

**Collusion via Signalling in Open Ascending Auctions
with Multiple Objects and Complementarities**

by

Sandro Brusco
and
Giuseppe Lopomo

EC-99-05

Collusion via Signalling in Open Ascending Auctions with Multiple Objects and Complementarities*

Sandro Brusco[‡]
and
Giuseppe Lopomo[§]

April 7, 1999

Abstract

Collusive equilibria exist in open ascending auctions with multiple objects, if the number of bidders is sufficiently small relative to the number of objects, even with large complementarities in the buyers' utility functions. The bidders collude by dividing the objects among themselves, while keeping the prices low. Hence the complementarities are not realized.

*We thank José Campa, Faruk Gül, Fabrizio Perri, Joel Sobel, Ennio Stacchetti, the participants to the October 1998 Midwestern Theory Conference, and the participants to seminars at the University of California-San Diego, UC college in London, and the European University Institute for helpful comments and suggestions.

[‡]Departamento de Economía de la Empresa, Universidad Carlos III de Madrid. E-mail: brusco@emp.uc3m.es.

[§]Stern School of Business, New York University. E-mail: glopomo@stern.nyu.edu

1 Introduction

It is well known that the English auction has many desirable properties when a single object is to be sold. With private values, this auction implements the efficient allocation uniquely in weakly dominant strategies, and maximizes the seller's expected revenue within a large class of 'simple' selling procedures (Lopomo [10]). However, the properties of generalized versions of the English auction in situations in which many objects are to be sold, and the buyers have use for more than one object, are yet to be fully understood.

By and large, most of the existing work on simultaneous multiple objects auctions has focused on the case with nonincreasing marginal willingness to pay in the bidders' utility function. Most closely related to the present paper in terms of the auction rules, Milgrom [13] has analyzed the "simultaneous ascending auctions," which have been used by the US government to sell licenses for the use of radio frequency bands. Mostly under the hypothesis that the bidders' utility functions are common knowledge, Milgrom discusses issues surrounding the auction's performance in terms of its ability of generating efficient outcomes and its potential for maximizing the seller's expected revenue. In particular, he describes an equilibrium for the case of two bidders, two objects and no private information, that is similar to the one described in Proposition 1 of this paper: each bidder can buy one object for the minimum price allowed by the rules of the auction.

Proposition 1 allows for private information but restricts the bidders' utility functions to be additive: i.e. each bidder's willingness to pay for each object is independent of whether she is also buying the other object. This result has also been established independently by Engelbrecht-Wiggans and Kahn [6]. They also establish the existence of other 'low revenue' equilibria, always for the case with two bidders, two objects and additive utility functions.

Under a condition which rules out complementarities in the buyers' utility functions, Gül and Stacchetti [8] have studied a generalized version of the English auction akin to a *tatonnement* process, with emphasis on the relation between its equilibria and the Walrasian equilibria of the underlying economy. Ausubel and Cramton [1] have also studied environments with nonincreasing marginal values, but have focused mostly on sealed-bid auctions. Recently, Kwasnica [9] has done experimental work on collusion in multiple object sealed-bid auctions, with additive utility functions.¹

Environments in which the bidders have increasing marginal valuations have been considered in Chakraborty [3], who has studied properties of various sealed-bid auctions formats. His paper also contains a good survey of existing work on multiple object auctions.

¹The possibility of collusion in auctions has also been studied extensively in the single object case. (See Campbell [2], Graham and Marshall [7], Mailath and Zemsky [11], McAfee and McMillan [14] and Pesendorfer [16].)

In this paper we examine the claim that generalized English auctions can be more vulnerable to collusion in the multiple objects case than in the single object case. Concerns about collusive behavior of bidders have emerged, for example, in an article published in *The Economist* (1997). Most recently, Cramton and Schwartz [4] have indicated evidence of collusive behavior in the FCC spectrum auctions, and discussed the effectiveness of various modifications of the auction rules in hindering bidders' collusion.

In particular, the following conjectures are usually held about auctions with multiple objects:

- The presence of multiple objects facilitates collusion by allowing the bidders to signal their willingness to abstain from competing over certain objects, provided they are not challenged on others. In this way, the agents can allocate the objects among themselves without paying much.
- As the ratio of bidders to objects increases, the possibility of collusive schemes as the ones indicated in the previous conjecture tends to disappear.
- High complementarities among objects hinder collusion. This is because each bidder is less satisfied with owning only a subset of the objects on sale; she has therefore an incentive to break the collusion and compete for all the objects in order to fully realize the synergies.

We study how the signalling opportunities provided by the sequential nature of open ascending auctions can be exploited by the bidders in the presence of multiple objects to coordinate on equilibria which generate low revenue for the seller and implement socially inefficient allocations of the objects. For simplicity, we focus on the case of two objects, although the results carry over to the case with any number of objects.

The model is described in section 2. In section 3 we begin the analysis with the benchmark case of purely additive values, i.e. we assume that each bidder obtains no synergies from owning multiple objects, hence her willingness to pay for one object is independent of whether she is also buying other objects. We present conditions under which collusion-via-signaling can be sustained in equilibrium. Equilibria in this class can be described for the simple case with only two bidders as follows. Each bidder starts by placing the smallest possible bid on her most valued object, and no bid on the other object. If only one bid is placed on each object, it becomes common knowledge that the bidders rank the objects differently, and the bidders simply confirm their bids in the next round thus forcing the auction to end with each buying one object for the minimum price. If, instead, the initial bids reveal that the two bidders have a higher value for the same object, then the bidding continues according to some equilibrium strategy, which can entail, for example, a reversion to "bidding straightforwardly," i.e. raising the bid on each object if the value

is higher than the current highest bid and the bidder is not assigned the object.² Alternatively, the bidders may continue bidding according to some other continuation strategy in which they proceed to signal more detailed information about their values in order to coordinate with each other and buy only one object each for a relatively low price. In all equilibria of this kind, the outcome is socially inefficient — i.e. the objects are not always assigned in a way that maximizes the total bidders’ willingness to pay — but the bidders end up paying less than they would by bidding straightforwardly throughout the entire auction. And it turns out that the reduced payments make up for the loss of efficiency in assigning the objects, so that the each bidder’s interim expected surplus is increased.

We also show however that, for these equilibria, the probability that the bidders can collude via signaling decreases as their number increases relative to the number of objects. This result corroborates the conjecture that collusion is a ‘low numbers’ phenomenon.

In section 4 we consider the case in which the bidders’ utility functions exhibit large complementarities, i.e. their willingness to pay for the two objects together is much greater than the sum of the two objects’s “stand alone” values. We show that the sole presence of complementarities does not hinder collusion: the bidders can still manage to buy one object each, at low prices. In fact, in the extreme case in which the synergies are commonly known, and not too different across the bidders, the incentive structure for the bidders is essentially identical to the case with no complementarities. The efficiency loss however is much larger because it includes the unrealized complementarities.

When complementarities are not only large but also variable however, the possibility of collusion is seriously reduced. This suggests that what is crucial in determining the likelihood of collusion is not whether the complementarities are (on average) ‘large’, but more how variable they are. Section 5 contains concluding remarks, and an appendix collects all the proofs.

2 The Model

There are a set $N = \{1, \dots, n\}$ of bidders and a set $M = \{1, \dots, m\}$ of objects, with m, n finite. The bidders have quasi-linear utility functions, and the willingness to pay of bidder $i \in N$ for bundle $J \subset M$ is given by $u_i(J)$. Bidder i knows her values $\{u_i(J)\}_{J \in 2^M}$, while the rest of the world only knows that such values are drawn according to a probability function with support on a compact subset of $\mathfrak{R}_+^{2^m}$.

