# The Wealth-Consumption Ratio: A Litmus Test for Consumption-based Asset Pricing Models*

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**Abstract**

We propose a new method to measure the wealth-consumption ratio, the price-dividend ratio of a claim to aggregate consumption. It combines no-arbitrage restrictions with data on bond yields and stock returns. The estimated wealth-consumption ratio is much higher on average than the price-dividend ratio on stocks and has lower volatility. This implies that the consumption risk premium is substantially below the equity risk premium, or that total wealth is less risky than stock market wealth. Measuring the wealth-consumption ratio is important because changes in the wealth-consumption ratio enter as a second asset pricing factor besides consumption growth in the two leading representative-agent asset pricing models, the external habit model and the long-run risk model. The benchmark calibrations of these two asset pricing models have dramatically different implications for the wealth-consumption ratio, motivating our measurement exercise.

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From a macro-economist's perspective, stock market wealth (equity) is only a small fraction of total household wealth in the U.S. Other financial wealth, housing wealth, non-corporate business wealth, durable wealth, and especially human wealth constitute the bulk of total household wealth. In this paper we argue that total wealth has dramatically different risk-return characteristics than equity. Because it is less risky, it has both a lower mean return and a lower volatility. Correspondingly, the wealth-consumption ratio, which is the price-dividend ratio on a claim to aggregate consumption, is much higher on average and less volatile than the price-dividend ratio on equity.

Financial economists have written down models that were designed to match salient features of equity returns. The canonical consumption-based asset pricing model has spawned a large literature that seeks to solve its empirical shortcomings. Within the representative agent context, two main paradigms have emerged. The first approach introduces time-varying risk-aversion in preferences. The external habit model of Campbell and Cochrane (1999), henceforth EH model, is a prominent exponent.[1] The EH model was designed to show that equilibrium asset prices can be made to look like the data in a world without predictability in cash-flows, i.e. aggregate consumption and dividend growth are i.i.d. The second approach introduces predictability in aggregate consumption growth. The long-run risk model of Epstein and Zin (1991) and Bansal and Yaron (2004), henceforth LRR, is the leading exponent in this class.[2,3] The LRR model embodies a different philosophy: it tries to make sense of asset prices in a world where persistent shocks to cash-flows are the driving force. Because these shocks are small, predictability in consumption and dividend growth is hard to detect. These two models are the workhorses of modern finance, because reasonably calibrated versions deliver a large equity premium, a low risk-free rate, and time-varying expected returns.

Since consumption-based asset pricing models take a stance on aggregate consumption growth, they have implications for the price-dividend ratio on a claim to aggregate consumption, the wealth-consumption ratio. The wealth-consumption ratio is a key moment of interest in both models, because the log stochastic discount factor (SDF) is a function of the change in log consumption and the change in the log wealth-consumption ratio. Thus, the properties of the wealth-consumption ratio are intimately linked to the conditional market prices of risk generated in each model. Our

[1]Early contributions in the habit literature include Abel (1990), Constantinides (1990), Ferson and Constantinides (1991), Abel (1999). See Menzly, Santos, and Veronesi (2004) and Wachter (2006) for more recent contributions. Chen and Ludvigson (2007) estimate the habit process for a class of EH models.

[2]Hansen, Heaton, and Li (2006), Parker and Julliard (2005) and Malloy, Moskowitz, and Vissing-Jorgensen (2005) measure long-run risk based on leads and long-run impulse responses of consumption growth. Bansal, Kiku, and Yaron (2006) estimate the long-run risk model. Piazzesi and Schneider (2006) study its implications for the yield curve, Bansal, Dittmar, and Lundblad (2005) and Yang (2007) study the implications for the cross-section of equity portfolios, and Benzoni, Goldstein, and Collin-Dufresne (2005) for credit spreads.

[3]Bekaert, Engstrom, and Grenadier (2005) are the first to combine features of both models. It shares the focus on affine pricing models with ours and with Lettau and Wachter (2007). Bansal, Gallant, and Tauchen (2007) estimate both long-run risk and external habit models.

first contribution is to investigate the "macro" properties of total wealth in these two models. Section 1 documents that the benchmark calibrations of the EH and the LRR models imply wealth-consumption ratios with dramatically different properties, further motivating our measurement exercise.

Our second and main contribution is to measure the wealth-consumption ratio in US data. This is the price-dividend ratio on total wealth, which consists of human wealth, housing wealth, and broadly-defined financial wealth (private business wealth, durable wealth, stocks, bonds, life insurance). While we observe the cash flow on human wealth, labor income, we do not observe the discount rate (expected return), and therefore not the price. With housing wealth, as well as with other parts of broad financial wealth, such as private business wealth, the Flow of Funds' measurement may not accurately reflect market prices. Our approach in this paper is to (i) not take a stance on expected returns on human wealth, and (ii) not to use the Flow of Funds' measures of housing and financial wealth. Rather, we use data on aggregate consumption and labor income and put our trust in well-measured stock and bond prices to infer the economy's market prices of risk. Once market prices of risk are estimated, we value a claim to aggregate consumption. Its price-dividend ratio is the wealth-consumption ratio. Likewise, human wealth is measured as the expected present discounted value of future labor income.

Our work embeds the methodology of Campbell (1991, 1993, 1996) into the no-arbitrage framework of Ang and Piazzesi (2003). As Campbell (1993), we take a stance on the state variables that are in the investor's information set and assume that their dynamics are given by a vector autoregressive system. As in Ang and Piazzesi (2003), we assume that the log SDF is affine in innovations to the state vector, with market prices of risk that are also affine in the same state vector (Section 3). In a first step we estimate the dynamics of the state. In a second step, we estimate the market prices of risk. The second estimation imposes three sets of moments. The first set contains Euler equation for all traded assets in the state space. The second set imposes restrictions on assets that pay one unit of consumption, labor income, broad financial income, or dividend income. We impose the consistency requirements that the sum of these "strip" prices equals the price of the entire cash-flow stream. The third set of restrictions are Euler equations of assets we measure precisely: the cross-section of equity portfolio returns, a cross-section of returns on bonds of different maturities, and a cross-section of nominal bond yields.

Our estimation reveals that total wealth is considerably less risky than equity. The consumption risk premium, the expected excess return on total wealth, is 3.3% per year, half the size of the equity risk premium. This corresponds to an average wealth-consumption ratio of 46, much higher than the average price-dividend ratio on equity of 26. The wealth-consumption ratio is also less volatile than the price-dividend ratio on equity (17.9% versus 26.7%). Total wealth has very much the risk-return profile of a real bond, not that of a stock.

Using the same procedure, we value a claim to aggregate labor income. Human wealth has risk-return properties that closely resemble those of total wealth, not in the least because human wealth is estimated to be 89% of total wealth. This is consistent with Jorgenson and Fraumeni (1989). In contrast to the literature (Campbell (1996), Shiller (1995), Jagannathan and Wang (1996), Baxter and Jermann (1997), and Lustig and Van Nieuwerburgh (2007)), our approach avoids having to take a stance on the expected returns on human wealth. We find that human wealth is bond-like, an assumption typically made in the portfolio literature. Lettau and Ludvigson (2001a, 2001b) measure the cointegration residual between log consumption, broadly-defined financial wealth, and labor income, "*cay*". Their construction does not take into account the contribution of the volatility of price-dividend ratio on human wealth to the volatility of the wealth-consumption ratio.

Our methodology delivers a closed-form variance decomposition of the wealth-consumption ratio, the analog to Campbell and Shiller (1988)'s decomposition of the price-dividend ratio. We find that most of the variance in the $wc$ ratio is accounted for by the variance of total wealth returns rather than by the variance of consumption growth. While the modest variability of the $wc$ ratio implies only modest predictability, almost all predictability is concentrated in returns rather than consumption growth rates. Most of the predictability of future returns is predictability of future real interest rates rather than future risk premia. These properties contrast with predictability properties of equity returns. First, there is a lot more predictability as witnessed by the more volatile price-dividend ratio. Second, most predictability is concentrated in returns not in cash-flows (which is similar to the $wc$ ratio). Third, most predictability in returns in predictability in future risk premia rather than future risk-free rates.

Both models can account for some of the features of the measured wealth-consumption ratio. The LRR model delivers the observed dichotomy between total wealth and equity by assigning more long-run cash-flow risk to dividends than to consumption. Its benchmark calibration generates a much lower and less volatile wealth-consumption ratio than a price-dividend ratio on equity. On the predictability side, it delivers more cash-flow predictability than observed. The EH model replicates the variance decomposition of the wealth-consumption ratio very well. It also generates a lot of action in expected returns. However, the wealth-consumption ratio seems too volatile.

Section 2 argues that our results extend to a world with heterogeneous households where human wealth (or housing or private business wealth for that matter) are non-tradeable or carry idiosyncratic risk that cannot be insured away. We show that, as long as there is a non-zero set of households that participates in the equity and in the bond market, the no-arbitrage SDF that prices stocks and bonds also prices both individual and aggregate labor income (or housing or proprietary business income) streams. This is true even when most households only hold a bank account (one-period nominal bonds) and in the presence of generic borrowing or wealth constraints.

# 1 The Wealth-Consumption Ratio in Leading Asset Pricing Models

The wealth-consumption ratio plays a crucial role in the two leading asset pricing models, the external habit model and the long-run risk model. In this section, we show that the log stochastic discount factor in each of these models can be written as a linear function of log changes in consumption and log changes in the wealth-consumption ratio. This two-factor representation highlights the importance of the *wc* ratio dynamics for the models' respective asset pricing implications. Interestingly, the external habit (EH) and long-run risk (LRR) models turn out to have dramatically different implications for the wealth-consumption ratio, at least under their benchmark parameterizations. This discrepancy further motivates the efforts in this paper to measure the wealth-consumption ratio in the data.

## 1.1 The Total Wealth Return

We start from the budget constraint

$$W_{t+1}^T = R_{t+1}^c (W_t^T - C_t^T)$$

which states that the beginning-of-period (or cum-dividend) total wealth $W_t^T$ which is not spent on consumption $C_t^T$ earns a gross return $R_{t+1}^c$ and leads to beginning-of-next-period total wealth of $W_{t+1}^T$. Total wealth consists of human wealth, housing wealth, durable wealth, and financial wealth (stocks, bonds net of credit card and housing debt, pensions and life insurance, private business wealth) of the household sector. The return on a claim to aggregate consumption, the *total wealth return*, is defined as

$$R_{t+1}^c = \frac{W_{t+1}}{W_t - C_t} = \frac{C_{t+1}}{C_t} \frac{WC_{t+1}}{WC_t - 1}.$$

The *total wealth return* $R_{t+1}^c$ is a weighted combination of the returns on these wealth categories. Total consumption is the sum of non-durable and services consumption, which includes housing services consumption, and durable consumption. In what follows, we use lower-case letters to denote natural logarithms.

We start by using the Campbell, (1991, 1993) approximation of the log total wealth return $r_t^c = \log(R_t^c)$ around the long-run average log wealth-consumption ratio $A_0^c$.

$$r_{t+1}^c = \kappa_0^c + \Delta c_{t+1} + wc_{t+1} - \kappa_1^c wc_t, \tag{1}$$

where we define the log wealth-consumption ratio as

$$wc_t = \log\left(\frac{W_t^T}{C_t^T}\right) = w_t^T - c_t^T,$$

The linearization constants $\kappa_0^c$ and $\kappa_1^c$ are non-linear functions of the unconditional mean wealth-consumption ratio $A_0^c \equiv E[w_t^T - c_t^T]$:

$$\kappa_1^c = \frac{e^{A_0^c}}{e^{A_0^c} - 1} > 1 \text{ and } \kappa_0^c = -\log\left(e^{A_0^c} - 1\right) + \frac{e^{A_0^c}}{e^{A_0^c} - 1} A_0^c. \tag{2}$$

## 1.2 The Long-Run Risk Model

**Setup** The long-run risk literature works off the class of preferences due to Kreps and Porteus (1978), Epstein and Zin (1989, 1991), and Duffie and Epstein (1992); see equation (38) in Appendix A.1. These preferences impute a concern for the timing of the resolution of uncertainty. A first parameter $\alpha$ governs risk aversion and a second parameter $\rho$ governs the willingness to substitute consumption inter-temporally. In particular, $\rho$ is the inverse of the inter-temporal elasticity of substitution (EIS). We adopt the consumption growth specification of Bansal and Yaron (2004):

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \sigma_t \eta_{t+1}, && (3) \\
x_{t+1} &= \rho_x x_t + \varphi_e \sigma_t e_{t+1}, && (4) \\
\sigma_{t+1}^2 &= \sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}, && (5)
\end{aligned}$$

where $(\eta_t, e_t, w_t)$ are i.i.d. mean-zero, variance-one, normally distributed innovations. Consumption growth contains a low-frequency component $x_t$ and is heteroscedastic, with conditional variance $\sigma_t^2$. These two state variables capture time-varying growth rates and time-varying economic uncertainty.

**SDF Representation** The first proposition shows that the log SDF is a linear function of the growth rate of consumption and the growth rate of the log wealth-consumption ratio. The log wealth-consumption ratio itself is a linear function of the two state variables $x_t$ and $\sigma_t^2$, as noted in Bansal and Yaron (2004).

**Proposition 1.** *For $\rho \neq 1$, the log SDF in the long-run risk model can be stated as*

$$m_{t+1}^{LRR} = \left\{\frac{1-\alpha}{1-\rho}\log\beta + \frac{\rho-\alpha}{1-\rho}\kappa_0^c\right\} - \alpha\Delta c_{t+1} - \frac{\alpha-\rho}{1-\rho}\left(wc_{t+1} - \kappa_1^c wc_t\right) \tag{6}$$

*where the log wealth-consumption ratio is linear in the two state variables* $z_t^{LRR} = [x_t, \sigma_t^2 - \bar{\sigma}^2]$*:*

$$wc_t = A_0^{LRR} + A_1^{LRR\prime} z_t^{LRR}. \tag{7}$$

Appendix A.2 proves this proposition. The result relies on the Campbell approximation of returns and the joint log-normality of consumption growth and the two state variables.[4,5] The same appendix also spells out the (non-linear) system of equations that solves for the mean $wc$ ratio $A_0^{LRR}$, and its dependence on the state $A_1^{LRR}$ in equation (7) as a function of the structural parameters of the model. This system imposes the non-linear dependence of $\kappa_1^c$ and $\kappa_0^c$ on $A_0^{LRR}$ (equation 2). This proposition highlights how central the properties of the wealth-consumption ratio are for the LRR model's asset pricing implications.

**Calibration** We calibrate the long-run risk model choosing the benchmark parameter values of Bansal and Yaron (2004).[6] We use $\rho = 2/3$, $\alpha = 10$, and $\beta = .997$ for preferences in (6); and $\mu_c = .45e^{-2}$, $\bar{\sigma} = 1.35e^{-2}$, $\rho_x = .938$, $\varphi_e = .126$, $\nu_1 = .962$, and $\sigma_w = .39 * 10^{-5}$ for the cash-flow processes in (3)-(5). The vector $\Theta^{LRR} = (\alpha, \rho, \beta, \mu_c, \bar{\sigma}, \varphi_e, \rho_x, \nu_1, \sigma_w)$ stores these parameters.[7] We then solve for the loadings of the state variables in the log wealth-consumption ratio expression (7) and find: $A_0^{LRR} = 5.85$, $A_1^{LRR} = [5.16, -175.10]$. The corresponding linearization constants are $\kappa_0^c = .0198$ and $\kappa_1^c = 1.0029$. Since $\kappa_1^c$ is essentially 1, the second asset pricing factor in the SDF is essentially the log change in the wealth-consumption ratio.

**Simulation** We run 5,000 simulations of the model for 236 quarters each, corresponding to the period 1948-2006. In each simulation we draw a $236 \times 3$ matrix of mutually uncorrelated standard normal random variables for the cash-flow innovations $(\eta, e, w)$ in (3)-(5). We start off each run at the steady-state ($x_0 = 0$ and $\sigma_t^2 = \bar{\sigma}^2$). For each run, we form log consumption growth $\Delta c_t$, the two state variables $[x_t, \sigma_t^2 - \bar{\sigma}^2]$, the log wealth-consumption ratio $wc_t$ and its first difference, and the log total wealth return $r_t^c$. We compute their first and second moments.[8] These moments are based on

[4] Appendix A.1 shows that the ability to write the SDF in the LRR model as a function of consumption growth and the consumption-wealth ratio is general. It does not depend on the linearization of returns, nor on the assumptions on the stochastic process for consumption growth in equations (3)-(5).

[5] When $\rho$ equals 1, the wealth-consumption ratio is constant, and the SDF does not satisfy (6). Appendix F.1 shows that the consumption risk premium equals the risk premium in a model without long-run risk when $\rho = 1$. Appendix F.1 also discusses the implications for the dividend claim.

[6] Since their model is calibrated at monthly frequency but the data are quarterly, we work with a quarterly calibration instead. We have also simulated the model at monthly frequency or quarterly frequency and computed annualized statistics. The results were very similar. Appendix A.7 describes the mapping from monthly to quarterly parameters.

[7] The corresponding monthly values are $\Theta^{LRR} = (10, .6666, .998985, .0015, .0078, .044, .979, .987, .23 * 10^{-5})$.

[8] Most population moments are known in closed-form, so that we do not have to simulate. However, the simulation approach has the advantages of generating small-sample biases that may also exist in the data and delivering (bootstrap) standard errors.

the last 220 quarters only, for consistency with the length for our data for consumption growth and the growth rate of the wealth-consumption ratio (1952.I-2006.IV).[9] Column 1 of Table 1 reports the moments for the long-run risk model under the benchmark calibration. All reported moments are averages of the statistics across the 5,000 simulations. The standard deviation of these statistics across the 5,000 simulations can be interpreted as a small-sample bootstrap standard error on the moments, and is reported it in parentheses below the point estimate.

### 1.3 The External Habit Model

**Setup** We use the specification of preferences proposed by Campbell and Cochrane (1999), henceforth CC. The log SDF is

$$m_{t+1} = \log \beta - \alpha \Delta c_{t+1} - \alpha (s_{t+1} - s_t),$$

where $X_t$ is the external habit, the log surplus-consumption ratio $s_t = \log(S_t) = \log\left(\frac{C_t - X_t}{C_t}\right)$ measures the deviation of consumption from the habit, and has the following law of motion:

$$s_{t+1} - \bar{s} = \rho_s (s_t - \bar{s}) + \lambda_t (\Delta c_{t+1} - \mu_c).$$

The steady-state log surplus-consumption ratio is $\bar{s} = \log\left(\bar{S}\right)$. The parameter $\alpha$ continues to capture risk aversion. The "sensitivity" function $\lambda_t$ governs the conditional covariance between consumption innovations and the surplus-consumption ratio and is defined below in (11). To stay with the spirit of the CC exercise, we assume an i.i.d. consumption growth process:

$$\Delta c_{t+1} = \mu_c + \bar{\sigma} \eta_{t+1}, \tag{8}$$

where $\eta$ is mean zero, variance one, i.i.d., and normally distributed. It is the only shock in this model. The following proposition shows that the log SDF in the EH model is a linear function of the same two asset pricing factors as in the LRR model: the growth rate of consumption and the growth rate of the consumption-wealth ratio.

**Proposition 2.** *The log SDF in the external habit model can be stated as*

$$m_{t+1} = \log \beta - \alpha \Delta c_{t+1} - \frac{\alpha}{A_1} (wc_{t+1} - wc_t) \tag{9}$$

[9]This has the added benefit that the first 16 quarters are "burn-in," so that the first observation we use for the state vector is different in each run.

*where the log wealth-consumption ratio is linear in the sole state variable* $z_t^{EH} = s_t - \bar{s}$*,*

$$wc_t = A_0^{EH} + A_1^{EH} z_t^{EH}, \tag{10}$$

*and the sensitivity function takes the following form*

$$\lambda_t = \frac{\bar{S}^{-1}\sqrt{1-2(s_t-\bar{s})}+1-\alpha}{\alpha - A_1} \tag{11}$$

Appendix B.1 proves this proposition. The result relies on three assumptions: (1) the Campbell approximation of returns, (2) the joint log-normality of consumption growth and the state variable, and (3) the particular form of the sensitivity function in equation (11). Just like CC's sensitivity function delivers a risk-free rate that is linear in the state $s_t - \bar{s}$, our sensitivity function delivers a log wealth-consumption ratio that is linear in $s_t - \bar{s}$. To minimize the deviations with the CC model, we pin down the steady-state surplus-consumption level $\bar{S}$ by matching the steady-state risk-free rate to the one in the CC model. Taken together with the expressions for $A_0^{EH}$ and $A_1^{EH}$, this restriction amounts to a system of three equations in three unknowns $\left(A_0, A_1, \bar{S}\right)$.[10] The formulation of SDF in function of the wealth-consumption ratio suggests that, for the EH model to matter for asset prices, it needs to alter the properties of the $wc$ ratio in the right way.

**Calibration** We calibrate the long-run risk model choosing the benchmark parameter values of Campbell and Cochrane (1999). Since their model is calibrated at monthly frequency but our data are quarterly, we work with a quarterly calibration instead. Appendix B.8 describes the mapping from monthly to quarterly parameters. We use $\alpha = 2$, $\rho_s = .9658$, and $\beta = .971$ for preferences, and $\mu_c = .47e^{-2}$ and $\bar{\sigma} = .75e^{-2}$ for the cash-flow process (8), and summarize the parameters in the vector $\Theta^{EH} = (\alpha, \rho_s, \beta, \mu_c, \bar{\sigma})$.[11] After having found the quarterly parameter values, we solve for the loadings of the state variables in the log wealth-consumption ratio and find: $A_0^{EH} = 3.86$, $A_1^{EH} = 0.778$, and $\bar{S} = .0474$. The corresponding Campbell-Shiller linearization constants are $\kappa_0^c = .1046$ and $\kappa_1^c = 1.021583$. The simulation method is parallel to the one described for the LRR model. We note that the riskfree rate is nearly constant in the benchmark calibration; its volatility is .03% per quarter.

[10] Details are in Appendix B.2. Appendix B.7 discusses an alternative way to pin down $\bar{S}$. Appendix F.4 shows how to relax the Campbell-Shiller approximation of returns by including a second-order term in the approximation of $\log(\exp(wc_t)-1)$. The proposition remains unchanged, and the coefficients $A_0$ and $A_1$ are unchanged as well for all practical purposes. This suggests that our arguments does not hinge on the accuracy of the Campbell-Shiller approximation.

[11] The corresponding monthly values are $\Theta^{EH} = (\alpha, \rho_s, \beta, \mu_c, \bar{\sigma}) = \left(2, .9885, .990336, .1575e^{-2}, .433e^{-2}\right)$.

## 1.4 Properties of the Wealth-Consumption Ratio

The LRR and EH models have dramatically different implications for the wealth-consumption ratio, as summarized in Table 1. The first column is for the LRR model and the second column is for the EH model. Starting with the LRR model, we notice that the log wealth-consumption ratio is not that volatile. Its quarterly (and annual) volatility is 2.35%. Almost all the volatility in the wealth-consumption ratio comes from volatility in the persistent component of consumption (the volatility of $x$ is about 0.5% and the loading of $wc$ on $x$ is about 5). The persistence of both state variables induces substantial persistence in the $wc$ ratio: its auto-correlation coefficient is 0.91 at the 1-quarter horizon, 0.70 at the 4-quarter horizon, and 0.47 at the 8-quarter horizon (not reported). The standard errors indicate low sampling uncertainty.

The change in the $wc$ ratio, which is the second asset pricing factor, has a volatility of 0.90. For comparison, aggregate consumption growth, the first asset pricing factor, has a higher volatility of 1.45%. The change in the log $wc$ ratio has near-zero autocorrelation. The correlation between the two asset pricing factors is -.06, statistically indistinguishable from zero. The log total wealth return, defined below in (17), has a volatility of 1.64% per quarter in the LRR model. The low autocorrelation in $\Delta wc$ and $\Delta c$ generate low autocorrelation in total wealth returns. The total wealth return is strongly positively correlated with consumption growth (+.84) because most of the action in the total wealth return comes from consumption growth.

The final panel reports the consumption risk premium, the expected return on total wealth in excess of the risk-free rate (including a Jensen term). Appendix A.4 provides the expression and a decomposition for the consumption risk premium. Total wealth is not very risky in the LRR model; the quarterly risk premium is 40 basis points, which translates into 1.6% per year. Each asset pricing factor contributes about half of the risk premium. A low consumption risk premium indicates that the average wealth-consumption ratio must be very high. Indeed, expressed in annual levels ($e^{A_0^{LRR}-\log(4)}$), the mean wealth-consumption ratio is 87.

[Table 1 about here.]

The second column of Table 1 reports the moments of the wealth-consumption ratio under the benchmark calibration of the EH model. First and foremost, the $wc$ ratio is volatile in the EH model: it has a standard deviation of 29.3%, which is 12.5 times larger than in the LRR model. This volatility comes from the high volatility of the surplus consumption ratio (38%). The persistence in the surplus-consumption ratio drives the persistence in the wealth-consumption ratio: its auto-correlation coefficient is 0.93 at the 1-quarter horizon, 0.74 at the 4-quarter horizon, and 0.55 at the 8-quarter horizon (not reported).

The change in the $wc$ ratio has a volatility of 9.46%. This is more than 10 times higher than the volatility of the first asset pricing factor, consumption growth, which has a standard deviation of

0.75%. The high volatility of the change in the $wc$ ratio translates into a highly volatile total wealth return. The log total wealth return has a volatility of 10.26% per quarter in the EH model, six times the value of the LRR model. The change in the log $wc$ ratio has near-zero autocorrelation, as does the change in consumption. As in the LRR model, the total wealth return is strongly positively correlated with consumption growth (.91). In the habit model this happens because most of the action in the total wealth return comes from changes in the $wc$ ratio. The latter are highly positively correlated with consumption growth (.90, in contrast with the LRR model).[12]

The consumption risk premium is high in the EH model because total wealth is risky; the quarterly risk premium is 267 basis points, which translates into 10.7% per year. Most of the risk compensation in the EH model is for bearing $\Delta wc$ risk. The high consumption risk premium implies a low mean log wealth-consumption ratio of 3.86. Expressed in annual levels, the mean wealth-consumption ratio is 12.

To sum up, total wealth is not very risky in the LRR model and the $wc$ ratio is smooth. The opposite is true in the EH model. Essentially, the LRR model drives a wedge between the riskiness of total wealth and equity, whereas the EH model does not. The stark differences in the properties of the wealth-consumption ratio in the two workhorse models of modern asset pricing makes proper measurement of the wealth-consumption ratio imperative.

# 2 Measuring Human Wealth

The return on total wealth is a portfolio return that aggregates the returns on human wealth, and non-human wealth (housing, durable, and financial wealth). An important question is under which assumptions one can measure the returns on human wealth, and by extension on total wealth. The easiest way to derive these results in our paper is under the assumption that the representative agent can trade her human wealth. Starting with Campbell (1993), the literature makes this assumption explicitly. However, in reality, households cannot directly trade claims to their labor income. The securities they do trade cannot be used to hedge against idiosyncratic labor income shocks, i.e., markets are incomplete. A similar argument holds for the idiosyncratic risk they carry in the form of housing wealth or certain components of financial wealth, such as private business wealth. To aggravate matters, a substantial fraction of households only trades in a one-period bond (a bank account). This raises the question under what assumptions our approach of backing out market prices of risk from traded assets (stocks and bonds), and using them to price a claim to non-tradeable, aggregate labor income (or aggregate consumption) is a valid one.

Appendix E argues that these assumptions are rather mild. Our approach (and that of the entire Campbell (1993) machinery) applies to a setting with heterogeneous agents who face non-

[12] This correlation does not diminish much when we time-aggregate quarterly data. The corresponding correlation between the annualized series is .87.

tradeable, non-insurable labor income risk, as well as potentially binding borrowing constraints. We can allow for many of these households to be severely constrained in the menu of assets they trade. For example, they could just have access to a one-period bond. We show that, as long as there exists a non-zero set of households who trade in the stock market (securities that are contingent on the aggregate state of the economy) and the bond market, then the claim to aggregate labor income is priced off the same SDF that prices traded assets such as stocks and bonds. In other words, if there exists a SDF that prices stocks, it also prices aggregate labor income. This broadens the validity of our approach, and gives much more content to the measurement exercise that is about to follow.

# 3 Measuring the Wealth-Consumption Ratio in the Data

## 3.1 Estimation Strategy

In this section, we measure the wealth-consumption ratio in the data, proceeding in two broad steps. In a first step we define the state variables in the agent's and econometrician's information set, and posit a law of motion for them.

