Asymptotics for Duration-Driven Long Range Dependent Processes

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Abstract

We consider processes with second order long range dependence resulting from heavy tailed durations. We refer to this phenomenon as duration-driven long range dependence (DDLRD), as opposed to the more widely studied linear long range dependence based on fractional differencing of an *iid* process. We consider in detail two specific processes having DDLRD, originally presented in Taqqu and Levy (1986), and Parke (1999). For these processes, we obtain the limiting distribution of suitably standardized discrete Fourier transforms (DFTs) and sample autocovariances. At low frequencies, the standardized DFTs converge to a stable law, as do the standardized autocovariances at fixed lags. Finite collections of standardized autocovariances at a fixed set of lags converge to a degenerate distribution. The standardized DFTs at high frequencies converge to a Gaussian law. Our asymptotic results are strikingly similar for the two DDLRD processes studied. We calibrate our asymptotic results with a simulation study which also investigates the properties of the semiparametric log periodogram regression estimator of the memory parameter.

1 Introduction

The renewal-reward process of Taqqu and Levy (1986) and the error duration model of Parke (1999) are nonlinear models with long memory. Both models embody useful features not shared by traditional linear long-memory models such as ARFIMA. The renewal-reward process has been applied in long-memory analysis of internet traffic data (for a review, see Willinger *et. al.*, 2003). Liu (2000) has applied a modified version of this process to financial returns exhibiting simultaneous long memory and structural change in the volatility. The error duration model of Parke (1999) has drawn considerable recent attention among practitioners in finance and economics. By focusing on the duration of shocks rather than on fractional differencing of the shocks, the model provides an appealing paradigm for long memory in economic time series and

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in volatility of financial series. Both models exhibit a feature which may be viewed as structural change. In the Taqqu-Levy process, the value of the process stays constant at some random level throughout regimes of durations governed by a sequence of i.i.d. random variables with finite mean but infinite variance. In the model of Parke (1999), the process is written as a sum of present and past shocks, where shocks survive in the sum for durations governed by a long-tailed i.i.d. sequence. In both models, there is a one-to-one correspondence between the tail index of the i.i.d. duration sequence and the memory parameter of the process. Therefore, we will say that both of these models possess *duration-driven long range dependence* (DDLRD).

As these models gain increasingly widespread application, practitioners may feel that, if faced with data generated by a model having DDLRD, they can safely use the standard methods of data analysis and statistical inference for long-memory series. In particular, they may wish to examine the sample autocovariances, or to construct the log-periodogram regression estimator (GPH) of the memory parameter, due to Geweke and Porter-Hudak (1983), or to use the Gaussian semiparametric estimator (GSE) of Künsch (1987). Some caution may be in order here, however, since most of the existing theory assumes that the series is either Gaussian (see Robinson 1995a and Hurvich, Deo and Brodsky 1998 for GPH), linear in an i.i.d. sequence (see Velasco 2000 for GPH), linear in a Martingale difference sequence (See Robinson 1995b for GSE, Chung 2002 for autocovariances), or, in the case of volatility, that the observations can be transformed into a sum of linear series (see Deo and Hurvich 2001, Hurvich and Soulier 2002, for GPH applied to long memory stochastic volatility models). If the Taqqu-Levy and Parke models are to be widely accepted and used, it is necessary to build a theory for the currently-standard methodology of long-memory data analysis and inference that applies to such series. The present paper represents a first step in that direction. We will explore the asymptotic properties of the discrete Fourier transforms (DFTs) and sample autocovariances from both processes. Some of the results are surprising, and tend to confirm that caution was indeed warranted.

One surprising result we find is that both the sample autocovariances at a fixed lag and the DFT at a fixed Fourier frequency, if suitably standardized, have limiting non-Gaussian stable distributions. This implies that a data analysis based on examination of the sample autocovariances may be misleading. It also implies that data analytic methods that rely on the very low frequency behavior of the DFT of a long memory series will not have the same asymptotic properties as in the linear long-memory case. (See, e.g., Chen and Hurvich (2003 a,b) on fractional cointegration of linear processes). On a more positive note, but still surprisingly, we find for the DFT at the j'th Fourier frequency $x_j = 2\pi j/n$ where n is the sample size, that if j tends to ∞ sufficiently quickly, then the DFT is asymptotically normal. This indicates that the DFT at not-too-low frequencies has some robustness to the type of long-memory generating mechanism. It also suggests that standard estimation methods such as GPH and GSE may retain the same properties they are already known to have in the linear case, although some trimming of very low frequencies may be needed. Our theoretical results will be augmented with a Monte Carlo study, both to calibrate the finite-sample applicability of our theorems, and to briefly explore the properties of the GPH for models with DDLRD, a topic which we do not attempt to handle theoretically here.

The organization of the remainder of this paper is as follows. In Section 2, we review some of the existing theory on second order long memory processes, so as to contrast it with the theory we will develop for the DDLRD processes. In Section 3, we give the precise formulation of the Taqqu-Levy and Parke models, exhibit their autocovariance functions, present a proposition which shows that Parke's process is well defined only in the stationary case, and present some basic theory for these models. In particular, in Section 3.3 we consider the weak convergence of partial sums for both processes, and in Section 3.4 we consider asymptotics for the empirical process in the Taqqu-Levy case. In Section 4, we present the asymptotics for the discrete Fourier transforms for both series, treating the cases of low frequencies and high frequencies separately, as the limiting distribution is different in these two cases. In Section 5, we consider the asymptotics for the sample autocovariances of the Parke and Taqqu-Levy processes. Interestingly, the joint limiting distribution of a collection of standardized sample autocovariances at a fixed finite set of lags is degenerate. In Section 6 we present the results of a simulation study. In Section 7, we present some concluding remarks. Section 8 presents some lemmas useful for establishing our main results.

2 Second order long memory

We start by recalling some classical definitions and facts about long memory processes. A second order stationary process $X = \{X_t,\ t \in \mathbb{Z}\}$ is usually said to be long range dependent if its autocovariance function $\gamma(t) = \mathrm{cov}(X_0, X_t)$ is not absolutely summable. This definition is too wide to be useful. A more practical condition is that the autocovariance is regularly varying: there exist $H \in (1/2, 1)$ and a slowly varying function L such that

$$\gamma(t) = L(t)|t|^{2H-2}. \tag{2.1}$$

Under this condition, it holds that:

$$\lim_{n\to\infty} n^{-2H} L(n)^{-1} \mathrm{var}\left(\sum_{t=1}^{n} X_t\right) = 1/(2H(2H-1)). \tag{2.2}$$

A second order stationary process satisfying (2.2) will be referred to as a second order long memory process, and the coefficient H is the long memory parameter of the process, often referred to as the Hurst coefficient of the process X. We will hereafter use this terminology.

A weakly stationary process with autocovariance function satisfying (2.1) has a spectral density, *i.e.* there exists a function f such that

$$\gamma(t) = \int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mathrm{i}tx}\,dx.$$

The function f is the sum of the series

$$\frac{1}{2\pi}\sum_{t\in\mathbb{Z}} \gamma(t)\mathrm{e}^{\mathrm{i}tx},$$

which converges uniformly on the compact subsets of $[-\pi,\pi]\setminus\{0\}$ and in $L^1([-\pi,\pi],dx)$. It is then well known that the behaviour of the function f at zero is related to the rate of decay of γ. More precisely, if we assume in addition that L is ultimately monotone, we obtain the following Tauberian result:

$$\lim_{x\to 0} L(x)^{-1}x^{2H-1}f(x) = \pi^{-1}\Gamma(2H-1)\sin(\pi H). \tag{2.3}$$

(Cf. for instance Taqqu (2003), Proposition 4.1). The usual tools of statistical analysis of weakly stationary processes are the empirical autocovariance function, the discrete Fourier transform (DFT) and the periodogram. We will focus here on the DFT and periodogram ordinates of a sample $X_1,\dots,X_n$, defined as

$$d_{X,k} = (2\pi n)^{-1/2}\sum_{t=1}^{n} X_t \mathrm{e}^{\mathrm{i}tx_k}, \quad I_{X,k} = |d_{X,k}|^2,$$

where $x_k = 2k\pi/n$, $1\le k< n/2$ are the so-called Fourier frequencies. (Note that for clarity the index n is omitted from the notation). In the classical weakly stationary short memory case (when the autocovariance function is absolutely summable), it is well known that the periodogram is an asymptotically unbiased estimator of the spectral density. This is no longer true for second order long memory processes. Hurvich and Beltrao (1993) showed (in the case where the function L is continuous at zero but the extension is straightforward) that for any fixed positive integer k, there exists a constant $c(k,H)$ such that

$$\lim_{n\to\infty}\mathbb{E}[I_{X,k}/f(x_k)] = c(k,H).$$

The previous results are true for any second order long memory process. We now describe some weak convergence result that are valid for Gaussian or linear processes.

If X is a second order long memory Gaussian process, then $L(n)^{-1/2}n^{-H}\sum_{k=1}^{[nt]}X_k$ converges weakly to the fractional Brownian motion $B_H(t)$ which is the zero mean Gaussian process with covariance function given by:

$$\mathbb{E}[B_H(s)B_H(t)] = \frac{1}{2}\left(|s|^{2H} - |t-s|^{2H} + |t|^{2H}\right).$$

Here weak convergence is in the space $\mathcal{D}$ of right-continuous and left-limited (càdlàg) functions on $[0,\infty)$.

This result can be extended to a strict sense linear process, *i.e.* a process X for which there exist a sequence $(\epsilon_j)_{j\in\mathbb{Z}}$ of i.i.d. random variables with zero mean and finite variance, and a square summable sequence of real numbers $(a_j)_{j\in\mathbb{Z}}$ such that for all $t\in\mathbb{Z}$,

$$X_t = \sum_{j\in\mathbb{Z}} a_j\epsilon_{t-j}.$$

If $a_j = L(j)|j|^{H-3/2}$, then X is a second order long memory process with Hurst coefficient H, and $L(n)n^{-H}\sum_{k=1}^{[nt]}X_k$ converges weakly, in the sense of weak convergence of finite dimensional distribution, to the fractional Brownian motion $B_H(t)$. This can be proved easily by applying the Central Limit Theorem for linear processes of Ibragimov and Linnick (1971, Theorem 18.6.4). Weak convergence in the space $\mathcal{D}$ can also be proved. Cf. Gorodeckii (1977) or Lang and Soulier (2000). The classical example of such a long memory linear process is the ARFIMA(p,d,q) process, introduced by Granger and Joyeux (1980), Hosking (1981), whose Hurst coefficient is $H = 1/2 + d$.

For Gaussian and linear processes, a weak convergence result can also be obtained for the periodogram and the DFT ordinates. For any fixed j, $f(x_j)^{-1/2} d_{X,j}$ converges to a complex Gaussian distribution with dependent real and imaginary parts. Cf. Terrin and Hurvich (1994). Chen and Hurvich (2003 a,b), Walker (2000), and Lahiri (2003).

The asymptotic behaviour described above is different from the behaviour of weakly dependent processes, such as sequences of i.i.d. or strongly mixing random variables, whose partial sum process, renormalised by the usual rate $\sqrt{n}$, converges to the standard Brownian motion. But these long memory processes share with weakly dependent processes the Gaussian limit and the fact that weak limits and L^2 limits have consistent normalisations, in the sense that, if ξ_n denotes one of the statistics considered above, there exists a sequence v_n such that $v_n \xi_n$ converges weakly to a non degenerate distribution and $v_n^2 \mathbb{E}[\xi_n^2]$ converges to a positive limit (which is the variance of the asymptotic distribution).

In the sequel we define two second order stationary models, which possess properties (2.1) and (2.3), but whose weak limit behaviour is extremely different of that of Gaussian or linear models. In section 3 we define these models.

3 Formulation of the Models

3.1 The Taqqu-Levy Model

Let $\{T_k\}$ be i.i.d. positive integer-valued random variables with mean μ, in the domain of attraction of a stable distribution with tail index $\alpha \in (1,2)$, *i.e.* there exists a function L, slowly varying at infinity such that for all $n \geq 1$,

$$\mathbb{P}(T_1 \geq n) = L(n) n^{-\alpha}. \tag{3.1}$$

To avoid trivialities, we also assume that $\mathbb{P}(T_1 = 1) > 0$. Let S_0 be a non-negative integer-valued random variable, independent of the $\{T_k\}$, with probability distribution

$$P(S_0 = u) = \mu^{-1} P(T_k \geq u+1), \ u = 0, 1, \dots. \tag{3.2}$$

Let $\{W_k\}$ be i.i.d. random variables with $E[W_k] = 0$ and $\mathrm{var}[W_k] = \sigma_W^2 < \infty$. Assume that the $\{W_k\}$ are independent of S_0 and $\{T_k\}$. We observe a process denoted by $\{X_t\}$ for $t = 0, \dots, n-1$. The observed process is constant on regimes (intervals) determined by S_0 and the interarrival times T_k. The constant value on each regime is given by one of the $\{W_k\}$. The time between the start of the sample and the first change of regime is S_0, and the subsequent waiting times are

$T_1, T_2, \ldots$. The total time up to the end of the k'th regime ($k = 0, 1, \ldots$) is given by $S_{-1} \equiv -1$, S_0 and

$$S_k = S_0 + T_1 + \ldots + T_k, \ k = 1, 2, \ldots.$$

The observed process $\{X_t\}$ is given by W_k if t lies in the k'th regime, so that

$$X_t = \sum_{k=0}^{\infty} W_k \mathbf{1}_{\{S_{k-1} \le t < S_k\}}, \tag{3.3}$$

where $\mathbf{1}_A$ is the indicator function of the set A. Let M_n be the counting process associated with the renewal process $\{S_0, S_1, \ldots\}$, *i.e.* a non-negative integer-valued random variable denoting the total number of regime changes in the series before the time $n-1$:

$$M_n = k \Leftrightarrow S_{k-1} \le n < S_k.$$

The renewal process $\{S_0, S_1, \ldots\}$ is called a stationary renewal process, in the sense that the counting process M_n has stationary increments, whence the following result.

Proposition 3.1. *The process X defined by* (3.3) *is strictly stationary with zero mean and covariances*

$$\mathrm{cov}(X_0, X_r) = \sigma_W^2 \mathbb{P}(S_0 \ge r) = \mu^{-1} \sigma_W^2 \mathbb{E}[(T_1 - r)\mathbf{1}_{\{T_1 \ge r\}}].$$

If (3.1) holds with $1 < \alpha < 2$ and L ultimately monotone, then X is second order long memory with Hurst coefficient $H = (3-\alpha)/2$ and spectral density f satisfying

$$\lim_{x \to 0} L(1/x)^{-1} x^{2H-1} f(x) = \frac{\sigma_W^2}{2\pi(1-H)\mu} \, \Gamma(2H-1) \sin(\pi H).$$

3.2 The Parke Model

Let $(\epsilon_t)_{t \in \mathbb{Z}}$ be a sequence of i.i.d. random variables and $(n_s)_{s \in \mathbb{Z}}$ be a sequence of i.i.d. non-negative integer valued random variables which is independent of $(\epsilon_t)_{t \in \mathbb{Z}}$. For $s \in \mathbb{Z}$, define

$$g_{s,t} = 1 \Leftrightarrow s \le t \le s + n_s.$$

Parke's error duration process is then defined as:

$$X_t = \sum_{s \le t} g_{s,t} \epsilon_s,$$

Let N be a generic random variable with the same distribution as the n_s, and define

$$p_k := \mathbb{P}(N \ge k) \quad k \ge 0.$$

$(p_k)_{k \ge 0}$ is then a non increasing sequence such that $p_0 = 1$ and $\lim_{k \to \infty} p_k = 0$.

Parke (1999) does not discuss the existence of this process. In his main result, he assumes that it is well defined and second order stationary. Since the terms in the sum defining the process are not vanishing, by well defined we mean that the sum is almost surely finite. We now give a necessary and sufficient condition for the process X to be well defined.

