Optimal Exercise of Executive Stock Options and Implications for Firm Cost[1]

Jennifer N. Carpenter	Richard Stanton	Nancy Wallace
New York University	U.C. Berkeley	U.C. Berkeley

December 5, 2006

[1]We would like to thank Peter Carr and Peter Lakner for helpful comments and suggestions. We also thank Rik Sen for valuable research assistance.

Corresponding author: Jennifer Carpenter, New York University Stern School of Business, 44 West Fourth St., Ste. 9-190, New York, NY 10012, carpenter@stern.nyu.edu.

Optimal Exercise of Executive Stock Options and Implications for Firm Cost

Abstract

The cost of executive stock options to shareholders has become a focus of attention in finance and accounting. The difficulty is that the value of these options depends on the exercise policies of the executives. Because these options are nontransferable, the usual theory does not apply. We analyze the optimal exercise policy for a general utility-maximizing executive and indicate when the policy is characterized by a critical stock price boundary. We provide a counterexample in which the executive exercises at low and high stock prices but not in between. We show how the policy varies with risk aversion, wealth, and volatility and explore implications for option value. For example, option value can decline as volatility rises.

With the explosive growth of executive stock options in corporate compensation, the cost of these options to shareholders has become a focus of attention in finance and accounting. Recent regulation requiring firms to recognize option expense after 2005 has intensified the demand for suitable valuation methods. The difficulty is that the value of these options depends crucially on the exercise policies of the option holders, but, because these options are nontransferable, the usual theory does not apply.

In the case of an ordinary call, the holder can sell the option at any time, so his goal is presumably to maximize the option's present value. The value-maximizing exercise policy in a Black-Scholes world has been researched extensively (see Merton (1973), Van Moerbeke (1976), Roll (1977), Geske (1979), Whaley (1981), Kim (1990)). It calls for exercising the option once the stock price rises above a critical level. This critical level is increasing in the riskless rate, the stock return volatility, and the time remaining to maturity, and it is decreasing in the dividend rate, with no early exercise if the dividend rate is zero.

By contrast, the holder of an executive stock option must bear the risk of the option payoff, so simply maximizing the option's present value is generally not optimal. Indeed, evidence indicates that executives systematically exercise options on non-dividend paying stocks well before expiration. The executive presumably chooses an option exercise policy as part of a greater utility maximization problem that includes other decisions, such as portfolio and consumption choice and managerial strategy.

This paper studies the optimal exercise policy for an executive stock option under simple but appealing assumptions about the executive's choice set. We analyze the optimal exercise policy for a general utility-maximizing executive and indicate when the policy is characterized by a critical stock price boundary. We provide a counterexample in which the executive exercises at low and high stock prices but not in between. We show how the policy varies with risk aversion, wealth, and volatility, and explore implications for option value. For example, we provide conditions under which the continuation region is larger the greater executive risk aversion and outside wealth, and the lower the stock dividend rate. On the other hand, the size of the continuation region is not monotonic in stock volatility, and option cost to shareholders can actually decline as volatility rises.

The intuition that the need for diversification can lead an executive to sacrifice some option value by exercising it early is well understood in the literature, but explicit theory of the optimal exercise of ESOs is still developing. Huddart (1994), Marcus and Kulatilaka (1994), and Carpenter (1998) build binomial models of the utility-maximizing exercise decision with exogenous assumptions about how non-option wealth is invested. Detemple and Sundaresan (1999) extend these to allow for simultaneous option exercise and portfolio choice decisions. These papers establish the economic approach to ESO valuation, focusing on the optimality of early exercise (and the fact that this makes ESOs worth less than their Black-Scholes value), rather than an in-depth analysis of the exercise policy itself. In a continuous-time framework, Ingersoll (2006) develops a subjective option valuation methodology, assuming the option is a marginal component of the executive's portfolio. More recently, several papers have solved versions of the problem we describe here for the case of constant absolute risk averse utility, where the optimal exercise policy is independent of the executive's wealth. Leung and Sircar (2006) solve the finite horizon problem, and include the risk of job termination and the possibility of partial option exercise. Kadam, Lakner, and Srinivasan (2003) model the optimal exercise policy for an infinite horizon option, but the model links the manager's consumption date to the option exercise date, which can distort the exercise decision, even in the absence of trading restrictions. Similarly, Henderson (2004) models the optimal exercise policy for an infinite horizon real option, and links the manager's consumption date to the option exercise date, but uses a specialized utility function so that this link does not distort the exercise policy.

A number of papers model option value using exogenous specifications of the exercise policy. Jennergren and Näslund (1993), Carr and Linetsky (2000), and Cvitanić, Wiener, and Zapatero (2004) derive analytic formulas for option value assuming exogenously specified exercise boundaries and forfeiture rates. Hull and White (2004) propose a binomial model in which exercise occurs when the stock price reaches an exogenously specified multiple of the stock price and forfeiture occurs at an exogenous rate. Rubinstein (1995) and Cuny and Jorion (1995) also compute option value under exogenous assumptions about the timing of exercise.

Other authors have focused on the executive's private valuation of the option using certainty equivalents rather than on the market value of the option from the viewpoint of shareholders. These include Lambert, Larcker, and Verrecchia (1991), Hall and Murphy (2002), Cai and Vijh (2005), and Miao and Wang (2005).

1 General framework

This section lays out the general model of the executive's optimal exercise problem and defines the resulting option cost to shareholders.

1.1 General formulation of the executive's problem

In the general version of the problem we consider, the executive has n finite-lived options with strike price K and expiration date T

and additional wealth that can be invested subject to a prohibition on short sales of the stock. The investment set includes riskless bonds with constant riskless rate r, the underlying stock with price S_t, and a market portfolio with price M_t. These prices satisfy

$$\frac{dS_t}{S_t} = (\lambda - \delta)\, dt + \sigma\, dB_t \ , \tag{1}$$

$$\frac{dM_t}{M_t} = \mu\, dt + \sigma_m\, dB_t \ , \tag{2}$$

where B_t is a standard two-dimensional Brownian motion on a probability space equipped with the natural filtration and σ and σ_m are two-dimensional row vectors. The stock return volatility, σ, the stock dividend rate δ, and the mean and volatility of the market return, μ and σ_m are constant, and the mean stock return λ is equal to the normal return for the stock given its correlation with the market,

$$\lambda = r + \frac{\sigma\sigma_m'}{||\sigma_m||^2}(\mu - r) \ . \tag{3}$$

In particular, in the absence of the option, an optimal portfolio would contain no stock position beyond what is implicitly included in the market portfolio.

The executive simultaneously chooses an option exercise time τ, which is a stopping time of the filtration generated by the Brownian motion, and an investment strategy in the market and the stock, $\pi_t \equiv (\pi_t^m, \pi_t^s)$ satisfying $\mathrm{E}\int_{t=0}^{T} ||\pi_t||^2\, dt < \infty$. His goal is to maximize the expected utility of time T wealth:

$$\max_{\{\tau \le T, \pi^m, \pi^s \ge 0\}} \mathrm{E}\{V(W_\tau + n(S_\tau - K)^+, \tau)\} \tag{4}$$

subject to

$$dW_t = rW_t\, dt + \pi_t^m((\mu - r)\, dt + \sigma_m\, dB_t) + \pi_t^s((\lambda - r)\, dt + \sigma\, dB_t) \ , \tag{5}$$

where

$$V(W_t, t) \equiv \max_{\pi^m} \mathrm{E}_t\{U(W_T)\} \text{ s.t. } dW_u = rW_u\, du + \pi_u^m((\mu - r)\, du + \sigma_m\, dB_u) \ , \tag{6}$$

and the utility function U is strictly increasing, strictly concave, and twice continuously differentiable.

This formulation entails a number of simplifications. The executive's portfolio does not include a position in restricted shares of stock (see Kaul, Liu, and Longstaff (2003) for a model of portfolio choice with restricted stock). It allows only for a single block exercise of the option, although the executive would probably prefer to exercise the options at a stochastic rate over time. The model also considers only a single grant of options when in practice, executives are granted new ten-year options every year and typically build up large inventories of options with different strikes and expiration dates. It would be useful to understand which options are most attractive to exercise first and how the anticipation of future grants of options and other forms of compensation affects current exercise decisions. In addition, the model does not account for any control the executive has over the underlying stock price process through the exertion of effort and through project and leverage choices; these choices may interact with the exercise decision. Despite these simplifications, we believe this formulation captures the essence of the executive stock option problem.