**State Evolution Equation** The $N \times 1$ vector of state variables in the data, $z_t$, follows a Gaussian VAR with one lag:

$$z_t = \Psi z_{t-1} + \Sigma^{\frac{1}{2}} \epsilon_t$$

with $\epsilon_t \sim IID\mathcal{N}(0, I)$ and $\Psi$ is a $N \times N$ matrix. The vector $z$ is demeaned. The covariance matrix of the innovations is $\Sigma$. We use a Cholesky decomposition of the covariance matrix, $\Sigma = \Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}\prime}$, where $\Sigma^{\frac{1}{2}}$ is has non-zero elements on and below the diagonal. The Cholesky decomposition makes the order of the variables in $z$ important. The state $z$ contains (in order of appearance): the Cochrane-Piazzesi factor, the nominal short rate (yield on a 3-month Treasury bill), realized inflation, the spread between the yield on a 5-year Treasury note and a 3-month Treasury bill, the log price-dividend ratio on the CRSP stock market, real dividend growth on the CRSP stock market, the return on a factor mimicking portfolio for consumption growth, the return on a factor mimicking portfolio for labor income growth, real per capita consumption growth, and real per capita labor income growth:[13]

$$z_t = [CP_t, y_t^{\$}(1), \pi_t, y_t^{\$}(20) - y_t^{\$}(1), pd_t^m, \Delta d_t^m, r_t^{fmpc}, r_t^{fmpy}, \Delta c_t, \Delta y_t]'$$

Our data are quarterly and run from 1952.I until 2006.IV (220 observations). Appendix C describes data sources and definitions in detail. The VAR structure implies that $\Delta c_t = \mu_c + e_c' z_t$, where $\mu_c$

[13]The factor mimicking portfolio returns are defined below.

denotes the unconditional mean consumption growth rate and $e_c$ is $N \times 1$ and denotes the column of an $N \times N$ identity matrix that corresponds to the position of $\Delta c$ in the state vector. Likewise, the nominal short rate dynamics satisfy $y_t^{\$}(1) = y_0^{\$}(1) + e_{yn}' z_t$, where $y_0^{\$}(1)$ is the unconditional average nominal short rate and $e_{yn}$ selects the second column of the identity matrix. $\pi_{t+1}$ is the (log) inflation rate between $t$ and $t+1$; inflation has an unconditional mean $\pi_0$.

To keep the analysis tractable, we impose substantial structure on the companion matrix $\Psi$. For example, expected returns on stocks are only allowed to vary with the price-dividend ratio. We specify these restrictions below. We estimate $\Psi$ by OLS, equation-by-equation. We form each innovation $z_{t+1}(\cdot) - \Psi(\cdot,:)z_t$ and compute their (full rank) covariance matrix $\Sigma$.

**Stochastic Discount Factor** We adopt the SDF methodology used in the no-arbitrage term structure literature, following Ang and Piazzesi (2003). The nominal pricing kernel $M_{t+1}^{\$} = \exp(m_{t+1}^{\$})$ is conditionally log-normal, where lower case letters continue to denote logs:

$$m_{t+1}^{\$} = -y_t^{\$}(1) - \frac{1}{2} L_t' L_t - L_t' \varepsilon_{t+1}. \tag{12}$$

The real pricing kernel is $M_{t+1} = \exp(m_{t+1}) = \exp(m_{t+1}^{\$} + \pi_{t+1})$.[14] Each element of the VAR innovation $\varepsilon_{t+1}$ has a market price of risk associated with it. The $N \times 1$ market price of risk vector $L_t$ is assumed to be an affine function of the state:

$$L_t = L_0 + L_1 z_t,$$

for an $N \times 1$ vector $L_0$ and a $N \times N$ matrix $L_1$. The real short yield $y_t(1)$, or *risk-free rate*, satisfies $E_t[\exp\{m_{t+1} + y_t(1)\}] = 1$. Solving out this Euler equation, we get:

$$\begin{aligned} y_t(1) &= y_t^{\$}(1) - E_t[\pi_{t+1}] - \frac{1}{2} e_\pi' \Sigma e_\pi + e_\pi' \Sigma^{\frac{1}{2}} L_t \\ &= y_0(1) + \left[ e_{yn}' - e_\pi' \Psi + e_\pi' \Sigma^{\frac{1}{2}} L_1 \right] z_t \end{aligned} \tag{13}$$

The real short yield is the nominal short yield minus expected inflation minus a Jensen adjustment minus the inflation risk premium. We do not impose the expectations hypothesis. The unconditional average risk-free rate $y_0(1)$ is defined in (14):

$$y_0(1) \equiv y_0^{\$}(1) - \pi_0 - \frac{1}{2} e_\pi' \Sigma e_\pi + e_\pi' \Sigma^{\frac{1}{2}} L_0 \tag{14}$$

[14] It too is conditionally Gaussian. Note that the consumption-CAPM is a special case of this where $m_{t+1} = \log \beta - \gamma \mu_c - \gamma \eta_{t+1}$. Appendices A.5 and B.5 show that an (essentially) affine representation also exists for the LRR and EH models.

**The Wealth-Consumption Ratio and Total Wealth Returns** In a second step, we use no-arbitrage conditions mostly on stock returns and bond yields to estimate the market price of risk parameters $L_0$ and $L_1$ (Section 3.2). With the prices of risk in hand, we can evaluate any claim and in particular a claim to aggregate consumption. In this exponential-Gaussian setting, the log wealth-consumption ratio is an affine function of the state variables, just as in the two leading asset pricing models:

**Proposition 3.** *The log wealth-consumption ratio is a linear function of the state vector $z_t$*

$$wc_t = A_0^c + A_1^{c\prime} z_t$$

*where the mean log wealth-consumption ratio $A_0^c$ is a scalar and $A_1^c$ is the $N \times 1$ vector which jointly solve:*

$$\begin{aligned} 0 &= \kappa_0^c + (1-\kappa_1^c)A_0^c + \mu_c - y_0(1) + \frac{1}{2}(e_c + A_1^c)'\Sigma(e_c + A_1^c) - (e_c + A_1^c)'\Sigma^{\frac{1}{2}}\left(L_0 - \Sigma^{\frac{1}{2}\prime}e_\pi\right) \quad (15) \\ 0 &= (e_c + e_\pi + A_1^c)'\Psi - \kappa_1^c A_1^{c\prime} - e_{yn}' - (e_c + e_\pi + A_1^c)'\Sigma^{\frac{1}{2}} L_1. \quad (16) \end{aligned}$$

The proof is in Appendix D. Once we have estimated the market prices of risk $L_0$ and $L_1$, equations (15) and (16) allow us to solve for the mean log wealth-consumption ratio $A_0^c$ and its dependence on the state $A_1^c$. They form a non-linear system of $N+1$ equations and $N+1$ unknowns (recall equation 2), which can be solved numerically and turns out to have a unique solution.

This solution implies that the log real total wealth return equals:

$$\begin{aligned} r_{t+1}^c &= \Delta c_{t+1} + wc_{t+1} + \kappa_0^c - \kappa_1^c wc_t, \quad (17) \\ &= r_0^c + \left[(e_c + A_1^c)'\Psi - \kappa_1^c A_1^{c\prime}\right] z_t + (e_c' + A_1^{c\prime})\Sigma^{\frac{1}{2}}\varepsilon_{t+1}, \end{aligned}$$

with unconditional average total wealth return

$$r_0^c = \kappa_0^c + (1-\kappa_1^c)A_0^c + \mu_c. \quad (18)$$

The Euler equation $E_t[exp\{m_{t+1} + r_{t+1}^c\}] = 1$ implies a consumption risk premium given by:

$$\begin{aligned} E_t\left[r_{t+1}^{c,e}\right] \equiv E_t\left[r_{t+1}^c - y_t(1)\right] + \frac{1}{2}V_t[r_{t+1}^c] &= -E_t\left[\left(r_{t+1}^c - E_t\left[r_{t+1}^c\right]\right)\left(m_{t+1} - E_t\left[m_{t+1}\right]\right)'\right] \quad (19) \\ &= E_t\left[\left((e_c + A_1^c)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right)\left((-L_t + e_\pi'\Sigma^{\frac{1}{2}})\varepsilon_{t+1}\right)'\right] \\ &= (e_c + A_1^c)'\Sigma^{\frac{1}{2}}\left(L_0 - \Sigma^{\frac{1}{2}\prime}e_\pi\right) + (e_c + A_1^c)'\Sigma^{\frac{1}{2}}L_1 z_t \end{aligned}$$

where $r^{c,e}$ denotes the log expected return on total wealth in excess of the risk-free rate and corrected for a Jensen term. The first term on the last line is the average consumption risk premium. This is a key object of interest; it measures how risky total wealth is. The second term, which has mean-zero, governs time variation in the consumption risk premium.

**The Price-Dividend Ratio on Human Wealth and Human Wealth Returns** The same way we priced a claim to aggregate consumption, we price a claim to aggregate labor income. We impose that the conditional Euler equation for human wealth returns is satisfied. Given market prices of risk $L_0$ and $L_1$, (20) and (21) pin down $A_0^y$ and $A_1^y$ in the log price-dividend ratio on human wealth, $pd_t^y = A_0^y + A_1^y z_t$:

$$0 = \kappa_0^y + (1-\kappa_1^y)A_0^y + \mu_y - y_0(1) + \frac{1}{2}(e_2' + A_1^{y\prime})\Sigma(e_2 + A_1^y) - (e_2' + A_1^{y\prime})\Sigma^{\frac{1}{2}}\left(L_0 - \Sigma^{\frac{1}{2}\prime}e_\pi\right) \quad (20)$$

$$0 = (e_2 + e_\pi + A_1^y)'\,\Psi - \kappa_1^y A_1^{y\prime} - e_{yn}' - (e_2 + e_\pi + A_1^y)'\Sigma^{\frac{1}{2}}L_1. \quad (21)$$

where $\mu_y$ is unconditional labor income growth. We set $\mu_y = \mu_c$ to impose stationarity on the labor income share.[15] The constants $\kappa_0^y$ and $\kappa_1^y$ relate to $A_0^y$ the same way $\kappa_0^c$ and $\kappa_1^c$ relate to $A_0^c$. We recall that labor income growth is the second element of the state. The derivation is parallel to the proof of Proposition 3. The returns on human wealth are given by

$$\begin{aligned} r_{t+1}^y &= \kappa_0^y + \Delta y_{t+1} + pd_{t+1}^y - \kappa_1^y pd_t^y \\ &= r_0^y + [(e_2 + A_1^y)'\Psi - \kappa_1^y A_1^{y\prime}]z_t + (e_2' + A_1^{y\prime})\Sigma^{\frac{1}{2}}\varepsilon_{t+1} \\ r_0^y &= \kappa_0^y + (1-\kappa_1^y)A_0^y + \mu_y \end{aligned}$$

Finally, the conditional risk premium on the labor income claim is given by:

$$E_t\left[r_{t+1}^{y,e}\right] = (e_2 + A_1^y)'\Sigma^{\frac{1}{2}}\left(L_0 - \Sigma^{\frac{1}{2}\prime}e_\pi\right) + (e_2 + A_1^y)'\Sigma^{\frac{1}{2}}L_1 z_t. \quad (22)$$

## 3.2 Estimating Market Prices of Risk

The second step estimates the market price of risk parameters in $L_0$ and $L_1$. We identify them off three sets of moments. The first set of moments prices the term structure of interest rates. The second set of moments contains the price-dividend ratio and the expected excess return on the overall stock market. The third set of moments prices two portfolios of stocks that are maximally correlated with consumption and labor income growth. Finally, we impose on the estimation that the human wealth share resides between zero and one.

[15]This is a cointegration assumption which prevents that human wealth becomes 0% or 100% of total wealth in finite time with probability 1. We rescale the level of consumption to end up with the same average labor income share (after imposing $\mu_y = \mu_c$) than in the data (before rescaling). As explained below, we also impose that the human wealth share stays above 0% and below 97%.

### 3.2.1 Step 1: The Term Structure of Interest Rates

The first four elements in the state, the Cochrane-Piazzesi factor, the nominal 3-month T-bill yield, the inflation rate, and the yield spread (5-year T-bond minus the 3-month T-bill), govern the term structure of interest rates. In the first four rows of the companion matrix $\Psi$, only the elements in the first four columns are non-zero. Note that this delivers a three-factor term structure model, with bond risk premia driven by the Cochrane-Piazzesi factor. All factors are observable.

**Nominal Yield Curve** The price of a $\tau$-year nominal zero-coupon bond satisfies:

$$p_t^{\$}(\tau) = E_t\left[\exp\left\{m_{t+1}^{\$} + p_{t+1}^{\$}(\tau - 1)\right\}\right].$$

This defines a recursion with $p_t^{\$}(0) = 1$. The corresponding bond yield is $y_t^{\$}(\tau) = -\log(p_t^{\$}(\tau))/\tau$. The following proposition shows that bond yields can be written as linear function of the state:

**Proposition 4.** *Nominal bond yields are affine in the state vector:*

$$y_t^{\$}(\tau) = -\frac{A^{\$}(\tau)}{\tau} - \frac{B^{\$}(\tau)'}{\tau} z_t,$$

*where the coefficients $A^{\$}(\tau)$ and $B^{\$}(\tau)$ follow ODEs:*

$$\begin{aligned} A^{\$}(\tau+1) &= -y_0^{\$}(1) + A^{\$}(\tau) + \frac{1}{2}\left(B^{\$}(\tau)\right)'\Sigma\left(B^{\$}(\tau)\right) - \left(B^{\$}(\tau)\right)'\Sigma^{\frac{1}{2}}L_0, \\ \left(B^{\$}(\tau+1)\right)' &= \left(B^{\$}(\tau)\right)'\Psi - e_{yn}' - \left(B^{\$}(\tau)\right)'\Sigma^{\frac{1}{2}}L_1, \end{aligned}$$

*and are initialized at $A^{\$}(0) = 0$ and $B^{\$}(0) = 0$.*

The proof is in Appendix D, and follows Ang and Piazzesi (2003). At the one-quarter horizon, we have $A^{\$}(1) = -y_0^{\$}(1)$ and $B^{\$}(1) = -e_{yn}$. This guarantees that the one-quarter nominal yield is priced correctly, on average and state-by-state. Because the state also contains the nominal yield spread, the restrictions

$$y_t^{\$}(20) = y_0^{\$}(20) + (e_{yn}' + e_{spr}')z_t \Leftrightarrow \frac{-1}{20}A^{\$}(20) = y_0^{\$}(20) \tag{23}$$

$$\frac{-1}{20}\left(B^{\$}(20)\right)' = (e_{yn} + e_{spr})' \tag{24}$$

impose that the model prices the 20-quarter nominal bond correctly. Equation (23) imposes that the model matches the unconditional average 5-year nominal yield $y_0^{\$}(20)$. This provides one restriction on $L_0$, more precisely it identifies the element $L_0[2]$. The dynamics of the 5-year yield imply restrictions on $L_1$ as in equation (24). Given the block structure of $\Psi$, the latter implies four restrictions on $L_1$, one element in each column.[16] We choose to estimate $L_1[4,1], L_1[2,2], L_1[2,3], L_1[2,4]$. We impose these restrictions by minimizing the squared distance $\left(y_t^{\$}(20) + \frac{A^{\$}(20)}{20} + \frac{\left(B^{\$}(20)\right)'}{20} z_t\right)^2$.

**Real Yield Curve** There is a similar proposition for *real* bond yields, which turns out to be useful for valuing real claims such as the claim to real dividends (equity) or the claim to real consumption (total wealth). Real yields $y_t(\tau)$, denoted without the \$ superscript, are also affine in the state, with coefficients following similar ODEs:

$$\begin{aligned}
A(\tau+1) &= -y_0(1) + A(\tau) + \frac{1}{2}(B(\tau))' \Sigma (B(\tau)) - (B(\tau))' \Sigma^{\frac{1}{2}} \left(L_0 - \Sigma^{\frac{1}{2}\prime} e_\pi\right), \\
(B(\tau+1))' &= (e_\pi + B(\tau))' \Psi - e_{yn}' - (e_\pi + B(\tau))' \Sigma^{\frac{1}{2}} L_1,
\end{aligned}$$

The proof is omitted for brevity. Note that for $\tau = 1$, we recover the expression for the risk-free rate in (13)-(14). The difference between $y_t^{\$}(\tau)$ and $y_t(\tau)$ is the sum of expected inflation averaged over the next $\tau$ periods and the $\tau$-period inflation risk premium.

**Additional Nominal Yields** We also minimize the squared distance between the observed and model-implied yields on nominal bonds of maturities 1,2, 3, 7, 10, and 20 years. The data are constant-maturity yields from the St.-Louis Federal Reserve Bank. Since the 5-year yield is the only one that features in the state space, we give its squared-distance moment a weight that is twice as high as the weight on the pricing error moments for other yields. These additional yields are potentially helpful to identify the decomposition of the long-term nominal bond risk premium into an inflation risk premium and a real rate risk premium component. They allow us to identify two more elements in $L_0$, $L_0[2]$ and $L_0[3]$. To avoid over-fitting, we estimate no further elements in $L_1$. In sum, the term structure component of the model pins down three elements in $L_0$ and four elements in $L_1$.[17]

### 3.2.2 Step 2: The Stock Market

The fifth and sixth row of the state space are the log price-dividend ratio and the log dividend growth on the CRSP value-weighted stock market portfolio. We match the expected excess stock market return and the $pd^m$ ratio. The corresponding rows of $\Phi$ have non-zero elements in the first six columns. This implies a rich model for expected stock return, which depends on the first six elements of the state space.

[16] All four elements are strictly necessary to match the yield dynamics implied by the VAR.

[17] We have experimented with freeing up four additional elements of $L_1$ in the term-structure block, but this did not lead to a better overall fit of the model.

**Stock Market Return** We define the return on equity conform the literature as $R^m_{t+1} = \frac{P_{t+1}+D^m_{t+1}}{P_t}$, where $P^m_t$ is the end-of-period price on the equity market. A log-linearization delivers:

$$r^m_{t+1} = \kappa^m_0 + \Delta d^m_{t+1} + \kappa^m_1 pd^m_{t+1} - pd^m_t. \tag{25}$$

The unconditional mean stock return is $r^m_0 = \kappa^m_0 + (\kappa^m_1 - 1)A^m_0 + \mu_m$, where $A^m_0 = E[pd^m_t]$ is the unconditional average log price-dividend ratio on equity and $\mu_m = E[\Delta d^m_t]$ is the unconditional mean dividend growth rate. The linearization constants $\kappa^m_0$ and $\kappa^m_1$ are different from the other wealth concepts because the timing of the return is different:

$$\kappa^m_1 = \frac{e^{A^m_0}}{e^{A^m_0}+1} < 1 \text{ and } \kappa^m_0 = \log\left(e^{A^m_0}+1\right) - \frac{e^{A^m_0}}{e^{A^m_0}+1}A^m_0. \tag{26}$$

Even though these constants arise from a linearization, we *define* log dividend growth so that the return equation holds exactly, given the CRSP series for $\{r^m_t, pd^m_t\}$. Our state vector $z$ contains the (demeaned) dividend growth on the stock market, $\Delta d^m_{t+1} - \mu_m$, and the (demeaned) log price-dividend ratio $pd^m - A^m_0$. We impose that the model prices excess stock returns correctly; we minimize the squared distance between VAR- and SDF-implied excess returns:

$$\begin{aligned} E^{VAR}_t[r^{m,e}_{t+1}] &= r^m_0 - y_0(1) + \frac{1}{2}\left(e_d + \kappa^m_1 e_{pdm}\right)\Sigma\left(e_d + \kappa^m_1 e_{pdm}\right) \\ &\quad + \left(\left(e_d + \kappa^m_1 e_{pdm} + e_\pi\right)'\Psi - e'_{pdm} - e'_{yn}\right) z_t & (27) \\ E^{SDF}_t[r^{m,e}_{t+1}] &= \left(e_d + \kappa^m_1 e_{pdm}\right)\Sigma^{\frac{1}{2}}\left(L_0 - \Sigma^{\frac{1}{2}\prime}e_\pi\right) + \left(e_d + \kappa^m_1 e_{pdm} + e_\pi\right)'\Sigma^{\frac{1}{2}}L_1 z_t, & (28) \end{aligned}$$

Matching the unconditional equity risk premium in model and data allows us to pin down $L_0[6]$. Matching the risk premium dynamics pins down six elements in $L_1$: $L1[6,1]$ through $L_1[6,6]$.

**Price-Dividend Ratio** While we imposed that equity returns satisfy their Euler equation, we have not yet imposed that the return on stocks reflects cash-flow risk in the equity market. We insist that the SDF correctly prices the claim to dividends on equity. In other words, we require that the price-dividend ratio in the model, which is the expected present discounted value of all future dividends, matches the price dividend ratio in the data, period by period.

The price-dividend ratio on equity must equal the sum of the price-dividend ratios on dividend strips of all horizons. A dividend strip of maturity $\tau$ pays 1 unit of consumption at period $\tau$, and nothing in the other periods.

$$\frac{P^m_t}{D^m_t} = \exp\{pd^m_t\} = \sum_{\tau=0}^{\infty} p^d_t(\tau) \tag{29}$$

where $p^d_t(\tau)$ denotes the price of a $\tau$ period dividend strip *divided by the current dividend.* The dividend strip price satisfies the following recursion:

$$p_t^d(\tau) = E_t\left[\exp\left\{m_{t+1} + \Delta d_{t+1} + \log\left(p_{t+1}^d(\tau-1)\right)\right\}\right],$$

with $p_t^d(0) = 1$. Appendix D proofs the following proposition:

**Proposition 5.** *Log strip prices are affine in the state vector:*

$$\log p_t^m(\tau) = A^m(\tau) + B^{m\prime}(\tau) z_t,$$

*where the coefficients $A^m(\tau)$ and $B^m(\tau)$ follow ODEs:*

$$\begin{aligned}
A^m(\tau+1) &= A^m(\tau) + \mu_m - y_0(1) + \frac{1}{2}\left(e_1 - \kappa_1^m e_3 + B^c(\tau)\right)' \Sigma \left(e_1 - \kappa_1^m e_3 + B^m(\tau)\right) \\
&\quad - \left(e_1 - \kappa_1^m e_3 + B^m(\tau)\right)' \Sigma^{\frac{1}{2}} \left(L_0 - \Sigma^{\frac{1}{2}} e_\pi\right), \\
B^m(\tau+1)' &= \left(e_1 - \kappa_1^m e_3 + e_\pi + B^m(\tau)\right)' \Psi + e_3' - e_{yn}' - \left(e_1 - \kappa_1^m e_3 + e_\pi + B^m(\tau)\right)' \Sigma^{\frac{1}{2}} L_1
\end{aligned}$$

*and are initialized at $A^m(0) = 0$ and $B^m(0) = 0$.*

The proof is in Appendix D. Using (34) and the affine structure, we obtain the restriction that the price-dividend ratio in the data equals the price-dividend ratio in the model:

$$0 = \left(pd_t^m - \sum_{\tau=0}^{T} \exp\{A^m(\tau) + (B^m(\tau))' z_t\}\right)^2. \tag{30}$$

Satisfying (30) implies equating (27) and (28) because dividend growth dynamics are fully described by the VAR and because of the relationship (25). The reverse is not true.

It turns out to be important to jointly estimate the market price of risk parameters that govern the term structure and the stock market blocks. The insight is that the observed price-dividend ratio on stocks contains important information about the real term structure, once that information is imposed in the form of a present-value model. That real term structure information is critical in pricing the claim to any real asset, such as a claim to real dividend or consumption growth. In other words, the price-dividend ratio on stocks is useful in separating out inflation and the real term structure.

#### 3.2.3 Step 3: Factor Mimicking Portfolios

Since our goal is to price a claim to aggregate consumption and labor income growth, and to use information about traded assets to do so, it is very helpful to have an asset whose returns are highly correlated with consumption growth and income growth, resp. The stock market portfolio only has a modest correlation with consumption growth (26%). Therefore, we use a broad cross-section of stock and bond portfolio returns to construct a traded portfolio that has maximal correlation with consumption and income growth, resp.[18] This results in two factor mimicking portfolios (fmp), whose returns we include in the state. The consumption (labor income) growth fmp has a correlation with consumption (labor income) growth of 63% (66%). These two fmp have a mutual correlation of 58%, suggesting non-trivial differences between the return to the consumption and income claims. The fmp returns are much lower on average than the stock return (2.3% and 2.3% versus 7.3% per annum) and are much less volatile (0.5% and 1.2% versus 16.7% volatility per annum). This suggests that a claim to consumption or labor income may be substantially less risky than a claim to equity dividends.

We include the fmp returns in the VAR as its seventh and eighth element and have non-zero elements in the corresponding rows of $\Phi$ in columns one through six. The estimation imposes that the risk premia on the fmp coincide between the VAR and the SDF model. In the same fashion as above, this implies one additional restriction on $L_0$ and $N$ additional restrictions on $L_1$:

$$E_t^{VAR}[r_{t+1}^{fmp,e}] = r_0^{fmp} - y_0(1) + \frac{1}{2}e'_{fmp}\Sigma e_{fmp} + \left((e_{fmp}+e_\pi)'\Psi - e'_{yn}\right) z_t \tag{31}$$

$$E_t^{SDF}[r_{t+1}^{fmp,e}] = e'_{fmp}\Sigma^{\frac{1}{2}}\left(L_0 - \Sigma^{\frac{1}{2}\prime}e_\pi\right) + (e_{fmp}+e_\pi)'\Sigma^{\frac{1}{2}}L_1 z_t \tag{32}$$

where $r_0^{fmp}$ is the unconditional average fmp return. There are two sets of such restrictions, one set for the consumption growth and one set for the labor income growth fmp. Again we minimize squared distances to identify $L_0[7]$, $L_0[8]$, $L_1[7,1]$ through $L_1[7,6]$, and $L_1[8,1]$ through $L_1[8,6]$.

### 3.2.4 Adding-Up Constraint

**Human Wealth Share** We define the labor income share as the ratio of labor income to consumption:

$$\overline{lis} = E\left[lis_t\right] = E\left[\frac{Y_t}{C_t^T}\right].$$

Total (human) wealth $W_t^T$ ($W_t^y$) is the expected present discounted value of current and future consumption (labor income). Therefore, the human wealth share is

$$\overline{hws} = E\left[hws_t\right] = E\left[\frac{W_t^y}{W_t^T}\right].$$

[18] We use 25 size and value portfolios, 10 industry portfolios, 25 size and long-term reversal portfolios, and bond returns of maturities 1, 2, 5, 7, 10, 20, and 30 years. The stock portfolio return data are from Kenneth French, the bond return data from CRSP. We project consumption (labor income) growth on these 67 traded assets and a constant to form factor mimicking portfolios.

We impose on the estimation that the time series for the human wealth share lies between zero and one.

**Adding-Up Constraint** Furthermore, the difference between total and human wealth, which we call broad financial wealth, is the present discounted value of broad financial income $D_t^a$. Indeed, the resource constraint in the economy states that

$$C_t^T = Y_t + D_t^a.$$

Imposing that broad financial and human wealth add up to total wealth on average amounts to imposing that

$$\exp\{A_0^c\} = \overline{lis}\exp\{A_0^y(\tau)\} + (1-\overline{lis})\exp\{A_0^a(\tau)\}, \tag{33}$$

I.e., we impose that the mean price-dividend ratio on broad financial wealth, $A_0^a$, is consistent with the mean price-dividend ratios on human and total wealth.

**Starting Values** In order to produce starting values for the estimation, we first estimate the term structure step in isolation. Then, we estimate the market prices of risk identified by the stock market block, taking as given the market prices of risk from the term structure block. Next, we identify the fmp market prices of risk taking as given all previously estimated parameters. This delivers a full set of starting values. We then add the human wealth share constraint and the adding-up constraint to all previous constraints and re-estimate the 6 elements of $L_0$ and the 22 elements of $L_1$. In approximate the price-dividend ratio in the model, which is a sum over an infinite number of strips, by a finite sum of 4,000 strips, or 1,000 years out.

## 3.3 Estimation Results

### 3.3.1 Bonds and Stocks

Before we study the estimation results for the wealth-consumption ratio, it is important to establish that the model succeeds in pricing the nominal term structure of interest rates and that it matches the stock return moments we discussed above. Starting with the term structure, recall that we match the 3-month yield by construction. The first two panels of Figure 1 plot the observed and model-implied *average* yield curve while Figure 2 plots the entire time-series for the 1-, 3-, 5-, 7-, 10-, and 20-year yields. The model provides a reasonably close fit. For the 5-year yield, which we insist on matching more precisely because it also features in the state vector, the average pricing error is -22 basis points (bp) per year. The standard deviation of the pricing error is 15 basis points, and the root mean squared error (RMSE) is 26bp. For the other 6 yields, the mean annual pricing errors range from -21bp to +83bp, the volatility of the pricing errors range from 32-59 bp, and the RMSE from 39-97bp.[19] While these pricing errors are higher than the ones obtained by term-structure models with latent factors, the model does a good job capturing the level and dynamics of long yields. For example, the annual volatility of the nominal yield on the 5-year bond is 1.36% in the data and 1.34% in the model.

[Figure 1 about here.]

[Figure 2 about here.]

The model also manages the capture the dynamics of stock returns quite well. The top panel of Figure 3 shows the dynamics of the price-dividend ratio on the stock market. This suggests that the present value model is a good data generating process for stock prices. The bottom panel shows that the model matches the equity risk premium that arises from the VAR structure. The average equity risk premium is 6.53% per annum in the data, and 6.54% in the model. Its annual volatility is 4.43% in the model and 4.42% in the model.

