Economic Trade: a Solution to the Production Frontier of Two Economies in Trade

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Abstract

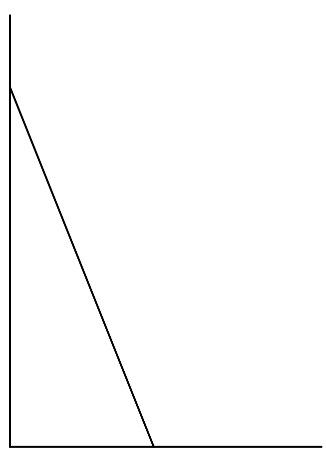
It is often possible to intuit the function describing the production frontier of two economies in trade. For example, when the two individual frontiers are linear, the aggregate frontier can be rendered by an intuitive sense of the two derivatives, as will be shown below. But when the two functions are complex, no amount of judgment will suffice to sketch the aggregate production frontier. To solve this conundrum, we will employ polar coordinates and Lagrange Multipliers to yield an analytical function describing the aggregate production frontier.

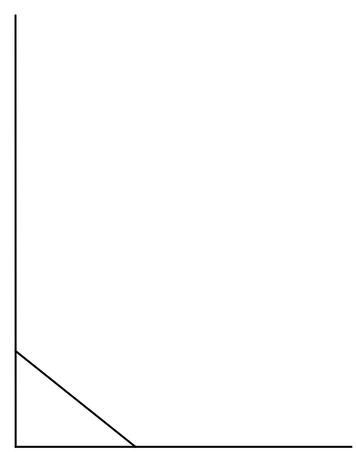
Introduction

Consider two economies that produce the same two goods. This can be represented with two functions on Cartesian coordinates, as will be shown below. Separately, they will optimize their economy by choosing the production point that intersects the highest utility curve. But introductory economics always tells us that they can profit by each choosing a production level and then trading among themselves. The aggregate production curve is the set of points obtained by adding the individual productions of the two economies. It is this curve that we will be interested in deriving.

Formulation

Some notation is now useful. Consider the two economies that produce the same two goods. They can be represented thusly.





The first curve is $y=5-2x$. Its limits are 5 on the y axis and 5/2 on the x axis. This will be economy 1. The second curve is $y=2-x$. The limits are 2 on the y axis and 2 on the x axis. The two axes represent the two goods that are being produced. So, for example, the first economy can produce point (1,3). Economy two can produce point (1,1). There is, of course, a trade off. The more of good 1 produced demands that less of good 2 be produced.

Now for the notation.

x1 = the amount of good x produced by economy 1

y1 = the amount of good y produced by economy 1

x2 = the amount of good x produced by economy 2

y2 = the amount of good y produced by economy 2

y1 and y2 will sometimes be referred to as f1(x1) and f2(x2) to illustrate that they are functions.

Aggregation

Therefore, if economy 1 is producing amount (x1,y1) and economy 2 is producing (x2,y2), then the aggregate amount produced by both economies is (x1+x2,y1+y2). In general, this is the sum of two point sets. Consider economy 1 as some blob on the x-y plane and likewise for economy 2. Their aggregate production is a sum of all of the possible points of economy 1 paired with all of the possible points of economy 2. The resulting blob, or point set, is the production set for the two economies together. As an example, we said that economy 1 can produce point (1,3) and economy 2 can produce point (1,1). Therefore, point (2,4) is somewhere in the aggregate production set. The object, however, is to find the *frontier* of the aggregate production set. This proves to be not as elementary as it would seem.

Convexity

Some ease of treatment of the notion of an aggregate production set is helpful. For example, we would like to show that if the two original sets are convex, then the aggregate production set is convex as well. This is very simple. We are almost inclined to leave it as an exercise for the reader, but we won't.

Let z1, z2 be two vectors in the aggregate production set. We would like to show that…

a*z1 + (1-a)*z2 is also in the set, where 0<=a<=1.

a*z1 + (1-a)*z2 can be rewritten…

a*(p1+q1) + (1-a)*(p2+q2) where the p's are two vectors in economy 1 and the q's are two vectors in economy 2. We can write this since any vector in the aggregate production set is a sum of some two vectors in the original sets.

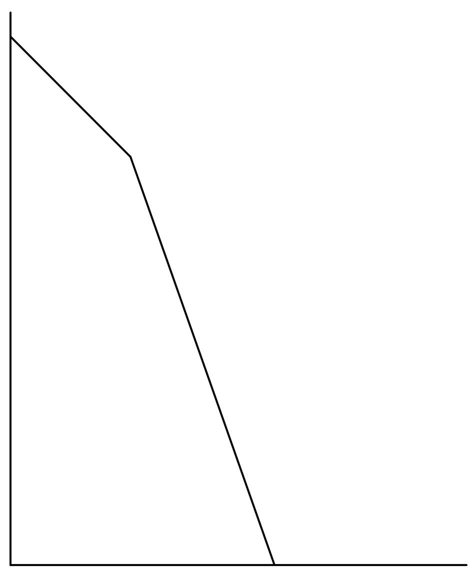
Rewrite…

a*p1 + (1-a)*p2 + a*q1 + (1-a)*q2

The two halves of the expression are each in their respective sets, by virtue of the convexity of the individual sets. The sum is in the aggregate production set since it is the sum of two vectors in the original sets. And the theorem is proved.