Intuition suggests that the optimal outside position in the stock in problem (4) is $\pi^s \equiv 0$, however this remains to be proved. The example in Evans, Henderson, and Hobson (2005) shows that results from traditional portfolio theory may fail to hold in the presence of an optimal stopping problem.

If the optimal investment policy π_t and the indirect utility function V satisfy, respectively, linear and polynomial growth conditions in W and S, then Theorem 3.1.8 of Krylov (1980) implies that the value function for the executive's problem,

$$f(W_t, S_t, t) \equiv \max_{\{t \leq \tau \leq T, \pi^m, \pi^s \geq 0\}} \mathrm{E}_t\{V(W_\tau + n(S_\tau - K)^+, \tau)\} \tag{7}$$

subject to

$$dW_u = rW_u\, dt + \pi_u^m((\mu - r)\, du + \sigma_m\, dB_u) + \pi_u^s((\lambda - r)\, du + \sigma\, dB_u)\ , \tag{8}$$

is continuous and satisfies $f(W_t, S_t, t) \geq V(W_\tau + n(S_\tau - K)^+, \tau)$ and $f(W_T, S_T, T) = U(W_T + n(S_T - K)^+)$.

1.2 Option cost to shareholders

The solution to the executive's optimal exercise problem, that is, the optimal exercise policy τ, defines the option payoff, $(S_\tau - K)^+$ that occurs at time τ. The cost of the option to shareholders who can trade freely is the present value, or replication cost, of that payoff, which can be represented as the risk-neutral expectation of its risklessly discounted value,

$$\mathrm{E}^*\{e^{-r\tau}(S_\tau - K)^+\}\ , \tag{9}$$

where E^* means the expectation is taken with respect to the probability measure under which the expected returns on both the market and the stock are equal to the riskless rate.

Standard theory for tradeable options assumes the option holder chooses the exercise policy to maximize the option's present value, because when the option is tradeable, maximizing present value is consistent with maximizing expected utility. When the option is nontransferable these objectives are different, and the utility-maximizing payoff typically has a lower present value.

In addition, when the option is nontransferable, its value to the executive is different from its present value or cost to shareholders. The value to the executive may be defined as the amount of freely investable cash that would make the executive equally happy as having the nontransferable option. Since the having the option present value tied up in the option can be no better for the executive than having that value invested freely, the amount of freely investable cash that would make the executive as happy as the having the option can be no more than the option's present value and is typically less. That discount in the executive's private valuation of the option relative to its cost to shareholders is the price shareholders pay for any performance incentive benefits the option creates relative to cash compensation. While the executive's private valuation is an important consideration in the more general problem of optimal contracting, our focus in this paper is on the option cost to shareholders.

2 Special case with outside wealth in riskless bonds

We start by analyzing the case in which the stock appreciates at the riskless rate,

$$\frac{dS_t}{S_t} = (r - \delta)\, dt + \sigma\, dB_t \ , \tag{10}$$

and there is no other risky asset available. In this case, it is clear that after the executive exercises the options, his optimal portfolio contains only riskless bonds, so

$$V(W_t, t) = U(W_t e^{r(T-t)}) \ . \tag{11}$$

Intuition suggests that even before the option is exercised, the executive's optimal outside portfolio contains no stock, since he would choose to short stock in the absence of a short sale constraint. A sketch of a proof of this conjecture appears in the Appendix. We proceed with the assumption that the outside wealth is invested in riskless bonds. The executive's problem at each time $t < T$ becomes

$$f(S_t, t) \equiv \max_{\{t \le \tau \le T\}} \mathrm{E}_t\{U(n(S_\tau - K)^+ e^{r(T-\tau)} + W)\} \ , \tag{12}$$

where the constant W is outside wealth at time T with $W > nKe^{rT}$ and $f : (0, \infty) \times [0, T] \to \mathcal{R}$ is a continuous function satisfying $f(S_t, t) \ge U(n(S_t - K)^+ + W)$ and $f(S_T, T) = U(n(S_T - K)^+ + W)$.

Note that

$$\begin{aligned}
\mathrm{E}[\sup_{0 \le t \le T} U(n(S_t - K)^+ e^{r(T-t)} + W)] &= \mathrm{E}[U(\max_{0 \le t \le T}(n(S_t - K)^+ e^{r(T-t)} + W))] & (13)\\
&\le U(\mathrm{E}[\max_{0 \le t \le T}(n(S_t - K)^+ e^{r(T-t)} + W)]) & (14)\\
&< \infty \ , & (15)
\end{aligned}$$

so Theorem D.12 of Karatzas and Shreve (1998) implies that an optimal exercise time is

$$\tau^* \equiv \inf\{t \in [0, T] : f(S_t, t) = U(n(S_t - K)^+ e^{r(T-t)} + W)\} \tag{16}$$

and the continuation region for the problem is

$$D \equiv \{(s, t) \in (0, \infty) \times [0, T] : f(s, t) > U(n(s - K)^+ e^{r(T-t)} + W)\} \ . \tag{17}$$

2.1 Existence of a critical stock price boundary

This section explores whether a single critical stock price boundary $\bar{s}(t)$ separates the continuation region below from the exercise region above, as is the case for ordinary American calls. This is often assumed to be true in executive stock option models with exogenously specified exercise policies, however, it remains to be proved that the utility-maximizing policy has this structure.

To formalize intuition about the various effects of waiting to exercise, let $g(s, t) \equiv U(n(s - K)^+ e^{r(T-t)} + W)$ denote the payoff function for the optimal stopping problem and note that on $(K, \infty) \times [0, T]$, g is $C^{2,1}$ and Itô's lemma implies that g has drift equal to $H(S_t, t)$ where

$$H(s, t) \equiv U'(h(s, t))(rK - \delta s)ne^{r(T-t)} + \frac{1}{2}U''(h(s, t))n^2 e^{2r(T-t)}\sigma^2 s^2 \tag{18}$$

and $h(s, t) \equiv n(s - K)e^{r(T-t)} + W$ is total time T wealth given exercise at time t and stock price s. This expression shows that when the option is in the money, the effects of waiting to exercise include the benefits of delaying payment of the strike price, the cost of losing dividends, and the cost of bearing stock price risk.

Proposition 2.1 *Suppose that H is nonincreasing in the stock price s. Then for each time $t \in [0, T)$, if there is any stock price at which exercise is optimal, then there exists a critical stock price $\bar{s}(t)$ such that it is optimal to exercise the option if and only if $S_t \geq \bar{s}(t)$.*

Proof Fix $t \in [0, T)$. Suppose (s_1, t) is a continuation point. We show that if $s_2 < s_1$ then (s_2, t) is also a continuation point. First note that it must be optimal to continue holding the option if $S_t \leq K$. Stopping then would guarantee a reward of $U(W)$, which is less than the expected utility of continuing, for example, until the first time the stock price rises to $K + c$, for some $c > 0$, or until expiration T.

So assume $s_1 > s_2 > K$. For $u \geq t$, let $S_u^{(i)}$ denote the stock price process starting from s_i at time t and note that $S_u^{(1)} > S_u^{(2)}$. Finally, let τ be the optimal stopping time given $S_t = s_1$. Since τ is a feasible strategy if $S_t = s_2$,

$$
\begin{aligned}
f(s_2, t) - f(s_1, t) &\geq \mathrm{E}_t\{U(n(S_\tau^{(2)} - K)^+ e^{r(T-\tau)} + W) - U(n(S_\tau^{(1)} - K)^+ e^{r(T-\tau)} + W)\} \\
&\geq \mathrm{E}_t\{U(n(S_\tau^{(2)} - K) e^{r(T-\tau)} + W) - U(n(S_\tau^{(1)} - K) e^{r(T-\tau)} + W)\} \\
&= g(s_2, t) - g(s_1, t) + \mathrm{E}_t \int_t^\tau (H(S_u^{(2)}, u) - H(S_u^{(1)}, u)) du \\
&\geq g(s_2, t) - g(s_1, t) \ . \qquad (19)
\end{aligned}
$$

Therefore, $f(s_2, t) - g(s_2, t) \geq f(s_1, t) - g(s_1, t) > 0$.