[Figure 3 about here.]

Including the price-dividend moment in the estimation turns out to be valuable for disentangling real rate and inflation risk premia. The (long-run) nominal risk premium on a 5-year bond, defined as the difference between the 5-year yield and the average expected future short term yield over the next 5 years, is the sum of a real rate risk premium (defined the same way for real bonds) and the inflation risk premium. We do not have good data for real bonds, but stocks are real assets that contain information about the term structure of real rates. The third panel of Figure 1 shows that our model implies real yields that range from 1.7% per year for 1-year real bonds to 2.6% per year for 20-year real bonds. The left panel of Figure 4 decomposes the 5-year yield into the real 5-year yield (which itself consists of the expected real short rate plus the real rate risk premium), expected inflation over the next 5-years, and the inflation risk premium. The inflationary period in the late 1970s-early 1980s was accompanied by high inflation expectations and an increase in the (long-run) inflation risk premium, but also by a substantial increase in the 5-year real yield. Intuitively, higher long real yields lower the price-dividend ratio on stocks, which indeed was low in that period. The right panel decomposes the average nominal bond risk premium into the average real rate risk premium and inflation risk premium for maturities ranging from 1 to 120 quarters. The inflation risk premium hovers around 60 basis points for maturities around 5 years and gradually goes down for longer bonds while the real rate risk premium become increasingly important at longer horizons.

[19] Note that the 2-year yield data only start in 1976.II, the 7-year yield only in 1969.II, and that the 20-year yield is unavailable between 1986.IV and 1993.II. The pricing errors are largest on these three bonds with missing data.

[Figure 4 about here.]

Finally, the model matches expected returns on the consumption and labor income growth factor mimicking portfolios (fmp) very closely, as Figure 5 shows. The annual risk premium on the consumption growth fmp is 0.71% in the data (VAR) and 0.78% in the model (SDF). Their volatilities are 1.44 and 1.45%. Likewise, the risk premium on the labor income growth fmp is 0.67% in the data and 0.62% in the model with volatilities of 1.46 and 1.42%.

[Figure 5 about here.]

### 3.3.2 The Wealth-Consumption Ratio

With the estimates for $L_0$ and $L_1$ in hand, it is straightforward to solve for $A_0^c$ and $A_1^c$ from equations (15)-(16). The third column of Table 1 summarizes the key moments of the log wealth-consumption ratio. The log wealth-consumption ratio has a volatility of 18% in the data. This number is in between the low volatility of the LRR model and the high volatility of the EH model. Just like in the models, the $wc$ ratio in the data is a persistent process. Its 1-quarter (4-quarter) serial correlation is .96 (.88). The volatility of the change in the wealth consumption ratio is 4.57%, again in between the two models. The same holds for the volatility of the total wealth return. The volatility of the second asset pricing factor is ten times larger than the volatility of the first asset pricing factor, consumption growth. The change in the $wc$ ratio has weak autocorrelation (-.07 and +.09 at the 1 and 4 quarter horizons). The correlation between the total wealth return and consumption growth is also mildly positive (.23), whereas in both models it is close to 1. How risky is total wealth in the data? According to our estimation, the consumption risk premium (calculated from equation 19) is 3.33% per year or 83 basis points per quarter. This results in a mean wealth-consumption ratio of 5.21 in logs, or 45.8 in annual levels. Total wealth is riskier in the data than in the LRR model, but much less risky than in the EH model. Also, the wealth-consumption ratio is much higher than the price-dividend ratio on equity, suggesting an important difference between the riskiness of stock market wealth and total wealth. Figure 6 plots the time-series for the annual wealth-consumption ratio, $\exp\{wc_t - log(4)\}$. The wealth consumption ratio dynamics are to a large extent inversely related to the long real yield dynamics in the left panel of Figure 4.

[Figure 6 about here.]

**Consumption Strips** Total wealth is a claim on future consumption. Therefore, the wealth-consumption ratio must equal the sum of the wealth-consumption ratios on consumption strips of all horizons. A consumption strip of maturity $\tau$ pays 1 unit of consumption at period $\tau$, and nothing in the other periods.

$$\frac{W_t^T}{C_t^T} = \exp\{wc_t\} = \sum_{\tau=0}^{\infty} p_t^c(\tau) \tag{34}$$

where $p_t^c(\tau)$ denotes the price of a $\tau$ period consumption strip *divided by the current consumption.* The consumption strip price satisfies the following recursion:

$$p_t^c(\tau) = E_t \left[ \exp \left\{ m_{t+1} + \Delta c_{t+1} + \log \left( p_{t+1}^c(\tau - 1) \right) \right\} \right],$$

with $p_t^c(0) = 1$. Appendix D proofs the following proposition:

**Proposition 6.** *Log strip prices are affine in the state vector:*

$$\log p_t^c(\tau) = A^c(\tau) + B^{c\prime}(\tau) z_t,$$

*where the coefficients $A^c(\tau)$ and $B^c(\tau)$ follow ODEs:*

$$\begin{aligned} A^c(\tau+1) &= A^c(\tau) + \mu_c - y_0(1) + \frac{1}{2}(e_c + B^c(\tau))' \Sigma (e_c + B^c(\tau)) - (e_c + B^c(\tau))' \Sigma^{\frac{1}{2}} \left( L_0 - \Sigma^{\frac{1}{2}} e_\pi \right), \\ B^c(\tau+1)' &= (e_c + e_\pi + B^c(\tau))' \Psi - e_{yn}' - (e_c + e_\pi + B^c(\tau))' \Sigma^{\frac{1}{2}} L_1. \end{aligned}$$

*and are initialized at $A^c(0) = 0$ and $B^c(0) = 0$.*

**Is the consumption claim like a stock or like a bond?** Each of these consumption strip prices $\tilde{p}_t(\tau)$ can be decomposed into the price of a real bond $p_t(\tau)$, a deterministic consumption dividend $\tau\mu_c$, and the price of a security that carries consumption cash-flow risk $p_t^{ccr}(\tau)$:

$$\log \tilde{p}_t(\tau) = \log p_t(\tau) + \tau \times \mu_c + \log p_t^{ccr}(\tau)$$

Since the log prices of the strips and the real bonds are both affine, so is the log price of the consumption cash-flow risk security: $\log p_t^{ccr}(\tau) = A^{ccr}(\tau) + B^{ccr}(\tau) z_t$.[20] The yield on a consumption strip therefore equals the yield on a real bond of the same maturity minus consumption growth plus the yield on the consumption cash-flow risk security

$$\widetilde{y}_t^\tau = y_t(\tau) - \mu_c + \left( -\frac{A_t^{ccr}}{\tau} - \frac{B_t^{ccr}}{\tau} z_t \right) = y_t(\tau) - \mu_c + y_t^{ccr},$$

We find that the consumption cash-flow risk premium is the dominant ingredient at shorter horizons, whereas the real yield component become more important ta longer horizons (Figure 7).

[20] Its coefficients follow ODEs:

$$\begin{aligned} A^{ccr}(\tau+1) &= A^{ccr}(\tau) + \frac{1}{2}(e_c + B^{ccr}(\tau))' \Sigma (e_c + B^{ccr}(\tau)) + (e_c + B^{ccr}(\tau))' \Sigma B(\tau) - (e_c + B^{ccr}(\tau))' \Sigma^{\frac{1}{2}} \left( L_0 - \Sigma^{\frac{1}{2}} e_\pi \right), \\ B^{ccr}(\tau+1)' &= (e_c + B^{ccr}(\tau))' \Psi - (e_c + B^{ccr}(\tau))' \Sigma^{\frac{1}{2}} L_1 \end{aligned}$$

Therefore, the long-run risk in a consumption claim seems to be mostly discount rate risk rather than cash-flow risk. At longer horizons, the consumption claim looks like a real perpetuity (with deterministically growing coupons).

[Figure 7 about here.]

#### 3.3.3 Human Wealth Returns

With the estimates for $L_0$ and $L_1$ in hand, we can easily calculate human wealth returns according to (22). Our estimates imply an equally low risk premium on human wealth of 3.29% per year. The mean price-dividend ratio on human wealth is 48.7. The volatility of $pd^y$ is 17.2%. The price-dividend ratios on human wealth and total wealth are almost perfectly correlated. Human wealth looks similar to total wealth because our estimation implies that most wealth is human wealth; the average human wealth share is 89.0%.[21] Interestingly, Jorgenson and Fraumeni (1989) have argued that the human wealth share is 90% of total wealth.

**Connection with Literature** The existing approaches to human wealth returns explicitly take a stance on expected returns on human wealth. The model of Campbell (1996) assumes that expected human wealth returns are equal to expected returns on financial assets (either equity or non-human wealth). This is a natural benchmark when financial wealth is a claim to a constant fraction of aggregate consumption. Shiller (1995) models a constant discount rate on human wealth: $E_t[r^y_{t+1} - r^y_0] = 0, \forall t$. Jagannathan and Wang (1996) assume that expected returns on human wealth equal the expected labor income growth rate. In this model, the price-dividend ratio on human wealth is constant. The construction of *cay* in Lettau and Ludvigson (2001a) effectively makes the same assumption. These models can be thought of as special case of ours, imposing additional restrictions on the market prices of risk $L_0$ and $L_1$ which we do not impose. Our estimation results indicate that expected excess human wealth returns have an annual volatility of 2.6%. This is substantially higher than zero, higher than expected labor income growth (1.8%), but lower than the expected excess returns on equity (4.4%). Lastly, average (real) human wealth returns (4.9%) are much lower than (real) equity returns (8.1%), but higher than (real) labor income growth (2.3%) and the (real) short rate (1.6%). Our findings therefore do not exactly fit any of these three assumptions on human wealth returns.

[21] The mean labor income share is 83.6%. The reason that the average human wealth share is higher than the average labor income share is because the mean price-dividend ratio on human wealth is higher than the mean price-dividend ratio on non-human wealth.

### 3.3.4 Predictability Properties

It is instructive to investigate the sources of variability in the wealth-consumption ratio using the standard Campbell and Shiller (1988) methodology. By iterating forward on the total wealth return equation (1), we can link the log wealth-consumption ratio at time $t$ to expected future total wealth returns and consumption growth rates:

$$wc_t = \frac{\kappa_0^c}{\kappa_1^c - 1} + \sum_{j=1}^{H} \left(\kappa_1^c\right)^{-j} \Delta c_{t+j} - \sum_{j=1}^{H} \left(\kappa_1^c\right)^{-j} r_{t+j}^c + \left(\kappa_1^c\right)^{-H} wc_{t+H}. \tag{35}$$

Because this expression holds both ex-ante and ex-post, one is allowed to add the expectation sign on the right-hand side. Imposing the transversality condition as $H \to \infty$ drops the last term, and delivers the familiar Campbell-Shiller decomposition of the "price-dividend" ratio for the consumption claim, the wealth-consumption ratio:

$$wc_t = \frac{\kappa_0^c}{\kappa_1^c - 1} + E_t\left[\sum_{j=1}^{\infty} \left(\kappa_1^c\right)^{-j} \Delta c_{t+j}\right] - E_t\left[\sum_{j=1}^{\infty} \left(\kappa_1^c\right)^{-j} r_{t+j}\right] \equiv \frac{\kappa_0^c}{\kappa_1^c - 1} + \Delta c_t^H - r_t^H. \tag{36}$$

We denote the cash-flow component by $\Delta c_t^H$ and the discount rate component by $r_t^H$. The wealth-consumption ratio fluctuates because it predicts consumption growth rates ($Cov\left[wc_t, \Delta c_t^H\right]$) or because it predicts future total wealth returns ($Cov\left[wc_t, -r_t^H\right]$):

$$V\left[wc_t\right] = Cov\left[wc_t, \Delta c_t^H\right] + Cov\left[wc_t, -r_t^H\right] = V\left[\Delta c_t^H\right] + V\left[r_t^H\right] - 2Cov\left[r_t^H, \Delta c_t^H\right]$$

The second equality suggests an alternative decomposition into the variance of expected future consumption growth, expected future returns, and their covariance. Finally, it is straightforward to break up $Cov\left[wc_t, r_t^H\right]$ into a piece that measures the predictability of future excess returns, and a piece that measures the covariance of $wc_t$ with future risk-free rates.

**Variance Decomposition in the Data** Our methodology delivers analytical expressions for all variance and covariance terms, spelled out in Appendix D equations (87-92). Our estimation has three implications. First, the mild variability of the $wc$ ratio implies only mild (total wealth) return predictability. Second, 100.2% of the variability in $wc$ is due to covariation with future total wealth returns, and -0.2% is due to covariation with future consumption growth. Hence, the wealth-consumption ratio predicts future returns (discount rates), not future consumption growth rates (cash-flows). Using the second equality above, the variability of future returns is 100.6%, the variability of future consumption growth is 0.2% and their covariance is -0.8% of the total variance of $wc$. Third, 71.2% of the 100.2% covariance with returns is due to covariance with future *risk-free* *rates*, and only 29% is due to covariance with future *excess* returns. The wealth-consumption ratio therefore not only predict future risk premia but mostly future variation in interest rates. What makes these results interesting is that they are quite different from the predictions implied by the leading asset pricing models.

**Variance Decomposition in the LRR Model** In the LRR model, the overall level of predictability is low because the wealth-consumption ratio is smooth. The (demeaned) log wealth-consumption ratio can be decomposed into a discount rate and a cash-flow component:

$$wc_t = \underbrace{\frac{1}{\kappa_1 - \rho_x} x_t}_{\Delta c_t^H} - \underbrace{\frac{\rho}{\kappa_1 - \rho_x} x_t - A_2^{LRR} \left(\sigma_t^2 - \overline{\sigma}^2\right)}_{r_t^H}.$$

Appendix A.3 derives this decomposition as well as the decomposition of the variance of $wc$. The discount rate component itself contains a risk-free rate component and a risk premium component. The persistent component of consumption growth $x_t$ drives only the risk-free rate effect (first term in $r_t^H$). It is governed by $\rho$, the inverse EIS. In the log case ($\rho = 1$), the cash flow loading on $x$ and the risk-free rate loading on $x$ exactly offset each other. The risk premium component is driven by the heteroscedastic component of consumption growth.[22] The expressions for the theoretical covariances of $wc_t$ with $\Delta c_t^H$ and $-r_t^H$ show that both cannot simultaneously be positive. When $\rho < 1$, the sign on the regression coefficient of future consumption growth on the log wealth-consumption ratio is positive, but the sign on the return predictability equation is negative (unless the heteroscedasticity mechanism is very strong). The opposite is true for $\rho > 1$ (low EIS). In the benchmark calibration of the LRR model, we are in the first case (high EIS). Most of the volatility in the wealth-consumption ratio arises from covariation with future consumption growth (297.5%). The other -197.5% is accounted for by the covariance with future returns. A calibration with an EIS below 1 would generate the same covariance signs as in the data. Alternatively, a positive (instead of zero) correlation between $x$ and $\sigma_t^2 - \bar{\sigma}^2$ may also help to generate a CS decomposition that is closer to the data. Finally, virtually all predictability in future total wealth returns arises from predictability in future risk-free rates. This is similar to what we find in the data.

**Variance Decomposition in the EH Model** In contrast to the LRR model, the EH model asserts that all variability in returns arises from variability in risk premia. The wealth-consumption ratio only has a discount rate component, because aggregate consumption growth is assumed to

[22]The heteroscedasticity also affects the risk-free rate component, see equation (59) in the Appendix. But without heteroscedasticity, there would be no time-variation in risk premia.

be i.i.d.:

$$wc_t = \underbrace{\frac{(1-\rho_s)}{\kappa_1 - \rho_s}\alpha\left(s_t - s\right) - \frac{\sigma^2 \bar{S}^{-2}}{\kappa_1 - \rho_s}\left(s_t - s\right)}_{r_t^H}$$

Since there is no cash flow predictability, 100% of the variability of $wc$ is variability of the discount rate component. The covariance between the wealth-consumption ratio and returns has the right sign: it is positive by construction. This variance decomposition is very close to the data. However, by overstating the variability of $wc$, the benchmark CC model overstates the predictability of the total wealth return. Also, the EH model implies that almost all the covariance with future returns comes from covariance with future excess returns, not future risk-free rates. This three-way comparison makes clear that both models account for some of the predictability features that we observed in the data.

## 3.4 Comparison with Equity

Compared to equity, total wealth is much less risky. The equity risk premium is 6.53% per year in the data, compared to an annual consumption risk premium of 3.33%. Consequently, the mean price-dividend ratio in levels is much lower than the mean wealth-consumption ratio (26.6 versus 45.8). The volatility of the log price-dividend ratio on equity, $pd^m$ is 26.7%, higher than the 17.9% volatility of $wc$. The dynamics of the wealth-consumption ratio and the price-dividend ratio on equity in Figure 3 are quite different; the two series have a correlation of only 0.53. Finally, the Campbell-Shiller decomposition of the "price-dividend" ratio for the equity dividend claim is:

$$pd_t^m = \frac{\kappa_0^m}{1-\kappa_1^m} + E_t\left[\sum_{j=0}^{\infty} \left(\kappa_1^m\right)^j \Delta d_{t+1+j}^m\right] - E_t\left[\sum_{j=0}^{\infty} \left(\kappa_1^m\right)^j r_{t+1+j}\right] \equiv \frac{\kappa_0^m}{1-\kappa_1^m} + \Delta dm_t^H - rm_t^H. \quad (37)$$

The VAR immediately delivers analytical expressions for all six components in the variance decomposition

$$V\left[pd_t^m\right] = Cov\left[pd_t^m, \Delta dm_t^H\right] + Cov\left[pd_t^m, -rm_t^H\right] = V\left[\Delta dm_t^H\right] + V\left[rm_t^H\right] - 2Cov\left[rm_t^H, \Delta dm_t^H\right]$$

The variance decomposition has the following features: (i) 79.1% of the variability in $pd^m$ comes from its covariance with future returns and 20.9% from covariation with future dividend growth, (ii) the variance can be broken into 10.4% dividend growth variability, 68.6% stock return variability, and 21% covariation between the two, and (iii) the 79.1% covariance of $pd^m$ with future returns can be broken out into 55.5% covariance with future excess returns and 23.5% covariance with future risk-free rates. That is, 70% of the predictability of stock returns is predictability of *excess* stock returns.

**Equity Returns in the LRR Model** The equity risk premium is the expected excess return on a claim to aggregate dividends in excess of the risk-free rate. We follow the specification and the calibration of dividend growth in Bansal and Yaron (2004):

$$\Delta d_{t+1} = \mu_m + \phi x_t + \varphi_d \sigma_t u_{t+1}$$

The shock $u_t$ is orthogonal to the other cash-flow innovations in (3)-(5). Just like the log wealth-consumption ratio, the log price-dividend ratio on stocks $pd^m$ is linear in the state vector $z_t^{LRR}$. Appendix A.6 proves the linearity, provides expressions for the coefficients, and describes the parameter choices in detail.[23] Dividend growth has the same mean as consumption growth in the model, but is more volatile (6.25% per quarter versus 1.45%). This greater volatility comes from a larger loading on the long-run risk component $x_t$ ($\phi = 3 > 1$) as well as from a larger loading on the heteroscedasticity component $\sigma_t^2 - \bar{\sigma}^2$. For these parameters, the equity risk premium is 139 basis points per quarter or 5.6% per year. It is 4% per year (or 3.5 times) higher than the consumption risk premium. More long-run risk translates into a higher risk premium on stocks. In the data, the risk premium difference is similar, at 3.3%. The volatility of the log price-dividend ratio is 16%, lower than the 25.7% in the data, but six times higher than that of the log wealth-consumption ratio in the LRR model. In the data, that volatility ratio is 2.65. Thus, the LRR model is able to drive a strong wedge between the riskiness of equity and total wealth. As was the case for the $wc$ ratio, the variance decomposition of the $pd^m$ ratio indicates mostly predictability of future cash-flows; 128% of the long-run variance comes from its covariance with future dividend growth and -28% from its covariance with future equity returns.

**Equity Returns in the EH Model** Finally, we look at the implications of the EH model for the equity risk premium. In Campbell and Cochrane (1999), dividend growth is i.i.d., with the same mean $\mu$ as consumption growth, and innovations that are correlated with the innovations in consumption growth. To make the dividend growth process more directly comparable across models, we write it as a function of innovations to consumption growth $\eta$ and innovations $u$ that are orthogonal to $\eta$:

$$\Delta d_{t+1} = \mu_d + \varphi_d \bar{\sigma} u_{t+1} + \varphi_d \bar{\sigma} \chi \eta_{t+1}.$$

[23] We continue to use the benchmark calibration of Bansal and Yaron (2004). Their dividend growth specification does not impose cointegration with consumption growth. Appendix F.2 derives stock returns in a world with cointegration. For our purposes, the results are similar with and without cointegration. Therefore, we focus on the case without cointegration. Bansal, Dittmar, and Lundblad (2005) deal with the case of cointegration. Bekaert, Engstrom, and Grenadier (2005) and Bekaert, Engstrom, and Xing (2005) also consider a version of the LRR model that imposes cointegration.

We choose parameters $\varphi_d$ and $\chi$ to match the volatility of dividend growth and its correlation with consumption growth to those in the benchmark calibration of Campbell and Cochrane (1999). The model's volatility of quarterly dividend growth is 5.6%, compared to .75% for consumption growth. We lose the linearity of the log price-dividend ratio in the state variables and solve for $pd^m$ using the numerical algorithm developed by Wachter (2005). Appendix B.6 contains the details of the dividend growth specification, the calibration, and the computation of the price-dividend ratio.[24] The equity risk premium in the EH model is somewhat higher than the total wealth risk premium: 3.30% per quarter.[25] The EH model's predictions for stock return predictability are the same as for total wealth return predictability: all variability in the $pd^m$ ratio comes from the discount rate channel. Variability in expected future stock returns itself is driven by risk premia rather than by risk-free rates. This characterization of stock return predictability is close to the one in the data.

# 4 Conclusion

The wealth-consumption ratio, the price-dividend ratio on total wealth, has different properties from the price-dividend ratio on equity. This has important implications for consumption-based asset pricing models financial economists work with. In particular, the same stochastic discount factor needs to price both a claim to aggregate consumption, which is not that risky and carries a low return, and a claim to equity dividends, which is much more risky and carries a high return. The long-run risk model solves this problem by making a claim to aggregate dividends have more long-run risky than a claim to aggregate consumption. Total wealth returns are less predictable than equity returns and most of their predictability is concentrated is for future returns, not for future consumption growth rates. The external habit model generates this feature by assuming i.i.d. consumption growth.

We have developed a new methodology for estimating the wealth-consumption ratio in the data, based on no-arbitrage conditions familiar from the term structure literature. This approach can readily be extended to incorporate additional asset classes. Future work could think about real exchange rate variation driven by differences in the wealth-consumption ratio across countries.

[24]The dividend growth specification in Campbell and Cochrane (1999) does not impose cointegration with consumption growth. Wachter (2006) and others assume that dividends are a levered-up version of consumption: $\Delta d_{t+1} = \phi \Delta c_{t+1}$, which also does not impose cointegration. In Appendix F.3, we develop a model with cointegration following Wachter (2005). Because the equity returns are similar in both cases, we focus on the case without cointegration.

[25]We use the average excess return, corrected for the Jensen term, as a proxy because we do not have a closed form expression for the expected excess return in the EH model.

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# A The Long-Run Risk Model

## A.1 The General Case

Let $V_t(C_t)$ denote the utility derived from consuming $C_t$, then the value function of the representative agent takes the following recursive form:

$$V_t(C_t) = \left[(1-\beta)C_t^{1-\rho} + \beta(\mathcal{R}_t V_{t+1})^{1-\rho}\right]^{\frac{1}{1-\rho}}, \tag{38}$$

where the risk-adjusted expectation operator is defined as:

$$\mathcal{R}_t V_{t+1} = \left(E_t V_{t+1}^{1-\alpha}\right)^{\frac{1}{1-\alpha}}.$$

For these preferences, $\alpha$ governs risk aversion and $\rho$ governs the willingness to substitute consumption intertemporally. It is the inverse of the inter-temporal elasticity of substitution. In the special case where $\rho = \alpha$, they collapse to the standard power utility preferences, used in Breeden (1979) and Lucas (1978). Epstein and Zin (1989) show that the stochastic discount factor can be written as:

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{V_{t+1}}{\mathcal{R}_t V_{t+1}}\right)^{\rho-\alpha} \tag{39}$$

The next proposition shows that the ability to write the SDF in the long-run risk model as a function of consumption growth and the consumption-wealth ratio is general. It does not depend on the linearization of returns, nor on the assumptions on the stochastic process for consumption growth.

**Proposition 7.** *The log SDF in the non-linear version of the long-run risk model can be stated as*

$$m_{t+1} = \frac{1-\alpha}{1-\rho}\log\beta - \alpha\Delta c_{t+1} + \frac{\rho-\alpha}{1-\rho}\log\left(\frac{e^{-cw_{t+1}}}{e^{-cw_t}-1}\right) \tag{40}$$

*Proof.* We start from the value function definition in equation (38) and raise both sides to the power $1-\rho$, and subsequently divide through by $(1-\beta)C_t^{-\rho}$ to obtain:

$$\frac{V_t^{1-\rho}}{(1-\beta)C_t^{-\rho}} = C_t + \beta \frac{\left(E_t V_{t+1}^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}}}{(1-\beta)C_t^{-\rho}} \tag{41}$$

Some algebra and the definition of the risk-adjusted expectation operator imply that

$$E_t(V_{t+1}^{1-\alpha})^{\frac{1-\rho}{1-\alpha}} = E_t(V_{t+1}^{1-\alpha})^{1-\frac{\rho-\alpha}{1-\alpha}} = \frac{E_t(V_{t+1}^{1-\alpha})}{E_t(V_{t+1}^{1-\alpha})^{\frac{\rho-\alpha}{1-\alpha}}} = \frac{E_t(V_{t+1}^{1-\alpha})}{(\mathcal{R}_t V_{t+1})^{\rho-\alpha}} = E_t\left[\frac{V_{t+1}^{\rho-\alpha} V_{t+1}^{1-\rho}}{(\mathcal{R}_t V_{t+1})^{\rho-\alpha}}\right]$$

Substituting this expression into the previous one, and multiplying and dividing inside the expectation operator by $C_{t+1}^{-\rho}$, we get:

$$\frac{V_t^{1-\rho}}{(1-\beta)C_t^{-\rho}} = C_t + E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left(\frac{V_{t+1}}{\mathcal{R}_t V_{t+1}}\right)^{\rho-\alpha}\frac{V_{t+1}^{1-\rho}}{(1-\beta)C_{t+1}^{-\rho}}\right]$$

Note that the first three terms inside the expectation are equal to the stochastic discount factor in equation (39). This is a no-arbitrage asset pricing equation of an asset with dividend equal to aggregate consumption. The price of this asset is $W_t$. Hence,

$$W_t = C_t + E_t\left[M_{t+1}W_{t+1}\right] \quad \text{and} \quad W_t = \frac{V_t^{1-\rho}}{(1-\beta)C_t^{-\rho}}. \tag{42}$$

This equation, together with $E[M_{t+1}R_{t+1}] = 1$ delivers the return on the total wealth portfolio:

$$R_{t+1} = \frac{W_{t+1}}{(W_t - C_t)} = \frac{\frac{V_{t+1}^{1-\rho}}{(1-\beta)C_{t+1}^{-\rho}}}{\frac{V_t^{1-\rho}}{(1-\beta)C_t^{-\rho}} - C_t} = \frac{\frac{V_{t+1}^{1-\rho}}{(1-\beta)C_{t+1}^{-\rho}}}{\frac{\beta}{(1-\beta)C_t^{-\rho}}(\mathcal{R}_t V_{t+1})^{1-\rho}} = \beta^{-1}\left(\frac{C_{t+1}}{C_t}\right)^{\rho}\left(\frac{V_{t+1}}{\mathcal{R}_t V_{t+1}}\right)^{1-\rho}, \tag{43}$$

where the first equality is a definition, the second one follows from the homogeneity of the value function, the third equality follows from equation (41), and the last one from algebraic manipulation.
Typically, one would rearrange this equation (after raising both sides to the power $\frac{\rho-\alpha}{1-\rho}$)

$$\left(\frac{V_{t+1}}{\mathcal{R}_t V_{t+1}}\right)^{\rho-\alpha} = \beta^{\frac{\rho-\alpha}{1-\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{-\rho\frac{\rho-\alpha}{1-\rho}} R_{t+1}^{\frac{\rho-\alpha}{1-\rho}},$$

and substitute it into the stochastic discount factor expression (39) to obtain an expression that depends only on consumption growth and the return to the wealth portfolio:

$$M_{t+1} = \beta^{\frac{1-\alpha}{1-\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{-\rho\frac{1-\alpha}{1-\rho}} R_{t+1}^{\frac{\rho-\alpha}{1-\rho}} \tag{44}$$

Instead, we rewrite the return on the wealth portfolio in terms of the wealth-consumption ratio $WC$

$$R_{t+1} = \frac{WC_{t+1}}{WC_t - 1}\frac{C_{t+1}}{C_t},$$

and use equation (43) to obtain

$$\left(\frac{V_{t+1}}{\mathcal{R}_t V_{t+1}}\right)^{\rho-\alpha} = \beta^{\frac{\rho-\alpha}{1-\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{\rho-\alpha}\left(\frac{WC_{t+1}}{WC_t - 1}\right)^{\frac{\rho-\alpha}{1-\rho}},$$

and substitute it into the stochastic discount factor expression (39) to obtain an expression that depends only on consumption growth and the wealth-consumption ratio:

$$M_{t+1} = \beta^{\frac{1-\alpha}{1-\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{-\alpha}\left(\frac{WC_{t+1}}{WC_t - 1}\right)^{\frac{\rho-\alpha}{1-\rho}} = \beta^{\frac{1-\alpha}{1-\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{-\alpha}\left(\frac{e^{wc_{t+1}}}{e^{wc_t} - 1}\right)^{\frac{\rho-\alpha}{1-\rho}} \tag{45}$$

$\square$

The log SDF expression in the Bansal and Yaron (2004) model is a first special case of this general, non-linear formulation. Indeed, one recovers equation (6) by using a first-order Taylor approximation of $wc_t$ in equation (40) around $A_0$. A second special case obtains by approximating the last term in (40) using a first-order Taylor expansion of $wc_{t+1}$ around $wc_t$ instead. In that case, we obtain a three-factor model:

$$m_{t+1} \approx \frac{1-\alpha}{1-\rho}\log\beta - \alpha\Delta c_{t+1} + \frac{\rho-\alpha}{1-\rho}\log\left(\frac{e^{wc_t}}{e^{wc_t} - 1}\right) - \frac{\rho-\alpha}{1-\rho}\Delta cw_{t+1}. \tag{46}$$

Expressions (46) and (6) are functionally similar because $\kappa_1^c$ is close to 1 and $\kappa_0^c$ equals $\frac{e^{wc_t}}{e^{wc_t}-1}$ when $wc_t$ is evaluated at its long-run mean $A_0$.