The m objects are sold with an open ascending auction, named here the Generalized English Auction, or GEA, which is a natural extension of the standard one-object English auction to

²The use of the expression “bidding straightforwardly” to denote a strategy that is similar to the standard strategy in the one-object English auction is due to Milgrom [13].

environments with multiple objects. The auction proceeds in rounds. At the initial round each bidder i submits a vector of bids $(b_i^1(1), \dots, b_i^m(1))$, where $b_i^j(1)$ denotes the amount that bidder i declares she is prepared to pay for object j at round 1. The case in which bidder i places no bid on object j is treated setting $b_i^j(1) = \emptyset$, and the minimum effective bid is normalized to zero. The auction ends with the seller keeping all objects if and only if each bidder places no bid on any object, i.e. if $b_i^j(1) = \emptyset$ for each $j \in M$, $i \in N$. Otherwise, both the highest bid $b^j(1) := \max_i b_i^j(1)$ ($\emptyset = -\infty$ by convention) and a potential winner among the bidders who have offered $b^j(1)$ are identified for each object $j \in M$, and the auction moves to round 2. At round 2, each bidder i submits a new vector of bids $(b_i^1(2), \dots, b_i^m(2))$. We model the condition that previous bids cannot be withdrawn by requiring that $(b_i^1(2), \dots, b_i^m(2)) \geq (b_i^1(1), \dots, b_i^m(1))$. The auction ends at stage 2 if no bidder revises her bid on any object, i.e. if $b_i^j(2) = b_i^j(1)$, all $j \in M$, $i \in N$. Otherwise a potential winner is selected again for each object j among all i such that $b_i^j(2) = b^j(2) := \max_{i'} b_{i'}^j(2)$, and the auction moves to the next round. Proceeding in this fashion, if round $t \geq 2$ is reached, and if $b_i^j(t) = b_i^j(t-1)$ for all $j \in M$ and $i \in N$, then the auction ends, and each object j is assigned to the buyer selected at the end of round $t-1$ among all i such that $b_i^j(t-1) = b^j(t-1) := \max_{i'} b_{i'}^j(t-1)$. The selected buyer pays his last bid $b^j(t-1) = b^j(t)$.

To keep the analysis and the notation as simple as possible, we establish our main results for the case $m = 2$, i.e. only two objects on sale. The main insights however apply to the more general case. We define:

- $v_i := u_i(\{1\})$, the value to bidder i of having object 1 only;
- $w_i := u_i(\{2\})$, the value to bidder i of having object 2 only.

We will use interchangeably the terms ‘object v ’ (object w) and ‘object 1’ (object 2). With only two objects, a bid by agent i in round t is just an ordered pair $(b_i^1(t), b_i^2(t))$.

Finally, we assume that the size of the complementarity is independent of the two objects’ ‘stand-alone’ values, i.e. the value to bidder i of having both objects is

$$u_i(\{1, 2\}) = v_i + w_i + k_i,$$

For each $i \in N$, the values (v_i, w_i, k_i) are drawn from a joint probability distribution with density $h(v_i, w_i, k_i)$ and support $[0, 1]^2 \times K$. We assume that v_i and w_i are independent and identically distributed. Thus, v_i and w_i have identical marginal distribution, whose density and c.d.f. we denote by f and F respectively. The marginal distribution of k_i is either degenerate on 0, or is represented by a density g (and c.d.f. G) with support over the interval $[\underline{k}, \bar{k}]$ with $\underline{k} > 0$. The values (v_i, w_i, k_i) are drawn independently of (v_j, w_j, k_j) for each $j \neq i$.

In the next section we examine the case with no complementarities, i.e. $k_i = 0$ for all $i \in N$. Section 4 will be devoted to the case with positive complementarities.

3 Collusive Equilibria with No Complementarities

In this section we consider the case of *purely additive values*

$$u_i(\{1, 2\}) = v_i + w_i, \quad i \in N,$$

or *no complementarities*, $k_i = 0$ for each i . In this case there is really no point in auctioning the objects simultaneously, since the efficient allocation of each object is independent of the allocation of the other objects. A sequence of single-object English auctions would allocate each object to the bidder with the highest value, and, with suitably chosen reserve prices, would also maximize the seller's expected revenue. However, the analysis of the bidders' equilibrium behavior in the GEA will provide a useful benchmark for the more realistic case in which complementarities are present. In particular, the purely additive case will shed light on the role played by multiple objects in facilitating collusion among the bidders.

We begin with the following simple but important observation: with no complementarities, it is a perfect Bayesian equilibrium for the GEA that each bidder i follow a 'Separated English Auctions' strategy (SEA), bidding on each object j until the price reaches the value $u_i(\{j\})$. It is clear that, if all other agents are following this strategy, then player i 's best reply is to follow the same strategy³. Note that the SEA strategies are weakly dominant, hence optimal for each bidder independent of her beliefs about her opponent's values. We state the result for an arbitrary number of objects in the following Proposition.

Proposition 0 *With no complementarities, for any n and m , the separated English auctions strategy (SEA) profile form a perfect Bayesian equilibrium (with some consistent belief system) after any history in the Generalized English auction.*

The SEA strategy profile can be used in the same way as Pareto inferior equilibria are used in repeated games to support collusive outcomes. They constitute the threat used to deter the bidders from using aggressive bidding strategies.

The next observation is an immediate implication of the well-known Revenue Equivalence Theorem (Myerson [15]). Since the bidder's types are drawn from independent probability distributions, by incentive compatibility the SEA equilibrium is the unique PBE which implements the socially

³It is worth pointing out here that this argument would fail in the presence of complementarities.

efficient allocation. The incentive compatibility constraints pin down the interim expected payment function of each bidder for any given objects' allocation rule. Therefore, in any equilibrium of any trading game with no complementarities which implements the socially efficient allocation the buyers' interim expected payments must be equal to the sum of the expected payments in m separate standard English auctions. This also implies that in any perfect Bayesian equilibrium of the GEA in which all bidders are better off than in the SEA equilibrium the objects cannot be allocated according to the socially efficient rule.

3.1 Two Bidders

We begin with the case in which there are only two bidders. Proposition 1 establishes the existence of a symmetric Perfect Bayesian equilibrium which dominates the SEA in terms of buyers's interim expected surplus⁴. Recall that F is the common marginal c.d.f. of v_i and w_i .

Proposition 1 *Suppose that $E(x) \equiv \int_0^1 x dF(x) \geq \frac{1}{2}$. Then the following strategy, together with some consistent belief system, form a symmetric perfect Bayesian equilibrium:*

- *Types (v_i, w_i) such that $v_i \geq w_i$ open with $(b_i^1(1), b_i^2(1)) = (0, \emptyset)$;*
- *Type (v_i, w_i) such that $v_i < w_i$ open with $(b_i^1(1), b_i^2(1)) = (\emptyset, 0)$;*
- *If the initial bids are different, all types confirm their bid in round 2.*
- *If at any round the bids differ from the ones given above, all types revert to the SEA strategy described in Proposition 0.*

This equilibrium can be described as follows. Each bidder opens by making the minimum bid (zero) only on her most preferred object. If, at the end of the first round, the bidders discover that they rank the objects differently, then they stop bidding, and each is able to buy her preferred object at the lowest possible price. If instead they discover that they rank the two objects in the same way, then they revert to the Separated English Auctions strategies. The condition $E(x) \geq \frac{1}{2}$ guarantees that, for each type of each bidder, the expected surplus from triggering the SEA strategies after an opening with different bids is lower than the surplus obtained by buying just her most preferred object for the minimum price.

In the next subsection, we show that the set of perfect Bayesian equilibria of the GEA contains other, "more collusive" equilibria, i.e. equilibria in which the bidders end up with a higher interim expected surplus.

⁴The existence of this equilibrium has been established independently by Englebrecht-Wiggans and Kahn [6].