Remark The hypothesis is satisfied for constant relative risk averse utility functions with relative risk aversion less than or equal to one. Similarly, in the value maximization problem for an ordinary option, the second order term in H does not appear, the drift is nonincreasing in the stock price, and it follows that it is optimal to exercise if and only if the stock price has risen above a critical level. For executive stock options however, the risk aversion of the option holder gives rise to the second order term, and the drift need no longer be monotonic in the stock price.

Example with a split continuation region Figure 1 shows the optimal exercise policy for utility function

$$
U(W) = \frac{W^{1-A}}{1-A} + cW \qquad (20)
$$

with $A = 10, c = 0.0001, K = 1, T = 10, r = 0.05, \sigma = 30\%$, and $\delta = 0$. The utility function is strictly increasing and strictly concave. As the figure shows, the executive continues for low and high stock prices, but exercises the option for intermediate stock prices. It is not clear, however, how much valuation error would be created by erroneously assuming the existence of a single critical exercise boundary. That would depend on how that single boundary was determined. In this example, if we ignore the presence of the upper boundary, the option cost is 0.408 instead of the correct value of 0.432.

2.2 Dependence of the continuation region on the parameters

Understanding how executive stock option value varies with stock return volatility, executive wealth, and other parameters requires an understanding of how these parameters affect the exercise policy. With an ordinary American call option, the exercise boundary, or in other words, the set of stock prices at which the option holder would continue at a given point in time, is increasing with the stock volatility and the time to expiration and decreasing with the dividend rate. With executive stock options, the dependence of the continuation region on the parameters is less clear cut. This section describes how the continuation region changes with executive risk aversion and wealth, the stock dividend rate, and the stock return volatility.

2.2.1 Monotonicity with respect to risk aversion and wealth

Intuition suggests that less risk averse managers are likely to continue longer. Similarly, one would expect that managers with decreasing absolute risk aversion will continue longer if they have more nonoption wealth. The following results verify this intuition and hold regardless of the actual shape of the continuation region.

Proposition 2.2 *An executive with less absolute risk aversion has a larger continuation region.*

Proof If U_1 and U_2 are utility functions and U_2 has everywhere less absolute risk aversion than U_1, then by Theorem 5 on page 40 of Ingersoll (1987),

$$U_2(W) = G(U_1(W)) \tag{21}$$

where the function G satisfies $G' > 0$ and $G'' > 0$. Now suppose a given state (s, t) is in the continuation region with utility U_1 and let τ be the optimal stopping time for U_1. Let $f_i(s, t)$ and $g_i(s, t)$ denote the value and payoff functions for the problem with utility U_i. Since τ is feasible for the problem with U_2,

$$\begin{aligned} f_2(s,t) - g_2(s,t) &\geq \mathrm{E}_t\{U_2(n(S_\tau - K)^+ e^{r(T-\tau)} + W)\} - U_2(n(S_t - K)^+ e^{r(T-t)} + W) \\ &= \mathrm{E}_t\{G(U_1(n(S_\tau - K)^+ e^{r(T-\tau)} + W))\} - G(U_1(n(S_t - K)^+ e^{r(T-t)} + W)) \\ &\geq G(\mathrm{E}_t\{U_1(n(S_\tau - K)^+ e^{r(T-\tau)} + W)\}) - G(U_1(n(S_t - K)^+ e^{r(T-t)} + W)) \end{aligned}$$

$$
\begin{aligned}
&= G(f_1(s,t)) - G(g_1(s,t)) \\
&> 0 \qquad (22)
\end{aligned}
$$

Therefore, (s, t) is also in the continuation region for U_2.

Corollary 2.1 *If the executive has decreasing absolute risk aversion, then the continuation region is larger with greater wealth.*

Proof Let $W_2 > W_1$ and note that $U(w + W_2 - W_1) = G(U(w))$ for some function G satisfying $G' > 0$ and $G'' > 0$.

Figures 2 and 3 illustrate these results and their implications for option value using examples with constant relative risk averse utility. In all of the numerical examples, the dividend rate is set to zero so that the only motive for early exercise is the ability to transfer the option value to a more efficient portfolio, in this case, the riskless asset. In addition, the number of options and initial stock price are each normalized to one. Even in examples in which the coefficient of relative risk aversion, A, is greater than one, we find that the continuation region is characterized by a single critical stock price boundary.

Figure 2 plots the exercise boundaries and option values for various levels of risk aversion. The exercise boundary is a plot of the critical stock price $\bar{s}(t)$ vs. time t. The option value labeled "ESO" is the present value or cost of the option under the executive's exercise policy, described by the boundary. Shown for comparison, the option value labeled "Max" is the value of the option under the present value-maximizing policy, which in this zero-dividend case is to hold the option to maturity. The figure shows that option cost to shareholders is greater the less risk averse the executive. Figure 3 plots exercise boundaries and option values for various levels of executive wealth. It shows that option cost to shareholders is greater the wealthier the executive.

2.2.2 Monotonicity with respect to the dividend rate

This section shows that the executive's continuation region is larger the smaller the dividend rate on the stock, as is the case for an ordinary American option. Again, this result holds regardless of the shape of the continuation region.

Proposition 2.3 *The executive's continuation region is larger the smaller the dividend rate on the stock.*

Proof Suppose a given state (s, t) is in the continuation when the dividend rate is δ_1 and let $\delta_2 < \delta_1$. Let $f(s, t; \delta)$ denote the value function and $S_t^{(\delta)}$ denote the stock price process when the dividend rate is δ. Let τ be the optimal stopping time for the problem with δ_1. Then, since τ is a feasible choice for the problem with δ_2 and $S_\tau^{(\delta_2)}/S_t^{(\delta_2)} > S_\tau^{(\delta_1)}/S_2^{(\delta_1)}$,

$$\begin{aligned} f(s,t;\delta_2) - f(s,t;\delta_1) &\geq \mathrm{E}\{U(n(S_\tau^{(\delta_2)} - K)^+ e^{r(T-\tau)} + W) - \\ &\quad U(n(S_\tau^{(\delta_1)} - K)^+ e^{r(T_1-\tau)} + W) | S_t^{(\delta_1)} = S_t^{(\delta_2)} = s\} \\ &\geq 0 \end{aligned} \tag{23}$$

Therefore, $f(s,t;\delta_2) \geq f(s,t;\delta_1) > g(s,t)$ so (s,t) is in the continuation region for δ_2.

2.2.3 Non-monotonicity with respect to the stock return volatility

A basic result in standard option pricing theory is that option value is increasing in volatility. This is also typically the case in executive stock option models with an exogenously specified exercise boundary that does not change with volatility (see, for example, Cvitanić, Wiener, and Zapatero (2004)). However, the utility-maximizing continuation region can shrink considerably with volatility and this can lead to option value declining in volatility.

Figure 4 illustrates these effects using examples with constant relative risk averse utility and a zero dividend rate. Again, in all examples, even those in which the coefficient of relative risk aversion, A, is greater than one, the continuation region is characterized by a single critical stock price boundary. Figure 4 plots the exercise boundaries and option values for various levels of stock return volatility.

As volatility rises from 10% to 200%, the exercise boundary tends to fall first and then rise slightly. This is shown most clearly in Figure 4a, with risk aversion coefficient $A = 0.5$. The risk averse utility of the option payoff as a function of the stock price has both a convex region and a concave region, so in principle, an increase in volatility could either lead the executive to continue longer or exercise sooner. Apparently the concave portion dominates at low levels of volatility, making the executive exercise sooner as volatility rises. At higher levels of volatility, the convex portion seems to dominate and the boundary rises slightly. Empirically, Bettis, Bizjak, and Lemmon (2005) find that options are exercised earlier at higher volatility firms.

At the lower levels of risk aversion shown in Figures 4a and 4b, executive stock option value is generally increasing in volatility. However, at the higher levels of risk aversion shown in Figures 4c and 4d, executive stock option value is decreasing in volatility at low levels of volatility. Here the negative effect on value of the drop in the boundary of offsets the positive effect of extreme stock prices becoming more likely.

