SOLVING A CLASS OF TRAVELING SALESMAN PROBLEMS ANALYTICALLY

RICHARD H. TASHJIAN, PH.D.
ADJUNCT PROFESSOR, STATISTICS GROUP
INFORMATION, OPERATIONS, & MANAGEMENT SCIENCES
STERN SCHOOL OF BUSINESS
NEW YORK UNIVERSITY

WORKING PAPER # SOR-2003-6
STATISTICS GROUP
INFORMATION, OPERATIONS, & MANAGEMENT SCIENCES
LEONARD N. STERN SCHOOL OF BUSINESS
NEW YORK UNIVERSITY

SOLVING A CLASS OF TRAVELING SALESMAN PROBLEMS ANALYTICALLY

RICHARD TASHJIAN

Abstract

This paper addresses a class of Traveling Salesman Problems (TSP) in which a route must be made to a series of nodes and return to the original location and attempts to solve it using analytical methods. The problem will be presented as a matrix of routes, much as might be seen in a national road map, excepting for there being in this case less entries. This familiar arrangement of routes will be cast as a matrix problem and solved using familiar formulations of quadratic forms. This solution, should it prove successful, can be contrasted with differing numeric or even iterative methods, such as the well-known Gomory cut method of solving integer linear programs. The advantage, should it prove tenable, will be theoretic in that a familiar and accessible form of quadratic forms can be readily applied to the problem and to similar cases.

The Problem Cast as a Matrix Problem

The series of nodes and arches comprising the network will be imagined, for ease of explanation, as the four bases of a baseball diamond, with 6 arches signifying an arch from every node to every other node. The matrix, clearly symmetric, will be designated by the character **R**.

$$\mathbf{R} = \begin{bmatrix} 0 & 3 & 4 & 12 \\ 3 & 0 & 5 & 10 \\ 4 & 5 & 0 & 8 \\ 12 & 10 & 8 & 0 \end{bmatrix}$$

An ij entry of this matrix is read as the distance from node i to j. So, for example, the 3,1 entry of the matrix, namely 4, says that the distance from node 3 to node 1 is 4. A diagram here, though difficult to format, would be appreciated. These numbers, as well, should abide by the Pythagorean Theorem, and it probably does not here. This does not affect the generality of the solution.

The distance of traveling from node 1 to 3 can be calculated in the following way. Let the following definition be made for easier expression. Let the vector **i** be a 4 vector with a 1 in the ith location. So, **3** will be the column vector (0 0 1 0). With this simple definition, a trip from node 1 to 3 can be calculated as follows.

$$\mathbf{1}^T*\mathbf{R}*\mathbf{3}$$

In this expression, the T is used to denote "Transpose," so that the first vector is converted from a column to a row. To then travel from node 3 to 2 would require the calculation…

$$\mathbf{3}^T*\mathbf{R}*\mathbf{2}$$

This cumbersome calculation can be consolidated in the following way. Let **Q** be a 16 X 16 block matrix with R on the diagonals. Therefore…

$$\mathbf{Q} = \begin{bmatrix} R & & & \\ & R & & \\ & & R & \\ & & & R \end{bmatrix}$$

Now, a complete trip to all of the nodes, and finishing at the starting node, could be found by using two different, but related, 16 X 1 vectors. An example is valuable here. To travel from 1 to 3 to 4 to 2 to 1, a complete circuit requires the following two vectors.

$$x = (1\,0\,0\,0\quad 0\,0\,1\,0\quad 0\,0\,0\,1\quad 0\,1\,0\,0)^T$$

$$y = (0\,0\,1\,0\quad 0\,0\,0\,1\quad 0\,1\,0\,0\quad 1\,0\,0\,0)^T$$

…where the T is again used to indicate that vectors are always column vectors.

The total cost of the trip is therefore….

$$\mathbf{x^T * Q * y}$$

It is useful to define a final set of expedient variables. Partition the variables x,y into 4 portions of 4 vectors. Without being concerned too much with precision but following intuition, the variables x,y can be rewritten as, in the above example…

x= (**1 3 4 2**)

y=(**3 4 2 1**)

Or, in general…

x=(**1 a b c**)

y=(**a b c 1**)

…where the four vectors a, b, c have a 1 in one location and 0's elsewhere.

Notice now that x and y are simply permutations of the other. Here another 16 X 16 matrix, a block permutation matrix, is needed. Therefore, let P be…

$$P = \begin{bmatrix} & & & I \\ I & & & \\ & I & & \\ & & I & \end{bmatrix}$$

…where I is the 4 X 4 identity matrix. Now it may be stated that x=Py and the above quadratic equal to $y^T * P^T * Q * y$. The objective function of the problem has therefore been stated as a quadratic form in the variable y, a particular 16 X 1 matrix.

Constraints

The object in this section is to state constraints on the variables y1 through y16 so that the vector y defines a route as described above. These constraints are surprisingly simple to yield such a complicated integer form. For example, simple constraints such as…

$Yi>=0$ …are implemented.

These constraints will prove to be unnecessary but are used here according to general formulation.

Now, dealing with each 4 yi's in turn, we have

$y1 + y2 + y3 + y4 = 1$

And

$y1*y2=0$
$y1*y3=0$
$y1*y4=0$
$y2*y3=0$
$y2*y4=0$
$y3*y4=0$

….and similarly for the others. The variables must be nonnegative, only one in four can be positive, and they must sum to 1. That indeed suffices.

The only other constraints insure that all of the nodes visited are different, so that the least of the routes cannot be chosen again. These constraints are similar.

$y1*y5=0$
$y1*y9=0$
$y1*y13=0$
$y5*y9=0$
$y5*y13=0$
$y9*y13=0$

…and similarly for the others. These constraints assure that the y vector will have four different nodes visited.

Solution

The solution is now readily attainable with the above quadratic form and standard Legrange multipliers using the given constraints. So then is this particular class of Traveling Salesman Problems apparently solvable by analytical methods as explained here.