3.2 Getting More out of Collusion

The equilibrium described in Proposition 1 prescribe that the bidders revert to the SEA strategies when they open with the same bids, i.e. when it becomes common knowledge that their preferred object is the same. It is thus natural to ask whether in this case the bidders can do better by trying again to coordinate themselves and buy one object each at relatively low prices. The affirmative answer to this question is provided by the next proposition.

Proposition 2 *Let x, y be two independent random variables with c.d.f. F . Assume that, for each $a \in [0, 1]$, the following inequalities hold:*

$$E[x | 0 \leq x \leq 1 - a] + E[x | a \leq x \leq 1] \geq 1, \quad (1)$$

$$E[x | x \geq a + y, 1 - a \geq y] + E[x | y - a \geq x \geq 0, y \geq a] \geq 1. \quad (2)$$

Then the following strategy, together with some consistent belief system, form a symmetric perfect Bayesian equilibrium:

First round:

- *Types (v_i, w_i) such that $v_i \geq w_i$ open with $(b_i^1(1), b_i^2(1)) = (0, \emptyset)$;*
- *Types (v_i, w_i) such that $v_i < w_i$ open with $(b_i^1(1), b_i^2(1)) = (\emptyset, 0)$.*

Subsequent rounds:

- *If the initial bids are either $((0, \emptyset), (\emptyset, 0))$ or $((\emptyset, 0), (0, \emptyset))$ then all types confirm their bids;*
- *If the initial bids are $((0, \emptyset), (0, \emptyset))$, then types (v_i, w_i) such that $v_i - w_i = a_i$ keep raising their bid on object v while refraining from bidding on w until either i) the opponent stops, or ii) the bid reach the value a_i . In case i), they do not revise any bid for the next two rounds; and in case ii) they bid $(a_i, 0)$, for two consecutive rounds, thus moving the outstanding bid on w from \emptyset to 0. If the initial bids are $((\emptyset, 0), (\emptyset, 0))$, the strategy is symmetric, with the roles of v and w switched.*

Out-of-equilibrium paths:

- *If at any round a bid not in accordance to the above described strategy is observed, then each type reverts to the SEA strategy.*

The equilibrium formally stated in Proposition 2 can be described as follows. The bidders open by signalling which object they prefer. If they prefer different objects, then the game ends, as in the equilibrium of Proposition 1. If they prefer the same object, say v , then they keep raising the price on v while abstaining from competing on w , with bidder i prepared to bid up to the difference between her two values $a_i = v_i - w_i$. Thus, if $a_i > a_{3-i}$, bidder i ends up buying object v for the difference between her opponent's values a_{3-i} . The opponent stops competing on v when the price reaches the difference between her values, and buys w for the minimum bid.

In this equilibrium, each bidder's type set $[0, 1]^2$ is partitioned into lines with slope 1: types on the same line — i.e. with the same difference between the two objects' values — behave identically hence remain indistinguishable until the end of the auction. It is because of this pooling among low and high types that the bidders abstain from triggering the SEA strategies at the end of the bidding on one object.

It is worth noting that the equilibrium strategy is simple: only at the final round the bidders' beliefs play a role in sustaining the equilibrium. It is crucial however that no object is assigned before the end of the auction, i.e. any object can be still bought even after many rounds in which its outstanding bid has not moved. It is thus tempting to conclude that simply imposing an “activity rule” — i.e. a condition specifying that any object whose price does not move for a certain number of rounds be assigned to a bidder who has made the highest bid — can be effective in preventing the bidders from achieving this level of collusion. In light of Proposition 2 however, it is possible to observe that (under some conditions on the c.d.f. F) there exists another symmetric perfect Bayesian equilibrium yielding the outcome of Proposition 2 in at most three rounds: if the bidders open signaling that they both prefer the same object, say v , in the second round each places a bid on v equal to the expected difference between her opponent's values; and in the third round, the lowest bidder can buy w for the minimum price. The competition phase on v is thus compressed in just one round, hence this equilibrium cannot be destroyed by any rule which allows at least two rounds of inactivity before closing the bidding on an object.

If any positive weight is given to the seller's surplus, the outcome of this equilibrium is Pareto inferior even to the one generated by the equilibrium of Proposition 1. Social efficiency requires that each object be assigned to the agent who values it most, and this is what happens when the SEA strategies are triggered. In the equilibrium of Proposition 2 the SEA strategies are never triggered.

Conditions (1) and (2) imply $E(x) \geq \frac{1}{2}$, since the latter is obtained setting $a = 0$ in (1). The two conditions are satisfied, for example, by the uniform distribution over $[0, 1]$.

3.3 More than Two Bidders

The equilibria described in Propositions 1 and 2 may seem to rely heavily on the fact that the number of bidders is equal to the number of objects. However, some degree of collusion is still possible when there are more bidders than objects. The basic idea is that the bidders can follow the SEA strategy until only 2 players are left, and then adopt the strategies described in Propositions 1 or 2 to divide the objects.

Proposition 3 *If there are $n > 2$ bidders and the c.d.f. $F(x)$ satisfies $E[x|x \geq z] \geq \frac{1+z}{2}$ for each $z \in [0, 1]$, then the following strategy, together with some consistent belief system, form a symmetric perfect Bayesian equilibrium:*

- *Round 1: If $v_i \geq w_i$, open with $(0, \emptyset)$, otherwise open with $(\emptyset, 0)$;*
- *Round t : if more than two bidders were active at round $t - 1$, all types use the SEA strategy, i.e. they increase their bid on each object if their value is higher than the current highest bid and they are not assigned the object.*

If at round $t - 1$ only i and $j \neq i$ were active, and bidder j opened with $(\emptyset, 0)$, then types (v_i, w_i) such that $v_i \geq w_i$ rise the bid on v only by a small amount. Types (v_i, w_i) such that $v_i \leq w_i$ use symmetric strategy if j opened with $(0, \emptyset)$.

- *If the observed history of bids is not obtained according to the strategies previously described then adopt the SEA strategy.*

A family of c.d.f.'s which satisfies the condition $E[x|x \geq z] \geq \frac{1}{2}(1+z)$ for each $z \in [0, 1]$ is $F(x) = x^\alpha$, with $\alpha \geq 1$. In this case we have:

$$E(x|x \geq z) = \frac{\alpha}{\alpha + 1} \frac{1 - z^{\alpha+1}}{1 - z^\alpha}$$

and it can be checked that the inequality is satisfied for $z \in [0, 1]$.

The equilibrium of Proposition 2 can also be extended to the case of $n > 2$ bidders.

Proposition 4 *Suppose that there are $n > 2$ bidders and the c.d.f. F is such that for each pair (a, z) such that $z \in [0, 1]$ and $a \in [0, 1 - z]$ the two following conditions are satisfied:*

$$E(x|z \leq x \leq 1 - a) + E(x|z + a \leq x \leq 1) \geq 1 + z \quad (3)$$

$$E(x|x \geq a + y, 1 - a \geq y \geq z) + E(x|y - a \geq x \geq z, y \geq a + z) \geq 1 + z. \quad (4)$$

Then the following strategy is part of a symmetric perfect Bayesian equilibrium: Behave as in Proposition 3 except at the following point:

- If at round $t - 1$ only you and another bidder were active then:
- If $v_i \geq w_i$ and you opened with $(0, \emptyset)$ while the other bidder opened with $(\emptyset, 0)$ then increase the bid on v and not on w , then stop.
- If $v_i < w_i$ and you opened with $(\emptyset, 0)$ while the other bidder opened with $(0, \emptyset)$ then increase the bid on v and not on w , then stop.
- If both players opened with $(0, \emptyset)$ and z was the last offer for both objects then increase the bid on v up to $z + a_i$, while keeping the offer for w at z . If the other bidder offers more than $z + a_i$ then get w for z . Otherwise, get v at the price at which competition ends, and leave w to the other bidder.