Proposition 3.2. *Parke's process is well defined if and only if* $\mathbb{E}[N] < \infty$. *In that case it is strictly stationary. If moreover* ϵ_0 *has mean* μ_ϵ *and variance* σ_ϵ^2, *then Parke's process has mean* $\mu_\epsilon(1+\mathbb{E}[N])$, *finite variance and covariances*

$$\operatorname{cov}(X_0, X_r) = \sigma_\epsilon^2 \sum_{j \geq r} p_j = \sigma_\epsilon^2\, \mathbb{E}[(N+1-r)\mathbf{1}_{\{N \geq r\}}] = \sigma_\epsilon^2 \sum_{k=r}^{\infty} p_k.$$

If the survival probabilities p_k *are regularly varying with index* $\alpha \in (1,2)$, i.e. *if they satisfy*

$$p_j = \mathbb{P}(N \geq j) = L(j) j^{-\alpha}, \quad j \geq 1, \tag{3.4}$$

where L *is slowly varying and ultimately monotone at infinity, then Parke's error duration model* X *exhibits second order long memory with Hurst coefficient* $H = (3-\alpha)/2$ *and its spectral density* f *satisfies*

$$\lim_{x \to 0} L(1/x)^{-1} x^{2H-1} f(x) = \frac{\sigma_\epsilon^2}{2\pi(1-H)}\, \Gamma(2H-1)\sin(\pi H).$$

Proof. A necessary and sufficient condition for this process to be well defined is that almost surely, for all $t \in \mathbb{Z}$,

$$\inf\{s \in \mathbb{Z}, s + n_s \geq t\} > -\infty.$$

Since the random variables n_s are i.i.d. with the same distribution as N, by Borel-Cantelli's Lemma this condition is equivalent to

$$\sum_{s \leq t} \mathbb{P}(n_s \geq t-s) = \sum_{k=0}^{\infty} \mathbb{P}(N \geq k) < \infty.$$

Hence the necessary and sufficient condition for Parke's error duration process to be well defined is $\mathbb{E}[N] < \infty$. The expression of the autocovariance function is proved in Parke (1999) Proposition 1. □

3.3 Invariance principle

Let X denote either Parke's or Taqqu-Levy's process. The next proposition shows that although the process X is second order stationary and its autocovariance function exhibits long range dependence, the partial sum process of X converges to a stable Lévy process with independent increment, which implies that its behaviour mimics that of a sum of i.i.d. heavy tailed random variables. In the case of the Taqqu-Levy Process, it is stated without proof in Taqqu and Levy (1986); it can also be seen as a particular case of Theorem 2 in Mikosch *et al.* (2002).

Proposition 3.3. *Assume that (3.1) and* (3.4) *hold with* $1 < \alpha < 2$. *Denote* $\ell(n) = n^{-1/\alpha} \inf \{t > 0 : \mathbb{P}(U > t) < n^{-1}\}$ *with* $U = T_1$ *for the Taqqu-Levy process and* $U = N$ *for the Parke* *process. Then the finite dimensional distributions of $\ell(n)^{-1}n^{-1/\alpha}\sum_{k=1}^{[nt]}X_k$ converge weakly to those of the α-stable Levy process Λ_α with characteristic function*

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}u\Lambda_\alpha(t)}] = \exp\left\{-t\,|u|^\alpha\mu^{-1}\mathbb{E}[|\xi|^\alpha]\Gamma(1-\alpha)\cos(\pi\alpha/2)(1-\mathrm{i}\beta\mathrm{sign(u)}\tan(\pi\alpha/2))\right\}, \qquad (3.5)$$

with $\beta = (\mathbb{E}[\xi_+^\alpha]-\mathbb{E}[\xi_-^\alpha])/\mathbb{E}[|\xi|^\alpha]$ and $\xi = W_1$ in the case of Taqqu-Levy's process and $\xi=\epsilon_1$ and $\mu=1$ in the case of Parke's process.

Proof in the case of Parke's process. For any real numbers x,y, denote $x_+=\max(x,0)$, $x_-=\max(-x,0)$, $x\vee y=\max(x,y)$ and $x\wedge y=\min(x,y)$. Recall that $g_{s,t}=1$ if $s\le t\le s+n_s$ and 0 otherwise. Hence, we can write:

$$\begin{aligned}\sum_{k=1}^n X_k &= \sum_{k=1}^n\sum_{s\le k} g_{s,k}\epsilon_s = \sum_{s\le n}\sum_{k=1\vee s}^{(s+n_s)_+\wedge n} g_{s,k}\epsilon_s \\ &= \sum_{s\le 0}\{(s+n_s)_+\wedge n\}\epsilon_s + \sum_{s=1}^n\{(s+n_s)\wedge n - s+1\}\epsilon_s = U_n+V_n.\end{aligned}$$

Since $\sum_{s\le 0}\mathbb{P}(s+n_s>0)=\sum_{k\ge 0}\mathbb{P}(N>k)=\mathbb{E}[N]<\infty$, the number of terms in the sum $U=\sum_{s\le 0}(s+n_s)_+\epsilon_s$ is almost surely finite. Hence U_n converges almost surely to U and $U_n=O_P(1)$. We now split V_n into three terms: $V_n=V_{1,n}-V_{2,n}+V_{3,n}$, with

$$V_{1,n}=\sum_{s=1}^n(n_s+1)\epsilon_s,\quad V_{2,n}=\sum_{s=1}^n(n_s+1)\mathbf{1}_{\{s+n_s>n\}}\epsilon_s,$$
$$\text{and}\quad V_{3,n}=\sum_{s=1}^n(n-s+1)\mathbf{1}_{\{s+n_s>n\}}\epsilon_s.$$

Since the sequences (n_s) and (ϵ_s) are i.i.d. and independent of each other, $V_{3,n}$ has the same distribution as $W_n=\sum_{k=1}^n k\mathbf{1}_{\{n_k\ge k\}}\epsilon_k$. Since $\sum_{k=1}^\infty\mathbb{P}(n_k\ge k)<\infty$, by Borel-Cantelli's Lemma, almost surely there exists an integer K such that for all $k>K$, $n_k<k$. Hence W_n converges almost surely to $\sum_{k=1}^\infty k\mathbf{1}_{\{n_k\ge k\}}\epsilon_k$, which is almost surely a finite sum. This implies that $V_{3,n}=O_P(1)$.

Similarly, $V_{2,n}$ has the same distribution as $\sum_{k=1}^n n_k\mathbf{1}_{\{n_k\ge k\}}\epsilon_k$, which converges almost surely to the almost surely finite sum $\sum_{k=1}^\infty n_k\mathbf{1}_{\{n_k\ge k\}}\epsilon_k$. Hence $V_{2,n}=O_P(1)$.

Under assumption (3.4), N is in the domain of attraction of an α-stable law. Since $\mathbb{E}[\epsilon_0^2]<\infty$, by Breiman's (1965) theorem, $(n_s+1)\epsilon_s$ is an i.i.d. sequence in the domain of attraction of an α-stable law. Thus we obtain that $n^{-1/\alpha}\ell(n)^{-1}V_{1,n}$ converges weakly to the stable distribution with characteristic function given by (3.5) (cf. for instance Embrechts *et al.* (1997), Proposition 2.2.13). The convergence of finite dimensional distribution is obtained similarly. □

3.4 Empirical process of Taqqu-Levy's process

In the case of Taqqu-Levy's process, the invariance principle can be straightforwardly extended to an invariance principle for instantaneous functions of the process: if ϕ is a measurable function such that $\mathbb{E}[\phi^2(W_1)] < \infty$ and $\mathbb{E}[\phi(W_1)] = 0$, then the finite dimensional distributions of $\ell(n)^{-1}n^{-1/\alpha}\sum_{k=1}^{[nt]}\phi(X_k)$ converge weakly to those of an α-stable Levy process, where ℓ is the same slowly varying function as in proposition 3.3. For Parke's process, we conjecture that this is true for polynomial functions. It is actually shown in the case $\phi(x) = x^2 - \mathbb{E}[\epsilon_1^2]$ in Theorem 5.1, and a similar proof would probably work in the case of a higher order polynomial.

In the special case of an indicator function, we obtain the usual interval-indexed empirical process:

$$\hat{F}_n(x) = \frac{1}{n}\sum_{k=1}^{n}\mathbf{1}_{\{X_k \le x\}}.$$

Let $F_W(x) = \mathbb{P}(W_1 \le x)$ be the distribution function of W_1. Then $\hat{F}_n$ is an estimator of F_W and we have the following convergence result.

Theorem 3.1. *The finite dimensional distributions of the process $\ell(n)^{-1}n^{1-1/\alpha}(\hat{F}_n - F_W)$ converges weakly to those of the process $\Lambda_\alpha(F(\cdot))$, where Λ_α is the stable Levy process with characteristic function*

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}u\Lambda_\alpha(t)}] = \exp\left\{-t\,|u|^\alpha\mu^{-1}\Gamma(1-\alpha)\cos(\pi\alpha/2)(1-\mathrm{i}\,\mathrm{sign(u)}\tan(\pi\alpha/2))\right\},$$

and $\ell(n) = n^{1/\alpha}\inf\left\{t > 0 :\ \mathbb{P}(T_1 > t) < n^{-1}\right\}$.

Sketch of Proof. Neglecting the first and last renewal periods, write

$$\ell(n)^{-1}n^{1-1/\alpha}\{\hat{F}_n(x) - F_W(x)\} \approx \ell(n)^{-1}n^{-1/\alpha}\sum_{k=1}^{M_n}\mathbf{1}_{\{W_k \le x\}}(T_k - \mu)$$
$$+ \mu\ell(n)^{-1}n^{-1/\alpha}\sum_{k=1}^{M_n}\left\{\mathbf{1}_{\{W_k\le x\}} - F_W(x)\right\} + (\mu\ell(n)^{-1}n^{-1/\alpha}M_n - 1)F_W(x).$$

By Lemma 8.1, the finite dimensional distributions of $\ell(n)n^{-1/\alpha}\sum_{k=1}^{M_n}\mathbf{1}_{\{W_k\le x\}}(T_k - \mu)$ are asymptotically equivalent to those of $\ell(n)n^{-1/\alpha}\sum_{k=1}^{n/\mu}\mathbf{1}_{\{W_k\le x\}}(T_k-\mu)$ which converge to those of $\Lambda_\alpha(F(x))$. Since the variables W_k are independent of M_n, we have, by the renewal theorem,

$$\mathbb{E}\left[\left(\sum_{k=1}^{M_n}\left\{\mathbf{1}_{\{W_k\le x\}} - F_W(x)\right\}\right)^2\right] = F_W(x)\{1 - F_W(x)\}\mathbb{E}[M_n] = O(n).$$

Thus, $\sum_{k=1}^{M_n}\left\{\mathbf{1}_{\{W_k\le x\}} - F_W(x)\right\} = o_P(\ell(n)n^{1/\alpha})$. □

4 Asymptotics for the DFTs

Define

$$D_{n,j} = \sum_{t=0}^{n-1} X_t \, \mathrm{e}^{\mathrm{i}tx_j},$$

where X denotes either Taqqu-Levy's or Parke's process.

4.1 Low frequencies

Proposition 4.1. *Define $d_{n,j} = \sum_{k=1}^{[n/\mu]} \eta_k \mathrm{e}^{\mathrm{i}\mu k x_j}$ with $\eta_k = T_k W_k$ for the Taqqu-Levy process and $\eta_k = n_k \epsilon_k$ and $\mu = 1$ for Parke's process. If (3.1) and (3.4) hold and if $j \le n^\rho$ for some $\rho \in (0, 1 - 1/\alpha)$, then $\ell(n)^{-1} n^{-1/\alpha}(D_{n,j} - d_{n,j}) = o_P(1)$, where ℓ is defined as in Proposition 3.3.*

Since in both cases η_k belongs to the domain of attraction of an α-stable law, Proposition 4.1 implies if $j \le n^\rho$, then $\ell(n)^{-1} n^{-1/\alpha} D_{n,j}$ converges to a stable distribution. In the case of fixed frequencies, we can describe more precisely the asymptotic distribution.

Theorem 4.1. *Let $j_1 < \cdots < j_q$ be q fixed positive integers. Let ℓ be defined as in Proposition 3.3. Then $\ell(n)^{-1} n^{-1/\alpha}(D_{n,j_1}, \dots, D_{n,j_q})$ converge in law to the complex α-stable vector $\left(\int_0^1 \mathrm{e}^{2\mathrm{i}\pi j_1 s} d\Lambda_\alpha(s), \dots, \int_0^1 \mathrm{e}^{2\mathrm{i}\pi j_q s} d\Lambda_\alpha(s)\right)$, where Λ_α is the α-stable Levy process with characteristic function given by (3.5).*

Proof of Proposition 4.1 in the case of Parke's process.

$$\begin{aligned} D_{n,j} &:= \sum_{t=1}^{n} X_t \mathrm{e}^{\mathrm{i}tx_j} = \sum_{t=1}^{n} \sum_{s \le t} g_{s,t} \mathrm{e}^{\mathrm{i}tx_j} \epsilon_s \\ &= \sum_{s \le 0} \sum_{t=1}^{(s+n_s)_+ \wedge n} \mathrm{e}^{\mathrm{i}tx_j} \epsilon_s + \sum_{s=1}^{n} \sum_{t=s}^{(s+n_s) \wedge n} \mathrm{e}^{\mathrm{i}tx_j} \epsilon_s =: U_{n,j} + V_{n,j}. \end{aligned}$$

As in the proof of Proposition 3.3, the sum defining $U_{n,j}$ is almost surely finite. If $j/n \to 0$, then $U_{n,j}$ converges almost surely to the random variable $U = \sum_{s \le 0} (s + n_s)_+ \epsilon_s$. Split now $V_{n,j}$

into three terms: $V_{n,j} = W_{n,j} - R_{n,j} + T_{n,j}$, with

$$W_{n,j} = \sum_{s=1}^{n} \left(\sum_{t=s}^{(s+n_s)} \mathrm{e}^{\mathrm{i}tx_j} \right) \epsilon_s, \tag{4.1}$$

$$R_{n,j} = \sum_{s=1}^{n} \sum_{t=s}^{(s+n_s)} \mathrm{e}^{\mathrm{i}tx_j} \mathbf{1}_{\{s+n_s>n\}} \epsilon_s,$$

$$T_{n,j} = \sum_{s=1}^{n} \sum_{t=s}^{n} \mathrm{e}^{\mathrm{i}tx_j} \mathbf{1}_{\{s+n_s>n\}} \epsilon_s.$$

Consider first R_n. Since the sequences (n_s) and (ϵ_s) are i.i.d. and independent of each other, we have:

$$R_{n,j} = \sum_{s=1}^{n} \mathrm{e}^{\mathrm{i}sx_j} \frac{1-\mathrm{e}^{\mathrm{i}(n_s+1)x_j}}{1-\mathrm{e}^{\mathrm{i}x_j}} \mathbf{1}_{\{s+n_s>n\}} \epsilon_s \stackrel{(d)}{=} \sum_{k=1}^{n} \mathrm{e}^{-\mathrm{i}(k-1)x_j} \frac{1-\mathrm{e}^{\mathrm{i}(n_k+1)x_j}}{1-\mathrm{e}^{\mathrm{i}x_j}} \mathbf{1}_{\{n_k \geq k\}} \epsilon_k,$$

where $\stackrel{(d)}{=}$ denotes equality of laws. Since almost surely there is only a finite number of indices k such that $n_k \geq k$, if $j/n \to 0$, this last sum converges almost surely to $\sum_{k=1}^{\infty} (n_k+1) \mathbf{1}_{\{n_k \geq k\}} \epsilon_k$. Hence $R_{n,j} = O_P(1)$. Similarly, $T_{n,j}$ has the same distribution as

$$\sum_{k=1}^{n} \sum_{t=n-k+1}^{n} \mathrm{e}^{\mathrm{i}tx_j} \mathbf{1}_{\{n_k \geq k\}} \epsilon_k = \sum_{k=1}^{n} \sum_{u=0}^{k-1} \mathrm{e}^{-\mathrm{i}ux_j} \mathbf{1}_{\{n_k \geq k\}} \epsilon_k.$$