3 Summary and Conclusions

This paper seeks to advance the theory of executive stock option valuation with an in-depth study of the optimal exercise policy of a risk averse executive. Recent valuation models for executive stock options set the exercise policy exogenously, assuming a single critical stock price boundary. This paper shows that the optimal exercise policy need not be in that form. However, when riskless bonds are the only investment available and the stock underlying the option appreciates at the riskless rate, we provide a sufficient condition for the existence of a single critical boundary. This condition is satisfied by constant relative risk averse utility functions with risk aversion coefficient less than or equal to one and we find no counterexamples among our numerical results for constant relative risk averse utility functions with risk aversion coefficient greater than one.

We also prove that the continuation region is larger for executives with less absolute risk aversion and larger for wealthier executives with decreasing absolute risk aversion, and these results hold regardless of the exact shape of the continuation region. Our numerical examples with constant relative risk aversion show that option cost to shareholders is increasing in executive nonoption wealth and decreasing in executive risk aversion.

The examples also show how the exercise boundary and option value vary with volatility. In contrast to results from standard option theory, or from executive stock option valuation models with a fixed exercise boundary, executive stock option value can decline in stock return volatility when increases in volatility cause the optimal exercise boundary to drop sufficiently. These results underscore the importance of accurately characterizing the exercise policy for option valuation.

Appendix

Conjecture 1 *When the stock appreciates at the riskless rate and there is no other risky asset available, the executive's optimal investment of outside wealth contains no long position in the stock.*

Sketch of proof Suppose that an optimal stopping time is

$$\tau^* = \inf\{t \in [0, T] : f(W_t, S_t, t) = V(W_t + n(S_t - K)^+, t)\} \ , \tag{24}$$

the time of first exit from the continuation region

$$D \equiv \{(w, s, t) \in (0, \infty) \times (0, \infty) \times [0, T] : f(w, s, t) > V(w + n(s - K)^+, t)\} \ . \tag{25}$$

We apply the duality methods of Karatzas and Shreve (1998) to show that zero is the optimal constrained holding of stock for $t < \tau^*$. In particular, we construct a new market, in which holding zero stock is the unconstrained optimum and then use it demonstrate that zero is the optimal constrained stock holding in the original market.

The first step is to construct and characterize the pricing kernel for the new market. Let

$$Z(t) = \frac{\mathrm{E}\{U'(n(S_{\tau^*} - K)^+ e^{r(T-\tau^*)} + W_0 e^{rT})|\mathcal{F}_t\}}{\mathrm{E}\{U'(n(S_{\tau^*} - K)^+ e^{r(T-\tau^*)} + W_0 e^{rT})\}} \tag{26}$$

For $t \geq \tau^*, Z(t) = Z(\tau^*)$. For $t < \tau^*$, $Z(t)$ is a function of the stock price and time, $Z(t) = z(S_t, t)$, because we have assumed that τ^* is the first exit time from the region $\mathcal{D}$. If z is smooth enough for an application of Itô's lemma, then z satisfies

$$\frac{1}{2}\sigma^2 S^2 z_{SS} + rSz_S + z_t = 0 \tag{27}$$

because Z is a martingale. Following Carr (2000), we differentiate with respect to S to obtain a p.d.e. for $z'(S_t, T) \equiv z_S(S_t, t)$,

$$\frac{1}{2}\sigma^2 S^2 z'_{SS} + (r + \sigma^2)Sz'_S + z'_t = -rz' \ , \tag{28}$$

subject to the boundary condition

$$z'(S_{\tau^*}, \tau^*) = \frac{1_{\{S_{\tau^*} > K\}} n e^{r(T-\tau^*)} U''(n(S_{\tau^*} - K)^+ e^{r(T-\tau^*)} + W_0 e^{rT})}{\mathrm{E}\{U'(n(S_{\tau^*} - K)^+ e^{r(T-\tau^*)} + W_0 e^{rT})\}} \ . \tag{29}$$

From equation (28), it follows that

$$z'(S_t, t) = \mathrm{E}^{(1)}\{e^{r\tau^*} z'(S_{\tau^*}, \tau^*)|\mathcal{F}_t\} \ , \tag{30}$$

where the expectation $\mathrm{E}^{(1)}$ is taken with respect to the equivalent probability measure under which the stock price appreciates at rate $r + \sigma^2$. Therefore, $z'(S_t, t) < 0$ and thus the diffusion coefficient of z is strictly negative. Therefore, the strictly positive martingale Z can be represented as

$$Z(t) = Z_\nu(t) \equiv e^{-\int_0^t \frac{\nu(s)}{\sigma}\, dB(s) - \frac{1}{2}\int_0^t (\frac{\nu(s)}{\sigma})^2\, ds} \tag{31}$$

where $\nu(t)$ is a progressively measurable, square-integrable process that is strictly positive for $t < \tau^*$.

Next, imagine a market in which the stock price appreciates at rate $r + \nu - \delta$ instead of merely $r - \delta$ as in equation (10). For any nonnegative portfolio process $\pi^s(t)$, the corresponding wealth process $W_\nu^{\pi^s}(t)$ starting from initial wealth W_0 and traded in a given ν-market satisfies

$$dW_\nu^{\pi^s}(t) = rW_\nu^{\pi^s}(t)\, dt + \pi^s(t)\nu(t)\, dt + \pi^s\sigma\, dB_t \ , \tag{32}$$

which implies

$$W_\nu^{\pi^s}(t) \geq W^{\pi^s}(t) \ , \tag{33}$$

that is, under any nonnegative trading strategy, wealth is at least as much in the ν-market as it is in the original market. Now consider the optimal portfolio choice problem in the ν-market without the non-negativity constraint on π^s, which can be represented as

$$\max_{W_T \geq 0} \mathrm{E}\{U(n(S(\tau^*) - K)^+ e^{r(T-\tau^*)} + W_T)\} \text{ s.t. } \mathrm{E}e^{-rT} Z_\nu(T) W(T) \leq W_0 \ . \tag{34}$$

The necessary and sufficient first-order condition for an optimal $W(T)$ is

$$U'(n(S(\tau^*) - K)^+ e^{r(T-\tau^*)} + W(T)) = \gamma e^{-rT} Z_\nu(T)) \tag{35}$$

where the positive constant γ is such that the budget constraint is met with equality. Rearranging equation (26) for $Z_\nu(T)$ yields

$$U'(n(S_{\tau^*} - K)^+ e^{r(T-\tau^*)} + W_0 e^{rT}) = \mathrm{E}\{U'(n(S_{\tau^*} - K)^+ e^{r(T-\tau^*)} + W_0 e^{rT})\} Z_\nu(T) \ , \tag{36}$$

which shows that $W(T) = W_0 e^{rT}$ is the optimal terminal outside wealth and thus $\pi^s(t) = 0$ a.e. a.s. is the optimal portfolio strategy.

Finally, we show that $\pi^s(t) = 0$ a.e. a.s. is also the optimal strategy for the constrained problem in the original market. First note that $\pi^s(t) = 0$ a.e. a.s. generates the same expected utility in the original market as in the ν-market. Now suppose $\pi'(t)$ is some other nonnegative strategy. Since π' is feasible for the unconstrained problem in the ν-market, it can provide no greater expected utility than $\pi^s(t) = 0$ in the ν-market. Suppose $\pi'(t)$ is not equal to zero a.e. a.s. Then from equation (32) it follows that π' generates less wealth and strictly less expected utility in the original market than in the ν-market. Therefore, in the original market, π' provides strictly less expected utility than $\pi^s(t) = 0$ a.e. a.s. This completes the sketch of the proof of the conjecture.

References

Bettis, J. Carr, John M. Bizjak, and Michael L. Lemmon, 2005, Exercise behavior, valuation, and the incentive effects of employee stock options, *Journal of Financial Economics* 76, 445–470.

Black, Fischer, and Myron S. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637–659.

Cai, Jie, and Anand M. Vijh, 2005, Executive stock and option valuation in a two state-variable framework, *Journal of Derivatives* Spring, 9–27.

Carpenter, Jennifer N., 1998, The exercise and valuation of executive stock options, *Journal of Financial Economics* 48, 127–158.

Carr, Peter, Robert Jarrow, and Ravi Myneni, 1992, Alternative characterizations of American put options, *Mathematical Finance* 2, 87–106.

Carr, Peter, and Robert Jarrow, 1990, The stop-loss start-gain paradox and option valuation: A new decomposition into intrinsic and time value, *Review of Financial Studies* 3, 469–492.