## A.2 Proof of Proposition 1

**Setting Up Some Notation** The starting point of the analysis is the Euler equation $E_t[M_{t+1}R^i_{t+1}] = 1$, where $R^i_{t+1}$ denotes a gross return between dates $t$ and $t+1$ on some asset $i$ and $M_{t+1}$ is the SDF. In logs:

$$E_t\left[m_{t+1}\right] + E_t\left[r^i_{t+1}\right] + \frac{1}{2}Var_t\left[m_{t+1}\right] + \frac{1}{2}Var_t\left[r^i_{t+1}\right] + Cov_t\left[m_{t+1}, r^i_{t+1}\right] = 0. \tag{47}$$

The same equation holds for the real risk-free rate $y_t(1)$, so that

$$y_t(1) = -E_t\left[m_{t+1}\right] - \frac{1}{2}Var_t\left[m_{t+1}\right]. \tag{48}$$

The expected excess return becomes:

$$E_t\left[r^{e,i}_{t+1}\right] = E_t\left[r^i_{t+1} - y_t(1)\right] + \frac{1}{2}Var_t\left[r^i_{t+1}\right] = -Cov_t\left[m_{t+1}, r^i_{t+1}\right] = -Cov_t\left[m_{t+1}, r^{e,i}_{t+1}\right], \tag{49}$$

where $r^{e,i}_{t+1}$ denotes the excess return on asset $i$ corrected for the Jensen term.

We adopt the consumption growth specification of Bansal and Yaron (2004), repeated from the main text:

$$\Delta c_{t+1} = \mu_c + x_t + \sigma_t \eta_{t+1}, \tag{50}$$

$$x_{t+1} = \rho_x x_t + \varphi_e \sigma_t e_{t+1}, \tag{51}$$

$$\sigma^2_{t+1} = \sigma^2 + \nu_1(\sigma^2_t - \sigma^2) + \sigma_w w_{t+1}, \tag{52}$$

where $(\eta_t, e_t, w_t)$ are i.i.d. mean-zero, variance-one, normally distributed innovations.

**Proof of Linearity** The proof closely follows the proof in Bansal and Yaron (2004), henceforth BY, but adjusts all expressions for our timing of returns.

*Proof.* In what follows we focus on the return on a claim to aggregate consumption, denoted $r$, where

$$r^c_{t+1} = \kappa^c_0 + \Delta c_{t+1} + wc_{t+1} - \kappa^c_1 wc_t,$$

and derive the five terms in equation (47) for this asset.

Taking logs on both sides of the non-linear SDF expression in equation (44) of Appendix A.1 delivers an expression of the log SDF as a function of log consumption changes and the log total wealth return

$$m_{t+1} = \frac{1-\alpha}{1-\rho}\log\beta - \frac{1-\alpha}{1-\rho}\rho\Delta c_{t+1} + \left(\frac{1-\alpha}{1-\rho} - 1\right) r^c_{t+1}. \tag{53}$$

We conjecture that the log wealth-consumption ratio is linear in the two states $x_t$ and $\sigma^2_t - \bar{\sigma}^2$,

$$wc_t = A_0 + A_1 x_t + A_2\left(\sigma^2_t - \bar{\sigma}^2\right).$$

As BY, we assume joint conditional normality of consumption growth, $x$, and the variance of consumption growth. We verify this conjecture from the Euler equation (47).

Using the conjecture for the wealth-consumption ratio, we compute innovations in the total wealth return, and its conditional mean and variance:

$$
\begin{aligned}
r^c_{t+1} - E_t\left[r^c_{t+1}\right] &= \sigma_t \eta_{t+1} + A_1 \varphi_e \sigma_t e_{t+1} + A_2 \sigma_w w_{t+1} \\
E_t\left[r^c_{t+1}\right] &= r_0 + \left(1 - (\kappa^c_1 - \rho_x) A_1\right) x_t - A_2(\kappa^c_1 - \nu_1)\left(\sigma^2_t - \bar{\sigma}^2\right) \\
V_t\left[r^c_{t+1}\right] &= (1 + (A_1\varphi_e)^2)\sigma^2_t + A^2_2 \sigma^2_w \\
r_0 &= \kappa^c_0 + A_0(1 - \kappa^c_1) + \mu_c
\end{aligned}
$$

These equations correspond to (A.8) and (A.9) in the Appendix of BY.

Substituting in the expression for the log total wealth return $r^c$ into the log SDF, we compute innovations, and the conditional mean and variance of the log SDF:

$$
\begin{aligned}
m_{t+1} - E_t\left[m_{t+1}\right] &= -\alpha \sigma_t \eta_{t+1} - \frac{\alpha - \rho}{1 - \rho} A_1 \varphi_e \sigma_t e_{t+1} - \frac{\alpha - \rho}{1 - \rho} A_2 \sigma_w w_{t+1}, \\
E_t\left[m_{t+1}\right] &= m_0 - \rho x_t + \frac{\alpha - \rho}{1 - \rho}(\kappa^c_1 - \nu_1) A_2 \left(\sigma^2_t - \bar{\sigma}^2\right) \\
V_t\left[m_{t+1}\right] &= \left(\alpha^2 + \left(\frac{\alpha - \rho}{1 - \rho}\right)^2 (A_1\varphi_e)^2\right)\sigma^2_t + \left(\frac{\alpha - \rho}{1 - \rho}\right)^2 A^2_2 \sigma^2_w \\
m_0 &= \frac{1 - \alpha}{1 - \rho}\log\beta - \frac{\alpha - \rho}{1 - \rho}\left[\kappa^c_0 + A_0(1 - \kappa^c_1)\right] - \alpha\mu_c \qquad (54)
\end{aligned}
$$

These expressions correspond to equations (A.10) and (A.27), and are only slightly different due to the different timing in returns.

The conditional covariance between the log consumption return and the log SDF is given by the conditional expectation of the product of their innovations

$$
Cov_t\left[r^c_{t+1}, m_{t+1}\right] = E_t\left[r^c_{t+1} - E_t\left[r^c_{t+1}\right], m_{t+1} - E_t\left[m_{t+1}\right]\right] = \left(-\alpha - \frac{\alpha - \rho}{1 - \rho}(A_1\varphi_e)^2\right)\sigma^2_t - \frac{\alpha - \rho}{1 - \rho} A^2_2 \sigma^2_w
$$

Using the method of undetermined coefficients and the five components of equation (47), we can solve for the constants $A_0$, $A_1$, and $A_2$:

$$
\begin{aligned}
A_1 &= \frac{1 - \rho}{\kappa^c_1 - \rho_x}, \qquad (55) \\
A_2 &= \frac{(1 - \rho)(1 - \alpha)}{2(\kappa^c_1 - \nu_1)}\left[1 + \frac{\varphi^2_e}{(\kappa^c_1 - \rho_x)^2}\right], \qquad (56) \\
0 &= \frac{1 - \alpha}{1 - \rho}\left[\log\beta + \kappa^c_0 + (1 - \kappa^c_1)A_0\right] + (1 - \alpha)\mu_c + \frac{1}{2}(1 - \alpha)^2\left[1 + \frac{\varphi^2_e}{(\kappa^c_1 - \rho_x)^2}\right]\bar{\sigma}^2 + \frac{1}{2}\left(\frac{1 - \alpha}{1 - \rho}\right)^2 A^2_2 \sigma^2_w \qquad (57)
\end{aligned}
$$

The first two correspond to equations (A.5) and (A.7) in BY. The last equation implicitly solves $A_0$ as a function of all parameters of the model. Because $\kappa^c_0$ and $\kappa^c_1$ are non-linear functions of $A_0$, this system of three equations needs to be solved simultaneously and numerically. Our computations indicate that the system has a unique solution. This verifies the conjecture that the log wealth-consumption ratio is linear in the two state variables.

It follows immediately from the above that the log SDF can be written as:

$$
m_{t+1} = \frac{1 - \alpha}{1 - \rho}\left[\log\beta + \kappa^c_0\right] - \kappa^c_0 - \alpha\Delta c_{t+1} - \frac{\alpha - \rho}{1 - \rho}\left(wc_{t+1} - \kappa^c_1 wc_t\right)
$$

This is the expression that arises in the proposition. $\square$

According to (49), the risk premium on the consumption claim is given by

$$E_t\left[r^e_{t+1}\right] = E_t\left[r^c_{t+1} - y_t(1)\right] + .5V_t[r^c_{t+1}] = -\lambda_{m,\eta}\sigma_t^2 + \lambda_{m,e}B\sigma_t^2 + \lambda_{m,w}A_2\sigma_w^2, \tag{58}$$

This corresponds to equation (A.11) in BY, with $\lambda_{m,\eta} = -\alpha$, $\lambda_{m,e} = \frac{\alpha-\rho}{1-\rho}A_1\varphi_e$, and $\lambda_{m,w} = \frac{\alpha-\rho}{1-\rho}A_2$.
According to equation (48), the expression for the risk-free rate is given by

$$\begin{aligned}
y_t(1) &= h_0 + \rho x_t + h_1(\sigma_t^2 - \bar\sigma^2) \\
h_0 &= -m_0 - .5\lambda^2_{m,w}\sigma_w^2 - .5\left(\lambda^2_{m,\eta} + \lambda^2_{m,e}\right)\bar\sigma^2 \\
h_1 &= -\frac{\alpha-\rho}{1-\rho}(\kappa_1^c - \nu_1)A_2 - .5\left(\lambda^2_{m,\eta} + \lambda^2_{m,e}\right) \\
&= .5(\rho-\alpha)\left(1 + \frac{\varphi_e^2}{(\kappa_1^c - \rho_x)^2}\right) - .5\left(\alpha^2 + (\alpha-\rho)^2\frac{\varphi_e^2}{(\kappa_1^c-\rho_x)^2}\right)
\end{aligned} \tag{59}$$

This corresponds to equation (A.25-A.27) in BY. Its unconditional mean is simply $h_0$ (see A.28).

## A.3 Campbell-Shiller Decomposition

Expected discounted future returns and consumption growth rates are given by:

$$r_t^H \equiv E_t\left[\sum_{j=1}^{\infty}(\kappa_1^c)^{-j}\, r_{t+j}\right] = \frac{r_0}{\kappa_1^c - 1} + \frac{\rho}{\kappa_1^c - \rho_x}x_t - A_2(\sigma_t^2 - \bar\sigma^2) \tag{60}$$

$$\Delta c_t^H \equiv E_t\left[\sum_{j=1}^{\infty}(\kappa_1^c)^{-j}\,\Delta c_{t+j}\right] = \frac{\mu}{\kappa_1^c - 1} + \frac{1}{\kappa_1^c - \rho_x}x_t \tag{61}$$

These expressions use the definitions of the total wealth return and consumption, as well as their dynamics.

These expressions enable us to go back and forth between the log wealth-consumption ratio expression in (7) and the Campbell-Shiller equation in (86). Starting from (7)

$$\begin{aligned}
wc_t &= A_0 + A_1 x_t + A_2(\sigma_t^2 - \bar\sigma^2) \\
&= A_0 + \frac{1}{\kappa_1^c - \rho_x}x_t - \left(\frac{\rho}{\kappa_1^c - \rho_x}x_t - A_2(\sigma_t^2 - \bar\sigma^2)\right) \\
&= A_0 + \left(\Delta c_t^H - \frac{\mu}{\kappa_1^c - 1}\right) - \left(r_t^H - \frac{r_0}{\kappa_1^c - 1}\right) \\
&= \frac{\kappa_0^c}{\kappa_1^c - 1} + \Delta c_t^H - r_t^H
\end{aligned}$$

we arrive at equation (86). The second equality uses the definition of $A_1$. The third equality uses the definition of $r_t^H$ and $\Delta c_t^H$. The fourth equality uses the definition of $r_0$.
The variance of the log wealth-consumption ratio can be written in two equivalent ways:

$$V\left[\Delta c_t^H\right] + V\left[r_t^H\right] - 2Cov\left[r_t^H, \Delta c_t^H\right] = V\left[wc_t\right] = Cov\left[wc_t, \Delta c_t^H\right] + Cov\left[wc_t, -r_t^H\right]$$

In the LRR model, the terms in this expression are given by

$$\begin{aligned}
V\left[\Delta c_t^H\right] &= \frac{1}{(\kappa_1^c-\rho_x)^2}\frac{\varphi_e^2}{1-\rho_x^2}\bar{\sigma}^2 > 0 \\
V\left[r_t^H\right] &= \frac{\rho^2}{(\kappa_1^c-\rho_x)^2}\frac{\varphi_e^2}{1-\rho_x^2}\bar{\sigma}^2 + A_2^2\frac{\sigma_w^2}{1-\nu_1^2} > 0 \\
Cov\left[r_t^H, \Delta c_t^H\right] &= \frac{\rho}{(\kappa_1^c-\rho_x)^2}\frac{\varphi_e^2}{1-\rho_x^2}\bar{\sigma}^2 > 0 \\
Cov\left[wc_t, \Delta c_t^H\right] &= \frac{1-\rho}{(\kappa_1^c-\rho_x)^2}\frac{\varphi_e^2}{1-\rho_x^2}\bar{\sigma}^2 > 0 \quad \Leftrightarrow \rho < 1 \\
Cov\left[wc_t, -r_t^H\right] &= \frac{\rho^2-\rho}{(\kappa_1^c-\rho_x)^2}\frac{\varphi_e^2}{1-\rho_x^2}\bar{\sigma}^2 + A_2^2\frac{\sigma_w^2}{1-\nu_1^2} > 0 \quad \Leftarrow \rho > 1
\end{aligned}$$

We can break up expected future returns into a risk-free rate component and a risk premium component. The former is equal to

$$E_t\left[\sum_{j=1}^{\infty}(\kappa_1^c)^{-j}\, r_{t+j-1}^f\right] = \frac{h_0}{\kappa_1^c-1} + \frac{\rho}{\kappa_1^c-\rho_x}x_t + \frac{h_1}{\kappa_1^c-\nu_1}(\sigma_t^2-\bar{\sigma}^2), \tag{62}$$

where the second equation uses the expression for the risk-free rate in equation (59) to compute future risk-free rates and takes their time-t expectations. The risk premium component is simply the difference between $r_t^H$ and the second expression.

## A.4 Risk Factor Representation

We can further rewrite the log SDF in terms of our two demeaned risk factors (denoted with a tilde):

$$m_{t+1} = m_0 - \alpha\widetilde{\Delta c}_{t+1} - \frac{\alpha-\rho}{1-\rho}\widetilde{\Delta wc}_{t+1} = m_0 - bf_{t+1},$$

where $m_0$ is defined in (54), the factor loadings are $b = \left[\alpha, \frac{\alpha-\rho}{1-\rho}\right]$, and the demeaned risk factors are defined as $f_{t+1} = \left[\widetilde{\Delta c}_{t+1}, \widetilde{\Delta wc}_{t+1}\right]' = [\Delta c_{t+1}-\mu, (wc_{t+1}-A_0) - \kappa_1^c(wc_t - A_0)]'$.

In the LRR model, the conditional and unconditional first and second moments of the two risk factors are

$$\begin{aligned}
E_t\left[\widetilde{\Delta c}_{t+1}\right] &= x_t & E\left[\widetilde{\Delta c}_{t+1}\right] &= 0 \\
V_t\left[\widetilde{\Delta c}_{t+1}\right] &= \sigma_t^2 & V\left[\widetilde{\Delta c}_{t+1}\right] &= \left(1+\frac{\varphi_e^2}{1-\rho_x^2}\right)\bar{\sigma}^2 \\
E_t\left[\widetilde{\Delta wc}_{t+1}\right] &= (\rho-1)x_t + A_2(\nu_1-\kappa_1^c)(\sigma_t^2-\bar{\sigma}^2) & E\left[\widetilde{\Delta wc}_{t+1}\right] &= 0 \\
V_t\left[\widetilde{\Delta wc}_{t+1}\right] &= A_1^2\varphi_e^2\sigma_t^2 + A_2^2\sigma_w^2 & V\left[\widetilde{\Delta wc}_{t+1}\right] &= A_1^2\varphi_e^2\left(1+\frac{(\rho_x-\kappa_1^c)^2}{1-\rho_x^2}\right)\bar{\sigma}^2 + A_2^2\left(1+\frac{(\nu_1-\kappa_1^c)^2}{1-\nu_1^2}\right)\sigma_w^2 \\
Cov_t\left[\widetilde{\Delta c}_{t+1}, \widetilde{\Delta wc}_{t+1}\right] &= 0 & Cov\left[\widetilde{\Delta c}_{t+1}, \widetilde{\Delta wc}_{t+1}\right] &= (\rho-1)\frac{\varphi_e^2}{1-\rho_x^2}\bar{\sigma}^2
\end{aligned}$$

The two risk factors are conditionally uncorrelated and have a positive unconditional correlation only if $\rho > 1$.

The expected excess return on the consumption claim can be written as the sum of the market prices of risk on the two risk factors.

$$E_t\left[r^e_{t+1}\right]=\ell^{LRR}_{1t}+\ell^{LRR}_{2t}=\left\{b_1V_t\left[\widetilde{\Delta c}_{t+1}\right]+b_2Cov_t\left[\widetilde{\Delta c}_{t+1},\widetilde{\Delta wc}_{t+1}\right]\right\}+\left\{b_1Cov_t\left[\widetilde{\Delta c}_{t+1},\widetilde{\Delta wc}_{t+1}\right]+b_2V_t\left[\widetilde{\Delta wc}_{t+1}\right]\right\}.$$

After some algebra, we obtain expressions for the conditional market prices of risk that are only a function of the structural parameters of the LRR model:

$$\ell^{LRR}_{1t} = \alpha\sigma_t^2 \tag{63}$$

$$\ell^{LRR}_{2t} = (\alpha-\rho)(1-\rho)\left\{\frac{\varphi_e^2}{(\kappa_1^c-\rho_x)^2}\sigma_t^2+\frac{(\alpha-1)^2}{4(\kappa_1^c-\nu_1)^2}\left[1+\frac{\varphi_e^2}{(\kappa_1^c-\rho_x)^2}\right]^2\sigma_w^2\right\} \tag{64}$$

The unconditional market prices of risk are the unconditional means of the conditional market prices of risk: $\ell_i^{LRR}=E[\ell_{it}^{LRR}]$, for $i=1,2$. This amounts to setting $\sigma_t^2=\bar{\sigma}^2$ in the above equations.

## A.5 Link to Affine SDF Representation

The log SDF in the LRR model has an affine representation

$$m_{t+1} = -y_t(1)-\frac{1}{2}L_t'L_t-L_t'\varepsilon_{t+1},$$

where $y_t(1)$ is the risk-free rate, the real market prices of risk are

$$L_t=\left[-\lambda_{m,\eta}\sigma_t,\lambda_{m,e}\sigma_t,\lambda_{m,w}\sigma_w\right],$$

and $\varepsilon_{t+1}=[\eta_{t+1},e_{t+1},w_{t+1}]$ are the Gaussian innovations that drive consumption growth. We recall that $\lambda_{m,\eta}=-\alpha$, $\lambda_{m,e}=\frac{\alpha-\rho}{1-\rho}A_1\varphi_e$, $\lambda_{m,w}=\frac{\alpha-\rho}{1-\rho}A_2$, and $wc_t=A_0+A_1x_t+A_2(\sigma_t^2-\bar{\sigma}^2)$. The market prices of risk in the long-run risk model vary because of heteroscedasticity in consumption growth ($\sigma_t$). They are affine in $\sigma_t$, not in the second state variable ($\sigma_t^2-\bar{\sigma}^2$).

*Proof.* Start from the canonical log SDF in the long-run risk model:

$$m_{t+1}=\frac{1-\alpha}{1-\rho}\log\beta-\frac{1-\alpha}{1-\rho}\rho\Delta c_{t+1}+\left(\frac{1-\alpha}{1-\rho}-1\right)r^c_{t+1}.$$

Recall the unconditional mean, the conditional mean, and the innovations in the total wealth return

$$\begin{aligned}
r^c_{t+1}-E_t\left[r^c_{t+1}\right] &= \sigma_t\eta_{t+1}+B\sigma_te_{t+1}+A_2\sigma_ww_{t+1}\\
E_t\left[r^c_{t+1}\right] &= r_0+\rho x_t-A_2(\kappa_1^c-\nu_1)\left(\sigma_t^2-\bar{\sigma}^2\right)\\
r_0 &= \kappa_0^c+A_0(1-\kappa_1^c)+\mu_c
\end{aligned}$$

Also recall the unconditional mean SDF

$$m_0 = \frac{1-\alpha}{1-\rho}\log\beta-\frac{\alpha-\rho}{1-\rho}\left[\kappa_0^c+A_0(1-\kappa_1^c)\right]-\alpha\mu_c$$

Substituting in for $\Delta c_{t+1}$ and $r^c_{t+1}$ into the log SDF, we obtain

$$
\begin{aligned}
m_{t+1} &= \frac{1-\alpha}{1-\rho}\log\beta - \frac{1-\alpha}{1-\rho}\rho(\mu_c + x_t + \sigma_t\eta_{t+1}) + \\
&\quad \left(\frac{1-\alpha}{1-\rho} - 1\right)(\sigma_t\eta_{t+1} + B\sigma_t e_{t+1} + A_2\sigma_w w_{t+1}) + \left(\frac{1-\alpha}{1-\rho} - 1\right)\left(r_0 + \rho x_t - A_2(\kappa^c_1 - \nu_1)\left(\sigma^2_t - \bar{\sigma}^2\right)\right). \\
&= \frac{1-\alpha}{1-\rho}(\log\beta - \rho\mu_c) - \rho x_t + \left(\frac{1-\alpha}{1-\rho} - 1\right)\left(r_0 - A_2(\kappa^c_1 - \nu_1)\left(\sigma^2_t - \bar{\sigma}^2\right)\right) \\
&\quad -\alpha\sigma_t\eta_{t+1} + \left(\frac{1-\alpha}{1-\rho} - 1\right)(B\sigma_t e_{t+1} + A_2\sigma_w w_{t+1}). \\
&= \frac{1-\alpha}{1-\rho}(\log\beta - \rho\mu_c) - \rho x_t + \left(\frac{1-\alpha}{1-\rho} - 1\right)\left(r_0 - A_2(\kappa^c_1 - \nu_1)\left(\sigma^2_t - \bar{\sigma}^2\right)\right) \\
&\quad -(-\lambda_{m,\eta}\sigma_t\eta_{t+1} + \lambda_{m,e}\sigma_t e_{t+1} + \lambda_{m,w}\sigma_w w_{t+1}),
\end{aligned}
$$

where $\lambda_{m,\eta} = -\alpha$, $\lambda_{m,e} = \frac{\alpha-\rho}{1-\rho}B$, $\lambda_{m,w} = \frac{\alpha-\rho}{1-\rho}A_2$, and $B = A_1\varphi_e$.
Recall that the risk-free rate is given by equation 59. Add and subtract the risk-free rate to the log SDF in a first step, and substitute in for the expressions for $y_0(1)$, $m_0$, and $r_0$ in a second step:

$$
\begin{aligned}
m_{t+1} &= -y_t(1) + y_0(1) + \frac{1-\alpha}{1-\rho}(\log\beta - \rho\mu_c) + \left(\frac{1-\alpha}{1-\rho} - 1\right) r_0 \\
&\quad -.5\left(\lambda^2_{m,\eta} + \lambda^2_{m,e}\right)\left(\sigma^2_t - \bar{\sigma}^2\right) - (-\lambda_{m,\eta}\sigma_t\eta_{t+1} + \lambda_{m,e}\sigma_t e_{t+1} + \lambda_{m,w}\sigma_w w_{t+1}). \\
&= -y_t(1) - .5\left[\left(\lambda^2_{m,\eta} + \lambda^2_{m,e}\right)\sigma^2_t + \lambda^2_{m,w}\sigma^2_w\right] \\
&\quad -\left[-\lambda_{m,\eta}\sigma_t\eta_{t+1} + \lambda_{m,e}\sigma_t e_{t+1} + \lambda_{m,w}\sigma_w w_{t+1}\right].
\end{aligned}
$$

This proofs the affine SDF representation with market prices of risk $L_t = [-\lambda_{m,\eta}\sigma_t, \lambda_{m,e}\sigma_t, \lambda_{m,w}\sigma_w]$ and $\varepsilon_{t+1} = [\eta_{t+1}, e_{t+1}, w_{t+1}]$.

□

## A.6 Pricing Stocks in the LRR Model

We discuss the case where dividends on equity and aggregate consumption are not cointegrated. Section F.2 discusses the case where cointegration is imposed.

**Dividend Growth Process** We start by pricing a claim to aggregate dividends, where the dividend process follows the specification in Bansal and Yaron (2004):

$$
\Delta d_{t+1} = \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1} \tag{65}
$$

The shock $u_t$ is orthogonal to $(\eta, e, w)$. This specification does not impose cointegration between consumption and dividends.
Defining returns ex-dividend and using the Campbell (1991) linearization, the log return on a claim to the aggregate dividend can be written as:

$$
r^m_{t+1} = \Delta d_{t+1} + pd_{t+1} + \kappa^m_0 - \kappa^m_1 pd_t,
$$

with coefficients

$$
\kappa^m_1 = \frac{e^{A^m_0}}{e^{A^m_0} - 1} > 1, \quad \text{and} \quad \kappa^m_0 = -\log\left(e^{A^m_0} - 1\right) + \frac{e^{A^m_0}}{e^{A^m_0} - 1} A^m_0
$$

which depend on the long-run log price-dividend ratio $A_0^m$. We denote the return on financial wealth by a superscript $m$.