If $j/n \to 0$, this last term converges to $\sum_{k=1}^{\infty} k \mathbf{1}_{\{n_k \geq k\}} \epsilon_k$, which is an almost surely finite sum, whence $T_{n,j} = O_P(1)$. In conclusion, as long as $j/n \to 0$, $W_{n,j}$ is the leading term in the decomposition of $D_{n,j}$. Consider now $W_{n,j}$. It can be written as

$$\begin{aligned} W_{n,j} &= \sum_{s=1}^{n} \mathrm{e}^{\mathrm{i}sx_j} \frac{1-\mathrm{e}^{\mathrm{i}(n_s+1)x_j}}{1-\mathrm{e}^{\mathrm{i}x_j}} \epsilon_s = \sum_{s=1}^{n} \mathrm{e}^{\mathrm{i}sx_j} \mathrm{e}^{\mathrm{i}n_s x_j/2} \frac{\sin((n_s+1)x_j/2)}{\sin(x_j/2)} \epsilon_s \\ &= \sum_{s=1}^{n} \mathrm{e}^{\mathrm{i}sx_j} \frac{\sin((n_s+1)x_j/2)}{\sin(x_j/2)} \epsilon_s + \sum_{s=1}^{n} \mathrm{e}^{\mathrm{i}sx_j} \left(\mathrm{e}^{\mathrm{i}n_s x_j/2} - 1 \right) \frac{\sin((n_s+1)x_j/2)}{\sin(x_j/2)} \epsilon_s \\ &= d_{n,j} + \sum_{s=1}^{n} \mathrm{e}^{\mathrm{i}sx_j} \left(\frac{\sin((n_s+1)x_j/2)}{\sin(x_j/2)} - n_s - 1 \right) \epsilon_s \\ &+ \sum_{s=1}^{n} \mathrm{e}^{\mathrm{i}sx_j} \left(\mathrm{e}^{\mathrm{i}n_s x_j/2} - 1 \right) \frac{\sin((n_s+1)x_j/2)}{\sin(x_j/2)} \epsilon_s = d_{n,j} + r_{n,j}. \end{aligned}$$

To deal with the remainder terms, we use the following bounds: there exists a constant C such that for all $u \in \mathbb{R}$ and for all $v \in (0,1)$,

$$\left| \mathrm{e}^{\mathrm{i}u} - 1 \right| \leq C(|u| \wedge 1)$$

$$\left| \frac{\sin(uv)}{\sin(v)} - u \right| \leq C|u|(|uv| \wedge 1) + |u|v^2.$$

For $p \in (1, \alpha)$, applying these bounds and the moment bound for independent zero mean random variables with finite p-th moment (cf. Petrov (1995), addendum 2.6.20), we have:

$$\mathbb{E}[|r_{n,j}|^p] \le C \sum_{s=1}^{n} \mathbb{E}[(n_s)^p((n_s j/n) \wedge 1)^p] = Cn\mathbb{E}[N^p((Nj/n) \wedge 1)^p] + Cn(j/n)^{2p}\mathbb{E}[N^p].$$

Let us compute $\mathbb{E}[N^p((Nj/n) \wedge 1)^p]$ for any $p > 1$.

$$\mathbb{E}[N^p((Nj/n) \wedge 1)^p] = (j/n)^p \sum_{k=1}^{n/j} k^{2p}\, \mathbb{P}(N = k) + \sum_{k=n/j}^{\infty} k^p\, \mathbb{P}(N = k) \le C(j/n)^{\alpha - p} L(n).$$

Hence, for any $p \in (1, \alpha)$, $\mathbb{E}[|r_{n,j}|] = O(L(n) n^{1+(1-\alpha)/p} j^{\alpha/p-1})$. If $j \le n^\rho$ for some $\rho \in (0, 1 - 1/\alpha)$, then p can be chosen close enough to α so that $\lim_{n\to\infty} h(n) n^{-1/\alpha} \mathbb{E}[|r_{n,j}|] = 0$, for any slowly varying function h. □

Proof of Proposition 4.1 in the case of Taqqu-Levy's process. For clarity, we denote in this proof $x_{n,j} = 2\pi j/n$. By summing over each regime separately, we can express $D_{n,j}$ as

$$\begin{aligned} D_{n,j} &= W_0 \sum_{t=0}^{S_0 - 1} \mathrm{e}^{itx_{n,j}} + \sum_{k=1}^{M_n} W_k \sum_{t=S_{k-1}}^{S_k - 1} \mathrm{e}^{\mathrm{i}tx_{n,j}} + W_{M_n+1} \sum_{t=S_{M_n}}^{n} \mathrm{e}^{\mathrm{i}tx_{n,j}} \\ &= r_{1,n,j} + w_{M_n,n,j} + r_{2,n,j}, \end{aligned}$$

where we have defined:

$$\begin{aligned} r_{1,n,j} &= W_0 \exp\{\mathrm{i}(S_0 - 1)x_{n,j}/2\} \frac{\sin(S_0 x_{n,j}/2)}{\sin(x_{n,j}/2)}, \\ r_{2,n,j} &= W_{M_n+1} \exp\{\mathrm{i}\{S_{M_n} + (n - S_{M_n})/2\}x_{n,j}\} \frac{\sin(\{n - S_{M_n} + 1\}x_{n,j}/2)}{\sin(x_{n,j}/2)}, \\ w_{m,n,j} &= \sum_{k=1}^{m} W_k \sum_{t=S_{k-1}}^{S_k - 1} \mathrm{e}^{\mathrm{i}tx_{n,j}} = \sum_{k=1}^{m} \mathrm{e}^{\mathrm{i}\{S_{k-1} + \frac{T_k - 1}{2}\}x_{n,j}} \frac{\sin(T_k x_{n,j}/2)}{\sin(x_{n,j}/2)} W_k. \end{aligned}$$

Obviously, $|r_{1,n,j}| \le |W_0| S_0$, hence $r_{1,n,j} = O_P(1)$, uniformly with respect to $j \le n/2$. To deal with $r_{2,n,j}$, note that $n - S_{M_n}$ is the forward recurrence time of the stationary renewal process $(S_n)_{n\ge 0}$, hence its marginal distribution is constant and is equal to that of S_0 (cf. Resnick (1992), Theorem 3.9.1). Thus, for $q < \alpha - 1$, $\mathbb{E}[|r_{2,n,j}|^q] \le \mathbb{E}[|W_0|^q]\mathbb{E}[S_0^q] < \infty$. $r_{2,n,j}$ is also $O_P(1)$, uniformly with respect to $j \le n/2$. Applying Lemma 8.1, we obtain that $w_{M_n,n,j} - w_{[n/\mu],n,j} = o_P(n^{-1/\alpha} h(n))$, uniformly with respect to the sequence j and for any slowly varying function h.

We now prove that $h(n) n^{-1/\alpha}(w_{[n/\mu],n,j} - d_{n,j}) = o_P(1)$. Define $\tilde{w}_{m,n,j} = \sum_{k=1}^{m} \mathrm{e}^{\mathrm{i}(S_{k-1} - 1/2)x_{n,j}}$ $T_k W_k$. Applying Lemma 8.2 with $m = [n/\mu]$, $H(u, v) = \mathrm{e}^{\mathrm{i}uv/2} \frac{\sin(uv/2)}{\sin(v/2)}$ and $\zeta_{n,k} = \mathrm{e}^{\mathrm{i}(S_{k-1} - 1/2)x_{n,j}}$, we obtain:

$$w_{[n/\mu],n,j} - \tilde{w}_{[n/\mu],n,j} = o_P(n^{1/\alpha} h(n)). \tag{4.2}$$

Define $\hat{w}_{m,n,j} = \sum_{k=1}^{m} \mathrm{e}^{\mathrm{i}\{(k-1)\mu-1/2\}x_{n,j}} T_k W_k$. Applying Lemma 8.3 with $\zeta_k = T_k W_k$, $K(u) = \mathrm{e}^{\mathrm{i}u}$ yields

$$\tilde{w}_{[n/\mu],n,j} - \hat{w}_{[n/\mu],n,j} = o_P(n^{1/\alpha} h(n)). \tag{4.3}$$

Finally, we bound $\hat{w}_{[n/\mu],n,j} - d_{n,j}$.

$$\begin{aligned}\hat{w}_{[n/\mu],n,j} - d_{n,j} &= \sum_{k=1}^{[n/\mu]} (\mathrm{e}^{\mathrm{i}k\mu x_{n,j}} \mathrm{e}^{-\mathrm{i}(\mu+1/2)x_{n,j}} - \mathrm{e}^{\mathrm{i}kx_{[n/\mu],j}})\zeta_k \\ &= \sum_{k=1}^{[n/\mu]} \mathrm{e}^{\mathrm{i}k\mu x_{n,j}} (\mathrm{e}^{-\mathrm{i}(\mu+1/2)x_{n,j}} - 1)\zeta_k + \sum_{k=1}^{[n/\mu]} (\mathrm{e}^{\mathrm{i}k\mu x_{n,j}} - \mathrm{e}^{\mathrm{i}kx_{[n/\mu],j}})\zeta_k.\end{aligned}$$

Since $1/[n/\mu] - 1/(n/\mu) = O(n^{-2})$ and $j \leq n^\rho$ with $\rho < 1 - 1/\alpha$, we obtain:

$$\mathbb{E}[|\hat{w}_{[n/\mu],n,j} - d_{n,j}|] \leq Cj/n = o(n^{1/\alpha} h(n)),$$

for any slowly varying function h. □

4.2 High frequencies

In the high frequency case, the asymptotic behaviour of the discrete Fourier transform is the same as it is for linear series.

Theorem 4.2. *If let j be a non decreasing sequence of integers such that $j/n \to 0$ and $j \geq n^\rho$ for some $\rho \in (1 - 1/\alpha, 1)$, then $(2\pi n f(x_j))^{-1/2} D_{n,j}$ is asymptotically complex Gaussian with independent real and imaginary parts, which are each zero mean Gaussian with variance $1/2$.*

Proof in the case of Parke's process. As seen in the proof of Proposition 4.1, the main term in the decomposition of $D_{n,j}$ is $W_{n,j}$, defined in (4.1). To prove convergence to a complex Gaussian law, we use the Wold device. For $a, b \in \mathbb{R}$, denote

$$\xi_{n,s}(a,b) = \{a\cos((s + n_s/2)x_j) + b\sin((s + n_s/2)x_j)\} \frac{\sin((n_s+1)x_j/2)\epsilon_s}{\sin(x_j/2)}.$$

Then $\sum_{s=1}^{n} \xi_{n,s}(a,b) = a\mathrm{Re}(W_{n,j}) + b\mathrm{Im}(W_{n,j})$. Denote $\sigma_n^2(a,b) = \sum_{s=1}^{n} \mathbb{E}[\xi_{n,s}^2(a,b)]$. To prove that $\sigma_n^{-1}(a,b) \sum_{s=1}^{n} \xi_{n,s}(a,b)$ is asymptotically Gaussian, it suffices to prove that

$$\sum_{s=1}^{n} \mathbb{E}[|\xi_{n,s}(a,b)|^q] = o(\sigma_n^q(a,b)), \tag{4.4}$$

for some $q > 2$. We first find an equivalent for $\sigma_n^2(a,b)$. To simplify the notation, without loss of generality, assume $\sigma_\epsilon^2 = 1$. We have

$$\begin{aligned}
\sin^2(x_j/2)\mathbb{E}[\xi_{n,s}^2] &= \mathbb{E}[\{a\cos((s+n_s/2)x_j) + b\sin((s+n_s/2)x_j)\}^2 \sin^2((n_s+1)x_j/2)] \\
&= a^2\mathbb{E}[\cos^2((s+n_s/2)x_j)\sin^2((n_s+1)x_j/2)] + ab\mathbb{E}[\sin((2s+n_s)x_j)\sin^2((n_s+1)x_j/2)] \\
&+ b^2\mathbb{E}[\sin^2((s+n_s/2)x_j)\sin^2((n_s+1)x_j/2)] \\
&= \frac{a^2+b^2}{2}\mathbb{E}[\sin^2((n_s+1)x_j/2)] + \frac{a^2-b^2}{2}\mathbb{E}[\cos((2s+n_s)x_j)\sin^2((n_s+1)x_j/2)] \\
&+ ab\mathbb{E}[\sin((2s+n_s)x_j)\sin^2((n_s+1)x_j/2)] \\
&= \frac{a^2+b^2}{2}\mathbb{E}[\sin^2((n_s+1)x_j/2)] \\
&+ \left\{\frac{a^2-b^2}{2}\cos(2sx_j) + ab\sin(2sx_j)\right\}\mathbb{E}[\cos(n_sx_j)\sin^2((n_s+1)x_j/2)] \\
&- \left\{\frac{a^2-b^2}{2}\sin(2sx_j) - ab\cos(2sx_j)\right\}\mathbb{E}[\sin(n_sx_j)\sin^2((n_s+1)x_j/2)].
\end{aligned}$$

Applying Lemma 8.4, we obtain that

$$\lim_{n\to\infty} x_j^{-\alpha}L(1/x_j)^{-1}\mathbb{E}[h(n_sx_j)\sin^2((n_s+1)x_j/2)] = \alpha\int_0^\infty h(t)\sin^2(t/2)t^{-\alpha-1}dt, \tag{4.5}$$

with either $h(t) = \cos(t)$, $h(t) = \sin(t)$ or $h(t) \equiv 1$. Now, since $j \to \infty$, we have:

$$\left|\frac{1}{n}\sum_{s=1}^n \mathrm{e}^{2\mathrm{i}sx_j}\right| \le \frac{2}{n|\mathrm{e}^{2\mathrm{i}x_j}-1|} = O(j^{-1}) = o(1).$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty} n^{-1}x_j^{2-\alpha}L(1/x_j)^{-1}\sum_{s=1}^n \mathbb{E}[\xi_{n,s}^2(a,b)] \\
&\qquad = 2\alpha(a^2+b^2)\int_0^\infty \sin^2(t/2)t^{-\alpha-1}dt = (a^2+b^2)\int_0^\infty \sin(t)t^{-\alpha}dt \\
&\qquad\qquad = (a^2+b^2)\Gamma(1-\alpha)\sin(\pi(\alpha-1)/2) = (a^2+b^2)\frac{\Gamma(2H-1)}{2-2H}\sin(\pi H).
\end{aligned}$$

Hence, applying (2.3), we obtain:

$$\lim_{n\to\infty}(2\pi nf(x_j))^{-1}\sum_{s=1}^n \mathbb{E}[\xi_{n,s}^2(a,b)] = \frac{a^2+b^2}{2}.$$

Hence $\sigma_n^2(a,b) \sim cnf(x_j) \to \infty$. Moreover, for any $q > 2$, we have:

$$\mathbb{E}[|\xi_{n,s}|^q] \le C(|a|+|b|)^q x_j^{-q}\mathbb{E}[|\sin((n_s+1)x_j/2)|^q] = O(x_j^{\alpha-q}L(1/x_j)),$$

$$\sigma_n^{-q}(a,b)\sum_{s=1}^n \mathbb{E}[\xi_{n,s}^q] = O\left((nx_j^\alpha)^{1-q/2}\right).$$

Since we have assumed that $j \gg n^{1-1/\alpha}$ and $q > 2$, we obtain that $nx_j^\alpha \to \infty$ and $\sum_{s=1}^n \mathbb{E}[\xi_{n,s}^q] = o(\sigma_n^q(a,b))$ and (4.4) holds. $\square$