Carr, Peter, and Vadim Linetsky, 2000, The valuation of executive stock options in an intensity based framework, *European Financial Review* 4, 211–230.

Carr, Peter, 2000, Deriving derivatives of derivative securities, *Journal of Computational Finance* 4, 5–29.

Cuny, Charles J., and Philippe Jorion, 1995, Valuing executive stock options with an endogenous departure decision, *Journal of Accounting and Economics* 20, 193–205.

Cvitanić, Jakša, Zvi Wiener, and Fernando Zapatero, 2004, Analytic pricing of executive stock options, Working paper, University of Southern California.

Detemple, Jérôme, and Suresh Sundaresan, 1999, Nontraded asset valuation with portfolio constraints: A binomial approach, *Review of Financial Studies* 12, 835–872.

Evans, Jonathan, Vicky Henderson, and David Hobson, 2005, The curious incident of the investment in the market: Real options and a fair gamble, Working paper, Princeton University.

Geske, Robert, 1979, A note on an analytic valuation formula for unprotected American call options on stocks with known dividends, *Journal of Financial Economics* 7, 375–380.

Hall, Brian, and Kevin Murphy, 2002, Stock options for undiversified executives, *Journal of Accounting and Economics* 33, 3–42.

Henderson, Vicky, 2004, Valuing the option to invest in an incomplete market, Working paper, Princeton University.

Huddart, Steven, 1994, Employee stock options, *Journal of Accounting and Economics* 18, 207–231.

Hull, John, and Alan White, 2004, How to value employee stock options, *Financial Analysts Journal* 60, 114–119.

Ingersoll, Jonathan E., Jr., 1987, *Theory of Financial Decision Making* (Rowman and Littlefield, Totowa, NJ).

Ingersoll, Jonathan, 2006, The subjective and objective evaluation of incentive stock options, Working paper, Yale University, forthcoming *Journal of Business*.

Jennergren, L. Peter, and Bertil Näslund, 1993, A comment on "Valuation of executive stock options and the FASB proposal", *Accounting Review* 68, 179–183.

Kadam, Ashay, Peter Lakner, and Anand Srinivasan, 2003, Executive stock options: Value to the executive and cost to the firm, Working paper, New York University.

Karatzas, Ioannis, and Steven E. Shreve, 1991, *Brownian Motion and Stochastic Calculus* (Springer-Verlag, New York, NY).

Karatzas, Ioannis, and Steven E. Shreve, 1998, *Methods of Mathematical Finance* (Springer-Verlag, New York).

Kaul, Matthias, Jun Liu, and Francis A. Longstaff, 2003, Paper millionaires: How valuable is stock to a stockholder who is restricted from selling it?, *Journal of Financial Economics* 67, 385–410.

Kim, In Joon, 1990, The analytic valuation of American options, *Review of Financial Studies* 3, 547–572.

Krylov, Nikolai, 1980, *Controlled Diffusion Processes* (Springer-Verlag, New York).

Lambert, Richard A., David F. Larcker, and Robert E. Verrecchia, 1991, Portfolio considerations in valuing executive compensation, *Journal of Accounting Research* 29, 129–149.

Leung, Tim, and Ronnie Sircar, 2006, Accounting for risk aversion, vesting, job termination risk and multiple exercises in valuation of employee stock options, Working paper, Princeton University.

Marcus, Alan, and Nalin Kulatilaka, 1994, Valuing employee stock options, *Financial Analysts Journal* 50, 46–56.

Merton, Robert C., 1973, Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Miao, Jianjun, and Neng Wang, 2005, Investment, consumption and hedging under incomplete markets, Working paper, Columbia University.

Øksendal, Bernt, 2002, *Stochastic Differential Equations: An Introduction with Applications*, fifth edition (Springer-Verlag, New York, NY).

Roll, Richard, 1977, An analytic valuation formula for unprotected American call options on stocks with known dividends, *Journal of Financial Economics* 5, 251–258.

Rubinstein, Mark, 1995, On the accounting valuation of employee stock options, *Journal of Derivatives* 3, 8–24.

Van Moerbeke, Pierre, 1976, On optimal stopping and free boundary problems, *Archives of Rational Mechanical Analysis* 60, 101–148.

Whaley, Robert E., 1981, On the valuation of American call options on stocks with known dividends, *Journal of Financial Economics* 9, 207–211.

Figure 1: Exercise Policy with Split Continuation Region

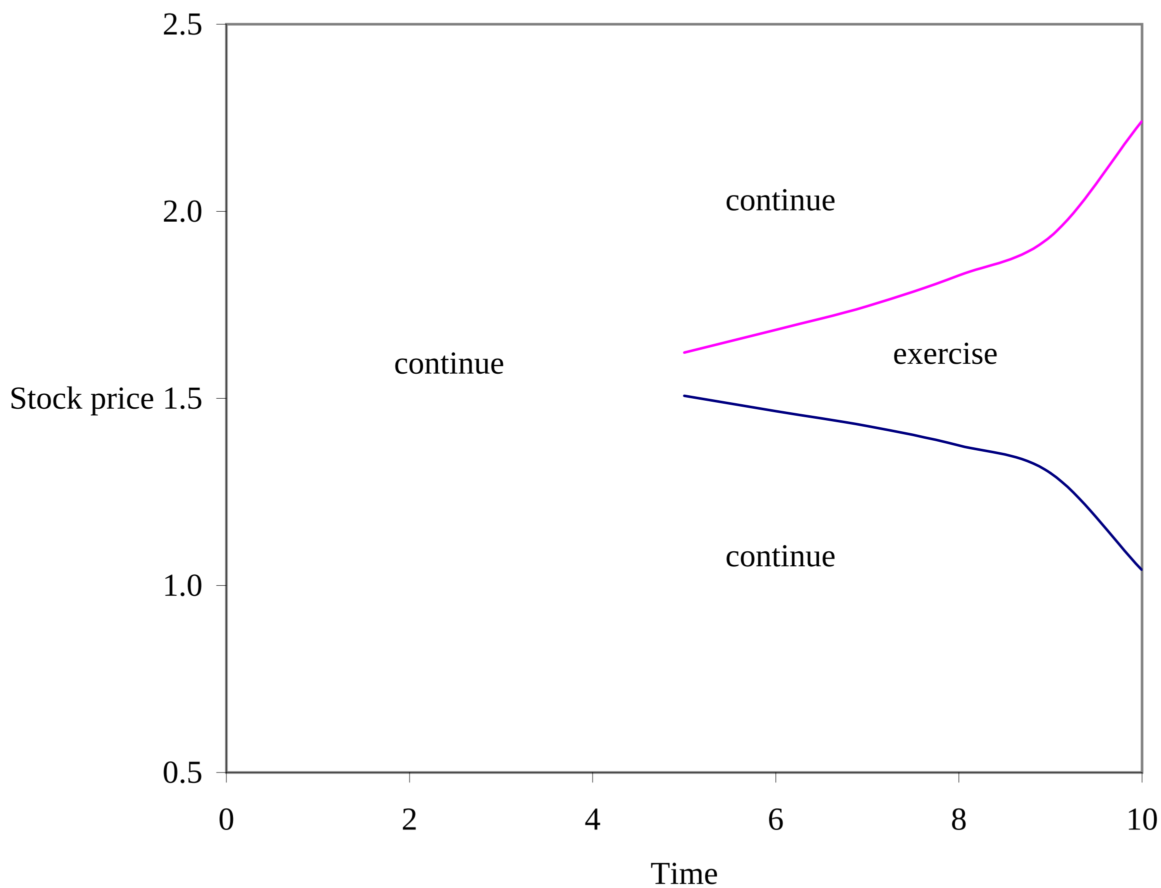
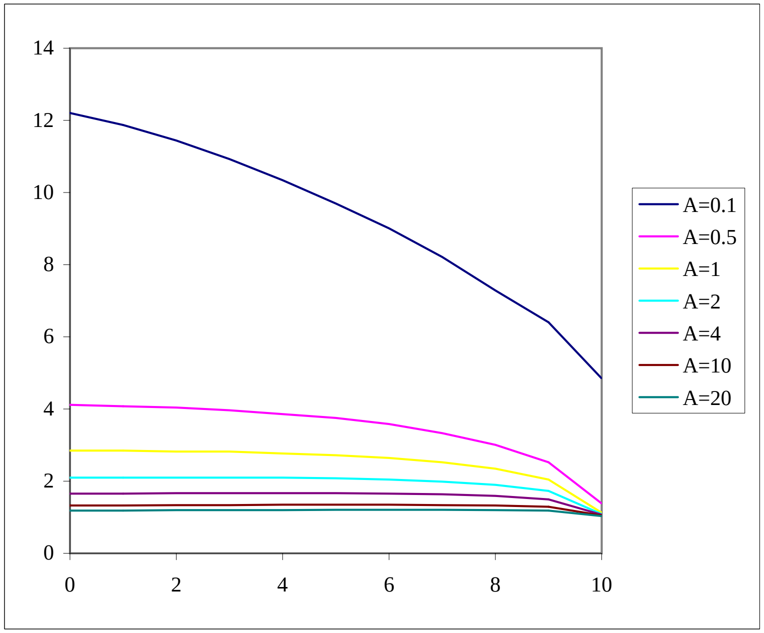
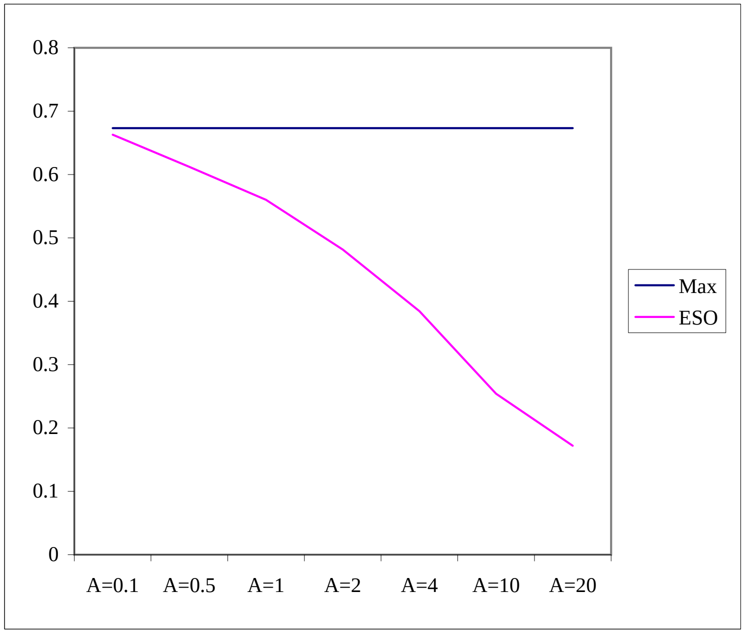