**Proof of Linearity** We conjecture, as we did for the wealth-consumption ratio, that the log price dividend ratio is linear in the two state variables:

$$pd_t^m = A_0^m + A_1^m x_t + A_2^m(\sigma_t^2 - \bar{\sigma}^2).$$

As we did for the return on the consumption claim, we compute innovations in the dividend claim return, and its conditional mean and variance:

$$\begin{aligned}
r_{t+1}^m - E_t\left[r_{t+1}^m\right] &= \varphi_d \sigma_t u_{t+1} + \beta_{m,e}\sigma_t e_{t+1} + \beta_{m,w}\sigma_w w_{t+1} \\
E_t\left[r_{t+1}^m\right] &= r_0^m + [\phi + A_1^m(\rho_x - \kappa_1^m)]x_t - A_2^m(\kappa_1^m - \nu_1)\left(\sigma_t^2 - \bar{\sigma}^2\right) \\
V_t\left[r_{t+1}^m\right] &= (\varphi_d^2 + \beta_{m,e}^2)\sigma_t^2 + \beta_{m,w}^2\sigma_w^2, \\
r_0^m &= \kappa_0^m + A_0^m(1-\kappa_1^m) + \mu_d
\end{aligned}$$

where $\beta_{m,e} = A_1^m\varphi_e$ and $\beta_{m,w} = A_2^m$. These equations correspond to (A.12) and (A.13) in the Appendix of Bansal and Yaron (2004). Finally, the conditional covariance between the log SDF and the log dividend claim return is

$$Cov_t\left[m_{t+1}, r_{t+1}^m\right] = -\lambda_{m,e}\beta_{m,e}\sigma_t^2 - \lambda_{m,w}\beta_{m,w}\sigma_w^2$$

From the Euler equation for this return $E_t\left[m_{t+1}\right] + E_t\left[r_{t+1}^m\right] + \frac{1}{2}V_t\left[m_{t+1}\right] + \frac{1}{2}V_t\left[r_{t+1}^m\right] + Cov_t[m_{t+1}, r_{t+1}^m] = 0$ and the method of undetermined coefficients, we can use the same procedure as described in A.2, and solve for the constants $A_0^m$, $A_1^m$, and $A_2^m$:

$$\begin{aligned}
A_1^m &= \frac{\phi - \rho}{\kappa_1^m - \rho_x}, \\
A_2^m &= \frac{\left[\frac{\alpha-\rho}{1-\rho}A_2(\kappa_1 - \nu_1) + .5H_m\right]}{\kappa_1^m - \nu_1}, \\
0 &= m_0 + \kappa_0^m + (1-\kappa_1^m)A_0^m + \mu_d + \frac{1}{2}H_m\bar{\sigma}^2 + \frac{1}{2}\left(A_2^m - A_2\frac{\alpha-\rho}{1-\rho}\right)^2\sigma_w^2
\end{aligned}$$

where

$$\begin{aligned}
H_m &= \lambda_{m,\eta}^2 + \left(\beta_{m,e} - \lambda_{m,e}\right)^2 + \varphi_d^2 \\
&= \alpha^2 + (\phi - \alpha)^2\frac{\varphi_e^2}{(\kappa_1 - \rho_x)^2} + \varphi_d^2
\end{aligned}$$

Again, this is a non-linear system in three equations and three unknowns, which we solve numerically. The first two equations correspond to (A.16) and (A.20) in BY.

**Equity Risk premium and CS Decomposition** The equity risk premium on the dividend claim (adjusted for a Jensen term) becomes:

$$E_t\left[r_{t+1}^{e,m}\right] \equiv E_t\left[r_{t+1}^m - y_t(1)\right] + .5V_t[r_{t+1}^m] = \lambda_{m,e}\beta_{m,e}\sigma_t^2 + \lambda_{m,w}\beta_{m,w}\sigma_w^2 \tag{66}$$

This corresponds to equation (A.14) in BY.

Expected discounted future equity returns and dividend growth rates are given by:

$$r_t^{m,H} \equiv E_t\left[\sum_{j=1}^{\infty}(\kappa_1^m)^{-j}\, r_{t+j}^m\right] = \frac{r_0^m}{\kappa_1^m - 1} + \frac{\rho}{\kappa_1^m - \rho_x}x_t - A_2^m(\sigma_t^2 - \bar{\sigma}^2) \tag{67}$$

$$\Delta d_t^H \equiv E_t\left[\sum_{j=1}^{\infty}(\kappa_1^m)^{-j}\, \Delta d_{t+j}\right] = \frac{\mu_d}{\kappa_1^m - 1} + \frac{\phi}{\kappa_1^m - \rho_x}x_t \tag{68}$$

From these expressions, it is easy to see that

$$pd_t = \frac{\kappa_0^m}{\kappa_1^m - 1} + \Delta d_t^H - r_t^{m,H},$$

and to compute the elements of the variance-decomposition:

$$V[pd_t^m] = Cov[pd_t^m, \Delta d_t^H] + Cov[pd_t^m, -r_t^{m,H}] = V[\Delta d_t^H] + V[r_t^{m,H}] - 2Cov[\Delta d_t^H, r_t^{m,H}].$$

## A.7 Quarterly Calibration LRR Model

The Bansal-Yaron model is calibrated and parameterized to monthly data. Since we want to use data on quarterly consumption and dividend growth, and a quarterly series for the wealth-consumption ratio, we recast the model at quarterly frequencies. We assume that the quarterly process for consumption growth, dividend growth, the low frequency component and the variance has the exact same structure than at the monthly frequency, with mean zero, standard deviation 1 innovations, but with different parameters. This appendix explains how the monthly parameters map into quarterly parameters. We denote all variables, shocks, and all parameters of the quarterly system with a tilde superscript.

**Preference Parameters** Obviously, the preference parameters do not depend on the horizon ($\tilde{\alpha} = \alpha$ and $\tilde{\rho} = \rho$), except for the time discount factor $\tilde{\beta} = \beta^3$. Also, the long-run average log wealth-consumption ratio at the quarterly frequency is lower than at the monthly frequency by approximately log(3), because log of quarterly consumption is the log of three times monthly consumption. When we simulate the quarterly model, we solve for the corresponding $A_0$, $A_1$, and $A_2$ from the system (55)-(57), but with the quarterly parameter values described in this appendix.

**Cash-flow Parameters** We accomplish this by matching the conditional and unconditional mean and variance of log consumption and dividend growth. Log quarterly consumption growth is the sum of log consumption growth of three consecutive months. We obtain $\Delta\tilde{c}_{t+1} \equiv \Delta c_{t+3} + \Delta c_{t+2} + \Delta c_{t+1}$

$$\Delta\tilde{c}_{t+1} = 3\mu_c + (1+\rho_x+\rho_x^2)x_t + \sigma_t\eta_{t+1} + \sigma_{t+1}\eta_{t+2} + \sigma_{t+2}\eta_{t+3} + (1+\rho_x)\varphi_e\sigma_t e_{t+1} + \varphi_e\sigma_{t+1}e_{t+2} \tag{69}$$

Log quarterly dividend growth looks similar:

$$\Delta\tilde{d}_{t+1} = 3\mu_d + \phi(1+\rho_x+\rho_x^2)x_t + \varphi_d\sigma_t u_{t+1} + \varphi_d\sigma_{t+1}u_{t+2} + \varphi_d\sigma_{t+2}u_{t+3} + \phi(1+\rho_x)\varphi_e\sigma_t e_{t+1} + \phi\varphi_e\sigma_{t+1}e_{t+2} \tag{70}$$

First, we rescale the long-run component in the quarterly system, so that the coefficient on it in the consumption growth equation is still 1:

$$\tilde{x}_t = (1 + \rho_x + \rho_x^2)x_t.$$

Second, we equate the unconditional mean of consumption and dividend growth :

$$\tilde{\mu} = 3\mu, \ \tilde{\mu}_d = 3\mu_d.$$

These imply that we also match the the conditional mean of consumption growth:

$$E_t[\Delta c_{t+3} + \Delta c_{t+2} + \Delta c_{t+1}] = 3\mu + (1 + \rho_x + \rho_x^2)x_t = \tilde{\mu} + \tilde{x}_t = E_t[\Delta \tilde{c}_{t+1}]$$

Third, we also match the conditional mean of dividend growth by setting the quarterly leverage parameter $\tilde{\phi} = \phi$. Fourth, we match the unconditional variance of quarterly consumption growth:

$$\begin{aligned} V[\Delta \tilde{c}_{t+1}] &= (1 + \rho_x + \rho_x^2)^2 V[x_t] + \sigma^2 \left[3 + (1 + \rho_x)^2 \varphi_e^2 + \varphi_e^2\right] \\ &= (1 + \rho_x + \rho_x^2)^2 \frac{\varphi_e^2 \sigma^2}{1 - \rho_x^2} + \sigma^2 \left[3 + (1 + \rho_x)^2 \varphi_e^2 + \varphi_e^2\right] \\ &= \frac{\tilde{\varphi}_e^2 \tilde{\sigma}^2}{1 - \tilde{\rho}_x^2} + \tilde{\sigma}^2 \end{aligned}$$

The first and second equalities use the law of iterated expectations to show that

$$V[\sigma_{t+j}\eta_{t+j+1}] \equiv E\left[E_{t+j}\left\{\sigma_{t+j}^2 \eta_{t+j+1}^2\right\}\right] - \left(E\left[E_{t+j}\left\{\sigma_{t+j}\eta_{t+j+1}\right\}\right]\right)^2 = E\left[\sigma_{t+j}^2\right] - 0 = \sigma^2$$

and the same argument applies to terms of type $V[\sigma_{t+j}e_{t+j+1}]$. Coefficient matching on the variance of consumption expression delivers expressions for $\tilde{\sigma}^2$ and $\tilde{\varphi}_e$:

$$\tilde{\sigma}^2 = \sigma^2 \left[3 + (1 + \rho_x)^2 \varphi_e^2 + \varphi_e^2\right]$$

$$\begin{aligned} \tilde{\varphi}_e^2 &= \varphi_e^2 \frac{(1 - \tilde{\rho}_x^2)(1 + \rho_x + \rho_x^2)^2}{1 - \rho_x^2} \frac{\sigma^2}{\tilde{\sigma}^2} \\ &= \frac{(1 - \rho_x^6)(1 + \rho_x + \rho_x^2)^2}{1 - \rho_x^2} \frac{\varphi_e^2}{3 + (1 + \rho_x)^2 \varphi_e^2 + \varphi_e^2}, \end{aligned}$$

where the third equality uses the first equality. Note that we imposed $\tilde{\rho}_x = \rho_x^3$, which follows from a desire to match the persistence of the long-run cash-flow component. Recursively substituting, we find that the three-month ahead $x$ process has the following relationship to the current value:

$$x_{t+3} = \rho_x^3 x_t + \varphi_e \sigma_{t+2} e_{t+3} + \rho_x \varphi_e \sigma_{t+1} e_{t+2} + \rho_x^2 \varphi_e \sigma_t e_{t+1}$$

which compares to the quarterly equation

$$\tilde{x}_{t+1} = \tilde{\rho}_x \tilde{x}_t + \tilde{\varphi}_e^2 \tilde{\sigma}_t \tilde{e}_{t+1}$$

The two processes now have the same auto-correlation and unconditional variance.

Fifth, we match the unconditional variance of dividend growth. Given the assumptions we have made sofar, this pins down $\varphi_d$:

$$\tilde{\varphi}_d^2 = \frac{3\varphi_d^2 + \phi^2(1+\rho_x)^2\varphi_e^2 + \phi^2\varphi_e^2}{3 + (1+\rho_x)^2\varphi_e^2 + \varphi_e^2}$$

Sixth, we match the autocorrelation and the unconditional variance of economic uncertainty $\sigma_t^2$. Iterating forward, we obtain an expression that relates variance in month $t$ to the one in month $t+3$:

$$\sigma_{t+3}^2 - \sigma^2 = \nu_1^3(\sigma_t^2 - \sigma^2) + \sigma_w \nu_1^2 w_{t+1} + \sigma_w \nu_1 w_{t+2} + \sigma_w w_{t+3}$$

By setting $\tilde{\nu}_1 = \nu_1^3$ and $\tilde{\sigma}_w^2 = \sigma_w^2(1+\nu_1^2+\nu_1^4)$, we match the autocorrelation and variance of the quarterly equation

$$\tilde{\sigma}_{t+1}^2 - \tilde{\sigma}^2 = \tilde{\nu}_1(\tilde{\sigma}_t^2 - \tilde{\sigma}^2) + \tilde{\sigma}_w \tilde{w}_{t+1}$$

A simulation of the quarterly model recovered the annualized cash-flow and asset return moments of the monthly simulation.

# B The External Habit Model

The organization of this EH model appendix exactly parallels the treatment of the LRR model in Appendix (A.2).

## B.1 Proof of Proposition 2

*Proof.* We conjecture that the log wealth-consumption ratio is linear in the sole state variable $(s_t - \bar{s})$,

$$wc_t = A_0 + A_1 (s_t - \bar{s}).$$

As Campbell and Cochrane (1999), henceforth CC, we assume joint conditional normality of consumption growth and the surplus consumption ratio. We verify this conjecture from the Euler equation (**??**).

We start from the canonical log SDF in the external habit model:

$$m_{t+1} = \log\beta - \alpha\Delta c_{t+1} - \alpha\Delta s_{t+1}.$$

Substituting in the expression for returns into the log SDF, we compute innovations, and the conditional mean and variance of the log SDF:

$$\begin{aligned} m_{t+1} - E_t[m_{t+1}] &= -\alpha(1+\lambda_t)\bar{\sigma}\eta_{t+1} \\ E_t[m_{t+1}] &= m_0 + \alpha(1-\rho_s)(s_t - \bar{s}) \\ V_t[m_{t+1}] &= \alpha^2(1+\lambda_t)^2\bar{\sigma}^2 \\ m_0 &= \log\beta - \alpha\mu_c \end{aligned} \tag{71}$$

Likewise, we compute innovations in the consumption claim return, and its conditional mean and variance:

$$\begin{aligned} r^c_{t+1} - E_t\left[r^c_{t+1}\right] &= (1 + A_1\lambda_t)\bar{\sigma}\eta_{t+1} \\ E_t\left[r^c_{t+1}\right] &= r_0 - A_1(\kappa^c_1 - \rho_s)\left(s_t - \bar{s}\right) \\ V_t\left[r^c_{t+1}\right] &= (1 + A_1\lambda_t)^2\bar{\sigma}^2 \\ r_0 &= \kappa^c_0 + A_0(1 - \kappa^c_1) + \mu_c \end{aligned} \tag{72}$$

The conditional covariance between the log consumption return and the log SDF is given by the conditional expectation of the product of their innovations

$$Cov_t\left[m_{t+1}, r^c_{t+1}\right] = -\alpha\left(1 + \lambda_t\right)\left(1 + A_1\lambda_t\right)\bar{\sigma}^2$$

We assume that the sensitivity function takes the following form

$$\lambda_t = \frac{\bar{S}^{-1}\sqrt{1 - 2(s_t - \bar{s})} + 1 - \alpha}{\alpha - A_1}$$

Using the method of undetermined coefficients and the five components of equation (**??**), we can solve for the constants $A_0$ and $A_1$:

$$A_1 = \frac{(1 - \rho_s)\alpha - \bar{\sigma}^2\bar{S}^{-2}}{\kappa^c_1 - \rho_s}, \tag{73}$$

$$0 = \log\beta + \kappa^c_0 + (1 - \kappa^c_1)A_0 + (1 - \alpha)\mu_c + .5\bar{\sigma}^2\bar{S}^{-2} \tag{74}$$

This verifies that our conjecture was correct. It follows immediately that the log SDF can be written as

$$m_{t+1} = \log\beta - \alpha\Delta c_{t+1} - \frac{\alpha}{A_1}\left(wc_{t+1} - wc_t\right),$$

which is the expression in the main text. □

The risk premium on the consumption claim is given by $Cov_t[r^c_{t+1}, -m_{t+1}]$:

$$E_t\left[r^e_{t+1}\right] \equiv E_t\left[r^c_{t+1} - y_t(1)\right] + .5V_t[r^c_{t+1}] = \alpha\left(1 + \lambda_t\right)\left(1 + A_1\lambda_t\right)\bar{\sigma}^2, \tag{75}$$

where the second term on the left is a Jensen adjustment. The expression for the risk-free rate appears in the next section B.2.

## B.2 The Steady-State Habit Level

Campbell and Cochrane (1999) engineer their sensitivity function $\lambda_t$ to deliver a risk-free rate that is linear in the state $s_t - \bar{s}$. (They mostly study a special case with a constant risk-free rate.) The linearity of the risk-free rate is accomplished by choosing

$$\lambda^{CC}_t = \bar{S}^{-1}\sqrt{1 - 2\left(s_t - \bar{s}\right)} - 1 \tag{76}$$

Note that if the risk aversion parameter $\alpha = 2$ and $A_1 = 1$, our sensitivity function exactly coincides with CC's. Instead, we engineer our sensitivity function to deliver a log wealth-consumption ratio that is linear in $s_t - \bar{s}$.
As a result of our choice, the risk-free rate, $y_t(1) = -E_t\left[m_{t+1}\right] - .5V_t\left[m_{t+1}\right]$, is no longer linear in the state, but

contains an additional square-root term:

$$
\begin{aligned}
y_t(1) &= h_0 + \left[\frac{\bar{\sigma}^2\alpha^2\bar{S}^{-2}}{(\alpha - A_1)^2} - \alpha(1-\rho_s)\right](s_t - \bar{s}) - \bar{\sigma}^2\alpha^2\frac{(1-A_1)\bar{S}^{-1}}{(\alpha - A_1)^2}\left(\sqrt{1-2(s_t-\bar{s})} - 1\right) && (77)\\
h_0 &= -\log\beta + \alpha\mu_c - .5\bar{\sigma}^2\alpha^2(1+\lambda(\bar{s}))^2, \quad \text{where} \quad \lambda(\bar{s}) = \left(\frac{\bar{S}^{-1} + 1 - \alpha}{\alpha - A_1}\right) && (78)
\end{aligned}
$$

where $\lambda(\bar{s})$ is obtained from evaluating our sensitivity function at $s_t = \bar{s}$.

CC obtain a similar expression, but without the last term. If $\alpha = 2$ and $A_1 = 1$, the expression collapses to the one in CC. A constant risk-free rate obtains in the CC model when $\bar{S}^{-1} = \bar{\sigma}^{-1}\sqrt{\frac{1-\rho_s}{\alpha}}$ because this choice makes the linear term vanish. While there is no $\bar{S}$ that guarantees a constant risk-less interest rate under our assumptions, we choose $\bar{S}$ to match the *steady-state* risk-free rate in CC, $\bar{r} = -\log\beta + \alpha\mu - .5\alpha(1-\rho_s)$. That is, we set $s_t = \bar{s}$ in the above equation, which then collapses to $h_0$. Setting $\bar{r} = h_0$ allows us to solve for $\bar{S}^{-1}$ as a function of $A_1$ and the structural parameters $\alpha$, $\rho_s$, and $\bar{\sigma}$:

$$
\bar{S}^{-1} = (\alpha - A_1)\left(\bar{\sigma}^{-1}\sqrt{\frac{1-\rho_s}{\alpha}}\right) - 1 + A_1. \tag{79}
$$

Substituting this expression back into the sensitivity function (11), we find that the steady-state sensitivity level $\lambda(\bar{s}) = \bar{\sigma}^{-1}\sqrt{\frac{1-\rho_s}{\alpha}} - 1$. This implies that we generate the same steady-state conditional covariance between the surplus consumption ratio and consumption growth as in CC.

As in CC, we define a maximum value for the log surplus consumption ratio $s_{max}$, as the value at which $\lambda_t$ runs into zero:

$$
s_{max} = \bar{s} + \frac{1}{2}\left(1 - (\alpha-1)^2\bar{S}^2\right)
$$

Note that if $\alpha = 2$, this coincides with equation (11) in CC. It is understood that $\lambda_t = 0$ for $s_t \geq s_{max}$.

## B.3 Campbell-Shiller Decomposition

Using (72) and the law of motion for $s_t$ and consumption growth, expected discounted future returns and consumption growth rates are given by:

$$
\begin{aligned}
r_t^H &\equiv E_t\left[\sum_{j=1}^{\infty} r_{t+j}\right] = \frac{r_0}{\kappa_1^c - 1} - A_1(s_t - \bar{s}) && (80)\\
\Delta c_t^H &\equiv E_t\left[\sum_{j=1}^{\infty} \Delta c_{t+j}\right] = \frac{\mu}{\kappa_1^c - 1} && (81)
\end{aligned}
$$

These expressions enable us to go back and forth between the log wealth-consumption ratio expression in (10) and the Campbell-Shiller equation in (86). Starting from (10)

$$
\begin{aligned}
wc_t &= A_0 + A_1(s_t - \bar{s})\\
&= A_0 + \left(\Delta c_t^H - \frac{\mu}{\kappa_1^c - 1}\right) - \left(r_t^H - \frac{r_0}{\kappa_1^c - 1}\right)\\
&= \frac{\kappa_0^c}{\kappa_1^c - 1} + \Delta c_t^H - r_t^H,
\end{aligned}
$$

we arrive at equation (86). The second equality uses the definitions of $r_t^H$ and $\Delta c_t^H$. The third equality uses the definition of $r_0$.

The variance of the log wealth-consumption ratio can be written in two equivalent ways:

$$V\left[\Delta c_t^H\right] + V\left[r_t^H\right] - 2Cov\left[r_t^H, \Delta c_t^H\right] = V\left[wc_t\right] = Cov\left[wc_t, \Delta c_t^H\right] + Cov\left[wc_t, -r_t^H\right]$$

In the EH model, the terms in this expression are given by

$$\begin{aligned}
V\left[\Delta c_t^H\right] &= 0, \;\; Cov\left[r_t^H, \Delta c_t^H\right] = 0, \;\; Cov\left[wc_t, \Delta c_t^H\right] = 0 \\
V\left[r_t^H\right] &= A_1^2\left(\frac{\bar{S}^{-1}+1-\alpha}{\alpha - A_1}\right)^2 \frac{1}{1-\rho_s^2}\bar{\sigma}^2 > 0 \\
Cov\left[wc_t, -r_t^H\right] &= A_1^2\left(\frac{\bar{S}^{-1}+1-\alpha}{\alpha - A_1}\right)^2 \frac{1}{1-\rho_s^2}\bar{\sigma}^2 > 0
\end{aligned}$$

Likewise, there is no predictability in dividend growth (see equation 84). Therefore, $V\left[pd_t\right] = V\left[r_t^{H,m}\right]$, where the latter is the unconditional variance of the expected return on the dividend claim.

## B.4 Risk Factor Representation

We can further rewrite the log SDF in terms of our two demeaned risk factors (denoted with a tilde):

$$m_{t+1} = m_0 - \alpha\widetilde{\Delta c}_{t+1} - \frac{\alpha}{A_1}\widetilde{\Delta wc}_{t+1} = m_0 - bf_{t+1},$$

where $m_0$ is defined in (71), the factor loadings are $b = \left[\alpha, \frac{\alpha}{A_1}\right]$, and the demeaned risk factors are defined as $f_{t+1} = \left[\widetilde{\Delta c}_{t+1}, \widetilde{\Delta wc}_{t+1}\right]' = \left[\Delta c_{t+1} - \mu, (wc_{t+1} - A_0) - (wc_t - A_0)\right]'$.

In the EH model, the conditional and unconditional first and second moments of the two risk factors are

$$\begin{aligned}
E_t\left[\widetilde{\Delta c}_{t+1}\right] &= 0 & E\left[\widetilde{\Delta c}_{t+1}\right] &= 0 \\
V_t\left[\widetilde{\Delta c}_{t+1}\right] &= \bar{\sigma}^2 & V\left[\widetilde{\Delta c}_{t+1}\right] &= \bar{\sigma}^2 \\
E_t\left[\widetilde{\Delta wc}_{t+1}\right] &= -A_1(1-\rho_s)(s_t - \bar{s}) & E\left[\widetilde{\Delta wc}_{t+1}\right] &= 0 \\
V_t\left[\widetilde{\Delta wc}_{t+1}\right] &= A_1^2\bar{\sigma}^2\lambda_t^2 & V\left[\widetilde{\Delta wc}_{t+1}\right] &= A_1^2\bar{\sigma}^2\left(1 + \frac{(1-\rho_s)(\kappa_1^c - \rho_s)}{1-\rho_s^2}\right)\left(\frac{\bar{S}^{-1}+1-\alpha}{\alpha - A_1}\right)^2 \\
Cov_t\left[\widetilde{\Delta c}_{t+1}, \widetilde{\Delta wc}_{t+1}\right] &= A_1\bar{\sigma}^2\lambda_t & Cov\left[\widetilde{\Delta c}_{t+1}, \widetilde{\Delta wc}_{t+1}\right] &= A_1\bar{\sigma}^2\left(\frac{\bar{S}^{-1}+1-\alpha}{\alpha - A_1}\right)
\end{aligned}$$

The two risk factors are conditionally and unconditionally positively correlated as long as $\lambda_t > 0$ (which is true for our calibrations).

The expected excess return on the consumption claim can be written as the sum of the market prices of risk on the two risk factors.

$$E_t\left[r_{t+1}^e\right] = \ell_{1t}^{EH} + \ell_{2t}^{EH} = \left\{b_1 V_t\left[\widetilde{\Delta c}_{t+1}\right] + b_2 Cov_t\left[\widetilde{\Delta c}_{t+1}, \widetilde{\Delta wc}_{t+1}\right]\right\} + \left\{b_1 Cov_t\left[\widetilde{\Delta c}_{t+1}, \widetilde{\Delta wc}_{t+1}\right] + b_2 V_t\left[\widetilde{\Delta wc}_{t+1}\right]\right\}.$$

After some algebra, we obtain expressions for these market prices of risk that are only a function of the structural parameters of the EH model:

$$\ell_{1t}^{EH} = \alpha(1+\lambda_t)\bar{\sigma}^2 \tag{82}$$
$$\ell_{2t}^{EH} = \alpha A_1 \lambda_t (1+\lambda_t)\bar{\sigma}^2 \tag{83}$$

The unconditional market prices of risk are the unconditional means of the conditional market prices of risk: $\ell_i^{LRR} = E[\ell_{it}^{LRR}]$, for $i = 1, 2$. This amounts to setting $\lambda_t = E[\lambda_t] = \lambda(\bar{s}) = \left(\frac{\bar{S}^{-1}+1-\alpha}{\alpha - A_1}\right)$ in the above equations.

## B.5 Link to Affine SDF Representation

The log SDF in the EH model has an affine representation

$$m_{t+1} = -y_t(1) - \frac{1}{2} L_t' L_t - L_t' \eta_{t+1},$$

where $y_t(1)$ is the risk-free rate, $\eta_{t+1} \sim \mathcal{N}(0,1)$ is the innovation to consumption growth, and the real market price of (consumption) risk $L_t$ is given by

$$L_t = \alpha\bar{\sigma}(1+\lambda_t)$$

The market price of risk in the external habit model varies because the surplus-consumption ratio varies. It is affine in the square root of the state variable:

$$L_t = \frac{\alpha\bar{\sigma}}{\alpha - A_1}\left(\bar{S}^{-1} + 1 - A_1\right) + \frac{\alpha\bar{\sigma}}{\alpha - A_1}\bar{S}^{-1}\left(\sqrt{1 - 2(s_t - \bar{s})} - 1\right)$$

To recover the expression for $L_t$ under the Campbell and Cochrane (1999) sensitivity function specification, simply set $\alpha = 2$ and $A_1 = 1$.

*Proof.* Start from the canonical log SDF in the external habit model:

$$m_{t+1} = \log\beta - \alpha\Delta c_{t+1} - \alpha\Delta s_{t+1}.$$

Use the law of motion of $s_t$:

$$s_{t+1} - \bar{s} = \rho_s(s_t - \bar{s}) + \lambda_t(\Delta c_{t+1} - \mu_c)$$

to obtain:

$$m_{t+1} = \log\beta - \alpha\mu_c + \alpha(1-\rho_s)(s_t - \bar{s}) - \alpha\bar{\sigma}(1+\lambda_t)\eta_{t+1}.$$

Recall that the risk-free rate equals:

$$y_t(1) = y_0(1) + \left[\frac{\bar{\sigma}^2\alpha^2\bar{S}^{-2}}{(\alpha - A_1)^2} - \alpha(1-\rho_s)\right](s_t - \bar{s}) - \bar{\sigma}^2\alpha^2\frac{(1-A_1)\bar{S}^{-1}}{(\alpha - A_1)^2}\left(\sqrt{1 - 2(s_t - \bar{s})} - 1\right)$$
$$y_0(1) = -\log\beta + \alpha\mu_c - .5\bar{\sigma}^2\alpha^2(1+\lambda(\bar{s}))^2, \quad \text{where} \quad \lambda(\bar{s}) = \left(\frac{\bar{S}^{-1}+1-\alpha}{\alpha - A_1}\right)$$

Add and subtract the risk-free rate to the log SDF:

$$\begin{aligned} m_{t+1} &= -y_t(1) + \log\beta - \alpha\mu_c + y_0(1) + \frac{\bar{\sigma}^2\alpha^2\bar{S}^{-2}}{(\alpha - A_1)^2}(s_t - \bar{s}) \\ &- \bar{\sigma}^2\alpha^2\frac{(1-A_1)\bar{S}^{-1}}{(\alpha - A_1)^2}\left(\sqrt{1 - 2(s_t - \bar{s})} - 1\right) - \alpha\sigma(1+\lambda_t)u_{t+1}. \end{aligned}$$

We can write out $y_0(1)$ as:

$$y_0(1) = -\log\beta + \alpha\mu_c - \frac{\bar{\sigma}^2\alpha^2}{2\bar{S}^2(\alpha - A_1)^2} - \frac{\sigma^2\alpha^2}{2}(\frac{1-A_1}{\alpha - A_1})^2 - \frac{\sigma^2\alpha^2}{\bar{S}}\frac{1-A_1}{(\alpha - A_1)^2}.$$

Substituting the last expression for $y_0(1)$ the log SDF:

$$\begin{aligned} m_{t+1} &= -y_t(1) - \frac{\bar{\sigma}^2\alpha^2}{2(\alpha - A_1)^2}(1-A_1)^2 - \frac{\bar{\sigma}^2\alpha^2}{2\bar{S}^2(\alpha - A_1)^2} + \frac{\bar{\sigma}^2\alpha^2}{\bar{S}^2(\alpha - A_1)^2}(s_t - \bar{s}) \\ &- \frac{\bar{\sigma}^2\alpha^2}{\bar{S}(\alpha - A_1)^2}(1-A_1)\sqrt{1 - 2\,(s_t - \bar{s})} - \alpha\bar{\sigma}(1+\lambda_t)\eta_{t+1}. \end{aligned}$$

Now define $L_t \equiv \alpha\bar{\sigma}(1+\lambda_t)$ and recall the sensitivity function:

$$\lambda_t = \frac{\bar{S}^{-1}\sqrt{1-2(s_t - \bar{s})} + 1 - \alpha}{\alpha - A_1}$$

This implies that $-\frac{1}{2}L_t^2$ equals:

$$\begin{aligned} -\frac{1}{2}L_t^2 &= -\frac{\sigma^2\alpha^2}{2(\alpha - A_1)^2}(1-A_1)^2 - \frac{\sigma^2\alpha^2}{2\bar{S}^2(\alpha - A_1)^2} + \frac{\sigma^2\alpha^2}{\bar{S}^2(\alpha - A_1)^2}(s_t - \bar{s}) \\ &- \frac{\sigma^2\alpha^2}{\bar{S}(\alpha - A_1)^2}(1-A_1)\sqrt{1 - 2\,(s_t - \bar{s})}. \end{aligned}$$

The affine representation follows immediately. □

## B.6 Pricing Stocks in EH Model

The main difference with the analysis in the long-run risk model, and the analysis for the total wealth return in the EH model is that the return to the aggregate dividend claim *cannot* be written as a linear function of the state variables. Our choice of the sensitivity function makes the log wealth-consumption ratio linear in the surplus consumption ratio. But, for that same sensitivity function, the log price-dividend ratio is not linear in the surplus-consumption ratio. As a result, we need to resort to a non-linear computation of the price-dividend ratio on the aggregate dividend claim. We focus here on the case where no cointegration is imposed between consumption and dividends on equity. Section F.3 discusses the case with cointegration.