This equilibrium works as the one of Proposition (3): the bidders start signalling which object they prefer and then push up both prices until only two players are left. The difference is that at that point the same strategies as in Proposition (2) are used: if bidders have opened showing that they rank the two objects in the same way, then they compete only on the top ranked object. The stopping point for each player is $z + a_i$, that is the last bid plus the difference between the two values.

An important observation is that in the equilibria exhibited in Propositions (1) and (3), or in Propositions (2) and (4), the probability of collusion decreases as the number of bidders increases. To be more precise, the probability of assigning each object to the bidder with the highest value increases as the number of bidders increases. Conditions (3) and (4) are also satisfied by the uniform distribution.

4 Collusive Equilibria with Large Complementarities

In this section we consider the case of complementarities, i.e. $u_i(A \cup B) > u_i(A) + u_i(B)$. As stated in Section 2, we define $u_i(1) = v_i$, $u_i(2) = w_i$ and $u_i(\{1, 2\}) = v_i + w_i + k_i$, $i = 1, 2$, and we assume that v_i and w_i are drawn from a symmetric distribution with support $[0, 1]^2$, marginal density f , and marginal c.d.f. F . We also assume that for each player $i = 1, 2$, the value of the complementarity k_i is drawn from a distribution with continuous density g and support over an interval $[\underline{k}, \bar{k}]$. Each random variable k_i is independent of (v_j, w_j, k_j) for each $j \neq i$.

Finding equilibria for the case in which complementarities are present is complicated by the fact that, at any given round of the auction, a bidder's willingness to pay for a given object depends on the probability of winning the other object. This destroys the 'belief-free' nature of the SEA equilibrium described in Proposition 0. We can show however that, if the complementarities are

commonly known to be ‘large’, in a sense to be made precise, then a result similar to the one found in Proposition 0 can be obtained. Define $\theta_i := v_i + w_i + k_i$, the total value of the bundle for agent i .

Proposition 5 *Suppose that there are n players, 2 objects, and $\underline{k} > 1$. Then there is a perfect Bayesian equilibrium with the following outcome: The two objects are allocated to the agent with the highest θ_i , at a price equal to the second highest valuation (i.e. $\max_{j \neq i} \theta_j$).*

The basic intuition here is as follows. Under the assumptions of Proposition 5, if the buyers compete on both objects, the auction cannot end with each bidder buying just one object because the value for each bidder of owning a second object is higher. Therefore, both players behave as if bidding for a single object, the bundle $\{v, w\}$.

The equilibrium of Proposition 5 can be used as a threat to sustain collusive equilibria when large complementarities are present. The next proposition establishes the existence of an equilibrium which yields a superior expected surplus for both players. Define

$$\Theta_v := \left\{ (v, w, k) \in [0, 1]^2 \times [\underline{k}, \bar{k}] \mid v > w \right\}$$

and

$$\Theta_w := \left\{ (v, w, k) \in [0, 1]^2 \times [\underline{k}, \bar{k}] \mid v \leq w \right\}.$$

Proposition 6 *There exist two sets $A_v \subset \Theta_v$ and $A_w \subset \Theta_w$ such that the following strategy form a PBE:*

- *Types $(v_i, w_i, k) \in [0, 1]^2 \times [\underline{k}, \bar{k}] \setminus A_v \cup A_w$ open with $(0, 0)$ and compete for both objects;*
- *Types $(v_i, w_i) \in A_w$ open with $(\emptyset, 0)$*
- *Types $(v_i, w_i) \in A_v$ open with $(0, \emptyset)$.*
- *If the initial bids are $\{(0, \emptyset), (\emptyset, 0)\}$ or $\{(\emptyset, 0), (0, \emptyset)\}$ then bidders do not place any further bid. In all other cases, the bidders play the SEA equilibrium described in Proposition 5.*
- *If, at any stage, the bids differ from the ones given above, the bidders play the SEA equilibrium described in Proposition 5.*

The set A_v and A_w have the property that if $(v, w, k) \in A_v$ implies $(w, v, k) \in A_w$ and vice-versa.

The equilibrium of Proposition 6 is a natural generalization of the equilibrium described in Proposition (1). The set of types of each bidder is divided into three subsets. The first subset consists of those types who cannot be induced to collude. These are the types who have very low stand-alone values for each object, and who therefore only value the two objects together. To illustrate, suppose that agent 1 is of type $(0, 0, k_1)$, and define $s_2 := v_2 + w_2 + k_2$. If 1 accepts to buy only one object at price zero, the utility is zero. On the other hand, competing for both objects yields a utility equal to $\Pr(s_2 \leq k_1)(k_1 - E(s_2 | s_2 \leq k_1))$, which is positive, although possibly small. It is clear that types like $(\varepsilon_1, \varepsilon_2, k)$, for ε_1 and ε_2 sufficiently small, will also be unwilling to collude.

However, types with a stand-alone value for v sufficiently high are in fact willing to collude. In particular assume that bidder 1 has type (v_1, w_1, k_1) , with $v_1 > w_1$. Define $s_1 := v_1 + w_1 + k_1$. Suppose that at the first round bidder 1 learns that her opponent's type lies in some subset $A_w \subset \Theta_w$. Then collusion is better than competition if:

$$v_1 \geq \int_{\underline{k}}^{s_1} (s_1 - s_2) dH(s_2 | (v_2, w_2, k_2) \in A_w) \quad (5)$$

where H denotes the conditional c.d.f. of s_2 . In equilibrium, the set A_v will be exactly the set of those types for whom inequality (5) is satisfied. A similar inequality will define A_w . In equilibrium the two sets A_v and A_w have to be defined simultaneously. It is intuitive from inequality (5) that the two sets will be symmetric.

The shape of the set A_v is roughly as follows. Suppose that bidder 1 has $v_1 \geq w_1$. Let us summarize the type of agent 1 by the pair (v_1, s_1) , with $v_1 \in [0, 1]$ and $s_1 \in [\underline{k}, 2 + \bar{k}]$. Then it is clear that if the pair (v_1^*, s_1^*) satisfies inequality (5) then all pairs (v_1^*, s_1) with $s_1 < s_1^*$ will also satisfy the inequality. The inequality is also satisfied by the types characterized by the pair $(0, \underline{k})$. This type has no use for a single object, but is also sure to lose the competition for the two objects. Thus, (5) holds with equality. It is also clear that all types characterized by pairs like $(v_1, v_1 + \underline{k})$ are willing to collude. These are types for whom $w_1 = 0$ and have the lowest possible value for the synergy. If they compete for both objects they pay at least \underline{k} (the lowest possible value for s_2), and receive less utility than v_1 , which is what they would get accepting collusion. In general, for a given v_1 there will be a corresponding value $s_1(v_1)$ such that types with $s_1 < s_1(v_1)$ are willing to accept collusion and types with $s_1 > s_1(v_1)$ prefer to compete for both objects rather than to accept collusion. The shape of the set A_v is thus similar to the one showed in figure 1.

INSERT FIGURE 1 HERE

One case which is particularly simple and striking is the one in which the extent of the complementarity is known and identical across bidders, i.e. the distribution of k_i is degenerate on

some value k^* . In this case, provided that the condition $E(x) \geq \frac{1}{2}$ holds, the strategies proposed in proposition 1 are still equilibrium strategies. In other words, the set A_v and A_w described in proposition (6) can be taken to be Θ_v and Θ_w respectively, when the complementarities k_i are known and identical across agents. The intuition is straightforward. If k_i is the same for each bidder, then it will be entirely competed away whenever the equilibrium of proposition 5 is triggered. This makes any attempt to get both objects unattractive, hence even types with very low ‘stand-alone’ values can be induced to collude.

We conclude this section by reconsidering the conjecture according to which collusion decreases when complementarities are present. We have shown that the presence of complementarities does not destroy collusion. In fact, we have seen that large complementarities which are known and common among the players do not reduce the possibility of collusion at all. What really matters in hindering collusion is the *variability* of the extent of complementarities, rather than their absolute values.