Proof in the case of Taqqu-Levy's process. If $j \gg n^{1-1/\alpha}$, then $\lim_{n\to\infty}(nf(x_{n,j}))^{-1/2}n^{1/\alpha} = 0$. Hence, as in the proof of Proposition 4.1, we obtain that $(2\pi n)^{-1/2} f(x_{n,j})^{-1/2}(D_{n,j} - w_{[n/\mu],n,j}) = o_P(1)$. We now prove that $w_{[n/\mu],n,j}$ is asymptotically complex Gaussian by the Wold device. Here again, without loss of generality, we assume $\sigma_W^2 = 1$. For arbitrary real numbers a and b, define $v_n^2 = 2\pi n \sin^2(x_{n,j}/2) f(x_{n,j})$ and

$$\begin{aligned}
\eta_{n,k} &= v_n^{-1}\{a\cos((S_{k-1} + (T_k-1)/2)x_{n,j}) + b\sin((S_{k-1} + (T_k-1)/2)x_{n,j})\}\sin(T_k x_{n,j}/2)W_k \\
&= v_n^{-1}\cos(S_{k-1}x_{n,j})\{a\cos((T_k-1)/2)x_{n,j}) + b\sin((T_k-1)/2)x_{n,j})\}\sin(T_k x_{n,j}/2)W_k \\
&\quad + v_n^{-1}\sin(S_{k-1}x_{n,j})\{-a\sin((T_k-1)/2)x_{n,j}) + b\cos((T_k-1)/2)x_{n,j})\}\sin(T_k x_{n,j}/2)W_k.
\end{aligned}$$

Then $(2\pi n)^{-1/2} f^{-1/2}(x_{n,j}) \left\{a\,\mathrm{Re}(w_{[n/\mu],n,j}) + b\,\mathrm{Im}(w_{[n/\mu],n,j})\right\} = \sum_{k=1}^{[n/\mu]} \eta_{n,k}$. Denote

$$\begin{aligned}
B_1(u) &= \{a\cos(u/2) + b\sin(u/2)\}\sin(u/2), \\
B_2(u) &= \{b\cos(u/2) - a\sin(u/2)\}\sin(u/2), \\
\tilde\eta_{n,k} &= v_n^{-1}\left\{\cos(S_{k-1}x_{n,j})B_1(T_k x_{n,j}) + \sin(S_{k-1}x_{n,j})B_2(T_k x_{n,j})\right\}W_k,
\end{aligned}$$

and $\tilde w_{m,n,j} = \sum_{k=1}^m \tilde\eta_{n,k}$. Then

$$\sum_{k=1}^{[n/\mu]} \eta_{n,k} - \tilde\eta_{n,k} = O_P(f(x_{n,j})^{-1/2}) = o_P(1).$$

Define $\mathcal{M}_j = \sum_{k=1}^j \tilde\eta_{n,k}$, $1 \le j \le [n/\mu]$ and $\mathcal{F} = (\mathcal{F}_k)_{k\ge1}$ with $\mathcal{F}_k = \sigma(T_j, W_j, j \le k)$. Then $\{\mathcal{M}_j\}$ is an $\mathcal{F}$-martingale and $\mathcal{M}_{[n/\mu]} = \tilde w_{[n/\mu],n,j}$. Hence, to prove that $\tilde w_{[n/\mu],n,j}$ is asymptotically Gaussian, we must prove the conditional Lindeberg conditions:

$$\text{there exists } \sigma^2 > 0 \text{ such that } \sum_{k=1}^{[n/\mu]} \mathbb{E}[\tilde\eta_{n,k}^2 \mid \mathcal{F}_{k-1}] \xrightarrow{P} \sigma^2, \tag{4.6}$$

$$\text{and } \forall\epsilon > 0,\quad \sum_{k=1}^{[n/\mu]} \mathbb{E}[\tilde\eta_{n,k}^2 \mathbf{1}_{\{|\tilde\eta_{n,k}|\ge\epsilon\}} \mid \mathcal{F}_{k-1}] \xrightarrow{P} 0. \tag{4.7}$$

To prove (4.6), note that

$$\begin{aligned}
\mathbb{E}[\tilde\eta_{n,k}^2 \mid \mathcal{F}_{k-1}] &= v_n^{-2}\{\cos^2(S_{k-1}x_{n,j})\mathbb{E}[B_1^2(T_1 x_{n,j})] \\
&\quad + \sin(2S_{k-1}x_{n,j})\mathbb{E}[B_1(T_1 x_{n,j})B_2(T_1 x_{n,j})] + \sin^2(S_{k-1}x_{n,j})\mathbb{E}[B_2^2(T_1 x_{n,j})]\}\sigma_W^2.
\end{aligned}$$

Applying Lemmas 8.4 and 8.5 and using similar computations as in the proof of the previous case, we obtain:

$$\sum_{k=1}^{[n/\mu]} \mathbb{E}[\tilde\eta_{n,k}^2 \mid \mathcal{F}_{k-1}] \xrightarrow{P} \frac{a^2+b^2}{2}. \tag{4.8}$$

To prove (4.7), since $\mathbb{E}[|W^q|] < \infty$ for some $q > 2$, it is sufficient to prove that:

$$\sum_{k=1}^{[n/\mu]} \mathbb{E}[|\tilde{\eta}_{n,k}|^q] = o(v_n^q). \tag{4.9}$$

Since $\mathbb{E}[|\tilde{\eta}_{n,k}|^q] \le 2^{q-1} v_n^{-q/2}\{|B_1(T_k x_{n,j})|^q + |B_2(T_k x_{n,j})|^q\}$ and $\mathbb{E}[|B_i(T_k x_{n,j})|^q] = O(x_{n,j}^2 f(x_{n,j}))$, $i = 1, 2$, we obtain:

$$\sum_{k=1}^{[n/\mu]} \mathbb{E}[|\tilde{\eta}_{n,k}|^q] = O(n v_n^{-q} x_{n,j}^2 f(x_{n,j})) = O(v_n^{1-q/2}) = o(1).$$

Hence (4.9) holds. Thus we have shown that $\{2\pi n f(x_{n,j}\}^{-1/2} D_{n,j}$ is asymptotically equivalent to $\{2\pi n f(x_{n,j}\}^{-1/2} w_{[n/\mu],n,j}$ which converges weakly to a standard complex normal law. □

5 Asymptotics for the Sample ACF

The empirical autocovariance is often used as a diagnostic of long memory, hence it is of importance to investigate its meaningfulness in the present context. For $k \ge 0$, define

$$\hat{\gamma}_n(k) = n^{-1} \sum_{t=1}^{n-k} X_t X_{t+k}. \tag{5.1}$$

Since in both cases, X is a second order stationary process, $\hat{\gamma}_n(k)$ is an asymptotically unbiased estimator of $\gamma(k) = \mathrm{cov}(X_0, X_k)$. In the next proposition, we show that it is also a consistent estimator and obtain its rate of convergence and asymptotic distribution.

Theorem 5.1. *Assume that* (3.1) *and* (3.4) *hold,* $\mathbb{E}[|\epsilon_0|^q] < \infty$ *and* $\mathbb{E}[|W_0|^q] < \infty$ *for some* $q > 2\alpha$. *Denote* $\xi_s = W_s^2\{T_s - \mathbb{E}[T_1]\}$ *or* $\xi_s = \epsilon_s^2\{n_s - \mathbb{E}[n_1]\}$. *Let* $\mu = 1$ *in the case of Parke's process. Then for any* $k \ge 0$ *and any slowly varying function* h,

$$\hat{\gamma}(k) - \gamma(k) = \frac{1}{n} \sum_{s=1}^{[n/\mu]} \xi_s + o_P(h(n) n^{1-1/\alpha}). \tag{5.2}$$

Define ℓ *as in Proposition 3.3. Then* $\ell(n)^{-1} n^{1-1/\alpha}(\hat{\gamma}_n(k) - \gamma(k))$ *converges weakly to an* α*-stable random variable* ζ *with characteristic function*

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}u\zeta}] = \exp\left\{-|u|^\alpha m_\alpha \Gamma(1-\alpha)\cos(\pi\alpha/2)(1 - \mathrm{i}\,\mathrm{sign(u)}\tan(\pi\alpha/2))\right\},$$

with $m_\alpha = \mathbb{E}[|\epsilon_1|^{2\alpha}]$ *in the case of Parke's process and* $m_\alpha = \mathbb{E}[|W_1|^{2\alpha}]/\mu$ *in the case of the Taqqu-Levy process.*

Remark 5.1. The o_P term in (5.2) is not uniform with respect to k, but (5.2) implies that for any fixed integers $q, k_1, \ldots, k_q$, the asymptotic distribution of the vector $\ell(n)^{-1}n^{1-1/\alpha}[\hat{\gamma}_n(k_1) - \gamma(k_1), \ldots, \hat{\gamma}_n(k_q) - \gamma(k_q)]$ is that of an α-stable vector whose components are equal. Thus, the joint limiting distribution of a finite collection of standardized sample autocovariances at fixed lags is degenerate.

Remark 5.2. The conclusions of Theorem 5.1 hold also for $\ell(n)^{-1}n^{1-1/\alpha}(\hat{\rho}_n(k) - \rho(k))$ where $\hat{\rho}_n(k)$ and $\rho(k)$ are the sample and population autocorrelations at lag k. This non-Gaussian limiting distribution (as well as the degeneracy described above) for the standardized sample autocorrelations will clearly affect the asymptotic properties of parametric method-of-moments estimators which are based on a finite number of sample autocorrelations.

Proof of Theorem 5.1 in the case of Taqqu-Levy's process.

$$\begin{aligned}
\hat{\gamma}_n(k) &= n^{-1}\sum_{t=1}^{n-k} W_{M_t}W_{M_{t+k}} = n^{-1}\sum_{j,j'=0}^{\infty} W_jW_{j'}\sum_{t=1}^{n-k}\mathbf{1}_{\{M_t=j\}}\mathbf{1}_{\{M_{t+k}=j'\}} \\
&= n^{-1}\sum_{j=0}^{\infty} W_j^2\sum_{t=1}^{n-k}\mathbf{1}_{\{M_t=M_{t+k}=j\}} + n^{-1}\sum_{j\neq j'=0}^{\infty} W_jW_{j'}\sum_{t=1}^{n-k}\mathbf{1}_{\{M_t=j\}}\mathbf{1}_{\{M_{t+k}=j'\}} \\
&= \tilde{\gamma}_n(k) + r_n.
\end{aligned}$$

Consider first r_n. Note that the sums in j and j' are limited to n since by definition, $M_t \leq t$. If $j' < j$ or $j' > k$, the event $\{M_t = j; M_{t+k} = j'\}$ is empty. Hence:

$$\begin{aligned}
\mathbb{E}[r_n^2] &= \frac{\sigma_W^4}{n^2}\sum_{j=0}^{\infty}\sum_{j'=j+1}^{j+k}\sum_{s,t=1}^{n-k}\mathbb{P}(M_s = M_t = j; M_{s+k} = M_{t+k} = j') \\
&= \frac{\sigma_W^4}{n^2}\sum_{j=0}^{\infty}\sum_{j'=j+1}^{j+k}\sum_{t=1}^{n-k}\mathbb{P}(M_t = j; M_{t+k} = j') \\
&+ \frac{\sigma_W^4}{n^2}\sum_{j=0}^{\infty}\sum_{j'=j+1}^{j+k}\sum_{1\leq s<t\leq n-k}\mathbb{P}(M_s = M_t = j; M_{s+k} = M_{t+k} = j')
\end{aligned}$$

For $s < t$ and $j < j'$, the set $\{M_s = M_t = j; M_{s+k} = M_{t+k} = j'\}$ is empty if $s + k \leq t$. Hence:

$$\begin{aligned}
\mathbb{E}[r_n^2] &= \frac{\sigma_W^4}{n^2}\sum_{t=1}^{n-k}\mathbb{P}(M_t < M_{t+k}) \\
&+ \frac{\sigma_W^4}{n^2}\sum_{s=1}^{n-k-1}\sum_{s+1<t<s+k-1}\mathbb{P}(M_s = M_t < M_{s+k} = M_{t+k}) = O(n^{-1}).
\end{aligned}$$

Thus $r_n(k) = O_P(n^{-1/2})$. Consider now $\tilde{\gamma}_n(k)$. By definition of the renewal process, $M_t = M_{t+k} = j$ if and only if $S_{j-1} \le t < S_j$ and $T_j \ge k$. Thus

$$\tilde{\gamma}_n(k) = \frac{1}{n} \sum_{j=1}^{M_{n-k}} W_j^2 \sum_{t=1}^{n-k} \mathbf{1}_{\{M_t = M_{t+k} = j\}} = \frac{1}{n} \sum_{j=1}^{M_{n-k}} W_j^2 (T_j - k) \mathbf{1}_{\{T_j \ge k\}}.$$

Define $\check{\gamma}_n(k) = \frac{1}{n} \sum_{j=1}^{[(n-k)/\mu]} W_j^2 (T_j - k)\mathbf{1}_{\{T_j \ge k\}}$. By Lemma 8.1, for any slowly varying function h, we have that $\check{\gamma}_n(k) - \tilde{\gamma}_n(k) = o_P(n^{1-1/\alpha} h(n))$. Note now that by definition, $\mathbb{E}[(T_1 - k)\mathbf{1}_{\{T_1 \ge k\}}] = \mu \mathbb{P}(S_0 \ge k)$. Thus:

$$\begin{aligned}
\check{\gamma}_n(k) - \gamma(k) &= \frac{1}{n} \sum_{j=1}^{[(n-k)/\mu]} W_j^2 (T_j - k)\mathbf{1}_{\{T_j \ge k\}} \\
&= \frac{1}{n} \sum_{j=1}^{[(n-k)/\mu]} W_j^2 \{(T_j - k)\mathbf{1}_{\{T_j \ge k\}} - \mathbb{E}[(T_1 - k)\mathbf{1}_{\{T_1 \ge k\}}]\} \\
&+ \frac{\mu \mathbb{P}(S_0 \ge k)}{n} \sum_{j=1}^{[(n-k)/\mu]} \{W_j^2 - \sigma_W^2\} + \gamma(k)\{\mu \frac{[(n-k)/\mu]}{n} - 1\} \\
&= \frac{1}{n} \sum_{j=1}^{[(n-k)/\mu]} W_j^2 \{(T_j - k)\mathbf{1}_{\{T_j \ge k\}} - \mathbb{E}[(T_1 - k)\mathbf{1}_{\{T_1 \ge k\}}]\} + O_P(n^{-1/2}) \\
&= \frac{1}{n} \sum_{j=1}^{[(n-k)/\mu]} W_j^2 \{T_j - \mathbb{E}[T_1]\} + O_P(n^{-1/2}).
\end{aligned}$$

Thus we conclude that for any slowly varying function h,

$$\hat{\gamma}_n(k) - \gamma(k) = \frac{1}{n} \sum_{j=1}^{[n/\mu]} W_j^2 \{T_j - \mathbb{E}[T_1]\} + o_P(n^{1-1/\alpha} h(n)).$$

The rest of the proof is straightforward, given the other proofs in this paper, and is omitted to save space. □

Proof in the case of Parke's process.

$$\begin{aligned}
\hat{\gamma}_n(k) &= n^{-1} \sum_{t=1}^{n-k} \sum_{s \le t} \sum_{s' \le t+k} g_{s,t} g_{s',t+k} \epsilon_s \epsilon_{s'} = n^{-1} \sum_{s \le n-k} \sum_{t=1}^{n-k} \mathbf{1}_{\{s \vee 1 \le t \le (s+n_s-k) \wedge (n-k)\}} \epsilon_s^2 \\
&+ n^{-1} \sum_{\substack{s \le n-k; s' \le n \\ s \ne s'}} \sum_{t=1}^{n-k} \mathbf{1}_{\{s \vee 1 \le t \le (s+n_s) \wedge (n-k)\}} \mathbf{1}_{\{(s'-k) \vee 1 \le t \le (s'+n_{s'}-k) \wedge n\}} \epsilon_s \epsilon_{s'} \\
&= \tilde{\gamma}_n(k) + r_n(k).
\end{aligned}$$

We first consider $r_n(k)$. It is split into four terms as follows.