**Dividend Growth Process** In Campbell and Cochrane (1999), dividend growth is i.i.d., with the same mean $\mu$ as consumption growth, and innovations that are correlated with the innovations in consumption growth. To make the dividend growth process more directly comparable across models, we write it as a function of innovations to consumption growth $\eta$ and innovations $u$ that are orthogonal to $\eta$:

$$\Delta d_{t+1} = \mu_d + \varphi_d\bar{\sigma}u_{t+1} + \varphi_d\bar{\sigma}\chi\eta_{t+1}. \tag{84}$$

It follows immediately that its (un)conditional variance equals $\varphi_d^2\bar{\sigma}^2(1+\chi^2)$ and its (un)conditional covariance with consumption growth is $\varphi_d\bar{\sigma}^2\chi$. If correlation between consumption and dividend growth is *corr*, then $\chi = \sqrt{corr^2/(1-corr^2)}$. We set $\varphi_d$ and $\chi$ to replicate the unconditional variance of dividend growth and the correlation of dividend growth and consumption growth *corr* in Campbell and Cochrane (1999). We set $\mu_d = \mu$, $\varphi_d = 7.32$, and $\chi = 0.20$.

**Computation of Price-Dividend Ratio** Wachter (2005) shows that the price-dividend ratio on a claim to aggregate dividends can be written as the sum of the price-dividend ratios on strips to the period-$n$ dividend, for $n = 1, \cdots, \infty$:

$$\frac{P_t}{D_t} = \sum_{n=1}^{\infty} \frac{P^d_{nt}}{D_t} \tag{85}$$

We adopt her methodology and show it continues to hold for our slightly different dividend growth process in equation (84).

The Euler equation for the period-$n$ strip delivers the following expression for the price-dividend ratio

$$\frac{P^d_{nt}}{D_t} = E_t \left[ M_{t+1} \frac{P^d_{n-1,t+1}}{D_{t+1}} \frac{D_{t+1}}{D_t} \right]$$

We conjecture that the price-dividend ratio on the period-$n$ strip equals a function $F_n(s_t)$, which follows the recursion

$$F_n(s_t) = \beta e^{\mu_d - \alpha\mu_c + \alpha(1-\rho_s)(s_t - \bar{s}) + \frac{1}{2}\varphi_d^2 \bar{\sigma}^2} E_t \left[ e^{[\varphi_d \chi - \alpha(1+\lambda_t)]\bar{\sigma}\eta_{t+1}} F_{n-1}(s_{t+1}) \right],$$

starting at $F_0(s_t) = 1$. We now verify this conjecture.

*Proof.* Substituting in the conjecture $\frac{P^d_{nt}}{D_t} = F_n(s_t)$ into the Euler equation for the period-$n$ strip, we get

$$F_n(s_t) = E_t \left[ M_{t+1} F_{n-1}(s_{t+1}) \frac{D_{t+1}}{D_t} \right].$$

Substituting in for the stochastic discount factor $M$ and the dividend growth process (84), this becomes

$$F_n(s_t) = \beta e^{\mu_d - \alpha\mu_c + \alpha(1-\rho_s)(s_t - \bar{s})} E_t \left[ e^{-\alpha(1+\lambda_t)\bar{\sigma}\eta_{t+1}} F_{n-1}(s_{t+1}) e^{\varphi_d \bar{\sigma} u_{t+1} + \varphi_d \chi \bar{\sigma} \eta_{t+1}} \right].$$

Because $u$ and $\eta$ are independent, we can write the expectation as a product of expectations. Because $u$ is standard normal, the expectation in the previous expression can be written as

$$e^{\frac{1}{2}\varphi_d^2 \bar{\sigma}^2} E_t \left[ e^{[\varphi_d \chi - \alpha(1+\lambda_t)]\bar{\sigma}\eta_{t+1}} F_{n-1}(s_{t+1}) \right].$$

This then verifies the conjecture for $F_n(s_t)$. □

Finally, let $g(\eta)$ be the standard normal pdf, then we can compute this function through numerical integration

$$F_n(s_t) = \beta e^{\mu_d - \alpha\mu_c + \alpha(1-\rho_s)(s_t - \bar{s}) + \frac{1}{2}\varphi_d^2 \bar{\sigma}^2} \int_{-\infty}^{+\infty} e^{[\varphi_d \chi - \alpha(1+\lambda(s_t))]\bar{\sigma}\eta_{t+1}} F_{n-1}(s_{t+1}) g(\eta_{t+1}) d\eta_{t+1},$$

starting at $F_0(s_t) = 1$. The grid for $s_t$ includes 14 very low values for $s_t$ (-300, -250, -200, -150, -100, -50, -40, -30, -20, -15, -10, -9, -8, -7), 100 linearly spaced points between -6.5 and $\bar{s} * 1.001 = -2.85$, and the log of 100 linearly spaced points between $\bar{S}$ and $\exp(1.0000001 s_{max})$. The function evaluation $F_{n-1}(s_{t+1})$ is done using linear interpolation (and extrapolation) on the grid for the log surplus-consumption ratio $s$. The integral is computed in matlab using quad.m. The price dividend ratio is computed as the sum of the price-dividend ratios for the first 500 strips.

## B.7 Alternative Way of Pinning Down $\bar{S}$

To conclude the discussion of the EH model, we investigate an alternative way to pin down $\bar{S}$. In our benchmark method, outlined in Appendix B.2, we chose it to match the steady state risk-free rate in Campbell and Cochrane (1999). Here, the alternative is to pin down $\bar{S}$ to match the average wealth-consumption ratio of 26.75 in Campbell and Cochrane (1999).

As before, we solve a system of three equations in $(A_0, A_1, \bar{S})$, only the third of which is different and simply imposes that $e^{A_0 - \log(4)} = 26.75$. We obtain the following solution: $A_0 = 4.673$, $A_1 = 0.447$, and $\bar{S} = 0.0339$. The wealth-consumption ratio is higher and less sensitive to the surplus-consumption ratio than in the benchmark case. The volatility of the surplus-consumption ratio is 41.6%, similar to the benchmark model. Because $A_1$ is lower, so is the volatility of the $wc$ ratio. It is 18.6% in the model, still much higher than the 8.4% in the data. The volatilities of the change in the $wc$ ratio and of the total wealth return are also lowered, but remain too high. Since, we are no longer pinning $\bar{S}$ down to match the steady-state risk-free rate, the risk-free rate turns negative: -33 basis points per quarter or -1.2% per year. It is also more volatile: .59% versus .03 in the main text and .55 in the data. The consumption risk premium is down from 2.67% per quarter to 1.97% per quarter and the equity premium is down from 3.30% per quarter to 2.23%. The main cost of this calibration is a price-dividend ratio that is too low. The volatility of $pd^m$ is now only 12.5% per quarter compared to 27% in the data.

## B.8 Quarterly Calibration EH Model

**Preference Parameters** Again, the preference parameter does not depend on the horizon ($\tilde{\alpha} = \alpha$, except for the time discount factor $\tilde{\beta} = \beta^3$). The surplus consumption ratio has the same law of motion as in the monthly model, but we set its persistence equal to $\tilde{\rho}_s = \rho_s^3$. When we simulate the quarterly model, we solve for the corresponding $A_0$, $A_1$, and $\bar{S}$ from the system (101), (102), and (79), but with the quarterly parameter values described in this appendix.

**Cash-flow Parameters** Following a similar logic, we can match mean and variance of quarterly consumption and dividend growth in the CC model. From matching the means we get:

$$\tilde{\mu} = 3\mu, \ \tilde{\mu}_d = 3\mu_d.$$

From matching the variances we get

$$\tilde{\sigma}^2 = 3\sigma^2, \ \tilde{\varphi}_d = \varphi_d, \ \tilde{\chi} = \chi.$$

A simulation of the quarterly model recovered the annualized cash-flow and asset return moments of the monthly simulation.

# C Data Appendix

## C.1 Macroeconomic Series

**Labor income** Our data are quarterly and span the period 1952.I-2006.IV. They are compiled from the most recent data available. Labor income is computed from NIPA Table 2.1 as wage and salary disbursements (line 3) + employer contributions for employee pension and insurance funds (line 7) + government social benefits to persons (line 17) - contributions for government social insurance (line 24) + employer contributions for government social insurance (line 8) - labor taxes. As in Lettau and Ludvigson (2001a), labor taxes are defined by imputing a share of personal current taxes (line 25) to labor income, with the share calculated as the ratio of wage and salary disbursements to the sum of wage and salary disbursements, proprietors' income (line 9), and rental income of persons with capital consumption adjustment (line 12), personal interest income (line 14) and personal dividend income (line 15). The series is seasonally-adjusted at annual rates (SAAR), and we divide it by 4. Because net worth of non-corporate business and owners' equity in farm business is part of financial wealth, it cannot also be part of human wealth. Consequently, labor income excludes proprietors' income.

**Consumption** Non-housing consumption consists of non-housing, non-durable consumption and non-housing durable consumption. Consumption data are taken from Table 2.3.5. from the Bureau of Economic Analysis' National Income and Product Accounts (BEA, NIPA). Non-housing, non-durable consumption is measured as the sum of non-durable goods (line 6) + services (line 13) - housing services (line14).

Non-housing durable consumption is unobserved and must be constructed. From the BEA, we observe durable *expenditures*. The value of the durables (Flow of Funds, see below) at the end of two consecutive quarters and the durable expenditures allows us to measure the implicit depreciation rate that entered in the Flow of Fund's calculation. We average that depreciation rate over the sample; it is $\delta$=5.293% per quarter. We apply that depreciation rate to the value of the durable stock at the beginning of the current period (= measured as the end of the previous quarter) to get a time-series of this period's durable *consumption*.

We use housing services consumption (BEA, NIPA, Table 2.3.5, line 14) as the dividend stream from housing wealth. The BEA measures rent for renters and imputes a rent for owners. These series are SAAR, so we divide them by 4 to get quarterly values.

Total consumption is the sum of non-housing non-durable, non-housing durable, and housing consumption.

**Population and deflation** Throughout, we use the disposable personal income deflator from the BEA (Table 2.1, implied by lines 36 and 37) as well as the BEA's population series (line 38).

## C.2 Financial Series

**Stock market return** We use value-weighted quarterly returns (NYSE, AMEX, and NASDAQ) from CRSP as our measure of the stock market return. In constructing the dividend-price ratio, we use the repurchase-yield adjustment advocated by Boudoukh, Michaely, Richardson, and Roberts (2004). We also add the dividends over the current and past three quarters, so as to obtain a price-dividend ratio that is comparable with an annual number.

**Bond yields** We use the nominal yield on a 3-month Treasury bill from Fama (CRSP file) as our measure of the risk-free rate. We also use the yield spread between a 5-year Treasury note and a 3-month Treasury bill as a return predictor. The 5-year yield is obtained from the Fama-Bliss data (CRSP file).

We also use monthly yield data for the period 1953.4-2006.12 from FRED II on bonds with maturities of 7, 10, 20, and 30 years. We only use the average yield over the sample. Since the 7-year yield data are missing from 1953.4-1969.6, we use spline interpolation (using the 1-, 2-, 5-, 10-, and 20-year yields) to fill in the missing data. The 30-year bond yield data are missing from 1953.4-1977.1 and from 2002.3-2006.1. We use the 20-year yield in those periods as a proxy. The 20-year yield data are missing in 1987.1-1993.9; we use the 30-year yield data in that period as a proxy. Once we formed the average yields, we calculate the yield difference of the of 7-, 10-, 20-, and 30-year yields with the 5-year yield from the FRED II file. We add this yield difference to the average 5-year yield from the Fama-Bliss file, to form our final numbers for the average yield. The average 5-year yield is 6.12% (from

Fama-Bliss), the average 7-year yield is 6.25%, 10-year yield is 6.32%, 20-year is 6.49%, and the average 30-year yield is 6.45%.

**Additional cross-sectional stock and bond returns** We also use the 25 size and value equity portfolio returns from Kenneth French and bond portfolios returns from CRSP, sorted by maturity (1-, 2-, 5-, 7-, 10-, 20-, and 30-year maturities). We form log real quarterly returns. The small value spread, which enters in our state vector is the log return difference between the S1B5 and S1B1 portfolios.

# D No Arbitrage Proofs

## Proof of Proposition 3

*Proof.* Now, to derive $A_0^c$ and $A_1^c$, we need to solve the Euler equation for a claim to aggregate consumption. This Euler equation can either be thought of as the Euler equation that uses the nominal log SDF $m_{t+1}^{\$}$ to price the nominal total wealth return $\pi_{t+1} + r_{t+1}^c$ or the real log SDF $m_{t+1}^{\$} + \pi_{t+1}$ to price the real return $r_{t+1}^c$:

$$\begin{aligned}
1 &= E_t[\exp\{m_{t+1}^{\$} + \pi_{t+1} + r_{t+1}^c\}] \\
&= E_t[\exp\{-y_t^{\$}(1) - \frac{1}{2}L_t'L_t - L_t'\varepsilon_{t+1} + \pi_0 + e_8'z_{t+1} + r_{t+1}^c\}] \\
&= E_t[\exp\{-y_t^{\$}(1) - \frac{1}{2}L_t'L_t - L_t'\varepsilon_{t+1} + \pi_0 + e_8'z_{t+1} + \mu_c + e_7'z_{t+1} + A_0^c + A_1^{c\prime}z_{t+1} + \kappa_0^c - \kappa_1^c\,(A_0^c + A_1^{c\prime}z_t)\}] \\
&= \exp\{-y_0^{\$}(1) + \pi_0 - e_4'z_t - \frac{1}{2}L_t'L_t + e_8'\Psi z_t + \kappa_0^c + (1-\kappa_1^c)A_0^c + \mu_c - \kappa_1^c A_1^{c\prime}z_t + (e_7' + A_1^{c\prime})\,\Psi z_t\} \times \\
&\quad E_t\left[\exp\{-L_t'\varepsilon_{t+1} + (e_7 + e_8 + A_1^c)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\}\right]
\end{aligned}$$

First, note that because of log-normality of $\varepsilon_{t+1}$, the last line equals:

$$\exp\left\{\frac{1}{2}\left(L_t'L_t + (e_7 + e_8 + A_1^c)'\Sigma(e_7 + e_8 + A_1^c)' - 2(e_7 + e_8 + A_1^c)'\Sigma^{\frac{1}{2}}L_t\right)\right\}$$

Substituting in for the expectation, as well as for the affine expression for $L_t$, we get

$$\begin{aligned}
1 &= \exp\{-y_0^{\$}(1) + \pi_0 - e_4'z_t + \kappa_0^c + (1-\kappa_1^c)A_0^c + \mu_c - \kappa_1^c A_1^{c\prime}z_t + (e_7 + e_8 + A_1^c)'\Psi z_t\} \times \\
&\quad \exp\{\frac{1}{2}(e_7 + e_8 + A_1^c)'\Sigma(e_7 + e_8 + A_1^c) - (e_7 + e_8 + A_1^c)'\Sigma^{\frac{1}{2}}(L_0 + L_1 z_t)\}
\end{aligned}$$

Taking logs on both sides, an collecting the constant terms and the terms in $z$, we obtain the following:

$$\begin{aligned}
0 &= \{-y_0^{\$}(1) + \pi_0 + \kappa_0^c + (1-\kappa_1^c)A_0^c + \mu_c + \frac{1}{2}(e_7 + e_8 + A_1^c)'\Sigma(e_7 + e_8 + A_1^c) - (e_7 + e_8 + A_1^c)'\Sigma^{\frac{1}{2}}L_0\} + \\
&\quad \{-e_4' - \kappa_1^c A_1^{c\prime} + (e_7 + e_8 + A_1^c)'\Psi - (e_7 + e_8 + A_1^c)'\Sigma^{\frac{1}{2}}L_1\}z_t
\end{aligned}$$

This equality needs to hold for all $z_t$. This is a system of $N+1$ equations in $N+1$ unknowns:

$$\begin{aligned}
0 &= -y_0^{\$}(1) + \pi_0 + \kappa_0^c + (1-\kappa_1^c)A_0^c + \mu_c + \frac{1}{2}(e_7 + e_8 + A_1^c)'\Sigma(e_7 + e_8 + A_1^c) - (e_7 + e_8 + A_1^c)'\Sigma^{\frac{1}{2}}L_0, \\
0 &= (e_7 + e_8 + A_1^c)'\Psi - \kappa_1^c A_1^{c\prime} - e_4' - (e_7 + e_8 + A_1^c)'\Sigma^{\frac{1}{2}}L_1
\end{aligned}$$

□

## Proof of Proposition 4

*Proof.* We conjecture that the $t+1$-price of a $\tau$-period bond is exponentially affine in the state

$$\log(p^{\$}_{t+1}(\tau)) = A^{\$}(\tau) + \left(B^{\$}(\tau)\right)' z_{t+1}$$

and solve for the coefficients $A^{\$}(\tau+1)$ and $B^{\$}(\tau+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned}
p^{\$}_t(\tau+1) &= E_t[\exp\{m^{\$}_{t+1} + \log\left(p^{\$}_{t+1}(\tau)\right)\}] \\
&= E_t[\exp\{-y^{\$}_t(1) - \frac{1}{2}L_t'L_t - L_t'\varepsilon_{t+1} + A^{\$}(\tau) + \left(B^{\$}(\tau)\right)' z_{t+1}\}] \\
&= \exp\{-y^{\$}_0(1) - e_4'z_t - \frac{1}{2}L_t'L_t + A^{\$}(\tau) + \left(B^{\$}(\tau)\right)' \Psi z_t\} \times \\
&\quad E_t\left[\exp\{-L_t'\varepsilon_{t+1} + \left(B^{\$}(\tau)\right)' \Sigma^{\frac{1}{2}}\varepsilon_{t+1}\}\right]
\end{aligned}$$

We use the log-normality of $\varepsilon_{t+1}$ and substitute for the affine expression for $L_t$ to get:

$$p^{\$}_t(\tau+1) = \exp\{-y^{\$}_0(1) - e_4'z_t + A^{\$}(\tau) + \left(B^{\$}(\tau)\right)' \Psi z_t + \frac{1}{2}\left(B^{\$}(\tau)\right)' \Sigma \left(B^{\$}(\tau)\right) - \left(B^{\$}(\tau)\right)' \Sigma^{\frac{1}{2}}(L_0 + L_1 z_t)\}$$

Taking logs and collecting terms, we obtain a linear equation for $\log(p_t(\tau+1))$:

$$\log\left(p^{\$}_t(\tau+1)\right) = A^{\$}(\tau+1) + \left(B^{\$}(\tau+1)\right)' z_t,$$

where

$$\begin{aligned}
A^{\$}(\tau+1) &= -y^{\$}_0(1) + A^{\$}(\tau) + \frac{1}{2}\left(B^{\$}(\tau)\right)' \Sigma \left(B^{\$}(\tau)\right) - \left(B^{\$}(\tau)\right)' \Sigma^{\frac{1}{2}} L_0, \\
\left(B^{\$}(\tau+1)\right)' &= \left(B^{\$}(\tau)\right)' \Psi - e_4' - \left(B^{\$}(\tau)\right)' \Sigma^{\frac{1}{2}} L_1
\end{aligned}$$

□

## Proof of Proposition 6

*Proof.* Let consumption growth be the seventh element in the state vector. We conjecture that the $t+1$-price of a $\tau$-period strip is exponentially affine in the state

$$\log(p^c_{t+1}(\tau)) = A^c(\tau) + B^c(\tau)' z_{t+1}$$

and solve for the coefficients $A^c(\tau+1)$ and $B^c(\tau+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned}
p_t^c(\tau+1) &= E_t[\exp\{m_{t+1}^{\$} + \pi_{t+1} + \Delta c_{t+1} + \log\left(p_{t+1}^c(\tau)\right)\}] \\
&= E_t[\exp\{-y_t^{\$}(1) - \frac{1}{2}L_t'L_t - L_t'\varepsilon_{t+1} + \pi_0 + e_8'z_{t+1} + \Delta c_{t+1} + A^c(\tau) + B^c(\tau)'z_{t+1}\}] \\
&= \exp\{-y_0^{\$}(1) - e_4'z_t - \frac{1}{2}L_t'L_t + \pi_0 + e_8'\Psi z_t + \mu_c + e_7'\Psi z_t + A^c(\tau) + B^c(\tau)'\Psi z_t\} \times \\
&\quad E_t\left[\exp\{-L_t'\varepsilon_{t+1} + (e_7 + e_8 + B^c(\tau))'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right]
\end{aligned}$$

We use the log-normality of $\varepsilon_{t+1}$ and substitute for the affine expression for $L_t$ to get:

$$\begin{aligned}
p_t^c(\tau+1) &= \exp\{-y_0^{\$}(1) - e_4'z_t + \pi_0 + \mu_c + A^c(\tau) + (e_7 + e_8 + B^c(\tau))'\,\Psi z_t + \frac{1}{2}\left(e_7 + e_8 + B^c(\tau)\right)'\Sigma\left(e_7 + e_8 + B^c(\tau)\right) \\
&\quad - \left(e_7 + e_8 + B^c(\tau)\right)'\Sigma^{\frac{1}{2}}\left(L_0 + L_1 z_t\right)\}
\end{aligned}$$

Taking logs and collecting terms, we obtain a log-linear expression for $p_t^c(\tau+1)$:

$$\log\left(p_t^c(\tau+1)\right) = A^c(\tau+1) + B^c(\tau+1)'z_t,$$

where

$$\begin{aligned}
A^c(\tau+1) &= A^c(\tau) + \mu_c - y_0^{\$}(1) + \pi_0 + \frac{1}{2}\left(e_7 + e_8 + B^c(\tau)\right)'\Sigma\left(e_7 + e_8 + B^c(\tau)\right) - \left(e_7 + e_8 + B^c(\tau)\right)'\Sigma^{\frac{1}{2}}L_0, \\
B^c(\tau+1)' &= \left(e_7 + e_8 + B^c(\tau)\right)'\Psi - e_4' - \left(e_7 + e_8 + B^c(\tau)\right)'\Sigma^{\frac{1}{2}}L_1
\end{aligned}$$

□

**Proof of Proposition 5** Recall that the definition of log equity returns allows us to back out dividend growth from the first and third elements of the state:

$$\Delta d_{t+1}^m = \mu^m + \left[(e_1 - \kappa_1^m e_3)'\Psi + e_3'\right]z_t + \left(e_1 - \kappa_1^m e_3\right)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}$$

*Proof.* We conjecture that the $t+1$-price of a $\tau$-period strip is exponentially affine in the state

$$\log(p_{t+1}^m(\tau)) = A^m(\tau) + B^m(\tau)'z_{t+1}$$

and solve for the coefficients $A^m(\tau+1)$ and $B^m(\tau+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned}
p_t^m(\tau+1) &= E_t[\exp\{m_{t+1}^{\$} + \pi_{t+1} + \Delta d_{t+1}^m + \log\left(p_{t+1}^m(\tau)\right)\}] \\
&= E_t[\exp\{-y_t^{\$}(1) - \frac{1}{2}L_t'L_t - L_t'\varepsilon_{t+1} + \pi_0 + e_8'z_{t+1} + \Delta d_{t+1}^m + A^m(\tau) + B^m(\tau)'z_{t+1}\}] \\
&= \exp\{-y_0^{\$}(1) - e_4'z_t - \frac{1}{2}L_t'L_t + \pi_0 + e_8'\Psi z_t + \mu_m + \left[(e_1 - \kappa_1^m e_3)'\Psi + e_3'\right]z_t + A^m(\tau) + B^m(\tau)'\Psi z_t\} \times \\
&\quad E_t\left[\exp\{-L_t'\varepsilon_{t+1} + \left(e_1 - \kappa_1^m e_3 + e_8 + B^m(\tau)\right)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right]
\end{aligned}$$

We use the log-normality of $\varepsilon_{t+1}$ and substitute for the affine expression for $L_t$ to get:

$$\begin{aligned} p_t^m(\tau+1) &= \exp\{-y_0^\$(1) - e_4' z_t + \pi_0 + \mu_m + A^m(\tau) + \left[(e_1 - \kappa_1^m e_3 + e_8 + B^m(\tau))' \Psi + e_3'\right] z_t + \\ &\quad \frac{1}{2}(e_1 - \kappa_1^m e_3 + e_8 + B^m(\tau))' \Sigma (e_1 - \kappa_1^m e_3 + e_8 + B^m(\tau)) - (e_1 - \kappa_1^m e_3 + e_8 + B^m(\tau))' \Sigma^{\frac{1}{2}} (L_0 + L_1 z_t)\} \end{aligned}$$

Taking logs and collecting terms, we obtain a log-linear expression for $p_t^m(\tau+1)$:

$$\log(p_t^m(\tau+1)) = A^m(\tau+1) + B^m(\tau+1)' z_t,$$

where

$$\begin{aligned} A^m(\tau+1) &= A^m(\tau) + \mu_m - y_0^\$(1) + \pi_0 + \frac{1}{2}(e_1 - \kappa_1^m e_3 + e_8 + B^c(\tau))' \Sigma (e_1 - \kappa_1^m e_3 + e_8 + B^m(\tau)) \\ &\quad - (e_1 - \kappa_1^m e_3 + e_8 + B^m(\tau))' \Sigma^{\frac{1}{2}} L_0, \\ B^m(\tau+1)' &= (e_1 - \kappa_1^m e_3 + e_8 + B^m(\tau))' \Psi + e_3' - e_4' - (e_1 - \kappa_1^m e_3 + e_8 + B^m(\tau))' \Sigma^{\frac{1}{2}} L_1 \end{aligned}$$

□

**Variance Decomposition in the Data** The familiar Campbell-Shiller decomposition of the wealth-consumption ratio reads:

$$wc_t = \frac{\kappa_0^c}{\kappa_1^c - 1} + E_t\left[\sum_{j=1}^{\infty} (\kappa_1^c)^{-j} \Delta c_{t+j}\right] - E_t\left[\sum_{j=1}^{\infty} (\kappa_1^c)^{-j} r_{t+j}\right] \equiv \frac{\kappa_0^c}{\kappa_1^c - 1} + \Delta c_t^H - r_t^H. \tag{86}$$

We denote the cash-flow component by $\Delta c_t^H$ and the discount rate component by $r_t^H$. The wealth-consumption ratio fluctuates because it predicts consumption growth rates ($Cov\left[wc_t, \Delta c_t^H\right]$) or because it predicts future total wealth returns ($Cov\left[wc_t, -r_t^H\right]$):

$$V[wc_t] = Cov\left[wc_t, \Delta c_t^H\right] + Cov\left[wc_t, -r_t^H\right] = V\left[\Delta c_t^H\right] + V\left[r_t^H\right] - 2Cov\left[r_t^H, \Delta c_t^H\right]$$

The second equality suggests an alternative decomposition into the variance of expected future consumption growth, expected future returns, and their covariance.