5 Conclusions

When sequential procedures are used to sell multiple objects, the buyers can collude in order to reduce their payments to the seller. The general feature of collusive equilibria in open ascending auctions is that each bidder signals to the others which object has the highest value to her. After the signaling round, the bidders implicitly promise each other not to compete on the objects that they value less, provided they are not challenged on the objects they value more. We have provided conditions under which this behavior can be made a perfect Bayesian equilibrium. We have also shown that at least some degree of collusion may still be present when the ratio of bidders to objects is high, and when there are complementarities in the bidders utility functions.

As a more general point, the set of equilibria in auctions with multiple objects appears to be much richer than in the single object case. In this paper, we have shown some of these equilibria. It is worth pointing out that in all equilibria in which collusion-via-signalling occurs it must be the case that not too much information is revealed by the equilibrium bidding strategy. To see this, suppose, for example, that the bidding strategy were to reveal that one bidder has very low values for both objects. Then the other bidder will decide to compete for both objects, i.e. to revert to the SEA strategies, since her expected payments on both objects will be low. A bidder with high values will accept a collusive outcome only if the information revealed is such that her expected payment in open competition is sufficiently high. But this must imply that there is always some pooling among low and high values. This in turn implies that in general collusion-via-signalling not only reduces the revenue to the seller, but also reduces the efficiency of the final allocation.

Appendix

Propositions 1 and 2 are special cases, with $z = 0$, of Propositions 3 and 4 respectively.

Proof of Proposition 3. Given the symmetry of the problem, it is enough to check the optimality of the strategy for types having $v_i \geq w_i$. We will do this proceeding backward.

Consider the first round at which only two agents remain, say 1 and 2. Suppose that bidder 1 has $v_1 \geq w_1$ and opened at round zero with $(0, \emptyset)$, while bidder 2 opened with $(\emptyset, 0)$. Suppose also that the outstanding pair of bids at round $t - 1$ was (z, z) . Let $F_V(v_2 | L_z)$ and $F_W(w_2 | L_z)$ denote the c.d.f. of v_2 and w_2 respectively, both conditional on the set $L_z := \{(v_2, w_2) \in [0, 1]^2 \mid z \leq v_2 \leq w_2\}$.

If bidder 1 changes her bids, then the SEA strategies are triggered. The expected utility in this case is:

$$S(v_1, w_1 | L_z) = \int_z^{v_1} (v_1 - v_2) dF_V(v_2 | L_z) + \int_z^{w_1} (w_1 - w_2) dF_W(w_2 | L_z)$$

and we have to check that the deviation is unprofitable, that is:

$$v_1 - z \geq S(v_1, w_1 | L_z)$$

for each pair (v_1, w_1) such that $v_1 \geq w_1$. Since $S(v_1, w_1 | L_z)$ is increasing in w_1 , it is enough to check the inequality for the types on the diagonal, i.e. types such that $v_1 = w_1$. Defining:

$$\gamma_z(v_1) \equiv S(v_1, v_1 | L_z), \quad v_1 \in [z, 1],$$

the inequalities to be checked are:

$$v_1 - z \geq \gamma_z(v_1), \quad \text{for each } v_1 \in [z, 1].$$

We start by noting that this holds at $v_1 = z$, since both sides are zero; and then observe that the derivative of the LHS with respect to v_1 is 1, while the RHS derivative

$$\gamma'_z(v_1) = \int_z^{v_1} dF_V(v_2 | L_z) + \int_z^{v_1} dF_W(w_2 | L_z),$$

is zero at $v_1 = z$, and increasing, hence positive for each $v_1 \in (z, 1]$. Thus the function $\gamma_z(v_1)$ is convex, and we are done if we can prove that

$$1 - z \geq \gamma_z(1).$$

This can be rewritten as:

$$1 - z \geq E[1 - v_2 | L_z] + E[1 - w_2 | L_z] = 2 - E[v_2 | L_z] - E[w_2 | L_z],$$

or, using the symmetry of the joint distribution of v_2 and w_2 , as

$$E[v_2 | L_z] + E[v_2 | U_z] \geq 1 + z, \quad (6)$$

where $U_z := \{(v_2, w_2) \in [0, 1]^2 \mid z \leq w_2 \leq v_2\}$. By symmetry, we have $\frac{1}{2} = \Pr(L_z \mid z \leq v_2, z \leq w_2) = \Pr(U_z \mid z \leq v_2, z \leq w_2)$, hence

$$E[v_2 | L_z] + E[v_2 | U_z] = 2E[v_2 \mid z \leq v_2, z \leq w_2].$$

Independence of v_2 and w_2 implies $E[v_2 \mid z \leq v_2, z \leq w_2] = E[v_2 \mid z \leq v_2]$, so that condition 6 can be written as:

$$E(v_2 \mid z \leq v_2, z \leq w_2) \geq \frac{1}{2}(1 + z).$$

This is the condition stated in the theorem, and we can therefore conclude that the agents will collude when the opportunity arises.

The optimality of the strategies when more than two agents are left follows from the fact that any other strategy simply destroys the opportunity of collusion should it arise, and does not improve the outcome otherwise.

The only thing which is left to show is that at the opening a player is willing to signal truthfully the triangle in which her type is. This is going to matter only when the player ends up being one of the two last players and both players are competing for both objects. We show that for any given z at which this may happen it is better to have announced the correct triangle at date 0.

If bidder 1 announces the correct triangle, then the expected payoff conditional on being one of the two last bidders, and on z being the last bid for both bidders, is:

$$\frac{1}{2}(v_1 - z) + \frac{1}{2}S(v_1, w_1 \mid L_z) \quad (7)$$

This is because, given the symmetry in the distributions of v_i and w_i for each i , with probability $\frac{1}{2}$ the opponent is of type $w_2 \geq v_2$, so that her initial bid is $(\emptyset, 0)$, and with probability $\frac{1}{2}$ the opponent is of type $v_2 \geq w_2$. In the first case the auction ends immediately, yielding a payoff $v_1 - z$, while in the second case bidders go on playing the SEA equilibrium.

If the bidder opens with $(\emptyset, 0)$ then the expected payoff conditional on being one of the two last players and both having valuation at least z for both objects is:

$$\frac{1}{2}(w_1 - z) + \frac{1}{2}S(v_1, w_1 \mid U_z) \quad (8)$$

(notice that now S is conditional to $v_2 \geq w_2$ rather than to $v_2 \leq w_2$). The expression in (8) does not exceed the one in (7) if

$$v_1 + S(v_1, w_1 \mid L_z) \geq w_1 + S(v_1, w_1 \mid U_z),$$

which holds with equality if $v_1 = w_1$. Moreover, the derivatives with respect to v_1 are

$$1 + F_V(v_1 | L_z)$$

for the LHS, and

$$F(v_1 | U_z)$$

for the RHS. Hence the LHS grows faster than the RHS as v_1 is increased, thus implying that the inequality holds for each $v_1 > w_1$. ■

Proof of Proposition 4. Again, because of symmetry it suffices to check the optimality of the strategy along the equilibrium path for a bidder whose type is in the ‘lower triangle.’ We proceed backward.