$$\begin{aligned} r_n(k) &= n^{-1} \sum_{\substack{s\le 0; s'\le k \\ s\ne s'}} \{(s+n_s)_+ \wedge (s'+n_{s'}-k)_+ \wedge (n-k)\}\epsilon_s\epsilon_{s'} \\ &+ n^{-1}\sum_{s\le 0}\sum_{s'=k+1}^{n} [\{(s+n_s)_+ \wedge (s'+n_{s'}-k) \wedge (n-k)\} - s' + k + 1]\epsilon_s\epsilon_{s'} \\ &+ n^{-1}\sum_{s=1}^{n-k}\sum_{s'\le k}[\{(s+n_s) \wedge (s'+n_{s'}-k)_+ \wedge (n-k)\} - s + 1]\epsilon_s\epsilon_{s'} \\ &+ n^{-1} \sum_{\substack{1\le s\le n-k; k+1\le s'\le n \\ s\ne s'}} [(s+n_s)\wedge(s'+n_{s'}-k)\wedge(n-k) - s\vee(s'-k)]\epsilon_s\epsilon_{s'} \\ &= r_{1,n} + r_{2,n} + r_{3,n} + r_{4,n}. \end{aligned}$$

By the usual Borel Cantelli argument, $nr_{1,n}$ converges to the almost surely finite sum $\sum_{\substack{s\le 0; t\le 0 \\ s\ne t+k}}$ $\{(s+n_s)_+ \wedge (t+n_{t+k})_+\}\epsilon_s\epsilon_{t+k}$. Hence $r_{1,n} = O_P(n^{-1})$. By independence of the i.i.d. sequences (ϵ_s) and (n_s), the terms $r_{2,n}$ and $r_{3,n}$ have the same distribution. We consider for instance the former. Let S be the set of nonpositive integers s such that $s+n_s \ge 0$. Then S is almost surely finite. Write $r_{2,n} = n^{-1}\sum_{s\in S}\xi_{n,s}\epsilon_s$, with

$$\xi_{n,s} = \sum_{t=1}^{n-k}[\{(s+n_s)_+ \wedge (t+n_{t+k}) \wedge (n-k)\} - t + 1]\epsilon_{t+k}$$

For each $s\in S$, we have:

$$\lim_{n\to\infty}\xi_{n,s} = \sum_{t=1}^{s+n_s}[\{(s+n_s)\wedge(t+n_{t+k})\} - t + 1]\epsilon_{t+k}$$

Since S is almost surely finite, we thus obtain that

$$\lim_{n\to\infty} nr_{2,n} = \sum_{s\in S}\sum_{t=1}^{s+n_s}[\{(s+n_s)\wedge(t+n_{t+k})\} - t + 1]\epsilon_{t+k}, \text{ almost surely.}$$

Hence $r_{2,n} = O_P(n^{-1})$ and similarly $r_{3,n} = O_P(n^{-1})$. Consider now the last term $r_{4,n}$.

$$\mathbb{E}[r_{4,n}^2] = \sigma_\epsilon^4 n^{-2} \sum_{\substack{1\le s\le n-k; 1\le t\le n-k \\ s\ne t+k}} \mathbb{E}[\{(s+n_s)\wedge(t+n_{t+k})\wedge(n-k) - s\vee t\}^2]. \tag{5.3}$$

This last expectation if finite, since the term inside is at most $n_s \wedge n_{t+k}$, and if N' is an independent copy of N, then $N\wedge N'$ is square integrable. Indeed, we have

$$\mathbb{P}(N\wedge N' \ge k) = \mathbb{P}(N\ge k)^2 = L^2(k)k^{-2\alpha}. \tag{5.4}$$

Since L is slowly varying, then so is L^2, and since $\alpha \in (1,2)$, then (5.4) implies that $N \wedge N'$ is square integrable. Let us now compute the expectation in the rhs of (5.3). Assume $s < t \le n-k$.

$$
\begin{aligned}
&\mathbb{E}[\{(s+N) \wedge (t+N') \wedge (n-k) - s \vee t\}^2] \\
&\qquad = \sum_{j=t-s}^{n-k-s} \sum_{j'=0}^{n-k-t} \{(s+j) \wedge (t+j') - t\}^2 \mathbb{P}(N=j)\mathbb{P}(N'=j') \\
&\qquad = \sum_{j=t-s}^{n-k-s} \sum_{j'=0}^{j-t+s} j'^2 \mathbb{P}(N=j)\mathbb{P}(N'=j') \\
&\qquad + \sum_{j=t-s}^{n-k-s} \sum_{j'=j-t+s+1}^{n-k-t} (j-t+s)^2 \mathbb{P}(N=j)\mathbb{P}(N'=j') \le CL^2(t-s)(t-s)^{2-2\alpha}.
\end{aligned}
$$

Plugging this bound into (5.3), we obtain:

$$
\mathbb{E}[r_{4,n}(k)^2] = \begin{cases} O(\tilde{L}(n)n^{2-2\alpha}) & \text{if } \alpha \in (1, 3/2], \text{ with } \tilde{L} \text{ slowly varying;} \\ O(n^{-1}) & \text{if } \alpha \in (3/2, 2). \end{cases}
$$

In conclusion, we have shown that $r_n(k) = O_P(n^{1-\alpha})$. Consider now $\tilde{\gamma}_n(k)$. Still by Borel Cantelli arguments, we have

$$
\begin{aligned}
\tilde{\gamma}_n(k) &= n^{-1} \sum_{s \le 0} \{(s+n_s-k)_+ \wedge (n-k)\}\epsilon_s^2 \\
&+ n^{-1} \sum_{s=1}^{n-k} \{(s+n_s-k) \wedge (n-k) - s + 1\}\mathbf{1}_{\{n_s \ge k\}}\epsilon_s^2 \\
&= n^{-1} \sum_{s=1}^{n-k} (n_s - k + 1)\mathbf{1}_{\{n_s \ge k\}}\epsilon_s^2 + O_P(n^{-1}).
\end{aligned}
$$

Altogether, we have

$$
\begin{aligned}
\hat{\gamma}_n(k) - \gamma(k) &= n^{-1} \sum_{s=1}^{n-k} (n_s - k + 1)\mathbf{1}_{\{n_s \ge k\}}\epsilon_s^2 - \gamma(k) + O_P(n^{1-\alpha}) \\
&= n^{-1} \sum_{s=1}^{n-k} \left\{(n_s - k + 1)\mathbf{1}_{\{n_s \ge k\}} - \mathbb{E}[(n_s - k + 1)\mathbf{1}_{\{n_s \ge k\}}]\right\} \epsilon_s^2 \\
&\qquad + \frac{\mathbb{E}[(N-1+k)\mathbf{1}_{\{N \ge k\}}]}{n} \sum_{s=1}^{n-k} \{\epsilon_s^2 - \sigma_\epsilon^2\} + O_P(n^{1-\alpha}) \\
&= n^{-1} \sum_{s=1}^{n-k} \left\{(n_s - k + 1)\mathbf{1}_{\{n_s \ge k\}} - \mathbb{E}[(n_s - k + 1)\mathbf{1}_{\{n_s \ge k\}}]\right\} \epsilon_s^2 + O_P(n^{-1/2}) + O_P(n^{1-\alpha}) \\
&= n^{-1} \sum_{s=1}^{n} \{n_s - \mathbb{E}[N]\} \epsilon_s^2 + O_P(n^{-1/2}) + O_P(n^{1-\alpha}).
\end{aligned}
$$

Thus, if $\mathbb{E}[|\epsilon_0|^q] < \infty$ for some $q > 2\alpha$, then $\ell(n)^{-1}n^{1-1/\alpha}(\hat{\gamma}_n(k) - \gamma(k))$ converges weakly to an α-stable distribution. □

6 Simulations

Throughout this section, we denote the long-memory parameter by $d \in (0, 0.5)$. Note that $d = H - 1/2 = 1 - \alpha/2$. In all of our simulations, we use a sample size of $n = 10000$. We chose to use $ARFIMA(0, d, 0)$ autocovariances in our simulations because they are nonnegative and monotone non-increasing for all t, which is consistent with the nonnegative and non-increasing autocovariances implied by both the Taqqu-Levy and Parke models. Let $\gamma(t)$ be the autocovariance sequence of an $ARFIMA(0, d, 0)$ process,

$$\gamma(t) = \frac{\Gamma(t+d)\Gamma(1-2d)}{\Gamma(t-d+1)\Gamma(1-d)\Gamma(d)}\ \sigma_0^2,\ t = 0, 1, \ldots \tag{6.1}$$

where σ_0 is the standard deviation of the $ARFIMA$ innovations. For the integer-valued interarrival time S_0 as well as the $\{T_k\}$ in the Taqqu-Levy process and the survival times $\{n_s\}$ in the Parke process, we use the following simulation algorithm : Let X denote either S_0, T_k or n_s and let $G(x) = P(X \geq x)$. We can simulate an observation x of X by drawing an observation u of a uniform random variable and setting x to be the integer such that

$$G(x) \geq u > G(x+1). \tag{6.2}$$

In all cases we consider here, $G(x)$ is expressed in terms of the Gamma function, so that there is an easily evaluated continuous increasing function $\tilde{G}(x)$ which is equal to $G(x)$ for all integer values at which $G(x)$ is defined. The solution to (6.2) can be written as

$$x = \lfloor \tilde{G}^{-1}(u) \rfloor, \tag{6.3}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than x. We obtain the solution x to (6.3) using a simple bisection algorithm.

6.1 Simulation of Taqqu-Levy Process

Before describing our sampling algorithm, we provide some convenient formulas for $P(S_0 \geq t)$ and $P(T_k \geq t)$. From (3.2) and Proposition 3.1, we have

$$\mu = \frac{1}{P(S_0 = 0)} \text{ and } \sigma_W^2 = \gamma(0)$$

and

$$P(S_0 \geq t) = \frac{\gamma(t)}{\gamma(0)},\ t = 0, 1, 2, \ldots \tag{6.4}$$

Thus, for $t \geq 1$, we have:

$$\begin{aligned} P(T_k \geq t) = \mu P(S_0 = t-1) &= \frac{P(S_0 = t-1)}{P(S_0 = 0)} \\ &= \frac{\mathbb{P}(S_0 \geq t-1) - \mathbb{P}(S_0 \geq t)}{\mathbb{P}(S_0 \geq 0) - \mathbb{P}(S_0 \geq 1)} = \frac{\gamma(t-1) - \gamma(t)}{\gamma(0) - \gamma(1)}. \end{aligned} \tag{6.5}$$

For all of our simulations of the Taqqu-Levy process, we assume that $\sigma_0^2 = 1$. From (6.4) and (6.5), we can sample S_0 and $\{T_k\}$ using the bisection algorithm. We also simulate *iid* normal random variables W_k with mean zero and variance $\sigma_W^2 = \gamma(0)$, independent of S_0 and $\{T_k\}$. The duration of the 0th regime is S_0 and the duration of the kth regime is T_k for $k \geq 1$. The value of the series X_t is constant at W_k throughout the kth regime. This yields the simulated realization $X_0, \ldots, X_{n-1}$. Occasionally, the entire simulated realization was constant, as there were no breaks before $n-1$. Such realizations were discarded.

6.2 Simulation of Parke's Process

By Proposition 3.2, Parke's process is well defined if and only if with probability one, for all t, there is a finite number of shocks surviving at time t. This allows us to simulate a process which is distributionally equivalent to Parke's using only a finite sum

$$X_t = \sum_{s=-J}^{t} g_{s,t}\epsilon_s, \;\; t = 1, 2, \ldots \tag{6.6}$$

where $-J$ is the time index of the oldest shock that survives at time $t = 0$. The non-negative integer-valued random variable J has a probability distribution

$$P(J \leq j) = \prod_{k=j+1}^{\infty} (1 - p_k). \tag{6.7}$$

In order to obtain the covariances (6.1), for $0 < d < 1/2$, the survival probabilities are defined by (see Parke, 1999)

$$p_k = \frac{\Gamma(2-d)}{\Gamma(d)} \frac{\Gamma(k+d)}{\Gamma(k+2-d)}, \;\; k = 0, 1, 2, \ldots \tag{6.8}$$

For each realization of Parke's process, we start by sampling J from the probability distribution determined by (6.7) truncated to the range $(0, 1, 2, \ldots, 10000)$. This was adequate for the values of d considered here, $d = 0.1$ and $d = 0.4$, since the sum of the probabilities up to that truncation point is extremely close to one in both cases. Next, we generate a sequence of standard normal shocks $\{\epsilon_s\}_{s=-J}^{n}$. The innovation variance σ_0^2 of the $ARFIMA(0, d, 0)$ process is related to σ_ϵ^2 (we have $\sigma_\epsilon^2 = 1$) by

$$\sigma_0^2 = \frac{\Gamma(1-d)\Gamma(2-d)}{\Gamma(2-2d)} \sigma_\epsilon^2. \tag{6.9}$$

Next we discuss the simulation of the $\{n_s\}$ sequence. Special attention must be paid to the survival time n_{-J} for the oldest shock ϵ_{-J}. It is not sampled from the probability distribution determined by $\{p_k\}$, but rather from the conditional distribution

$$P(N \geq i | N \geq J) = \frac{p_i}{p_J}, \; i \geq J. \tag{6.10}$$

We apply the bisection algorithm to sample n_{-J} and the other $\{n_s\}_{s=-J+1}^{n}$ from (6.8) and (6.10). Using the values $\{n_s\}_{s=-J}^{-1}$, we compute the "death time" for each prehistoric shock $\{\epsilon_s\}_{s=-J}^{-1}$. At each time $t \geq 0$, there may be some past shocks dying, so the time series X_t is generated by adding a new shock to the previous value X_{t-1} and subtracting the sum of those shocks dying at time t.

6.3 Simulation Results

We performed Monte Carlo simulations to assess the finite sample properties of the DFT coefficients in light of Theorems 4.1 and 4.2 for both the Taqqu-Levy and Parke processes. We generated 500 replications of length $n = 10000$ in each case. Recall that $d = 1 - \frac{1}{2}\alpha$, and $1 < \alpha < 2$. We used autocovariances corresponding to an $ARFIMA(0, d, 0)$ model as described earlier, with $d = 0.1$ and $d = 0.4$. For each value of d, the normalized Fourier coefficients were evaluated at frequency x_j with $j = 1, 2, \lfloor n^{0.2} \rfloor, \lfloor n^{0.4} \rfloor, \lfloor n^{0.6} \rfloor, \lfloor n^{0.8} \rfloor, \lfloor \frac{n}{2} \rfloor - 2, \lfloor \frac{n}{2} \rfloor - 1$. For the Taqqu-Levy process with $d = 0.4$, there were 60 constant realizations. We excluded these constant realizations from our analysis, while keeping the number of realizations used at 500.

Figures 1-2 present the normal Quantile-Quantile (QQ) plots of the normalized Fourier cosine coefficients $A_j / f(x_j)^{\frac{1}{2}}$ for the Parke process with $d = 0.1$ and $d = 0.4$, where

$$A_j = \frac{1}{(2\pi n)^{1/2}} \sum_{t=0}^{n-1} x_t \cos(x_j t). \tag{6.11}$$

The number inside the parenthesis at the bottom of each QQ plot represents the p-value for the Anderson-Darling test of normality. According to Theorems 4.1 and 4.2, if j increases sufficiently quickly with the sample size n, i.e. when $j \geq n^{\rho}$ for $\rho > 1 - 1/\alpha$, the normalized Fourier coefficients are asymptotically normal. Furthermore, as d increases, the value of α will decrease, and the condition on the rate of increase of j to ensure asymptotic normality becomes less stringent. When $d = 0.1$, we have $1 - 1/\alpha = 0.4444$, a number larger than $1 - 1/\alpha = 0.1667$ when $d = 0.4$. For the Parke process with $d = 0.1$, we do not reject the hypothesis of normality for $j \geq n^{0.4}$; while $d = 0.4$, we reject the hypothesis of normality for $j < n^{0.2}$. Thus our simulation results are essentially consistent with the results of Theorems 4.1 and 4.2. We found similar results for the Taqqu-Levy process. Since the results for the normalized Fourier sine coefficient

$$B_j = \frac{1}{(2\pi n)^{1/2}} \sum_{t=0}^{n-1} x_t \sin(x_j t) \tag{6.12}$$

are very similar to those we found here, we do not present them here.