Our methodology delivers analytical expressions for all variance and covariance terms:

$$\begin{aligned} V[wc_t] &= A_1^{c\prime} \Omega A_1^c & (87) \\ Cov\left[wc_t, \Delta c_t^H\right] &= A_1^{c\prime} \Omega (\kappa_1^c I - \Psi')^{-1} \Psi' e_c & (88) \\ Cov\left[wc_t, -r_t^H\right] &= A_1^{c\prime} \Omega \left[A_1^c - (\kappa_1^c I - \Psi)^{-1} \Psi' e_c'\right] & (89) \\ V\left[\Delta c_t^H\right] &= e_c' \Psi (\kappa_1^c I - \Psi)^{-1} \Omega (\kappa_1^c I - \Psi')^{-1} \Psi' e_c & (90) \\ V\left[r_t^H\right] &= \left[(e_c' + A_1^{c\prime})\Psi - \kappa_1^c A_1^{c\prime}\right] (\kappa_1^c I - \Psi)^{-1} \Omega (\kappa_1^c I - \Psi')^{-1} \left[\Psi'(e_c + A_1^c) - \kappa_1^c A_1^c\right] & (91) \\ Cov\left[r_t^H, \Delta c_t^H\right] &= \left[(e_c' + A_1^{c\prime})\Psi - \kappa_1^c A_1^{c\prime}\right] (\kappa_1^c I - \Psi)^{-1} \Omega (\kappa_1^c I - \Psi')^{-1} \Psi' e_c & (92) \end{aligned}$$

where $\Omega = E[z_t' z_t]$ is the second moment matrix of the state $z_t$.

# E Human Wealth Pricing

This appendix proves that our results carry over to a world where heterogeneous agents face labor income risk, which they cannot trade away because market incompleteness. We can allow for the fact that many of these households do not participate in the stock market, but only have a cash account. As long as there is a subset of agents of non-zero measure who can trade in the stock market, the claim to aggregate labor income, *human wealth*, is priced off the same stochastic discount factor that prices stocks and bonds.

**Environment** Let $z_t$ be the aggregate state vector. We use $z^t$ to denote the history of aggregate state realizations. Section 3.1 describes the dynamics of the aggregate state $z_t$ of this economy, including the dynamics of aggregate consumption $C_t(z^t)$ and aggregate labor income $Y_t(z^t)$.

Suppose the economy is populated by a continuum of heterogeneous agents, whose labor income is subject to idiosyncratic shocks. The idiosyncratic shocks are denoted by $y_t$, and we use $y^t$ to denote the history of these shocks. The household labor income process is given by:

$$\eta_t(y^t, z_t) = \widehat{\eta}_t(y^t, z^t) Y_t(z^t).$$

Let $\Phi_t(z^t)$ denote the distribution of household histories $y^t$ conditional on being in aggregate node $z^t$. The labor income shares $\hat{\eta}$ aggregate to one:

$$\int \widehat{\eta}_t(y^t, z_t) d\Phi_t(z^t) = 1.$$

Each period, households collect labor income, in addition to dividend income from stocks and bond payments (for those households who participate in financial markets).

**Trading in securities markets** A non-zero measure of these households can trade bonds and stocks in securities markets that open every period. These households are in partition 1. We assume the returns of these securities span $z_t$. Other households (in partition 2 for bonds) can only trade one-period risk-less one period discount bonds (a cash account). We use $A^i$ to denote the menu of traded assets for households in segment $i$. However, none of these households can insure directly against idiosyncratic shocks $y_t$ to their labor income by selling a claim to their labor income or by trading contingent claims on these idiosyncratic shocks.

**No Arbitrage Condition** Since there is a set of non-zero measure households that trade assets that span $z_t$, the absence of arbitrage opportunities implies the existence of the pricing kernel. We let $P_t$ be the arbitrage-free price of an asset with payoffs $\{D^i_t\}$:

$$P^i_t = E_t \sum_{\tau=t}^{\infty} \frac{M_\tau}{M_t} D^i_\tau. \tag{93}$$

for any non-negative stochastic dividend process $D^i_t$ that is measurable w.r.t $z^t$. The pricing kernel satisfies $\{M_t = \exp\left(\sum_{s\leq t} m_s\right)\}$. The log SDF $m_t$ satisfies the affine expression in (12).

**Household Problem** After collecting their labor income and their payoffs from the contingent bond market, households buy consumption in spot markets and take Arrow positions $a_{t+1}(y^{t+1}, z^{t+1})$ in the securities markets subject to a standard budget constraint:

$$c_t + E_t\left[\frac{M_{t+1}}{M_t} a_t(y^{t+1}, z^{t+1})\right] + \sum_{i\in A^i}^{N} P^i_t s^i_{t+1} \leq \theta_t$$

where $s$ denotes the shares in a security $i$ that is in the trading set of that agent. In the second term on the left-hand side, the expectations operator arises because we sum across all states of nature tomorrow and weight the price of each of the corresponding Arrow securities by the probability of that state arising.
Wealth evolves according to:

$$\theta_{t+1} = a_t(y^{t+1}, z^{t+1}) + \eta_{t+1} + \sum_{i \in A^i}^{N} \left[ P_{t+1}^i + D_{t+1}^i \right] s_t^i$$

subject to a measurability constraint:

$$a_t(y^{t+1}, z^{t+1}) \textit{ is measurable w.r.t. } A_t^j(y^{t+1}, z^{t+1}),\ j \in \{1, 2\}$$

and subject to a solvency constraint:

$$a_t(y^{t+1}, z^{t+1}) \geq B_t(y^t, z^t)$$

These measurability constraints limit the dependence of total household financial wealth on $(z^{t+1}, y^{t+1})$. For example, for those households in partition $b$ that only trade a risk-free bond, $A_t^2(y^t, z^t) = (y^{t-1}, z^{t-1})$, because their net wealth can only depend on the history of aggregate and idiosyncratic states up until $t-1$. The households that do trade in stock and bond markets can have net wealth depend on the state until time $t$: $A_t^1(y^t, z^t) = (y^{t-1}, z^t)$.

**Pricing of Household Human wealth** In the absence of arbitrage opportunities, we can eliminate trade in actual securities, and the budget constraint reduces to:

$$c_t + E_t \left[ \frac{M_{t+1}}{M_t} a_t(y^{t+1}, z^{t+1}) \right] \leq a_{t-1}(y^t, z^t) + \eta_t$$

By forward substitution of $a_t(y^{t+1}, z^{t+1})$ in the budget constraint, and by imposing the transversality condition on household net wealth:

$$\lim_{t \to \infty} M_t a_t(y^t, z^t) = 0,$$

it becomes apparent that the expression for financial wealth is :

$$\begin{aligned} a_{t-1}(y^t, z^t) &= E_t \left[ \sum_{\tau=t}^{\infty} \frac{M_\tau}{M_t} \left( c_\tau(y^\tau, z^\tau) - \eta_\tau(y^\tau, z^\tau) \right) \right] \\ &= E_t \left[ \sum_{\tau=t}^{\infty} \frac{M_\tau}{M_t} c_\tau(y^\tau, z^\tau) \right] - E_t \left[ \sum_{\tau=t}^{\infty} \frac{M_\tau}{M_t} \eta_\tau(y^\tau, z^\tau) \right] \end{aligned}$$

The second equation states that non-human wealth (on the left) equals the present discounted value of consumption minus human wealth. Put differently, the households current and future consumption is restricted by total wealth, the sum of human and non-human (financial) wealth. It is clear that an individual's human wealth, the price of a claim to her individual-specific labor income, is given by:

$$E_t \left[ \sum_{\tau=t}^{\infty} \frac{M_\tau}{M_t} \eta_\tau(y^\tau, z^\tau) \right]$$

In other words, each household's human wealth is priced off the pricing kernel that prices tradeable securities, such as stocks. This is despite the fact that this household cannot trade away its idiosyncratic human wealth risk, and neither can the other households. This result holds in the presence of generic borrowing constraints $B$. This result holds in the presence of generic borrowing constraints $B$. This is true for both kinds of households. Even for the non-participants in the stock market, the human wealth that enters the budget constraint at time 0 (or later) is priced off the pricing kernel $M_t$.

**Pricing of Aggregate Human Wealth** Let $\Phi_0$ denote the measure at time 0 over the history of idiosyncratic shocks. The (shadow) price of a claim to aggregate labor income is priced at time 0 is given by:

$$\begin{aligned} \int E_0\left[\sum_{t=0}^{\infty}\frac{M_t}{M_0}\widehat{\eta}_t(y^t,z_t)Y_t(z^t)\right]d\Phi_0 &= E_0\left[\sum_{t=0}^{\infty}\frac{M_t}{M_0}\int \widehat{\eta}_t(y^t,z_t)d\Phi_t(z^t)Y_t(z^t)\right] \\ &= E_0\left[\sum_{t=0}^{\infty}\frac{M_t}{M_0}Y_t(z^t),\right] \end{aligned}$$

where we have used $\int \widehat{\eta}_t(y^t,z_t)d\Phi_t(z^t)=1$, and where $\Phi_t(z^t)$ is the distribution of household histories $y^t$ conditional on being in aggregate node $z^t$. In the first equality, the integral and sum can be exchanged under regularity conditions. Aggregate human wealth, which is the sum of all individual households' human wealth is the present discounted value of aggregate labor income. The discount factor is the same one that prices tradeable securities, such as stocks and bonds. This is true despite the fact that human wealth is non-tradeable in this model. This result follows directly from the household budget constraint.

# F Supplementary Material

This section contains additional material that illustrates further details on the theory-side and robustness exercise on the empirical side.

## F.1 LRR Model: $\rho \to 1$

In this appendix we study the LRR model as the inverse intertemporal elasticity of substitution, $\rho$ goes to one. Holding $\kappa_1^c$ fixed, it is easy to see that

$$A_1 = \frac{1-\rho}{\kappa_1^c-\rho_x} \to 0 \text{ as } \rho \to 1$$

and

$$A_2 = \frac{(1-\rho)(1-\alpha)}{2(\kappa_1^c-\nu_1)}\left[1+\frac{\varphi_e^2}{(\kappa_1^c-\rho_x)^2}\right] \to 0 \text{ as } \rho \to 1$$

Note however that $\kappa_1^c$ depends on $A_0$ which in turn depends on $\rho$. We have solved the system of three non-linear equations (described in appendix A.2) for a sequence of values of $\rho$ approaching 1 (from above and from below) and verified that $A_1 \to 0$ and $A_2 \to 0$. Furthermore, we found that $A_0 \to \log\left(\frac{1}{1-\beta}\right)$, so that $\kappa_1^c \to \beta^{-1}$ and $\kappa_0^c \to -\log\left(\frac{\beta}{1-\beta}\right)+\frac{1}{\beta}\log\left(\frac{1}{1-\beta}\right)$.

As rho goes to one, the consumption risk premium converges to the one in the standard Lucas-Breeden economy:

$$E_t\left[r_{t+1}^c - y_t(1)\right] + .5V_t[r_{t+1}^c] \to \alpha\sigma_t^2 \text{ as } \rho \to 1.$$

This happens because the other two consumption risk premium components converge to zero. Holding $\kappa_1^c$ fixed, this can be seen in the expressions for these two components:

$$\begin{aligned}\lambda_{m,e}B &= (1-\rho)(\alpha-\rho)\frac{\varphi_e^2}{(\kappa_1^c-\rho_x)^2} \to 0 \text{ as } \rho \to 1 \\ \lambda_{m,w}A_2 &= (1-\rho)(\alpha-\rho)\frac{(1-\alpha)^2}{4(\kappa_1^c-\nu_1)^2}\left[1+\frac{\varphi_e^2}{(\kappa_1^c-\rho_x)^2}\right]^2 \to 0 \text{ as } \rho \to 1\end{aligned}$$

We have confirmed numerically that the two consumption risk premium components $\lambda_{m,e}B$ and $\lambda_{m,w}A_2$ go to zero as $\rho$ goes to one (from above or from below), when solving the system of equations. Explained differently the conditional market price of *wc* risk in equation (64) goes to zero, while the market price of standard consumption growth in equation (63) has a well-defined, non-zero limit $\alpha\sigma_t^2$. So, the only risk with a positive compensation associated to it that remains when $\rho \to 1$ is the standard high-frequency aggregate consumption growth risk.

The same is not true for the risk premium on the claim to the stream of aggregate dividends. We focus on the case without cointegration in Appendix A.6.

$$E_t\left[r_{t+1}^m - y_t(1)\right] + .5V_t[r_{t+1}^m] \to \xi_{m,e}\sigma_t^2 + \xi_{m,w}\sigma_w^2$$

where

$$\begin{aligned}\xi_{m,e} &\equiv \lim_{\rho\to 1}\lambda_{m,e}\beta_{m,e} = \frac{(\alpha-1)(\phi-1)\varphi_e^2}{(\kappa_1^c-\rho_x)(\kappa_1^m-\rho_x)} \\ \xi_{m,w} &\equiv \lim_{\rho\to 1}\lambda_{m,w}\beta_{m,w} = \frac{(\alpha-1)^2}{2(\kappa_1^c-\nu_1)}\left(1+\frac{\varphi_e^2}{(\kappa_1^c-\rho_x)^2}\right)\left[\frac{(\alpha-1)^2}{2(\kappa_1^m-\nu_1)}\left(1+\frac{\varphi_e^2}{(\kappa_1^c-\rho_x)^2}\right) - .5\frac{H_m}{(\kappa_1^m-\nu_1)}\right]\end{aligned}$$

As the second expression for $H_m$ in Appendix A.6 shows, $H_m$ does not depend on $\rho$. Clearly, for $\phi \neq 1$ and $\alpha \neq 1$, there are positive *equity* risk premia (on the dividend claim) over and above the ones that would arise in a Lucas-Breeden economy.

## F.2 LRR Model: Asset Pricing with Cointegration

**Dividend Growth Process** In the previous specification, consumption and dividends can drift arbitrarily away from each other. In this section, we follow Bansal, Dittmar, and Lundblad (2005) and modify the dividend growth process to impose cointegration between consumption and dividends. Log dividends are stochastically cointegrated with log consumption, and may have a deterministic trend:

$$\begin{aligned}d_{t+1} &= \varpi + \delta(t+1) + \phi c_{t+1} + q_{t+1} \\ \Delta d_{t+1} &= \delta + \phi\Delta c_{t+1} + \Delta q_{t+1},\end{aligned} \tag{94}$$

where the second equation is obtained by taking first differences of the first equation. The process $\{q\}$ denotes the dividend-consumption ratio, which we specify as a mean-zero, autoregressive process with heteroscedasticity:

$$q_{t+1} = \rho_q q_t + \varphi_q\sigma_t u_{t+1} \tag{95}$$

This is a generalization from the process in Bansal, Dittmar, and Lundblad (2005), who work with a homoscedastic model ($\sigma_t^2 = \bar{\sigma}^2, \forall t$). Equations (94) and (95) completely specify the dividend growth process in the cointegration case and replace equation (65) in the no cointegration case. The rest of the technology process is unaffected: the processes for $\Delta c_{t+1}$, $x_{t+1}$, and $\sigma_{t+1}^2$ remain unchanged from the main text. As a result, the stochastic discount factor, and the consumption-wealth ratio process all remain unaltered.

To facilitate comparison with the no-cointegration case, we use the same values for $\phi$ and $\varphi_d$ as in the no cointegration case. We match the unconditional mean and variance of dividend growth in the cases with and without cointegration. I.e., we choose the parameter $\delta$ to match the mean:

$$\delta = \mu_d - \phi\mu,$$

with $\mu_d = \mu$, and we choose $\varphi_q$ to match the variance:

$$\varphi_d^2 = \frac{2}{1+\rho_q}\varphi_q^2 + \phi^2 \Rightarrow \varphi_q = \sqrt{.5(1+\rho_q)(\varphi_d^2 - \phi^2)}.$$

We keep the parameter $\phi$ the same in both cases. Following Bansal and Yaron (2004), we choose $\mu_d = \mu$, $\phi = 3$ and $\varphi_d = 4.5$. The only other parameter is the persistence of the quarterly log dividend-consumption ratio $\rho_q$, which we set equal to 0.83. This follows Lettau and Ludvigson (2005), who document a persistence of .475 at annual frequency (or .83 at quarterly frequency) for the cointegration vector between log consumption, log stock dividends, and log labor income.

**Proof of Linearity** The only difference with the no-cointegration case is that $q_t$ becomes an additional state variable for the price-dividend ratio. That is, we conjecture:

$$pd_t^m = A_0^m + A_1^m x_t + A_2^m(\sigma_t^2 - \bar{\sigma}^2) + A_3^m q_t.$$

This leads to different expressions for the innovations in the dividend claim return, and the conditional mean and variance of the dividend claim return:

$$\begin{aligned}
r_{t+1}^m - E_t\left[r_{t+1}^m\right] &= \phi\sigma_t\eta_{t+1} + (1+A_3^m)\varphi_q\sigma_t u_{t+1} + \beta_{m,e}\sigma_t e_{t+1} + \beta_{m,w}\sigma_w w_{t+1} \\
E_t\left[r_{t+1}^m\right] &= r_0^m + [\phi + A_1^m(\rho_x - \kappa_1^m)]x_t - A_2^m(\kappa_1^m - \nu_1)\left(\sigma_t^2 - \bar{\sigma}^2\right) + (\rho_q - 1 - A_3^m(\kappa_1^m - \rho_q))q_t \\
V_t\left[r_{t+1}^m\right] &= \left((1+A_3^m)^2\varphi_q^2 + \beta_{m,e}^2 + \phi^2\right)\sigma_t^2 + \beta_{m,w}^2\sigma_w^2 \\
r_0^m &= \kappa_0^m + A_0^m(1-\kappa_1^m) + \delta + \phi\mu_c
\end{aligned}$$

Finally, the conditional covariance between the log SDF and the log dividend claim return is

$$Cov_t\left[m_{t+1}, r_{t+1}^m\right] = \left(\lambda_{m,\eta}\phi - \lambda_{m,e}\beta_{m,e}\right)\sigma_t^2 - \lambda_{m,w}\beta_{m,w}\sigma_w^2.$$

Using the method of undetermined coefficients, we obtain expressions for $A_0^m$, $A_1^m$, $A_2^m$, and $A_3^m$:

$$\begin{aligned}
A_1^m &= \frac{\phi - \rho}{\kappa_1^m - \rho_x}, \\
A_2^m &= \frac{\left[(1-\theta)A_2(\kappa_1^c - \nu_1) + .5\hat{H}_m\right]}{\kappa_1^m - \nu_1}, \\
A_3^m &= \frac{\rho_q - 1}{\kappa_1^m - \rho_q}, \\
0 &= m_0 + \kappa_0^m + (1-\kappa_1^m)A_0^m + \delta + \phi\mu_c + \frac{1}{2}\hat{H}_m\bar{\sigma}^2 + \frac{1}{2}\left(A_2^m - \frac{\alpha-\rho}{1-\rho}A_2\right)^2\sigma_w^2
\end{aligned}$$

where

$$\hat{H}_m = (\lambda_{m,\eta} + \phi)^2 + (\beta_{m,e} - \lambda_{m,e})^2 + (1 + A_3^m)^2 \varphi_q^2$$

The expressions for $A_1^m$ and $A_2^m$ are functionally identical to the ones in the no cointegration case, except that the definition of $\hat{H}_m$ is slightly different from that of $H_m$. This is a non-linear system in four equations and four unknowns, which we solve numerically.

**Equity Risk premium and CS Decomposition** The equity risk premium on the dividend claim (adjusted for a Jensen term) becomes:

$$E_t\left[r_{t+1}^{e,m}\right] \equiv E_t\left[r_{t+1}^m - y_t(1)\right] + .5V_t[r_{t+1}^m] = \left(-\lambda_{m,\eta}\phi + \lambda_{m,e}\beta_{m,e}\right)\sigma_t^2 + \lambda_{m,w}\beta_{m,w}\sigma_w^2. \tag{96}$$

Note that $q_t$ does not affect the equity risk premium. Its only driver is the conditional variance of consumption growth $\sigma_t^2 - \bar{\sigma}^2$.

Expected discounted future equity returns and dividend growth rates are given by:

$$r_t^{m,H} \equiv E_t\left[\sum_{j=1}^{\infty}\left(\kappa_1^m\right)^{-j} r_{t+j}^m\right] = \frac{r_0^m}{\kappa_1^m - 1} + \frac{\rho}{\kappa_1^m - \rho_x}x_t - A_2^m(\sigma_t^2 - \bar{\sigma}^2) \tag{97}$$

$$\Delta d_t^H \equiv E_t\left[\sum_{j=1}^{\infty}\left(\kappa_1^m\right)^{-j} \Delta d_{t+j}\right] = \frac{\delta + \phi\mu}{\kappa_1^m - 1} + \frac{\phi}{\kappa_1^m - \rho_x}x_t + \frac{\rho_q - 1}{\kappa_1^m - \rho_q}q_t \tag{98}$$

The only difference with the no-cointegration case is that expected future dividend growth rates now also depend on the current dividend-consumption ratio $q_t$. Discount rates remain unchanged. As before,

$$pd_t^m = \frac{\kappa_0^m}{\kappa_1^m - 1} + \Delta d_t^H - r_t^{m,H},$$

which allows us to compute the elements of the variance-decomposition:

$$V[pd_t^m] = Cov[pd_t^m, \Delta d_t^H] + Cov[pd_t^m, -r_t^{m,H}] = V[\Delta d_t^H] + V[r_t^{m,H}] - 2Cov[\Delta d_t^H, r_t^{m,H}].$$

## F.3 EH Model: Asset Pricing with Cointegration

**Dividend Growth under Cointegration** Just as in the LRR model, we impose cointegration and use the dividend growth specification

$$\Delta d_{t+1} = \delta + \phi\Delta c_{t+1} + \Delta q_{t+1} \tag{99}$$

instead of equation (84) in the case without cointegration. The process $\{q\}$ again denotes the log consumption-dividend ratio. We specify $q$ as an autoregressive process with homoscedastic innovations that are correlated with consumption growth innovations $\eta$. Relative to the LRR specification, we loose heteroscedasticity, but we gain correlation between consumption growth innovations and innovations to the dividend-consumption process. Since we prefer to work with independent innovations, we write:

$$q_{t+1} = \rho_q q_t + \varphi_q\bar{\sigma}u_{t+1} + \varphi_q\bar{\sigma}\chi\eta_{t+1}, \tag{100}$$

where, as usual, $\eta_t \perp u_t$.

The parameter choices for $\delta$, $\phi$, and $\varphi_q$ are the same as in the LRR model. The choice for $\chi$ is the same as in the no-cointegration case.

**Computation of Price-Dividend Ratio** Under the assumption of cointegration, the dividend growth process is given by equations (94) and (100). Closely following Appendix A in Wachter (2005), we conjecture that the price-dividend ratio can be written as the product of a function that only depends on the log surplus-consumption ratio and another function that only depends on the log dividend-consumption ratio:

$$\frac{P^d_{nt}}{D_t} = F^d_n(s_t)e^{A_n+B_nq_t}$$

The function that depends on $s_t$ follows a recursion

$$F^d_n(s_t) = E_t\left[M_{t+1}F^d_{n-1}(s_{t+1})e^{\phi\mu_c+X\bar{\sigma}\eta_{t+1}}\right] = \beta e^{\phi\mu-\alpha\mu+\alpha(1-\rho_s)(s_t-\bar{s})}E_t\left[e^{\{X-\alpha(1+\lambda_t)\}\bar{\sigma}\eta_{t+1}}F^d_{n-1}(s_{t+1})\right].$$

The verification of this conjecture delivers expressions for the constants $X$, $A_n$, and $B_n$.

*Proof.* The Euler equation for the period-$n$ strip delivers the following expression for the price-dividend ratio

$$\frac{P^d_{nt}}{D_t} = E_t\left[M_{t+1}\frac{P^d_{n-1,t+1}}{D_{t+1}}\frac{D_{t+1}}{D_t}\right] = E_t\left[M_{t+1}F^d_{n-1}(s_{t+1})e^{A_{n-1}+B_{n-1}q_{t+1}}e^{\delta+\phi\Delta c_{t+1}+\Delta q_{t+1}}\right],$$

where the second equality substituted in the expression for dividend growth, and the conjecture for the price-dividend ratio. Next we substitute in for the expressions for consumption growth, the log dividend-consumption ratio $q$, and $\Delta q$:

$$\frac{P^d_{nt}}{D_t} = e^{A_{n-1}+\delta+[B_{n-1}\rho_q+\rho_q-1]q_t}E_t\left[M_{t+1}F^d_{n-1}(s_{t+1})e^{(B_{n-1}+1)\varphi_q\bar{\sigma}u_{t+1}}e^{[(B_{n-1}+1)\varphi_q\chi+\phi]\bar{\sigma}\eta_{t+1}+\phi\mu_c}\right].$$

Because $u$ is independent of $\eta$ and standard normally distributed, we have

$$\frac{P^d_{nt}}{D_t} = e^{A_{n-1}+\delta+\frac{1}{2}(B_{n-1}+1)^2\varphi_q^2\bar{\sigma}^2+[B_{n-1}\rho_q+\rho_q-1]q_t}E_t\left[M_{t+1}F^d_{n-1}(s_{t+1})e^{[(B_{n-1}+1)\varphi_q\chi+\phi]\bar{\sigma}\eta_{t+1}+\phi\mu_c}\right].$$

Recursively define the coefficients $A_n$ and $B_n$ as

$$\begin{aligned} A_n &= A_{n-1}+\delta+\frac{1}{2}(B_{n-1}+1)^2\varphi_q^2\bar{\sigma}^2 \\ B_n &= B_{n-1}\rho_q+\rho_q-1, \end{aligned}$$

starting at $A_0 = B_0 = 0$, and define the constant $X$ as $X = (B_{n-1}+1)\varphi_q\chi+\phi$, then we obtain

$$\frac{P^d_{nt}}{D_t} = e^{A_n+B_nq_t}E_t\left[M_{t+1}F^d_{n-1}(s_{t+1})e^{X\bar{\sigma}\eta_{t+1}+\phi\mu_c}\right],$$

which verifies the conjecture. □

We use numerical integration to compute the sequence $\{F^d(s_t)\}$:

$$F^d_n(s_t) = e^{\log(\beta)+(\phi-\alpha)\mu_c-\alpha(1-\rho_s)(\bar{s}-s_t)}\int_{-\infty}^{+\infty}e^{[X-\alpha(1+\lambda(s_t))]\bar{\sigma}\eta_{t+1}}F^d_{n-1}(s_{t+1})g(\eta_{t+1})d\eta_{t+1},$$

where $g(\eta)$ is the standard normal pdf, and start from $F_0^d(s_t) = 1$.

## F.4 EH Model: Improving on the Campbell-Shiller Approximation

We start from the definition of the log total wealth return $r_{t+1}^c = \Delta c_{t+1} + wc_{t+1} - \log\left(e^{wc_t} - 1\right)$. Instead of a first-order Taylor approximation around the mean log wealth-consumption ratio $A_0$, we do a second-order approximation:

$$\begin{aligned}\log\left(e^{wc_t} - 1\right) &\approx \log\left(e^{A_0} - 1\right) + \kappa_1^c(wc_t - A_0) + .5\kappa_1^c(1-\kappa_1^c)(wc_t - A_0)^2 \\ &= -\kappa_0^c + \left[\kappa_1^c - A_0\kappa_1^c(1-\kappa_1^c)\right] wc_t + .5\kappa_1^c(1-\kappa_1^c)wc_t^2\end{aligned}$$

where

$$\kappa_1^c = \frac{e^{A_0}}{e^{A_0}-1} \text{ and } \kappa_0^c = -\log\left(e^{A_0} - 1\right) + \kappa_1^c A_0 - .5\kappa_1^c(1-\kappa_1^c)A_0^2$$

This leads to the return approximation

$$r_{t+1}^c \approx \Delta c_{t+1} + wc_{t+1} + \kappa_0^c - \kappa_1^c wc_t + \left\{A_0\kappa_1^c(1-\kappa_1^c)wc_t - .5\kappa_1^c(1-\kappa_1^c)wc_t^2\right\}$$

The term in accolades comes from adding second-order terms.

We conjecture that the log wealth-consumption ratio is linear in the sole state variable $(s_t - \bar{s})$,

$$wc_t = A_0 + A_1\left(s_t - \bar{s}\right).$$

As CC, we assume joint conditional normality of consumption growth and the surplus consumption ratio. We verify this conjecture from the Euler equation.