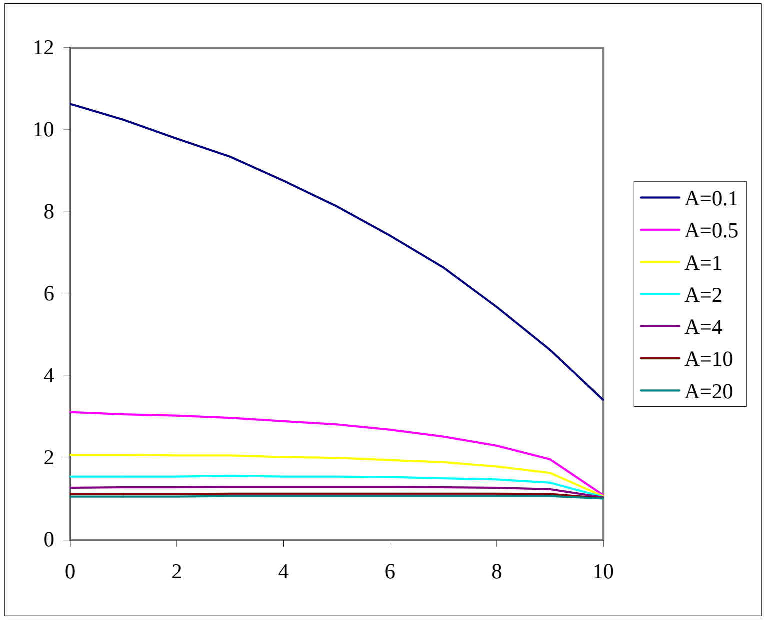
Figure 2: Exercise Boundaries and Option Values for Various Levels of Risk Aversion

a. Wealth = 2, Volatility = 50%





b. Wealth = 0.5, Volatility = 50%



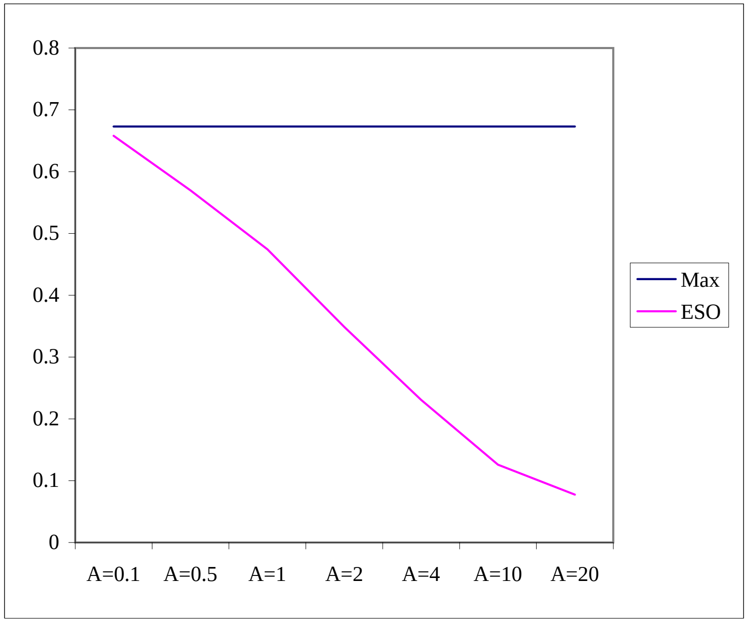
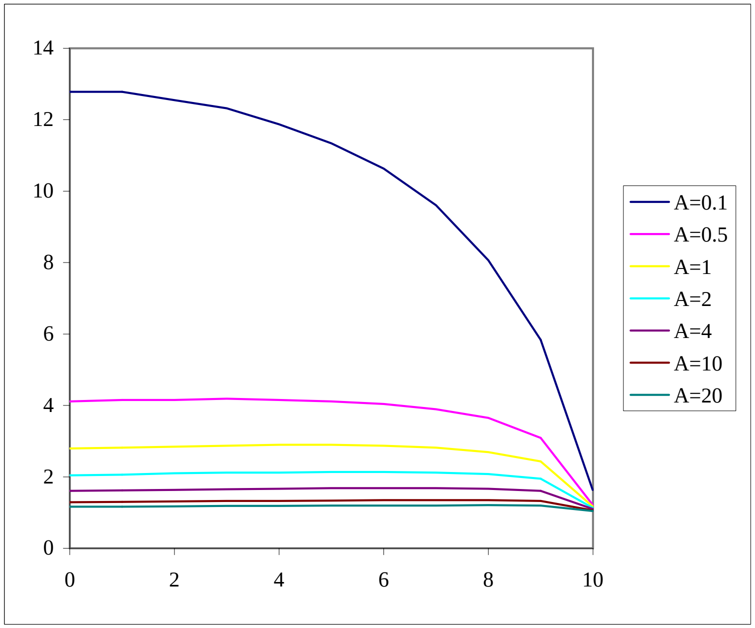
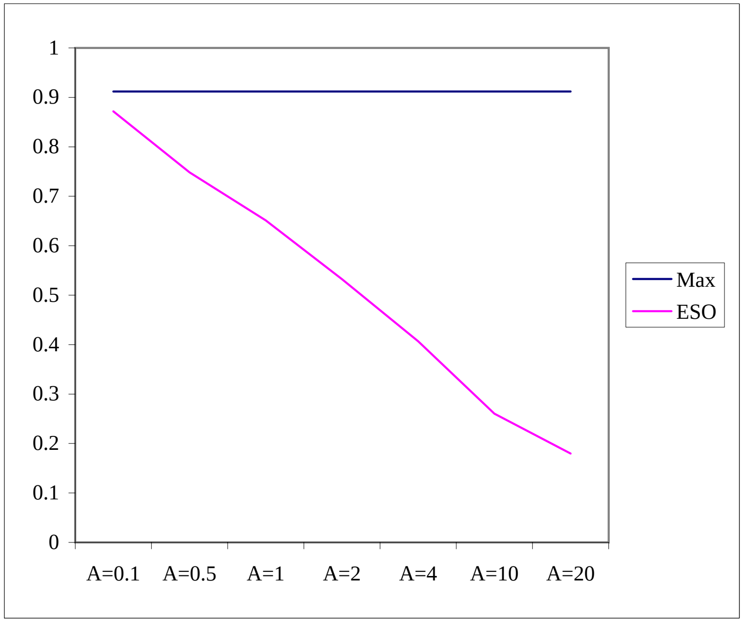