Figure 3 presents scatterplots of the average log normalized periodogram vs. $\log|2\sin(x_j/2)|$ at the Fourier frequencies from $j = 1, \ldots, 4999$. We would expect a horizontal line across all frequencies if $E\left[\log \frac{I(x_j)}{f(x_j)}\right]$ is constant for all j. The plots indicate that at low Fourier frequencies, the average log normalized periodogram is changing but approaches a constant as j increases. If $I(x_j)/f(x_j)$ were distributed as $(1/2)\chi_2^2$ as would be the case for a Gaussian white noise process, we would have $E[I(x_j)/f(x_j)] = -\gamma = -0.577216$ in Figure 3. There seems to be some evidence that the log normalized periodogram is biased upward for the Taqqu-Levy process with $d = 0.4$, but not for the other situations considered. Note that since the DFT coefficients converge weakly to an α stable law at fixed low Fourier frequencies, we should expect higher variability of the log normalized periodogram at these frequencies. This suggests that if we regress $\{\log(I(x_j))\}$ on $\{\log(f(x_j))\}$ without trimming a set of low Fourier frequencies, we may get a biased and/or highly variable GPH estimator. Further evidence is given in Figure 4, which presents scatterplots of the average of $\log(I(x_j))$ vs. $\log 2|\sin(x_j/2)|$ together with their fitted least-squares lines. We also found that there are several outliers at low frequencies for both processes with $d = 0.1$ as well as $d = 0.4$. However, there are more outliers in the case of $d = 0.1$ for both processes. This may be due to the more stringent condition required on the rate of increase of j to ensure asymptotic normality of the DFT coefficients when $d = 0.1$. The fact that the normalized periodogram behaves differently at the low Fourier frequencies may present a problem for the GPH estimator if we include all Fourier frequencies.

Figure 5 presents normal QQ plots for the sample autocorrelations based on the Taqqu-Levy process with $d = 0.1$. The Anderson-Darling p-values are extremely small so we reject the null hypothesis of normality in all cases. Furthermore, the plots indicate long-tailed distributions. These findings do not contradict Theorem 5.1 which states that the autocovariances for both processes will converge to an α-stable law. We found similar results for the Taqqu-Levy process with $d = 0.4$ as well as the Parke process for both values of d.

Tables 1 and 2 present simulation variances of the normalized DFT cosine coefficients and the corresponding normal-based 95% confidence intervals for the true variance, σ^2. We do not reject the null hypothesis that $\sigma^2 = 0.5$ for any j when $d = 0.1$ in the Taqqu-Levy process, but when $d = 0.4$, we reject the null hypothesis for $j = n/2 - 1$. For the Parke process, we accept the null hypothesis for all Fourier frequencies with both values of d except for $j = n^{0.2}$ in the case $d = 0.1$. Thus the results are essentially consistent with the theoretical variances stated in Theorem 4.2.

7 Concluding remarks

1. The main theoretical results we have obtained for the Parke and Taqqu-Levy models are strikingly similar. Also, it seems clear that the class of processes having DDLRD is much larger than the two processes we have considered in this paper. A specific example of another such process is the random coefficient autoregression studied in Leipus and

Surgailis (2002). We have so far been unable to find an overarching unification for DDLRD processes which would allow the development of a single set of theoretical results that applies to the entire class, although such a unification seems desirable, and may well be possible.

2. In Robinson (1995a), the theory of a modified GPH estimator was developed for Gaussian long-memory processes. One aspect of the modification was that an increasing number of low frequencies were trimmed (omitted) before constructing the estimate. Subsequently Hurvich, Deo and Brodsky (1998), who also assumed Gaussianity, showed that trimming can be avoided. More recently, Hurvich, Moulines and Soulier (2002) showed that trimming can also be avoided in a different log-periodogram regression estimator, assuming a linear, potentially non-Gaussian series. For linear series, it is known that the DFT at fixed j is asymptotically normal (Terrin and Hurvich, 1994), but that the periodogram is asymptotically neither independent, identically distributed, nor exponentially distributed (Künsch 1986, Hurvich and Beltrao 1993). Simulations, mostly from Gaussian long-memory series, indicate that trimming yields a very modest bias reduction, while inflating the variance of the GPH estimator substantially. (See also Deo and Hurvich 2001, in the context of LMSV models). Currently, there seems to be widespread agreement that trimming in the GPH estimator should be avoided.

   In contrast, the results of the present paper indicate that if the long memory is generated by DDLRD, then trimming of low frequencies may in fact be desirable. The DFT at fixed j converges in distribution to an infinite-variance stable distribution, but if j is allowed to increase suitably quickly a limiting normal distribution results. It is unclear at this moment whether trimming is needed to establish the asymptotic normality of the GPH estimator based on a process having DDLRD, but clearly the failure to trim low frequencies may adversely affect the finite-sample behavior of the GPH estimator. Paradoxically, the larger d is, the less stringent the conditions on the rate of increase of j to ensure asymptotic normality. This seems to indicate that when d is larger less trimming would be needed, both in theory and in practice. This runs counter to the effects studied by Hurvich and Beltrao (1993) (which concern only the second order structure of the process) which imply that the bias of the normalized periodogram increases as d increases from zero.

3. It is known (see Chung 2003 and the references therein) that for a long-memory process linear in martingale differences, the autocovariances are asymptotically normal if $d < 1/4$, but converge to a non-normal, finite-variance distribution if $d \in (1/4, 1/2)$. So the asymptotics for the sample autocovariances depend on d, which is an undesirable property from the point of view of statistical inference. Davis and Mikosch (1998) have shown that for short-memory ARCH and GARCH models, the asymptotic properties of the sample autocorrelations are more severe, as there is no convergence in distribution. Now, for DDLRD, the behavior is somewhere in between the linear long memory and ARCH/GARCH cases, since for DDLRD the sample autocorrelations do converge in distribution for all d with $0 < d < 1/2$, but the limiting distribution has infinite variance, and depends on d. Thus, the properties of parametric estimators of d which use a fixed number of sample autoco-

variances will be strongly affected by the presence of DDLRD.

8 Lemmas

We present some lemmas in this section. Most of these are presumably known, but we were unable to find references for them under the conditions we needed for our main results. We therefore include proofs for the sake of completeness.

Lemma 8.1. *Let* $(\zeta_k)_{k\in\mathbb{N}^*}$ *be a martingale difference sequence such that*$\sup_{k\geq 1}\mathbb{E}[|\zeta_k|^p] < \infty$ *for all* $p < \alpha$. *Then, for any slowly varying function* h,

$$\sum_{k=1}^{M_n}\zeta_k - \sum_{k=1}^{[n/\mu]}\zeta_k = o_P(h(n)n^{1/\alpha}).$$

Proof. To simplify the notation, without loss of generality, we can assume that $\mu = 1$. For all m, denote $S_m = \sum_{k=1}^m \zeta_k$. By Theorem 2.5.15 in Embrechts *et al.* (1997), there exists a slowly varying function ℓ such that $\ell(n)^{-1}n^{-1/\alpha}(M_n - n)$ converges in distribution to a stable law. Thus, for any sequence δ_n tending to infinity, we have:

$$\lim_{n\to\infty}\mathbb{P}\left(|M_n - n| \geq \delta_n\ell(n)n^{1/\alpha}\right) = 0. \tag{8.1}$$

Let $\epsilon > 0$ and δ_n be an arbitrary sequence tending to infinity. For any slowly varying function h, we can write:

$$\begin{aligned}
&\mathbb{P}\big(|S_{M_n} - S_n| \geq \epsilon n^{1/\alpha}h(n)\big) \\
&\quad\leq \mathbb{P}(|M_n - n| > \delta_n n^{1/\alpha}\ell(n)) + \mathbb{P}\left(|M_n - n| \leq \delta_n n^{1/\alpha}\ell(n); |S_{M_n} - S_n| \geq \epsilon n^{1/\alpha}h(n)\right) \\
&\qquad\leq \mathbb{P}(|M_n - n| > \delta_n n^{1/\alpha}\ell(n)) + \mathbb{P}\left(\max_{m:|m-n|\leq\delta_n n^{1/\alpha}\ell(n)}|S_n - S_m| \geq \epsilon n^{1/\alpha}h(n)\right).
\end{aligned}$$

Fix some $p \in (1,\alpha)$ and denote $C_p = \sup_{k\geq 1}\mathbb{E}[|\zeta_k|^p] < \infty$ by assumption. Denote By Kolmogorov's and Burkholder's inequalities (cf. Hall and Heyde (1980), Theorems 2.1 and 2.10), we obtain:

$$\begin{aligned}
&\mathbb{P}\big(\max_{m:|m-n|\leq n\delta_n n^{1/\alpha}\ell(n)}|S_n - S_m| \geq \epsilon n^{1/\alpha}h(n)\big) \\
&\qquad\leq c\epsilon^{-1}n^{-1/\alpha}h(n)^{-1}\mathbb{E}[|S_{n+\delta_n n^{1/\alpha}\ell(n)} - S_{n-\delta_n n^{1/\alpha}\ell(n)}|^p]^{1/p} \\
&\qquad\leq c\epsilon^{-1}n^{-1/\alpha}h(n)^{-1}\left(\sum_{k=n-\delta_n n^{1/\alpha}\ell(n)}^{n+\delta_n n^{1/\alpha}\ell(n)}\mathbb{E}[|\zeta_k|^p]\right)^{1/p} \\
&\qquad\leq c\,C_p\,\epsilon^{-1}n^{-1/\alpha}h(n)^{-1}(\delta_n n^{1/\alpha}\ell(n))^{1/p}.
\end{aligned}$$

Since $p > 1$, this last term is $o(1)$ is the sequence δ_n converges to infinity slowly enough. ☐

Lemma 8.2. *Let $(\zeta_{n,k})_{1\leq k\leq n}$ be uniformly bounded random variables. Let $(T_k)_{k\geq 1}$ be i.i.d. random variables that satisfy (3.4) for some $\alpha \in (1,2)$ and such that for all $n \geq 1$ and all $k \leq n$, T_k is independent of $\{\zeta_{n,j}, 1 \leq j < k\}$. Let W_k be i.i.d. random variables with zero mean and finite variance, independent of $\zeta_{n,k}$, $1 \leq k \leq n$ and T_k, $1 \leq k \leq n$. Let H be a bounded continuous function such that for all $u \in \mathbb{R}$ and $v \in (0,1)$:*

$$|H(u,v) - u| \leq C|u|\{u^2v^2 \wedge 1 + v^2\}. \tag{8.2}$$

If $m \leq cn$ and $j \leq n^\rho$ for some $\rho \in (0, 1/\alpha)$, then $\sum_{k=1}^m \zeta_{n,k}W_k\{H(T_k, x_{n,j}) - T_k\} = o_P(n^{1/\alpha}\ell(n))$ for any slowly varying function ℓ.

Proof. Define $\xi_n = \sum_{k=1}^n \zeta_{n,k}\{H(jT_k/n) - jT_k/n\}W_k$ and let $\mathbb{E}_{T,\zeta}$ denote the conditional expectation with respect to all the variables $\zeta_{n,k}$ and T_k. Since the variables $\zeta_{n,k}$ are uniformly bounded, and since, for $p \in [1, \alpha)$, the function $x \to x^{p/2}$ is concave, we obtain:

$$\mathbb{E}_{T,\zeta}\left[|\xi_n|^p\right] \leq C\left\{\sum_{k=1}^n \{H(T_k, x_{n,j}) - T_k\}^2\right\}^{p/2} \leq C\sum_{k=1}^n |H(T_k, x_{n,j}) - T_k|^p.$$

Hence, taking expectations on both sides and applying (8.2), we obtain:

$$\begin{aligned}
\mathbb{E}[|\xi_n|^p] &\leq C\sum_{k=1}^n \mathbb{E}[|H(T_k, x_{n,j}) - T_k|^p] \\
&\leq Cn\mathbb{E}[|T_1|^p(|jT_1/n|^2 \wedge 1)^p] + Cn(j/n)^{2p} \leq CnL(n)\{(j/n)^{\alpha-p} + (j/n)^{2p}\}.
\end{aligned}$$

Thus, $\xi_n = O_P(\{nL(n)\}^{1/p}\{j^{\alpha/p-1} + (j/n)^2\})$. If $\rho < 1 - 1/\alpha$, then p can be chosen such that $\{nL(n)\}^{1/p}\{j^{\alpha/p-1} + (j/n)^2\} = o(n^{1/\alpha}\ell(n))$. Hence $\xi_n = o_P(n^{1/\alpha}\ell(n))$ for any slowly varying function ℓ. □

Lemma 8.3. *Let ζ_k be a sequence of i.i.d. rv such that for all $p \in (1, \alpha)$, $\mathbb{E}[|\zeta_k|^p] < \infty$, $\mathbb{E}[\zeta_k] = 0$ and ζ_k is independent of $S_0, T_1, \dots, T_{k-1}$. Let K be a bounded continuously differentiable function on $\mathbb{R}$, with bounded derivative. Define $U_{m,n,j} = \sum_{k=1}^m K(S_{k-1}x_{n,j})\zeta_k$ and $V_{m,n,j} = \sum_{k=1}^m K((k-1)\mu x_{n,j})\zeta_k$. If $m \leq cn$ and $j \leq n^\rho$ for some $\rho \in (0, 1 - 1/\rho$, then $U_{m,n,j} - V_{m,n,j} = o_P(n^{1/\alpha}\ell(n))$ for any slowly varying function ℓ.*

Proof. Denote $R_k = T_1 + \cdots + T_k - k\mu$. Since K is differentiable, we can write:

$$\begin{aligned}
U_{m,n} - V_{m,n} &= \sum_{k=1}^m K'((k-1)\mu x_{n,j} + \varsigma_k(S_{k-1} - (k-1)\mu x_{n,j})\{S_{k-1} - (k-1)\mu\}x_{n,j}\zeta_k \\
&= x_{n,j}\sum_{k=1}^n \rho_{n,k}R_{k-1}\zeta_k + S_0 x_{n,j}\sum_{k=1}^n \rho_{n,k}\zeta_k,
\end{aligned}$$

where $\rho_{n,k} = K'(\{(k-1)\mu + \varsigma_k(S_{k-1} - (k-1)\mu\}/n)$. Since $\mathbb{E}[|\zeta_k|] < \infty$ and K' is bounded, the last term above is trivially $O_P(1)$. By assumption, $\{\sum_{j=1}^k \rho_{n,k}R_{k-1}\zeta_k,\ 1 \leq k \leq n\}$ is a martingale with finite p-th moment for $p < \alpha$. Hence by the Burkholder inequality for martingales, we have, for $p < \alpha$, $\mathbb{E}[|R_k|^p] = O(k)$ and

$$\mathbb{E}\left[\left|\sum_{k=1}^{n} \rho_{n,k} R_{k-1} \zeta_k\right|^p\right] \le C \sum_{k=1}^{n} \mathbb{E}[|R_{k-1}|^p] = O(n^2).$$

Thus $x_{n,j} \sum_{k=1}^{n} \rho_{n,k} R_{k-1} \zeta_k = O_P(jn^{2/p-1})$. If $\rho < 1 - 1/\alpha$, then p can be chosen so that $jn^{2/p-1} = o(n^{1/\alpha}\ell(n))$ for any slowly varying function ℓ. □

Lemma 8.4. *Let H be a bounded continuously differentiable function on $\mathbb{R}$ such that $H(x) = O(x^2)$ in a neighborhood of 0. If T_1 satisfies* (3.4)*, then*

$$\lim_{t\to\infty} t^\alpha L^{-1}(t)\mathbb{E}[H(T_1/t)] = \alpha \int_0^\infty H(s) s^{-\alpha-1} ds.$$