Suppose first that only two players are left, say 1 and 2. If 1 opened with $(0, \emptyset)$ and 2 opened with $(\emptyset, 0)$, then the analysis of Proposition (3) applies, since condition (3) implies $E(x | z \leq x \leq 1) \geq (1+z)/2$ for $a = 0$, hence deviating to the SEA strategy is not profitable. If both bidders have opened with $(0, \emptyset)$, then we have to show that bidder 1 with type $v_1 - w_1 = a_1$ is willing to raise the bid on the first object only if she is not assigned object v and the outstanding bid is (p, z) with $p < z + a_1$. There are two possible deviations from the equilibrium path:

1) Stop bidding on v , and raise the bid on w by a small amount if necessary, i.e. if 1 is not currently assigned w . This deviation yields at most $w_1 - z$. Define

$$T_z(p) = \left\{ (v_2, w_2) \in [z, 1]^2 \mid p + z \leq v_2 - w_2 \right\}.$$

The set $T_z(p)$ is the support of bidder 1’s beliefs about 2’s values conditional on the last round’s bids being (p, z) for each bidder. The expected utility from following the equilibrium strategy is:

$$\begin{aligned} U^*(v_1, w_1 | T_z(p)) &= \Pr\{a_2 \leq a_1 | T_z(p)\} (v_1 - z - E(a_2 | a_2 \leq a_1, T_z(p))) \\ &\quad + \Pr\{a_2 \geq a_1 | T_z(p)\} (w_1 - z), \end{aligned}$$

which can be written as:

$$U^*(v_1, w_1 | T_z(p)) = w_1 - z + \Pr\{a_2 \leq a_1 | T_z(p)\} (v_1 - w_1 - E(a_2 | a_2 \leq a_1, T_z(p))).$$

It is clear that the last expression higher than $w_1 - z$.

2) Raise the bid on w , without stopping the bidding on v . In this case, the SEA equilibrium is triggered and we have to verify that:

$$U^*(v_1, w_1 | T_z(p)) \geq S(v_1, w_1 | T_z(p))$$

It is enough to check the inequality at $p = z + a_1$. Triggering the SEA equilibrium before p reaches that level can only do worse.

Using $v_1 = w_1 + a_1$, the relevant inequality to be checked is therefore:

$$w_1 - z \geq \int_{a_1+z}^{w_1+a_1} (w_1 + a_1 - v_2) dF_V(v_2|T_z(a_1+z)) + \int_z^{w_1} (w_1 - w_2) dF_W(w_2|T_z(a_1+z))$$

The inequality is satisfied at $w_1 = z$ and the RHS is increasing and convex. Applying the same reasoning as in Proposition (3) we conclude that it is enough to check the inequality:

$$1 - a_1 - z \geq \int_{a_1+z}^1 (1 - v_2) dF_v(v_2|T_z(a_1+z)) + \int_z^{1-a_1} (1 - a_1 - w_2) dF_w(w_2|T_z(a_1+z))$$

where use is made of the fact that the highest possible value for w_1 when $v_1 - w_1 \geq a_1$ is $1 - a_1$. The inequality is equivalent to:

$$E(v_2|v_2 \geq a_1 + w_2, 1 - a_1 \geq w_2 \geq z) + E(w_2|v_2 - a_1 \geq w_2 \geq z, v_2 \geq a_1 + z) \geq 1 + z$$

or:

$$E(x|x \geq a + y, 1 - a \geq y \geq z) + E(x|y - a \geq x \geq z, y \geq a + z) \geq 1 + z$$

which is the condition stated in the Proposition.

At last, we check that a bidder wants to stop after the other bidder has stopped the bidding, rather than competing for both objects. Suppose that the bidder has $v_1 - w_1 = a$ and the other bidder stopped at $z + a'$ with $a' \leq a$. In this case define:

$$\bar{\Omega}_{a',z} = \left\{ (v_2, w_2) \in [z, 1]^2 \mid v_2 - w_2 = a' \right\}.$$

Then the inequality becomes:

$$v_1 - a' - z \geq \int_{a'+z}^{v_1} (v_1 - v_2) dF_v(v_2|\bar{\Omega}_{a',z}) + \int_z^{w_1} (w_1 - w_2) dF_w(w_2|\bar{\Omega}_{a',z})$$

Using $w_1 = v_1 - a$ we can rewrite the inequality as:

$$v_1 - a' - z \geq \int_{a'+z}^{v_1} (v_1 - v_2) dF_v(v_2|\bar{\Omega}_{a',z}) + \int_z^{v_1-a} (v_1 - a - w_2) dF_w(w_2|\bar{\Omega}_{a',z})$$

Again, the inequality is satisfied at $v_1 = z + a$, the RHS is increasing and convex and we have only to check:

$$1 - a' - z \geq \int_{a'+z}^1 (1 - v_2) dF_v(v_2|\bar{\Omega}_{a',z}) + \int_z^{1-a} (v_1 - a - w_2) dF_w(w_2|\bar{\Omega}_{a',z})$$

In order to compute the integrals observe:

$$\begin{aligned} \Pr(v_2 \leq x | v_2 = w_2 + a', v_2 \geq z, w_2 \geq z) &= \Pr(w_2 \leq x - a' | 1 - a' \geq w_2 \geq z) \\ &= \frac{\Pr(x - a' \geq w_2 \geq z)}{\Pr(1 - a' \geq w_2 \geq z)} = \begin{cases} \frac{F(x-a')-F(z)}{F(1-a')-F(z)} & \text{if } x \geq z + a' \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Therefore:

$$f(v_2 | v_2 = w_2 + a', v_2 \geq z, w_2 \geq z) = \begin{cases} \frac{f(v_2-a')}{F(1-a')-F(z)} & \text{if } v_2 \geq z + a' \\ 0 & \text{otherwise} \end{cases}$$

Similar computations lead to:

$$f(w_2 | v_2 = w_2 + a', v_2 \geq z, w_2 \geq z) = \begin{cases} \frac{f(w_2+a')}{1-F(z+a')} & \text{if } 1 - a' \geq w_2 \geq z \\ 0 & \text{otherwise} \end{cases}$$

We therefore have:

$$\int_{a'+z}^1 (1 - v_2) dF_v(v_2 | \bar{\Omega}_{a',z}) = 1 - \frac{\int_{a'+z}^1 v_2 f(v_2 - a') dv_2}{F(1 - a') - F(z)}$$

and

$$\frac{\int_{a'+z}^1 v_2 f(v_2 - a') dv_2}{F(1 - a') - F(z)} = \frac{\int_z^{1-a'} (y + a') f(y) dy}{F(1 - a') - F(z)} = E(x | z \leq x \leq 1 - a') + a'$$

Similarly, we have:

$$\begin{aligned} \int_z^{1-a} (1 - a - w_2) dF_w(w_2 | \bar{\Omega}_{a',z}) &= \frac{\int_z^{1-a} (1 - a - w_2) f(w_2 + a') dw_2}{1 - F(z + a')} \\ &= (1 - a) \frac{F(1 - (a - a')) - F(a' + z)}{1 - F(a' + z)} - \frac{\int_z^{1-a} w_2 f(w_2 + a') dw_2}{1 - F(z + a')} \end{aligned}$$

and

$$\begin{aligned} \frac{\int_z^{1-a} w_2 f(w_2 + a') dw_2}{1 - F(z + a')} &= \frac{\int_{z+a'}^{1-(a-a')} (y - a') f(y) dy}{1 - F(z + a')} \\ \frac{\int_{z+a'}^{1-(a-a')} (y - a') f(y) dy}{1 - F(z + a')} &= \frac{\int_{z+a'}^1 y f(y) dy}{1 - F(z + a')} - a \end{aligned}$$

Combining these results we obtain the following condition:

$$\begin{aligned} 1 - a' - z &\geq 1 - E(x | z \leq x \leq 1 - a') - a' \\ &+ (1 - a) \frac{F(1 - (a - a')) - F(a' + z)}{1 - F(a' + z)} - \frac{\int_z^{1-a} w_2 f(w_2 + a') dw_2}{1 - F(z + a')} \end{aligned}$$

The inequality has to hold for each $a \geq a'$. Noticing that the RHS is decreasing in a , the relevant condition is obtained setting $a = a'$. This yields:

$$E(x|z \leq x \leq 1-a) + E(x|z+a \leq x \leq 1) \geq 1+z$$

which is the condition stated in the Proposition.

