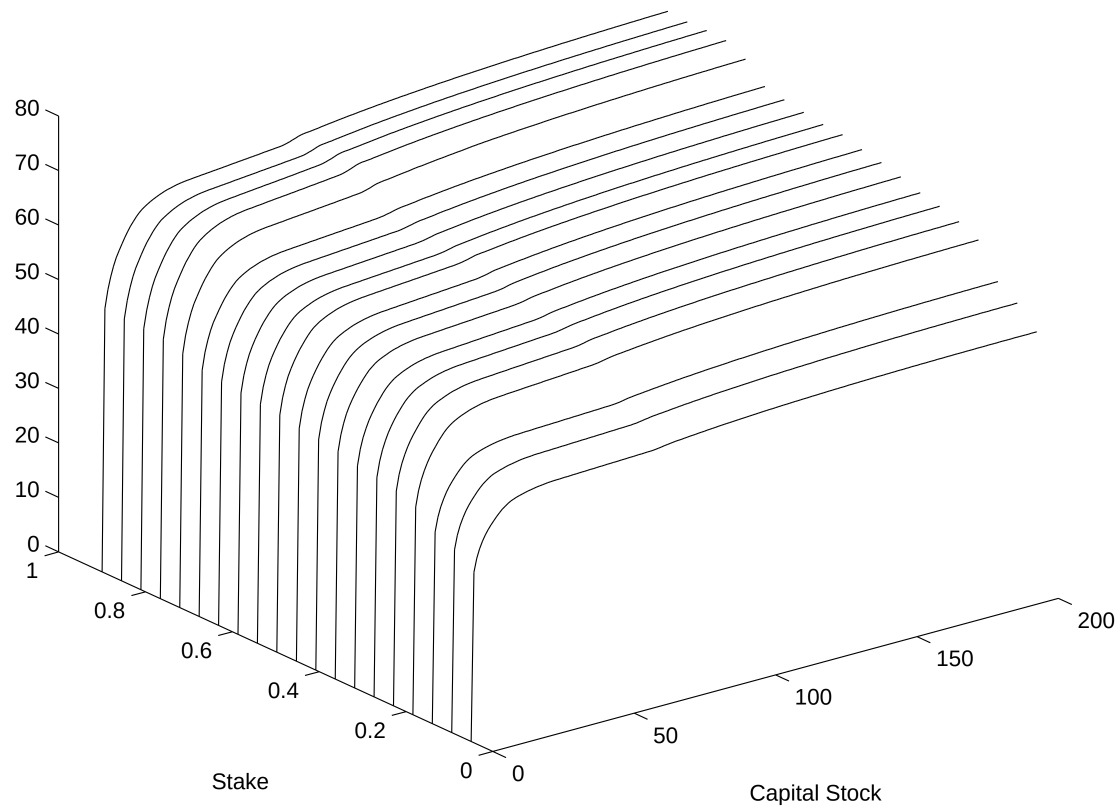

Figure 4: The Function $P(\alpha, k, h, z)$.

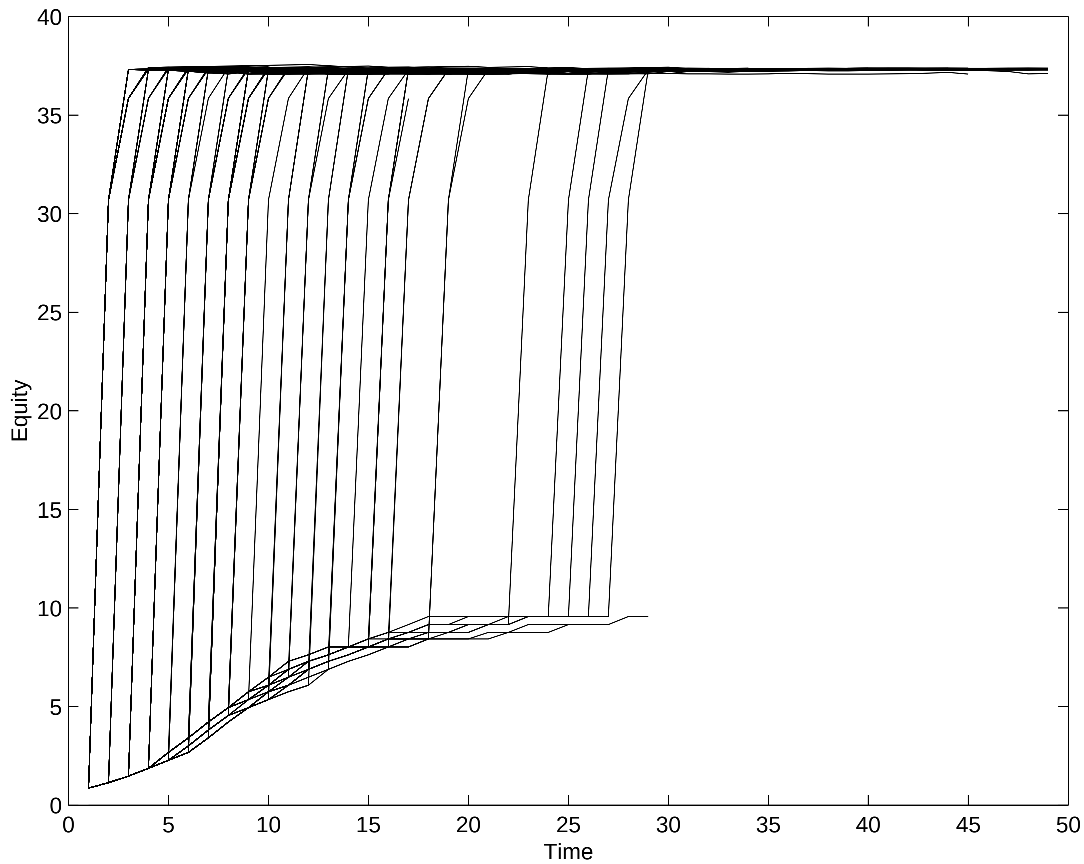

Figure 5: Paths for Equity.