Proof. Assume first that H has a compact support in $(0, \infty)$ and is continuously differentiable. Then:

$$\begin{aligned}
\mathbb{E}[H(T_1/t)] &= \sum_{k=1}^{\infty} H(k/t)\mathbb{P}(T = k) = \sum_{k=1}^{\infty} H(k/t)\{\mathbb{P}(T \ge k) - \mathbb{P}(T \ge k-1)\} \\
&= \sum_{k=1}^{\infty} \mathbb{P}(T \ge k)\{H(k/t) - H((k-1)/t)\} = \int_0^\infty \mathbb{P}(T > \lfloor s \rfloor] + 1) H'(s/t)\, ds/t \\
&= \int_0^\infty (\lfloor s \rfloor + 1)^{-\alpha} L(\lfloor s \rfloor + 1) H'(s/t)\, ds/t = \int_0^\infty (\lfloor tx \rfloor + 1)^{-\alpha} L(\lfloor tx \rfloor + 1) H'(x)\, dx,
\end{aligned}$$

Since L is slowly varying, by Karamata's Theorem, we know that $\lim_{t\to\infty} L(t)^{-1}L(\lfloor tx \rfloor + 1) = 1$, uniformly with respect to x in compact sets of $(0, \infty)$. Thus, since we have assumed that H has compact support in $(0, \infty)$, we obtain

$$\lim_{t\to\infty} t^\alpha L^{-1}(t)\mathbb{E}[H(T_1/t)] = \int_0^\infty x^{-\alpha} H'(x)\, dx = \alpha \int_0^\infty x^{-\alpha-1} H(x)\, dx.$$

To conclude, it is sufficient to prove that

$$\lim_{A\to\infty} \limsup_{t\to\infty} t^\alpha L^{-1}(t)\mathbb{E}[H(T/t)\mathbf{1}_{\{T > At \text{ or } T < t/A\}}] = 0. \tag{8.3}$$

This tightness property allows then to truncate the function H and apply the first part of the proof. For any $A > 0$ and t large enough, applying the assumption on the behaviour of the function H at zero, we have:

$$\mathbb{E}[H(T/t)\mathbf{1}_{\{T<t/A\}}] = \sum_{k=1}^{t/A} H(k/t)\mathbb{P}(T = k) \le Ct^{-2} \sum_{k=1}^{t/A} k^2 \mathbb{P}(T = k).$$

Applying summation by parts and Karamata's theorem, we obtain:

$$\sum_{k=1}^{t/A} k^2 \mathbb{P}(T=k) = 1 + \sum_{k=1}^{t/A} \mathbb{P}(T \geq n)\{k^2 - (k-1)^2\} \leq CA^{\alpha-2} L(At).$$

Thus, there exists a constant C such that:

$$\limsup_{t\to\infty} t^{\alpha} L^{-1}(t)\, \mathbb{E}[H(T/t)\mathbf{1}_{\{T<t/A\}}] \leq CA^{\alpha-2} \lim_{t\to\infty} L(At)/L(t) = CA^{\alpha-2}.$$

Similarly, we can show that

$$\limsup_{t\to\infty} t^{\alpha} L^{-1}(t)\, \mathbb{E}[H(T/t)\mathbf{1}_{\{T>At\}}] \leq CA^{-\alpha} \leq CA^{\alpha-2}.$$

This proves (8.3) and concludes the proof of Lemma 8.4. □

Lemma 8.5. *Let $j = j(n)$ be a sequence of integers such that $n^{\beta} \leq j \leq n^{\rho}$ for $0 < \beta \leq \rho < 1$. Then*

$$\mathbb{P} - \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \cos(x_j S_k) = \mathbb{P} - \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \sin(x_j S_k) = 0, \tag{8.4}$$

$$\mathbb{P} - \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \cos^2(x_j S_k) = \mathbb{P} - \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \sin^2(x_j S_k) = 1/2. \tag{8.5}$$

where $\mathbb{P} - \lim$ denotes convergence in probability.

Note that (8.5) follows from (8.4) by the relation $\cos^2(u) = (1+\cos(2u))/2$ and by replacing j by $2j$.

To prove Lemma 8.5, we use the following theorem, which adapts Theorem 2 in Yong (1971).

Theorem 8.1. *Let T be a non negative integer valued random variable in the domain of attraction of an α-stable law with $\alpha \in (1,2)$, such that $\mathbb{P}(T \geq k) = k^{-\alpha} L(k)$, where L is slowly varying at infinity. Let ϕ be the characteristic function of T. Then, for $z > 0$, $\phi(z) = 1 - z^{\alpha}\ell_1(z) + \mathrm{i}\ell_2(z)z$ where ℓ_1 and ℓ_2 are slowly varying at zero, positive in a neighborhood of zero and satisfy, for some finite nonzero constant $C(\alpha)$,*

$$\lim_{x\to 0} \ell_1(x)/L(1/x) = C(\alpha) \quad \textit{and} \quad \lim_{x\to 0} \ell_2(x) = \mathbb{E}[T] > 0.$$

We will use Theorem 8.1 through the following bound for the modulus of the characteristic function of T:

$$|\phi(z)|^2 \leq 1 - 2\ell(z) z^{\alpha}, \tag{8.6}$$

where $\ell(z) = \ell_1(z) - \frac{1}{2}\ell_1^2 z^{\alpha} - \frac{1}{2}\ell_2^2(z) z^{2-\alpha}$ is slowly varying and positive in a neighborhood of zero.

Proof of Lemma 8.5. We prove that the convergence holds in L^2. Write

$$\mathbb{E}\left[\left\{\frac{1}{n}\sum_{k=1}^{n}\cos(x_jS_k)\right\}^2\right] = \frac{1}{n^2}\sum_{k=1}^{n}\mathbb{E}[\cos^2(x_jS_k)] + \frac{2}{n^2}\sum_{k=2}^{n}\sum_{\ell=1}^{k-1}\mathbb{E}[\cos(x_jS_\ell)\cos(x_jS_k)]$$

$$= O(n^{-1}) + \frac{1}{n^2}\sum_{k=2}^{n}\sum_{\ell=1}^{k-1}\mathbb{E}[\cos(x_j(S_\ell+S_k)) + \cos(x_j(S_k-S_\ell))],$$

$$\mathbb{E}\left[\left\{\frac{1}{n}\sum_{k=1}^{n}\sin(x_jS_k)\right\}^2\right] = O(n^{-1}) + \frac{1}{n^2}\sum_{k=2}^{n}\sum_{\ell=1}^{k-1}\mathbb{E}[\cos(x_j(S_k-S_\ell)) - \cos(x_j(S_k+S_\ell))].$$

Thus we have to show that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=2}^{n}\sum_{\ell=1}^{k-1}\mathbb{E}[\cos(x_j(S_\ell+S_k))] = 0, \tag{8.7}$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=2}^{n}\sum_{\ell=1}^{k-1}\mathbb{E}[\cos(x_j(S_k-S_\ell))] = 0. \tag{8.8}$$

Proof of (8.8). Applying (8.6), for large enough n, we have

$$\left|\frac{1}{n^2}\sum_{k=2}^{n}\sum_{k'=1}^{k-1}\mathbb{E}[\cos(x_j(S_k-S_{k'}))]\right| \le \frac{1}{n^2}\sum_{k=2}^{n}\sum_{k'=1}^{k-1}|\phi(x_j)|^{k-k'} \le \frac{1}{n}\sum_{k=1}^{n-1}\{1-2\ell(x_j)x_j^\alpha\}^{k/2}.$$

For $z\in(0,1)$ and any real number $t\ge 1$, $(1-z)^{=}\mathrm{e}^{t\log(1-z)} \le \mathrm{e}^{-tz}$ and

$$\frac{1}{n}\sum_{k=1}^{n}(1-z)^{k/2} \le \frac{1}{n}\sum_{k=1}^{n}\mathrm{e}^{-kz/2} = \frac{1-\mathrm{e}^{-nz/2}}{n(\mathrm{e}^{z/2}-1)} \le \frac{1-e^{-nz/2}}{nz/2}.$$

Hence:

$$\left|\frac{1}{n^2}\sum_{k=2}^{n}\sum_{k'=1}^{k-1}\mathbb{E}[\cos(x_j(S_k-S_{k'}))]\right| \le \frac{1}{n}\sum_{k=1}^{n-1}\{1-2\ell(x_j)x_j^\alpha\}^{k/2} \le \frac{1-\mathrm{e}^{-\ell(x_j)nx_j^\alpha}}{n\ell(x_j)x_j^\alpha}. \tag{8.9}$$

Under the assumption on the sequence j, $\lim_{n\to\infty} n\ell(x_j)x_j^\alpha = \infty$. Thus the limit of the last term in (8.9) is 0. This concludes the proof of (8.8).

Proof of (8.7). Since $S_k + S_{k'} = 2S_0 + 2(T_1+\cdots+T_{k'}) + T_{k'+1}+\cdots+T_k$, and denoting ϕ_0 the characteristic function of S_0, we have

$$\left|\frac{1}{n^2}\sum_{k=2}^{n}\sum_{k'=1}^{k-1}\mathbb{E}[\cos(x_j(S_k+S_{k'}))]\right| \le \frac{1}{n^2}\sum_{k=2}^{n}\sum_{k'=1}^{k-1}|\phi_0(2x_j)||\phi(2x_j)|^{k'}|\phi(x_j)|^{k-k'}.$$

Applying (8.6), for large enough n, we obtain:

$$\left|\frac{1}{n^2}\sum_{k=2}^{n}\sum_{k'=1}^{k-1}\mathbb{E}[\cos(x_j(S_k+S_{k'}))]\right| \leq \frac{1}{n^2}\sum_{k=2}^{n}\sum_{k'=1}^{k-1}\{1-2\ell(2x_j)(2x_j)^{\alpha}\}^{k'}\{1-2\ell(x_j)x_j^{\alpha}\}^{k-k'}.$$

Since for any slowly varying function L and any $\alpha > 0$ the function $z^{\alpha}L(z)$ is ultimately non decreasing, we obtain, for n large enough:

$$\left|\frac{1}{n^2}\sum_{k=2}^{n}\sum_{k'=1}^{k-1}\mathbb{E}[\cos(x_j(S_k+S_{k'})]\right| \leq \frac{1}{n}\sum_{k=2}^{n}\{1-2\ell(x_j)x_j^{\alpha}\}^{k/2}.$$

The same line of reasoning as previously concludes the proof of (8.8) and of Lemma 8.5. $\square$

Figure 1: QQ Plots of the Normalized Fourier Cosine Coefficients $A_j/f(\omega_j)^{\frac{1}{2}}$ for Parke process; n=10000, d=0.1

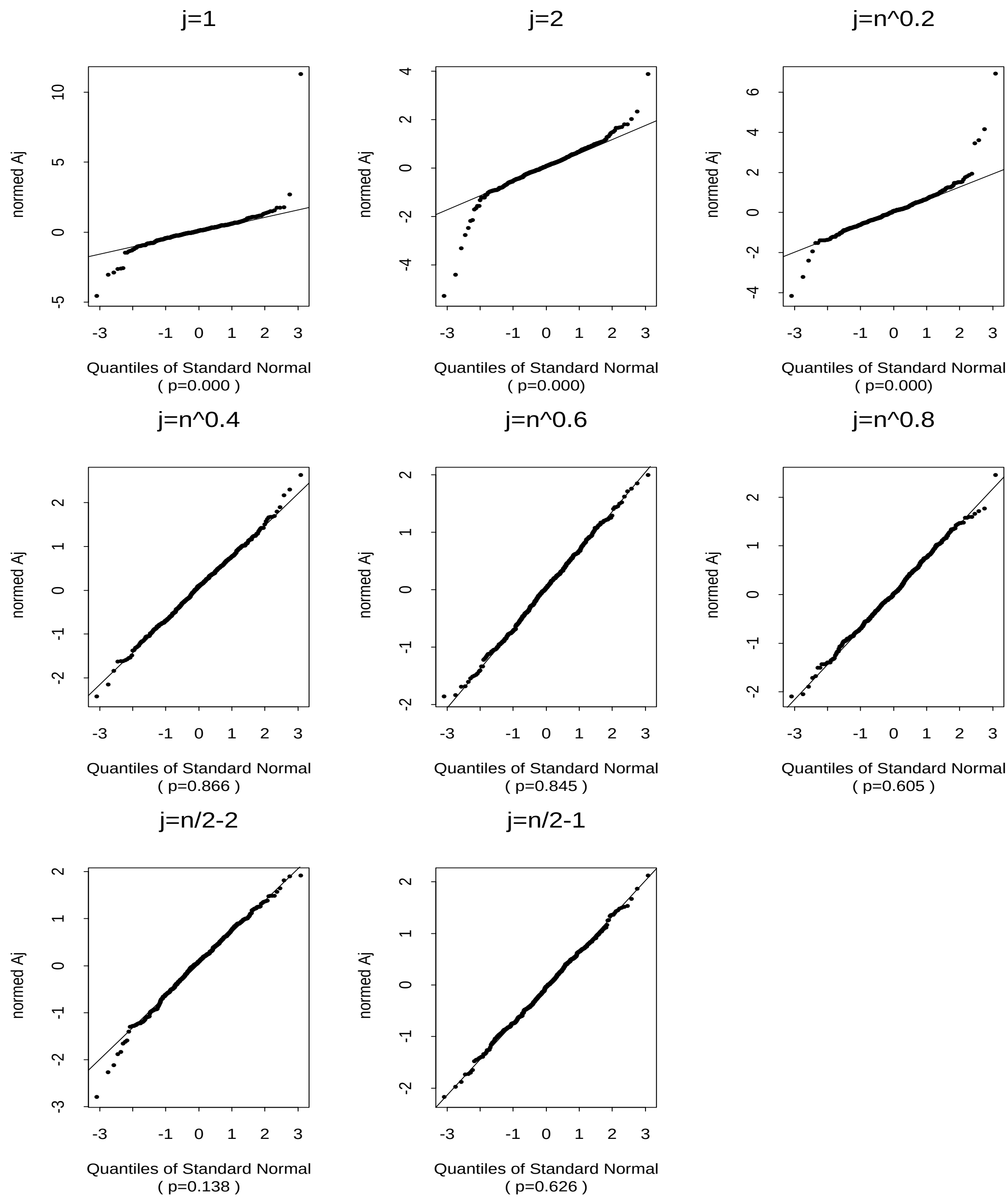

Figure 2: QQ Plots of the Normalized Fourier Cosine Coefficients $A_j/f(\omega_j)^{\frac{1}{2}}$ for Parke process; n=10000, d=0.4

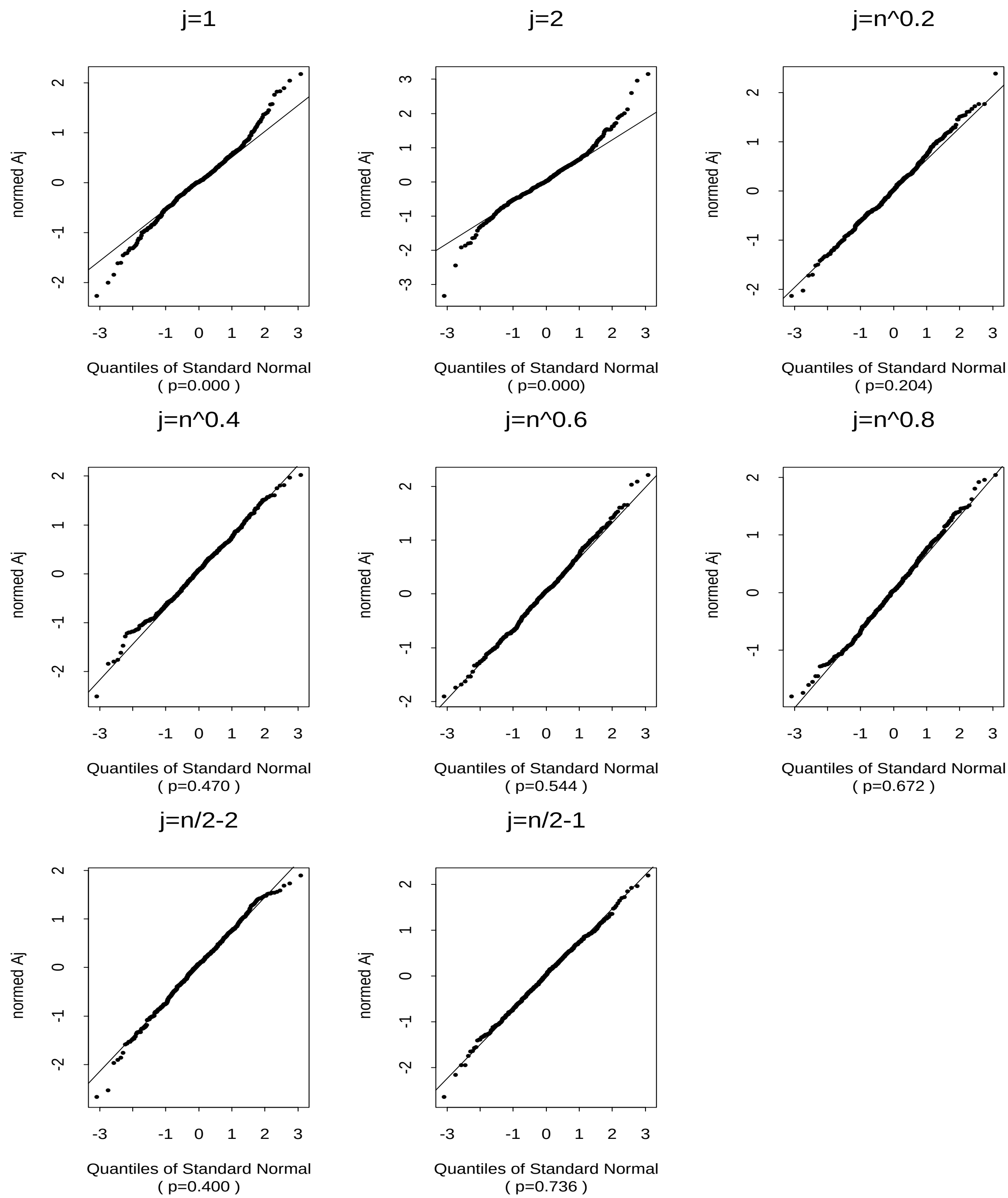

Figure 3: Scatterplots of Average Log Normalized Periodogram vs. $\log|2\sin(x_j/2)|$; j=1,2,...,4999 . Horizontal line represents $-\gamma = -0.577216$.