The argument for optimality when more than three bidders are active is identical to the one of Proposition (3): there is no point in triggering the SEA strategies at the opening, since the decision can always be taken later.

The only thing that remain to be proved is that it is convenient to open in the ‘true’ triangle. Possible deviations in this case are opening in the ‘wrong’ triangle or opening bidding on both objects, thus triggering the SEA equilibrium. The initial bid is only relevant if the bidder ends up among the two last bidders. We will show that for every z , and conditional on being one of the two last bidders, opening in the ‘true’ triangle gives a higher expected utility than any deviation.

The expected utility conditional on being one of the two remaining bidders at z for a type (v_1, w_1) such that $v_1 - w_1 = a_1 \geq 0$ is:

$$U^{(0,\emptyset)} = \frac{1}{2}v_1 + \frac{1}{2}(w_1 + \Pr(a_2 \leq a_1)(a_1 - E(a_2|a_2 \leq a_1))) - z \quad (9)$$

where $a_2 = v_2 - w_2$ and the probability distribution is conditional to $v_2 \geq z, w_2 \geq z$. This is because with probability $\frac{1}{2}$ the other bidder has opened in the upper triangle, so that the auction ends and 1 obtains v_1 at price z , while with probability $\frac{1}{2}$ the other bidder opens in the lower triangle. In the latter case the bidder pays at least z and obtains at least w_1 . It additionally obtains a_1 minus the price when the auction is won. Triggering the SEA equilibrium with an opening other than $(\emptyset, 0)$ or $(0, \emptyset)$ is obviously dominated, since the SEA equilibrium can be triggered later at no cost. We have therefore only to check that it is not convenient to open in the wrong triangle.

Suppose 1 opens bidding $(\emptyset, 0)$, i.e. signaling the ‘wrong’ triangle. If the other bidder also opens with $(\emptyset, 0)$ then the best strategy is to pretend to have $a_1 = 0$ and get v for z . This is clearly better than getting w for a price greater than z . The other possibility is to trigger the SEA strategies: To show that this cannot be optimal we have to check the inequality:

$$v_1 - z \geq S(v_1, w_1 | L_z)$$

Under the assumptions stated in the Proposition the inequality is satisfied (The analysis is the same as before).

If the other bidder opens with $(0, \emptyset)$ then any attempt to compete on good v triggers the SEA equilibrium. The payoff in this case is therefore whatever is best between obtaining w_1 at z and triggering the SEA equilibrium, that is $\max\{w_1 - z, S(v_1, w_1 | U_z)\}$. We therefore conclude that

the expected payoff, conditional on being one of the two players left at z , when the opening is in the wrong triangle is:

$$U^{(\emptyset,0)} = \frac{1}{2}(v_1 - z) + \frac{1}{2} \max \{w_1 - z, S(v_1, w_1 | U_z)\} \quad (10)$$

If $w_1 - z \geq S(v_1, w_1 | U_z)$ then this is clearly less than the utility obtained in equilibrium. If $w_1 - z < S(v_1, w_1 | U_z)$ the condition that the deviation be not profitable, that is (9) is greater than (10), can be written as:

$$w_1 + \Pr(a_2 \leq a_1)(a_1 - E(a_2 | a_2 \leq a_1)) - z \geq S(v_1, w_1 | U_z)$$

which is satisfied under the conditions stated in the Proposition because it is equivalent to the condition that it is optimal to follow the equilibrium strategy after opening in the ‘true’ triangle. ■

Proof of Proposition 5 . The following is a symmetric perfect Bayesian equilibrium yielding the desired outcome. The convention is that bids are a pair, with the first element referring to v .

- Open bidding $(1, 0)$.
- If the outstanding bids (\bar{b}_v, \bar{b}_w) are such that $\bar{b}_v + \bar{b}_w < v_j + w_j + k_j$ then keep the bid on v fixed and keep raising the bid on w .
- If the outstanding bids (\bar{b}_v, \bar{b}_w) are such that $\bar{b}_v + \bar{b}_w \geq v_j + w_j + k_j$ then:
 - Bid only on v if $v^i > \bar{b}_v, w \leq \bar{b}_w$.
 - Bid only on w if $v^i \leq \bar{b}_v, w > \bar{b}_w$.
 - Stop bidding otherwise.
- If at any point an agent makes an out of equilibrium bid then all the other agents believe that she is type $(1, 1, \bar{k})$ and they bid very high for both objects, confident that they don’t have to pay.

It is clear that the out of equilibrium strategies are optimal given the beliefs. We have to check optimality along the equilibrium path. Given the opening bid, it is clear that it is impossible to buy v for less than 1. Also, $v_i \leq \bar{b}_v$ from that point on. Therefore no agent will raise the bid on v , and each agent i will raise the bid on w up to $v_i + w_i + k_i - 1$. Any deviation gives zero utility, since the other agents will bid $(\bar{b}_v, 2 + \bar{k} - \bar{b}_v)$. ■

Proof of Proposition 6. Using the arguments of Proposition 5 we have that the strategies described in the last point of the Proposition constitute a perfect Bayesian equilibrium at any given stage. We are left with the task of finding the appropriate sets A_v , A_w , show that the prescribed strategy is optimal for all types at stage 0, and that for types in A_v , A_w it is optimal to stop bidding when the initial bids are $((0, \emptyset), (\emptyset, 0))$ or $((0, \emptyset), (\emptyset, 0))$.

Let

$$\Theta_v = \{(v, w, k) \mid v \in [0, 1], w \in [0, v], k \in [\underline{k}, \bar{k}]\}$$

and

$$\Theta_w = \{(v, w, k) \mid v \in [0, w], w \in [0, 1], k \in [\underline{k}, \bar{k}]\}.$$

Define $s \equiv v + w + k$, and let $H(s)$ be the c.d.f. on s , that is:

$$H(x) = \Pr \{v + w + k \leq x\}$$

Given our assumption on the support of v, w and k it is clear that $H(\underline{k}) = 0$ and $H(2 + \bar{k}) = 1$. Furthermore, given the symmetry of (v, w) and the independence of the distributions of v, w, k we have that $H(s \mid \Theta_v) = H(s \mid \Theta_w)$. Define the sets $A_v^0 = \Theta_v$, $A_w^0 = \Theta_w$, and define:

$$A_v^1 = \left\{ (v, w, k) \in \Theta_v \mid v \geq \int_{\underline{k}}^{v+w+k} (v+w+k-s) dH(s \mid \Theta_w) \right\}$$

$$A_w^1 = \left\{ (v, w, k) \in \Theta_w \mid w \geq \int_{\underline{k}}^{v+w+k} (v+w+k-s) dH(s \mid \Theta_v) \right\}$$

Thus, A_v^1 is the set of types in Θ_v who prefer to have v for free rather than competing for the bundle when it is known that the type of the other agent lies in Θ_w . A symmetric interpretation holds for A_w^1 . Observe that the sets A_v^1 and A_w^1 are compact and connected.

It is clear that the two sets are symmetric, meaning that if $(a, b, c) \in A_v^1$ then $(b, c, a) \in A_w^1$. Furthermore, it is also clear that $H(s \mid A_w^1) = H(s \mid A_v^1)$. Now, given two symmetric sets A_v^n and A_w^n with the property that $H(s \mid A_w^n) = H(s \mid A_v^n)$ define the sets:

$$A_v^{n+1} = \left\{ (v, w, k) \in \Theta_v \mid v \geq \int_{\underline{k}}^{v+w+k} (v+w+k-s) dH(s \mid A_w^n) \right\}$$

$$A_w^{n+1} = \left\{ (v, w, k) \in \Theta_w \mid w \geq \int_{\underline{k}}^{v+w+k} (v+w+k-s) dH(s \mid A_v^n) \right\}$$

If A_v^n and A_w^n are compact and connected then A_v^{n+1} and A_w^{n+1} are also compact and connected. We claim that the sequence $\{A_v^n\}$ has a converging subsequence, and that the set A_v to which the subsequence converges is the set we are looking for.