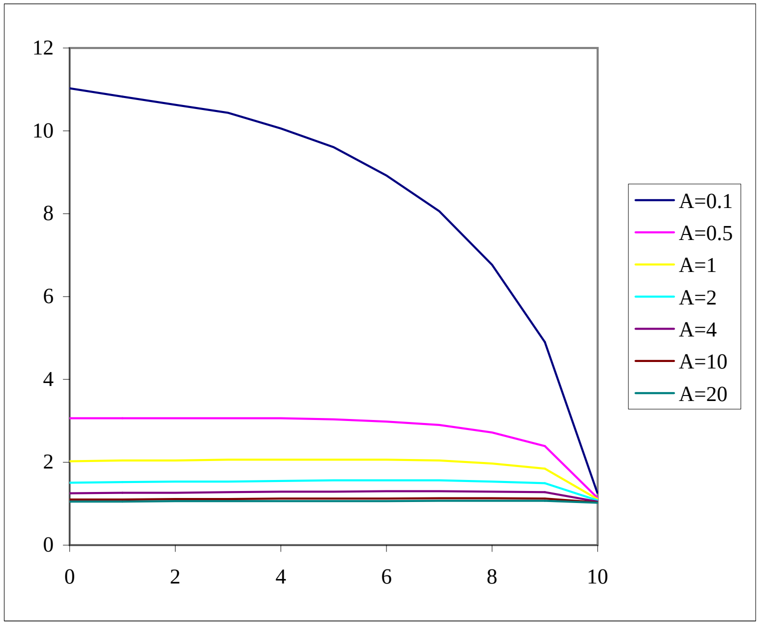
Figure 2 cont'd: Exercise Boundaries and Option Values for Various Levels of Risk Aversion

c. Wealth = 2, Volatility = 100%





d. Wealth = 0.5, Volatility = 100%



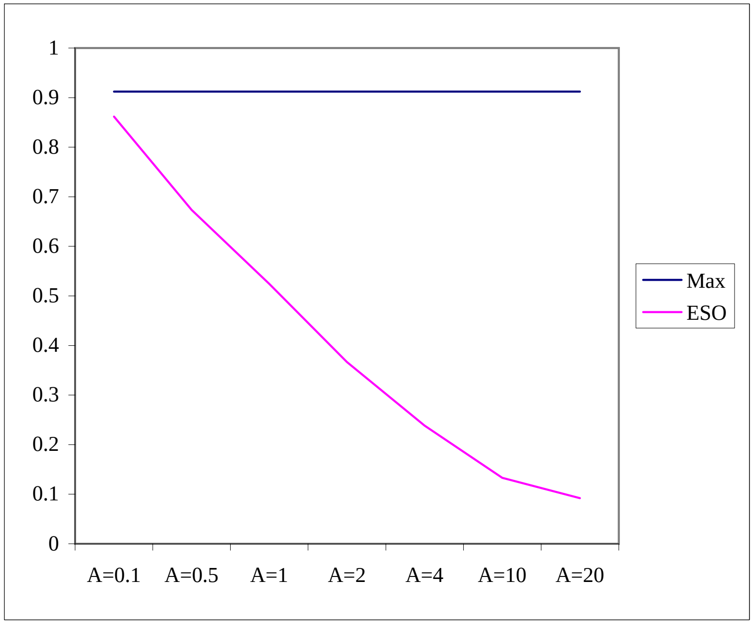
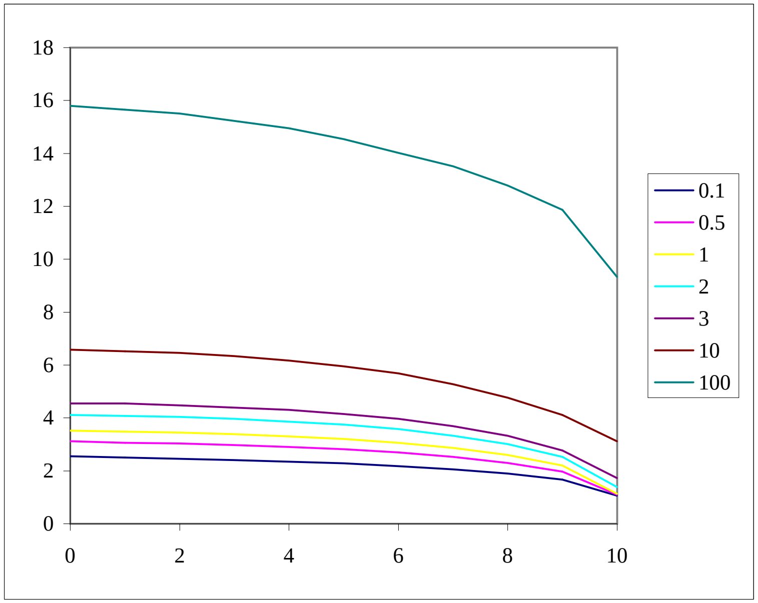
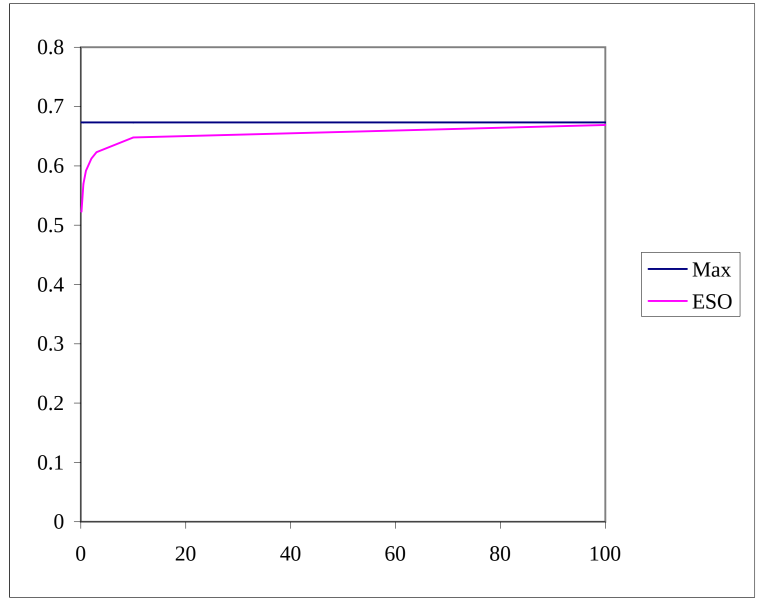


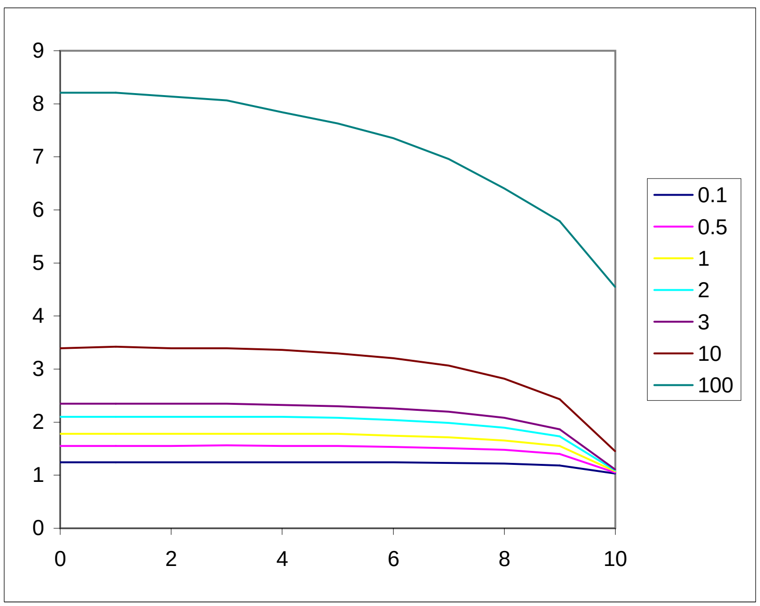
Figure 3: Exercise Boundaries and Option Values for Various Levels of Wealth