We slightly modify the preferences:

$$m_{t+1} = \log\beta - \alpha\Delta c_{t+1} - \alpha\Delta s_{t+1} + K\left(s_t - \bar{s}\right)^2.$$

The term $K\left(s_t - \bar{s}\right)^2$ is a linearity-inducing term, similar in spirit to Gabaix (2007), whose role will become clear below. As before, we compute innovations, and the conditional mean and variance of the log SDF:

$$\begin{aligned}m_{t+1} - E_t\left[m_{t+1}\right] &= -\alpha(1+\lambda_t)\bar{\sigma}\eta_{t+1}, \\ E_t\left[m_{t+1}\right] &= m_0 + \alpha(1-\rho_s)\left(s_t - \bar{s}\right) + K\left(s_t - \bar{s}\right)^2 \\ V_t\left[m_{t+1}\right] &= \alpha^2\left(1+\lambda_t\right)^2\bar{\sigma}^2 \\ m_0 &= \log\beta - \alpha\mu_c\end{aligned}$$

Likewise, we compute innovations in the consumption claim return, and its conditional mean and variance:

$$\begin{aligned}r_{t+1}^c - E_t\left[r_{t+1}^c\right] &= (1 + A_1\lambda_t)\bar{\sigma}\eta_{t+1} \\ E_t\left[r_{t+1}^c\right] &= \left[\kappa_0^c + A_0(1-\kappa_1^c) + .5A_0^2\kappa_1^c(1-\kappa_1^c)\right] + \mu_c - A_1(\kappa_1^c - \rho_s)\left(s_t - \bar{s}\right) - .5\kappa_1^c(1-\kappa_1^c)A_1^2\left(s_t - \bar{s}\right)^2 \\ V_t\left[r_{t+1}^c\right] &= (1 + A_1\lambda_t)^2\bar{\sigma}^2\end{aligned}$$

Again, the only difference with the previous version is the extra term is in $E_t[r_{t+1}^c]$; its intercept has an additional term, and it has an additional quadratic term in $(s_t - \bar{s})^2$.

The conditional covariance between the log consumption return and the log SDF is given by the conditional expectation of the product of their innovations

$$Cov_t\left[m_{t+1}, r_{t+1}^c\right] = -\alpha\left(1+\lambda_t\right)\left(1+A_1\lambda_t\right)\bar{\sigma}^2$$

We assume that the sensitivity function takes the following form

$$\lambda_t = \frac{\bar{S}^{-1}\sqrt{1-2(s_t-\bar{s})}+1-\alpha}{\alpha-A_1}$$

Using the method of undetermined coefficients and the five components of equation (**??**), we can solve for the constants $A_0$ and $A_1$:

$$A_1 = \frac{(1-\rho_s)\alpha-\bar{\sigma}^2\bar{S}^{-2}}{\kappa_1^c-\rho_s}, \tag{101}$$

$$0 = \log\beta+\kappa_0^c+(1-\kappa_1^c)A_0+.5A_0^2\kappa_1^c(1-\kappa_1^c)+(1-\alpha)\mu_c+.5\bar{\sigma}^2\bar{S}^{-2} \tag{102}$$

When we choose the constant $K = .5\kappa_1^c(1-\kappa_1^c)A_1^2$, the terms in $(s_t-\bar{s})^2$ cancel. This verifies that our conjecture was correct. Note that because $\kappa_1^c$ is close to 1, $K$ is close to zero.

Note also that the steady-state risk-free rate is unchanged. Even though $y_t(1) = -E_t[m_{t+1}] - .5V_t[m_{t+1}]$ will have the additional term $-.5\kappa_1^c(1-\kappa_1^c)A_1^2(s_t-\bar{s})^2$, this term is zero when evaluated at $s_t = \bar{s}$. That implies that the third equation of the system of three equations in three unknowns is the same as before.

The solution to this system is virtually identical to that of the linear system. I.e., $A_0$, $A_1$, $\bar{s}$, $s_{max}$, and $\kappa_1^c$ are identical up to the 9th decimal. Only $\kappa_0^c$ is different, as it should be, because of its changed definition. We conclude that the Campbell-Shiller approximation does an excellent job at approximating the log total wealth return.

# Table 1: Moments of the Wealth-Consumption Ratio

This table displays unconditional moments of the log wealth-consumption ratio $wc$, its first difference $\Delta wc$, and the log total wealth return $r^c$. The last but one row reports the time-series average of the conditional consumption risk premium, $E[E_t[r_t^{c,e}]]$, where $r^{c,e}$ denotes the expected log return on total wealth in excess of the risk-free rate and corrected for a Jensen term. The first column reports moments from the long-run risk (LRR) model, simulated at quarterly frequency. Each simulation is ran for 236 quarters and repeated 5,000 times. In each simulation, we discard the first 16 observations. All reported moments are averages of the quarterly statistics across the 5,000 simulations. The second column reports the same moments for the external habit (EH) model. The last column is for the data.

| Moments | LRR Model | EH model | data |
|---|---|---|---|
| $Std[wc]$ | 2.35% | 29.33% | 17.94% |
| (s.e.) | (.43) | (12.75) | |
| $AC(1)[wc]$ | .91 | .93 | .96 |
| (s.e.) | (.03) | (.03) | |
| $AC(4)[wc]$ | .70 | .74 | .88 |
| (s.e.) | (.10) | (.11) | |
| $Std[\Delta wc]$ | 0.90% | 9.46% | 4.57% |
| (s.e.) | (.05) | (2.17) | |
| $Std[\Delta c]$ | 1.43% | .75% | .44% |
| (s.e.) | (.08) | (.04) | |
| $Corr[\Delta c, \Delta wc]$ | -.06 | .90 | .14 |
| (s.e.) | (.06) | (.03) | |
| $Std[r^c]$ | 1.64% | 10.26% | 4.66% |
| (s.e.) | (.09) | (2.21) | |
| $Corr[r^c, \Delta c]$ | .84 | .91 | .23 |
| (s.e.) | (.02) | (.03) | |
| $E[E_t[r_t^{c,e}]]$ | 0.40% | 2.67% | 0.83% |
| (s.e.) | (.01) | (1.16) | |
| $E[wc]$ | 5.85 | 3.86 | 5.21 |
| (s.e.) | (.01) | (.17) | |

Figure 1: Average Term Structure of Interest Rates

The figure plots the observed and model-implied nominal bond yields for bonds of maturities 1-120 quarters. The data are obtained by using a spline-fitting function through the observed maturities. The third panel plots the model-implied real yields.

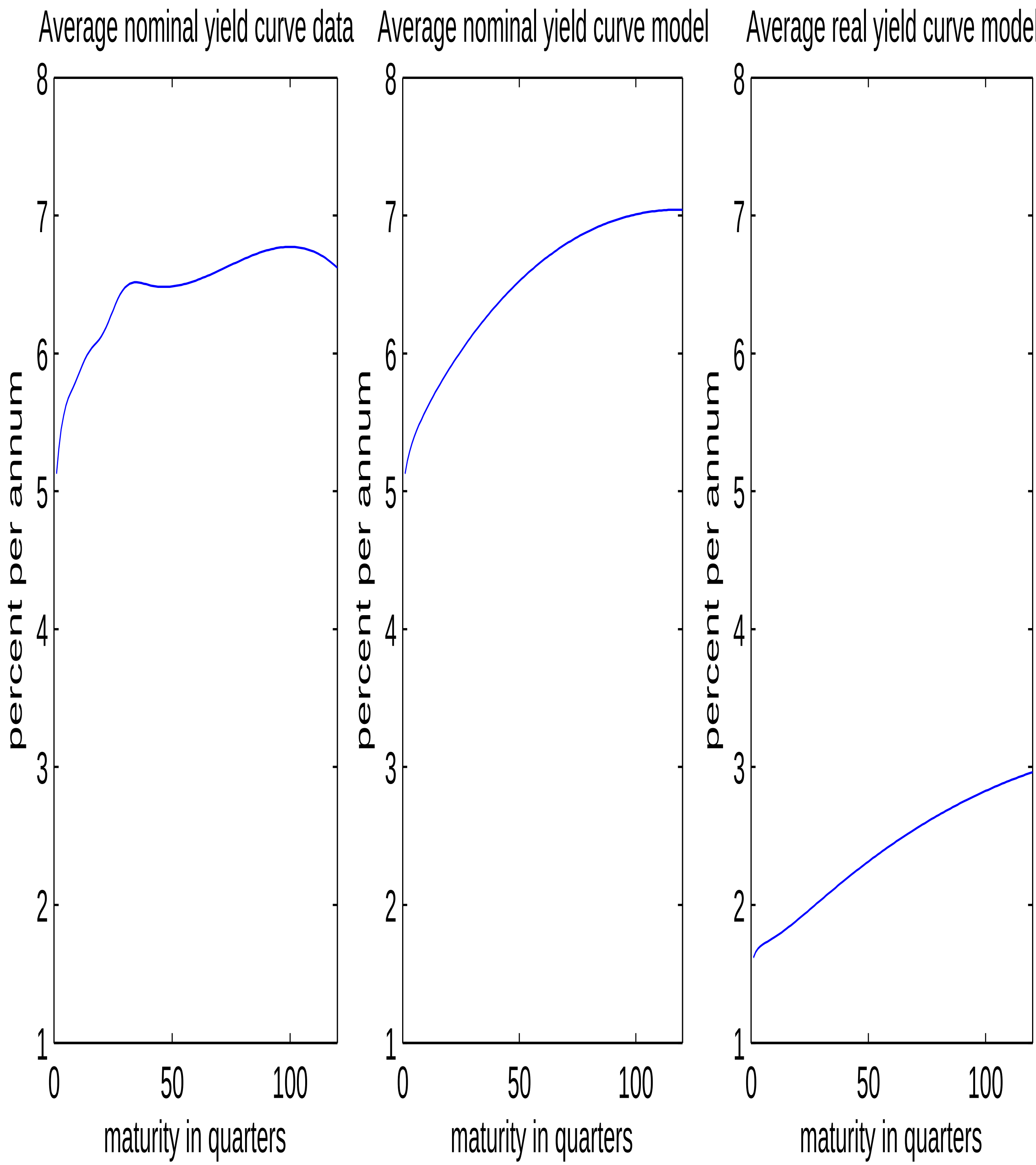

Figure 2: Dynamics of the Nominal Term Structure of Interest Rates

The figure plots the observed and model-implied 1-, 3-, 5-, 7-, 10-, and 20-year nominal bond yields.

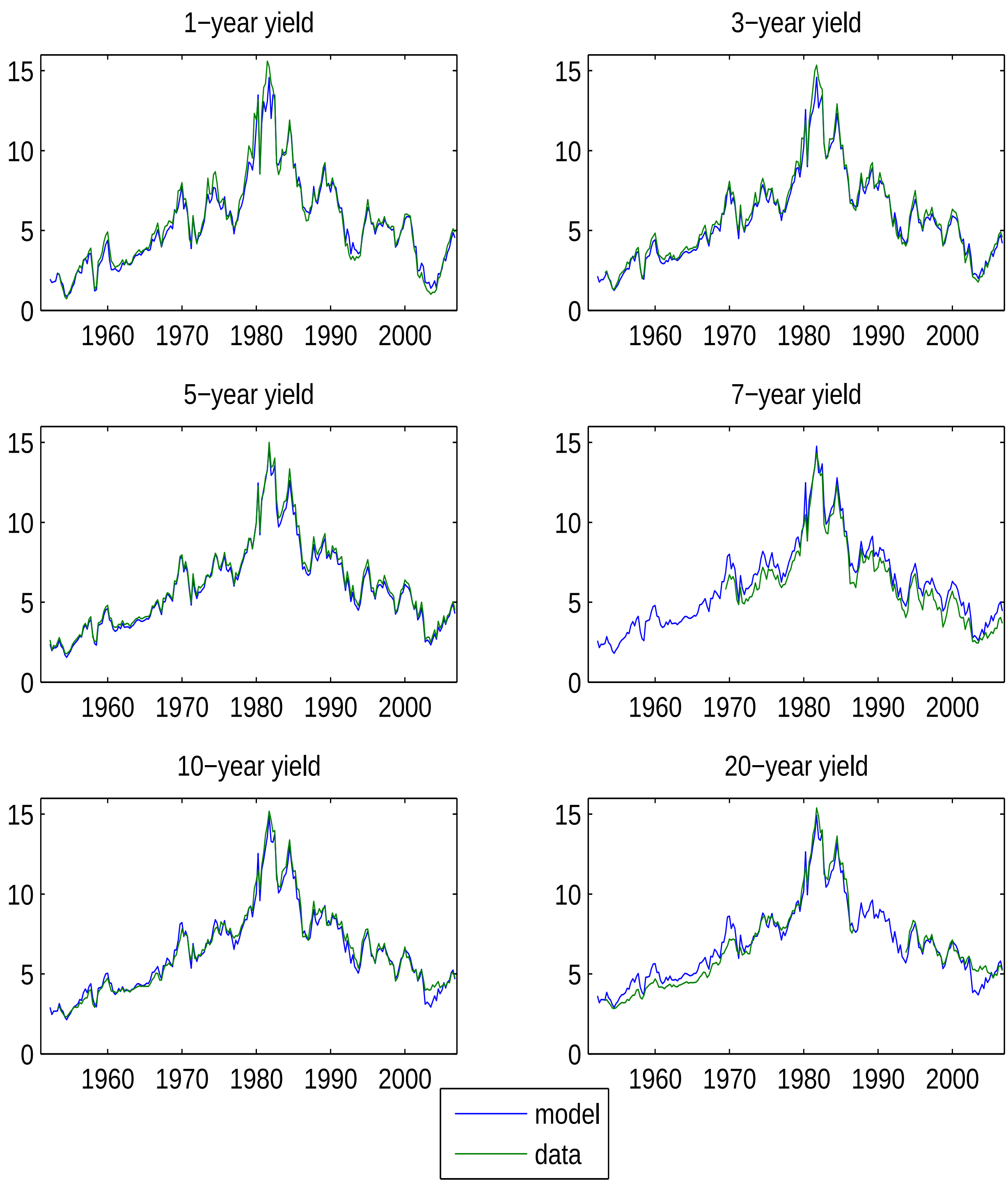

Figure 3: The Stock Market

The figure plots the observed and model-implied price-dividend ratio and expected excess return on the overall stock market.

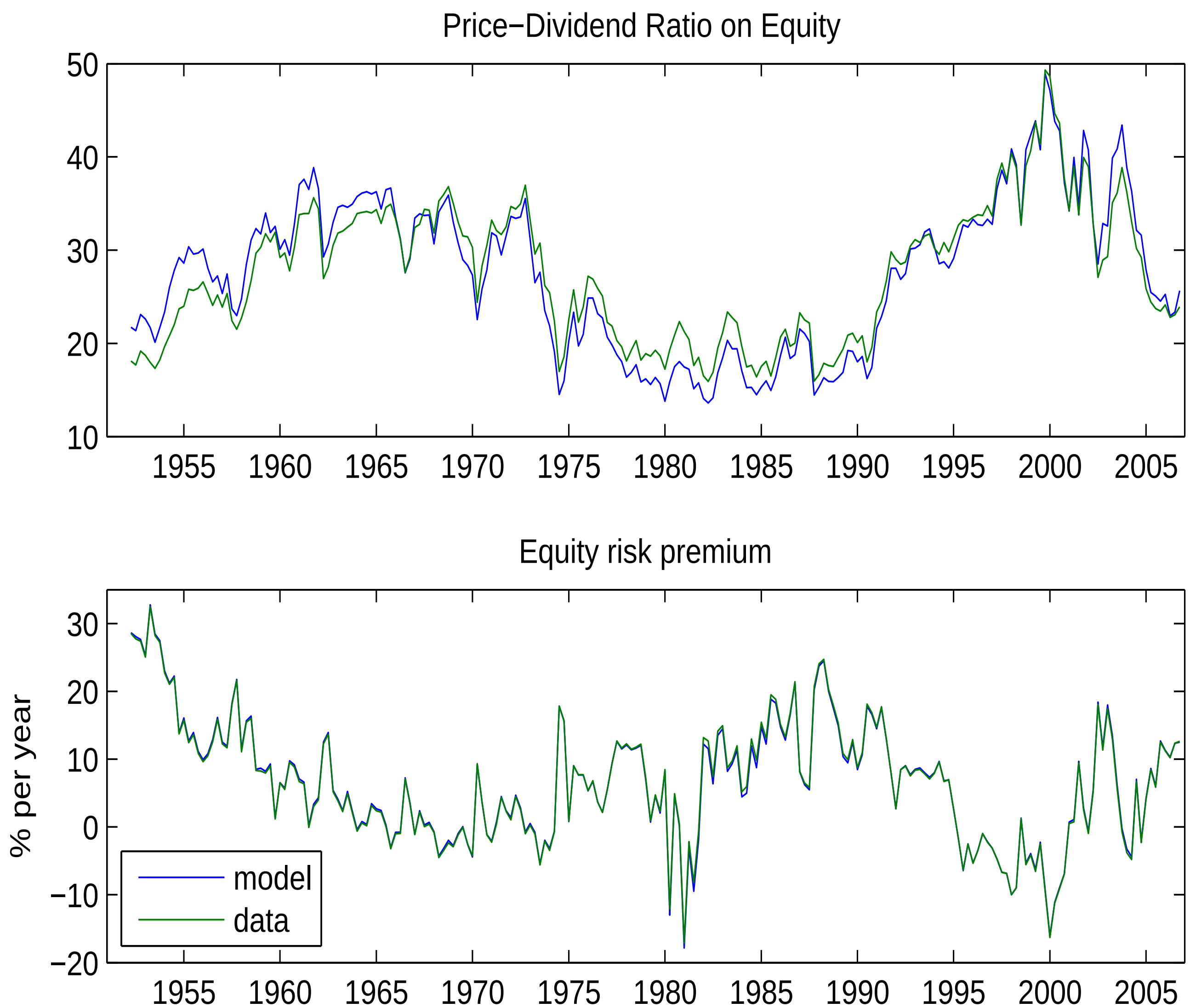

Figure 4: Decomposing the 5-Year Nominal Yield

The left panel decomposes the 5-year yield into the real 5-year yield, expected inflation over the next 5-years, and the inflation risk premium. The right panel decomposes the average nominal bond risk premium into the average real rate risk premium and inflation risk premium for maturities ranging from 1 to 120 quarters.

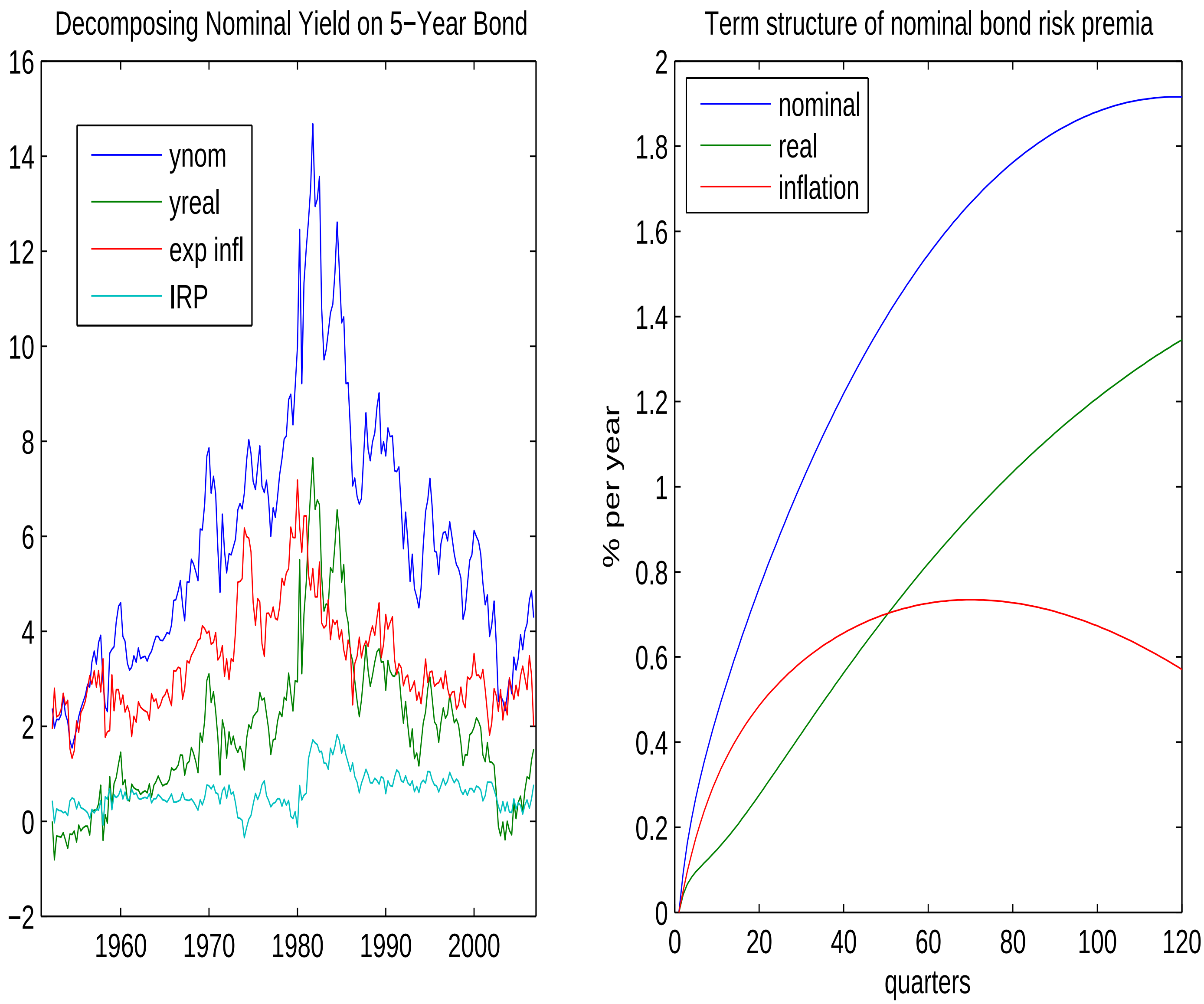

Figure 5: Factor Mimicking Portfolios

The left panel plots the expected excess return on the consumption growth factor mimicking portfolio. The right panel plots the expected excess return on the labor income growth factor mimicking portfolio

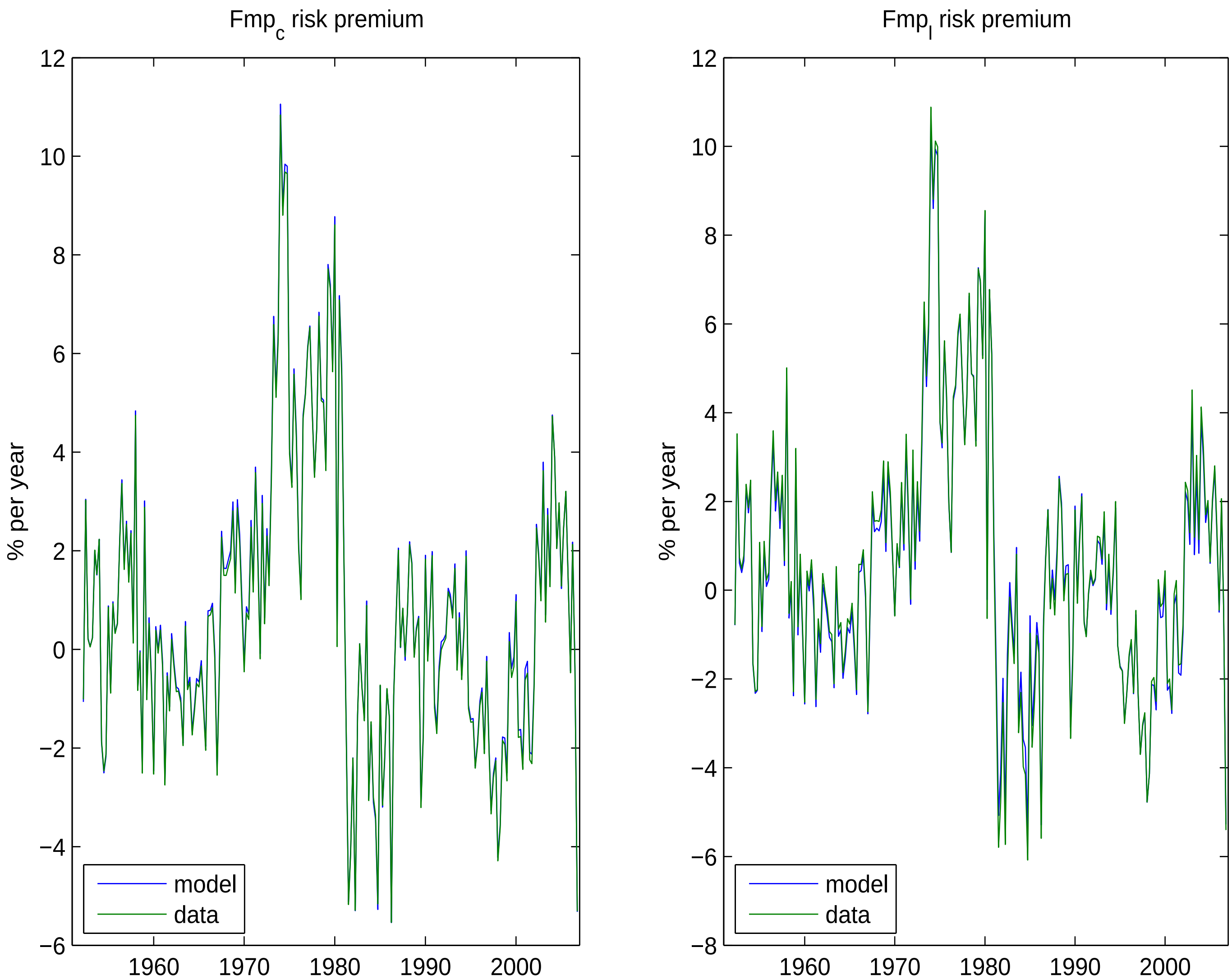

Figure 6: The Log Wealth-Consumption Ratio in the Data

The figure plots $\exp\{wc_t - log(4)\}$, where $wc_t$ is the quarterly log total wealth to total consumption ratio. The log wealth consumption ratio is given by $wc_t = A_0^c + (A_1^c)' z_t$. The coefficients $A_0^c$ and $A_1^c$ satisfy equations (15)-(16).

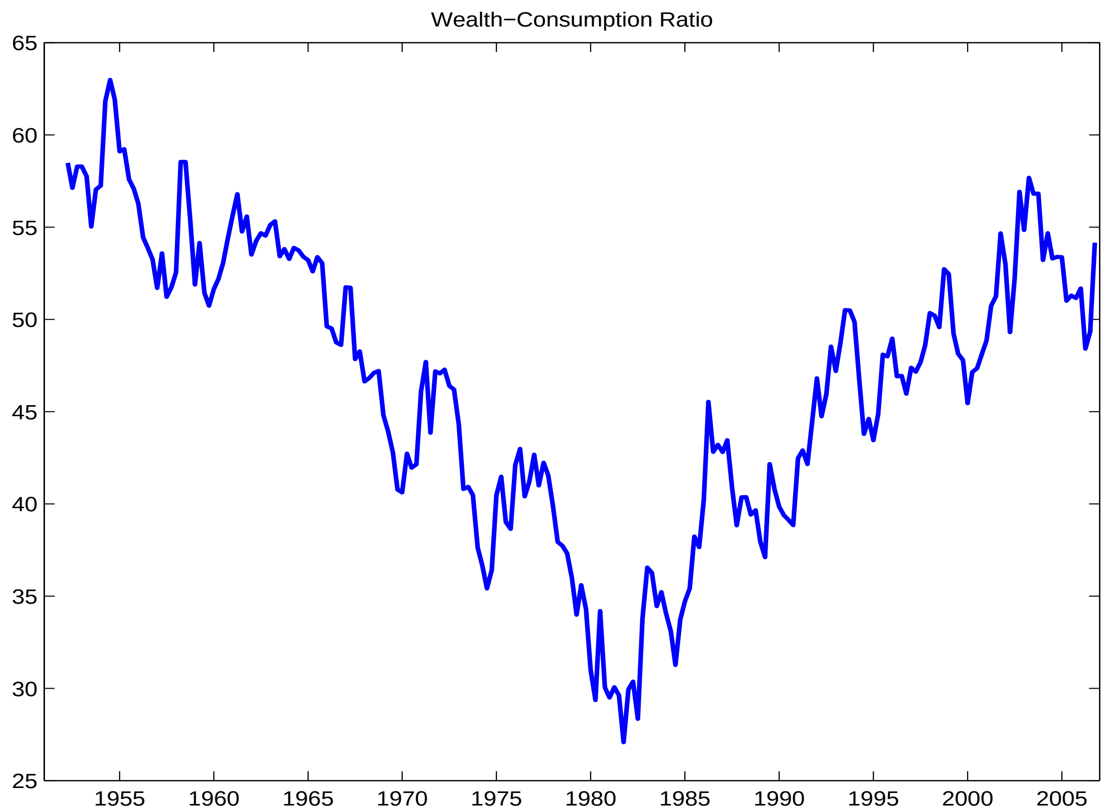

Figure 7: Decomposing the Yield on A Consumption Strip

The figure decomposes the yield on a consumption strip of maturity $\tau$, which goes from 1 to 120 quarters, into a real bond yield minus deterministic consumption growth and a cash-flow risk component: $\widetilde{y}_t^{\tau} = y_t(\tau) - \mu_c + \left(-\frac{A_t^{ccr}}{\tau} - \frac{B_t^{ccr}}{\tau} z_t\right) = y_t(\tau) - \mu_c + y_t^{ccr}$.