Let $H(\Theta_v)$ be the set of non-empty compact subsets of Θ_v . For a given set $F \in H(\Theta_v)$ define the set:

$$B_\varepsilon(F) = \{y \in \Theta_v \mid |y - x| < \varepsilon \text{ for some } x \in F\}$$

The space $H(\Theta_v)$ is a metric space when endowed with the Hausdorff distance:

$$\rho(F, G) = \min\{\varepsilon > 0 \mid F \subset B_\varepsilon(G) \text{ and } G \subset B_\varepsilon(F)\}$$

Since the set Θ_v is compact, the set $H(\Theta_v)$ is also compact (see e.g. Mas Colell (1985), Proposition A.5.1). The sequence $\{A_v^n\}$ is a sequence of elements in $H(\Theta_v)$, and since the set is compact there exists a converging subsequence. Let A_v be the element to which the subsequence converge, and observe that since all elements in $\{A_v^n\}$ are connected then A_v is connected too (see e.g. Mas Colell (1985), Proposition A.5.1). The set A_w can be obtained using exactly the same procedure.

The sets A_v and A_w satisfy the equilibrium conditions. Observe first that for each s and n we have $H(s \mid A_v^n) - H(s \mid A_w^n) = 0$. This implies that for each s :

$$\lim_{n \rightarrow \infty} H(s \mid A_v^n) - H(s \mid A_w^n) = H(s \mid A_v) - H(s \mid A_w) = 0 \quad (11)$$

Consider now that a type $(v, w, k) \in A_v$. The equilibrium strategy prescribes:

1. Open with $(0, \emptyset)$.
2. If the other bidder opens with $(\emptyset, 0)$ then stop bidding. In all other cases, use the SEA strategy.

Let us first check that the strategy after opening with $(0, \emptyset)$ and observing $(\emptyset, 0)$. The only possible deviation is to trigger the SEA equilibrium, which yields:

$$S(v_1, w_1, k \mid A_w) = \int_{\underline{k}}^{v_1 + w_1 + k} (v_1 + w_1 + k - s) dH(s \mid A_w)$$

Using (11) and the fact that $(v_1, w_1, k) \in A_v$ we obtain:

$$v \geq S(v_1, w_1, k \mid A_w)$$

We now check optimality at stage 0. It clearly makes no sense to trigger the SEA strategy. The only other possible deviation is to bid $(\emptyset, 0)$, thus signalling that the type belongs to A_w . It is not profitable to use the SEA equilibrium after the other type signals A_v , since this is equivalent to

triggering directly the SEA equilibrium with probability 1, which we know not to be profitable. Suppose now that collusion is accepted. Then we compare the expected utility of the deviation:

$$\Pr(A_v) w_1 + (1 - \Pr(A_v)) S(v_1, w_1, k_1 | \neg A_v)$$

with the expected utility of the equilibrium strategy:

$$\Pr(A_w) v_1 + (1 - \Pr(A_w)) S(v_1, w_1, k_1 | \neg A_w)$$

But now observe that the symmetry of A_v and A_w implies $\Pr(A_v) = \Pr(A_w)$ and:

$$S(v_1, w_1, k_1 | \neg A_v) = S(v_1, w_1, k_1 | \neg A_w)$$

Since $v_1 \geq w_1$ we conclude that the deviation is not profitable.

A symmetric reasoning shows that types $(v_1, w_1, k_1) \notin A_v \cup A_w$ are not better off announcing $(0, \emptyset)$ or $(\emptyset, 0)$. In this case the agent is going to trigger the SEA strategy no matter what the announcement of the other agent is, so that announcing $(0, 0)$ and triggering the SEA equilibrium from the very beginning is optimal. ■

References

- [1] Ausubel, L. and P. Cramton (1996) ‘Demand Reduction and Inefficiency in Multi-Unit Auctions’ mimeo, University of Maryland.
- [2] Campbell, C. (1998) ‘Coordination in Auction with Entry,’ mimeo, Ohio State University.
- [3] Chakraborty, I. (1998) ‘Equilibrium in Sealed-Bid Auctions for Multiple Units with Increasing Returns,’ mimeo.
- [4] Cramton, P. and J. Schwartz (1999) ‘Collusive Bidding: Lessons from the FCC Spectrum Auctions’ mimeo, University of Maryland.
- [5] *The Economist* (1997) ‘Learning to Play the Game’, May 17, page 94.
- [6] Engelbrecht-Wiggans, R. and C. Kahn (1998) ‘Low Revenue Equilibria in Ascending Price, Multi-Object Auctions’, mimeo, University of Illinois.
- [7] Graham, D. and R. Marshall (1987) ‘Collusive Bidder Behavior at Single-Object Second-Price and English Auctions’, *Journal of Political Economy*; **95**: 1217-39.
- [8] Gül, F. and E. Stacchetti (1997) ‘English and Double Auctions with Differentiated Commodities’, University of Michigan Working Paper 97-02.
- [9] Kwasnica, A. (1998) ‘Collusion in Multiple Object Simultaneous Auctions: Theory and Experiments’, working paper, Caltech SSWP1010.
- [10] Lopomo, G. (1998) ‘The English Auction is Optimal among Simple Sequential Auctions’, , *Journal of Economic Theory*; **82**: 144-66.
- [11] Mailath, G. and P. Zemsky (1991) ‘Collusion in Second Price Auctions with Heterogeneous Bidders’ *Games and Economic Behavior* **3**: 467-86.
- [12] Mas Colell, A. (1985) *The Theory of General Economic Equilibrium: A Differentiable Approach*, Econometric Society Monographs *n*^o 9, Cambridge University Press.
- [13] Milgrom, P. (1998) “Putting Auction Theory to Work: The Simultaneous Ascending Auction”, mimeo.
- [14] McAfee and McMillan (1992) ‘Bidding Rings’ *American Economic Review*; **93 (3)**: 579-99.
- [15] Myerson, R. (1981): “Optimal Auction Design,” *Mathematics of Operations Research* **6**: 58–73.

- [16] Pesendorfer, M., (1996) 'A Study of Collusion in First Price Auctions: Cartel Agreements' Ph. D. thesis, Northwestern University.

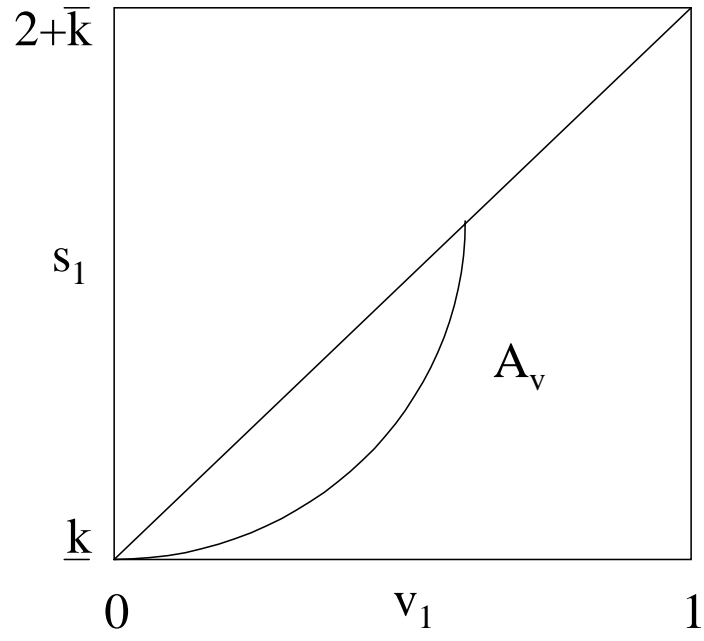


Figure 1