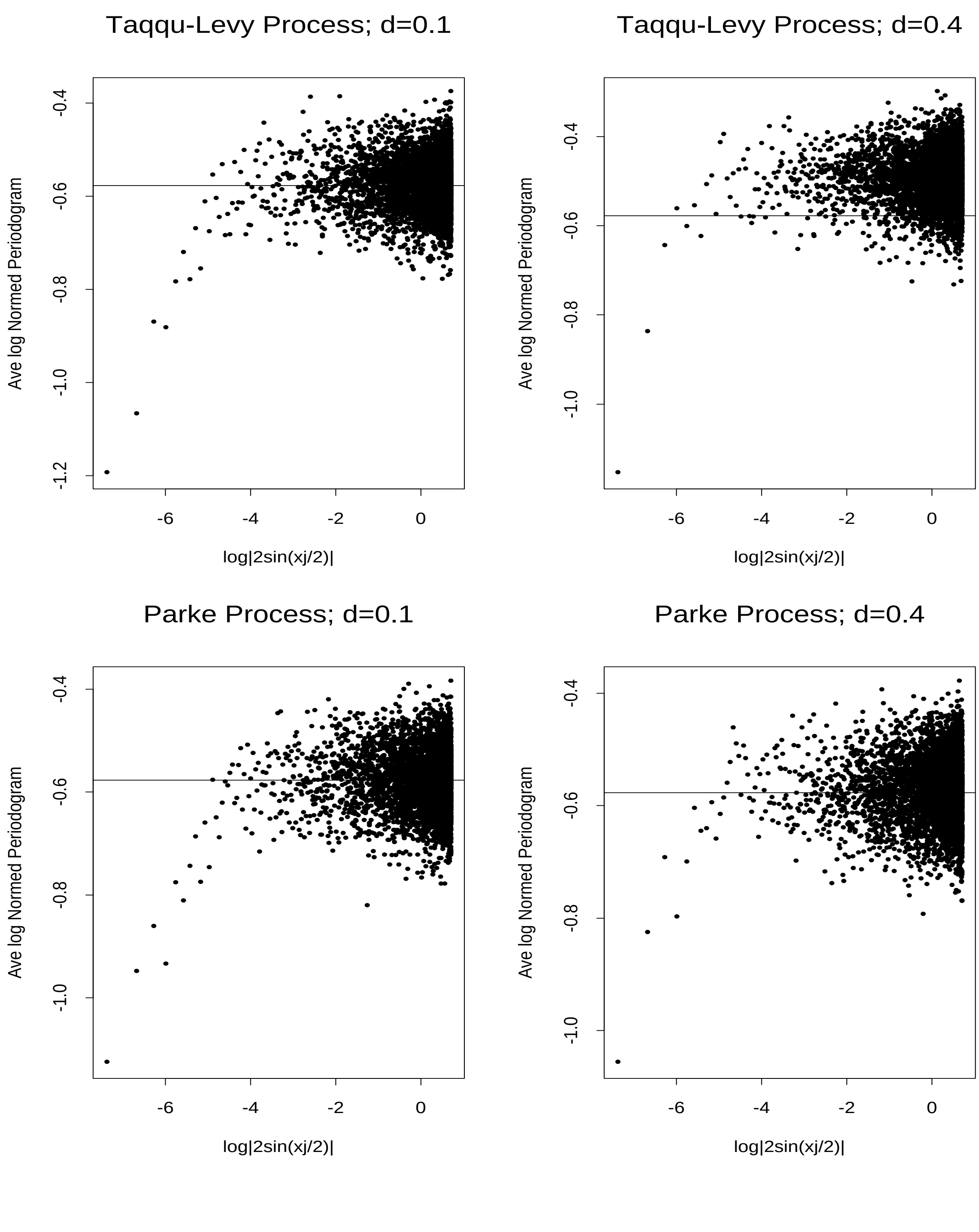

Figure 4: Scatterplots of Average Log Periodogram vs. $\log|2\sin(x_j/2)|$; j=1,2,...,4999

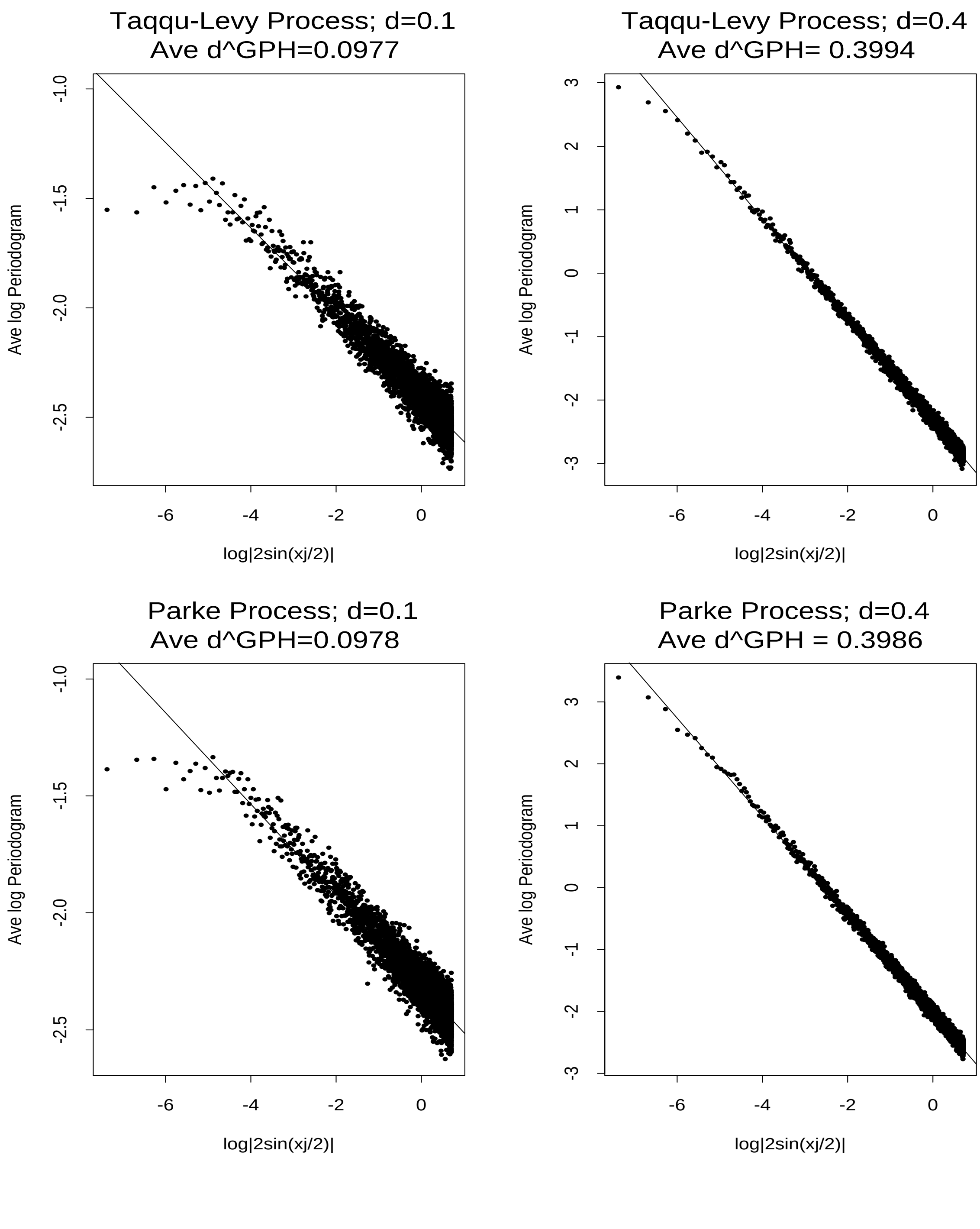

Figure 5: Normal QQ Plots of Sample Autocorrelations for Taqqu-Levy Process, d=0.1

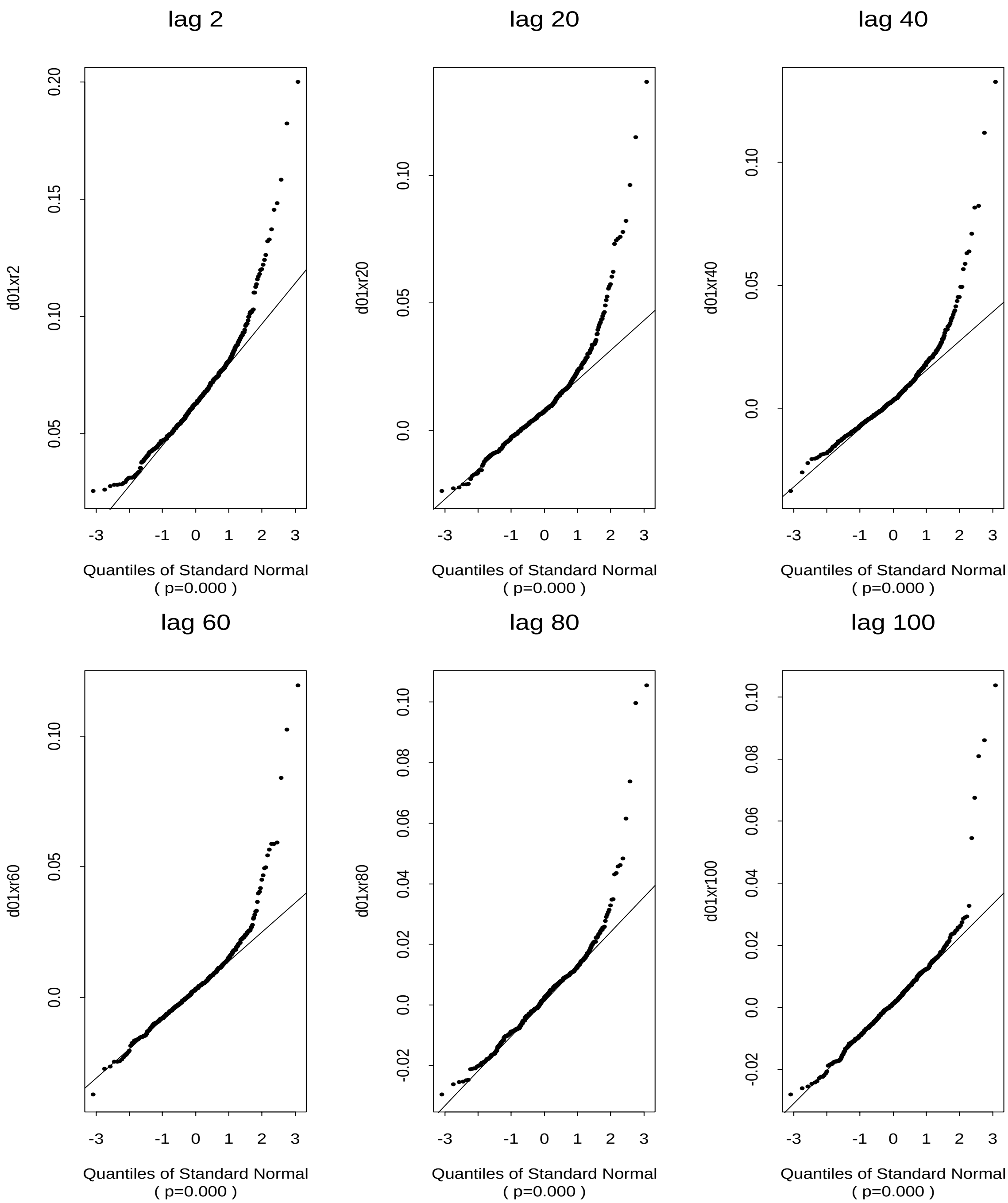

Table 1: Simulation variances for normalized DFT cosine coefficients at frequency x_j for Taqqu-Levy Process, with normal-based 95% Confidence Intervals. $\alpha = 0.05$. Intervals marked with $*$ reject the null hypothesis, $\sigma^2 = 0.5$.

d	x_j	Variance	Confidence Interval	
0.1	$n^{0.2}$	0.54	0.48	0.62
	$n^{0.4}$	0.54	0.47	0.61
	$n^{0.6}$	0.49	0.44	0.56
	$n^{0.8}$	0.53	0.47	0.60
	$\frac{n}{2} - 2$	0.51	0.45	0.58
	$\frac{n}{2} - 1$	0.50	0.44	0.56
0.4	$n^{0.2}$	0.56	0.49	0.63
	$n^{0.4}$	0.55	0.49	0.62
	$n^{0.6}$	0.52	0.46	0.59
	$n^{0.8}$	0.55	0.49	0.63
	$\frac{n}{2} - 2$	0.55	0.48	0.62
	$\frac{n}{2} - 1$	0.58	0.51	0.66*

Table 2: Simulation variances for normalized DFT cosine coefficients at frequency x_j for Parke Process, with normal-based 95% Confidence Intervals. $\alpha = 0.05$. Intervals marked with $*$ reject the null hypothesis, $\sigma^2 = 0.5$.

d	x_j	Variance	Confidence Interval	
0.1	$n^{0.2}$	0.66	0.58	0.75*
	$n^{0.4}$	0.54	0.48	0.61
	$n^{0.6}$	0.46	0.41	0.53
	$n^{0.8}$	0.51	0.45	0.58
	$\frac{n}{2} - 2$	0.49	0.44	0.56
	$\frac{n}{2} - 1$	0.46	0.41	0.53
0.4	$n^{0.2}$	0.47	0.42	0.54
	$n^{0.4}$	0.49	0.44	0.56
	$n^{0.6}$	0.46	0.41	0.53
	$n^{0.8}$	0.47	0.42	0.53
	$\frac{n}{2} - 2$	0.54	0.48	0.62
	$\frac{n}{2} - 1$	0.52	0.46	0.59

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