Aggregation--Again

The problem of aggregation is essentially this. If A and B are two sets describing the two production possibilities sets of the two economies, find the set of all vectors a+b such that a is contained in A and b is contained in B. To return to the simple case of two linear production possibilities sets, the aggregate production set can be intuited as such.



There is seemingly no amount of explanation that can justify this. It is only for the seasoned economist who simply understands the way curves agree and interact. With some practice, the reader might acquire this skill.

Lagrange and Descartes

Ideally, one would like to aggregate production possibilities functions of a more involved nature than simply linear. It is for this that we enlist some help from polar coordinates and the calculus of several variables. To begin,

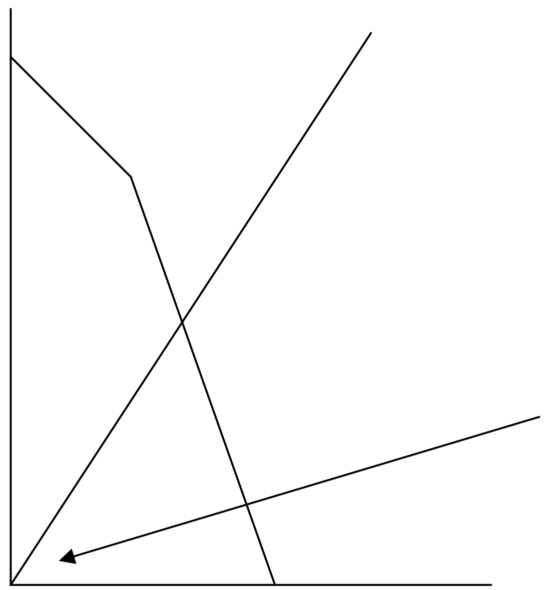
Let x1 be the x good in economy 1 and f1(x1) be the y good in economy 1.

Likewise, let x2 be the x good in economy 2 and f2(x2) be the y good in economy 2.

We wish to find the points (x1+x2, f1(x1)+f2(x2)) on the frontier of the aggregate production possibilities set. The *frontier* implies the farthest point in some sense. We plan to use the Euclidean distance that intuitively will supply the most remote point in the set.

Let Q be an angle in polar coordinates.

The strategy is to find the point in the aggregate set that is farthest along the ray of angle Q. This is represented as follows.



The arrow points to the angle Q. This constraint of insisting on the angle Q will prove essential in the formulation of the problem that can only be truly appreciated by anyone who has tried to solve the problem without it.

Consider the following optimization problem which tries to maximize the Euclidean distance of the points in the aggregate possibilities frontier along the ray of angle Q. We will consider a simple example first, the simple linear possibilities set from above. These are y1=5-2*x1 and y2=2-x2. The angle we will consider is the obliging 45 degrees.

Max (x1+x2)^2 + (5-2*x1+2-x2)^2
x1,x2

s.t. $\frac{5-2*x1+2-x2}{x1+x2} = 1$ (This means that the tangent of 45 degrees is 1.)

0<=x1<=5/2
0<=x2<=2

Simplifying the objective function yields…

(x1+x2)^2 + (7-2*x1-x2)^2

Then substituting the principal constraint into this equation and omitting all leading constants and arithmetic gives…

$\frac{1}{4}(7-x1)^2 + 1/4(7-x1)^2$

This should have been predictable since the angle is 45 degrees.

for x1 varying between 0 and 5/2.

Differentiate and write…

$-2+2*x1=0$

$x1=1$

Substituting to find x2 yields…

$x2 = 2$.

Consequently, y1=3 and y2=0. The aggregate production is thus x=3 and y=3. The determined geometer or the shrewd algebraist will confirm that this is the point that is implied by the illustration above. (The more sloping line is now y=-2x+9 and the ray is y=x.)

The general formulation of the problem is as follows.

$$\max_{x1,x2} \quad (x1+x2)^2 + (f1(x1) + f2(x2))^2$$

$$\text{s.t.} \quad \frac{f1(x1) + f2(x2)}{x1+x2} = \tan(Q)$$

and ranges of x1 and x2.

The reader will immediately recognize the tractability of this problem.

Conclusion

The point of this paper is to show the way to quantify the very abstract principle of adding all possibilities of points from two set points. The solution was somewhat heuristic, but the author's sense is that if the procedure were not true, then something Minkowski said would be violated. In all, it presents a solution that even the college economics student can understand and also the finest mathematical economists can debate.