a. Risk aversion coefficient = 0.5, Volatility = 50%





b. Risk aversion coefficient = 2, Volatility = 50%



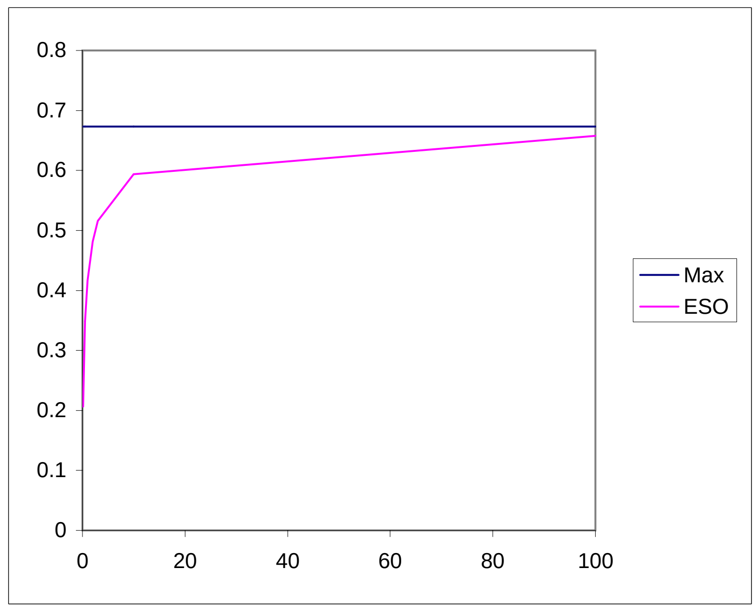
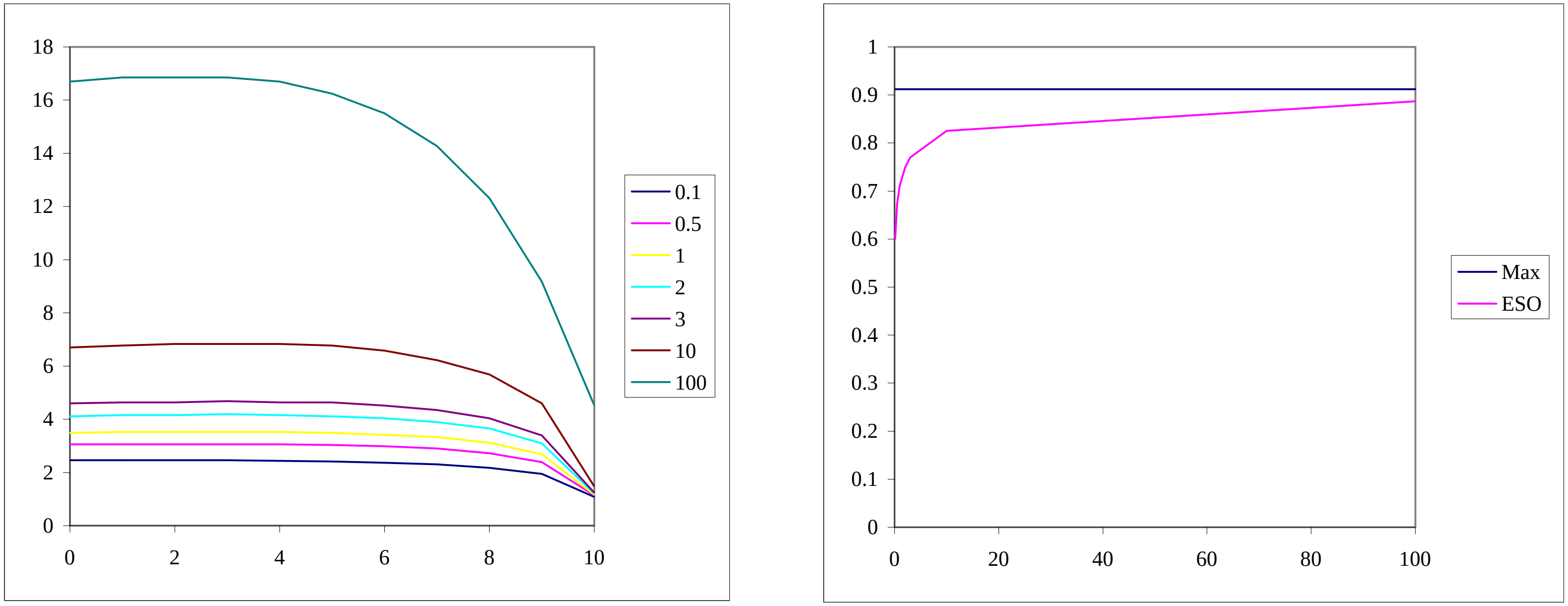


Figure 3 cont'd: Exercise Boundaries and Option Values for Various Levels of Wealth

c. Risk aversion coefficient = 0.5, Volatility = 100%



d. Risk aversion coefficient = 2, Volatility = 100%

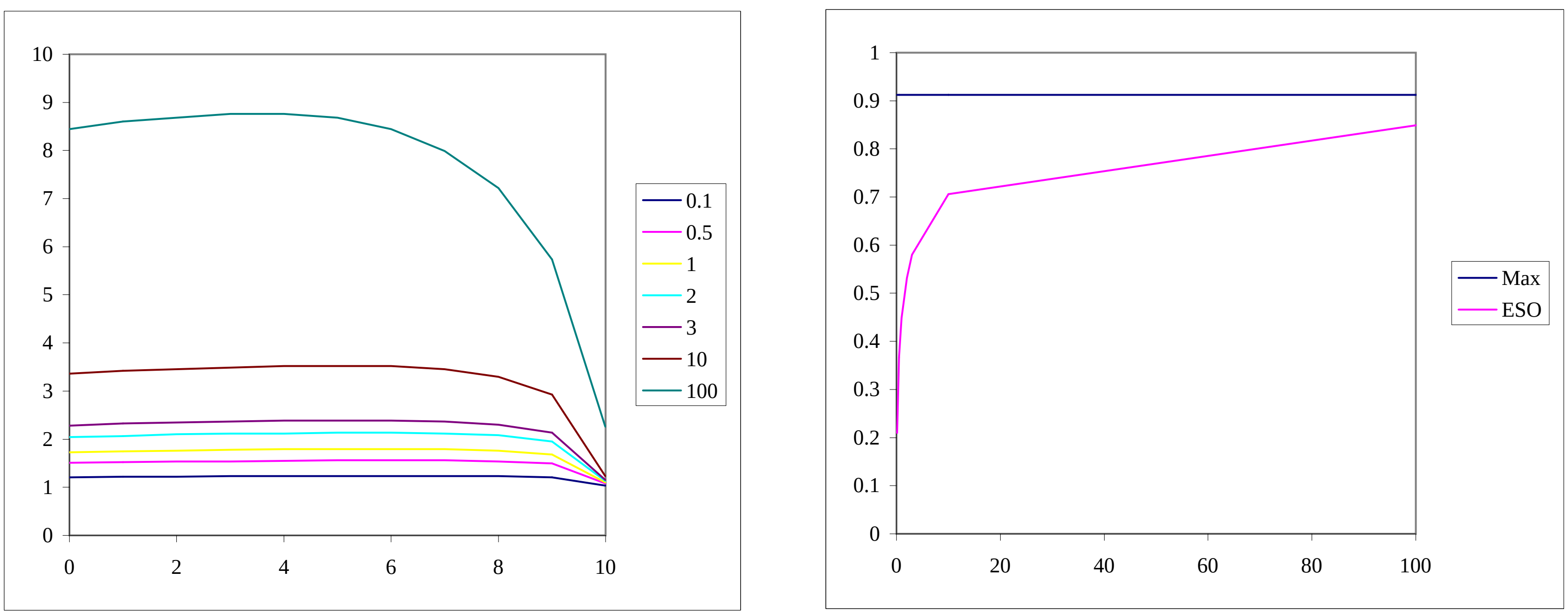
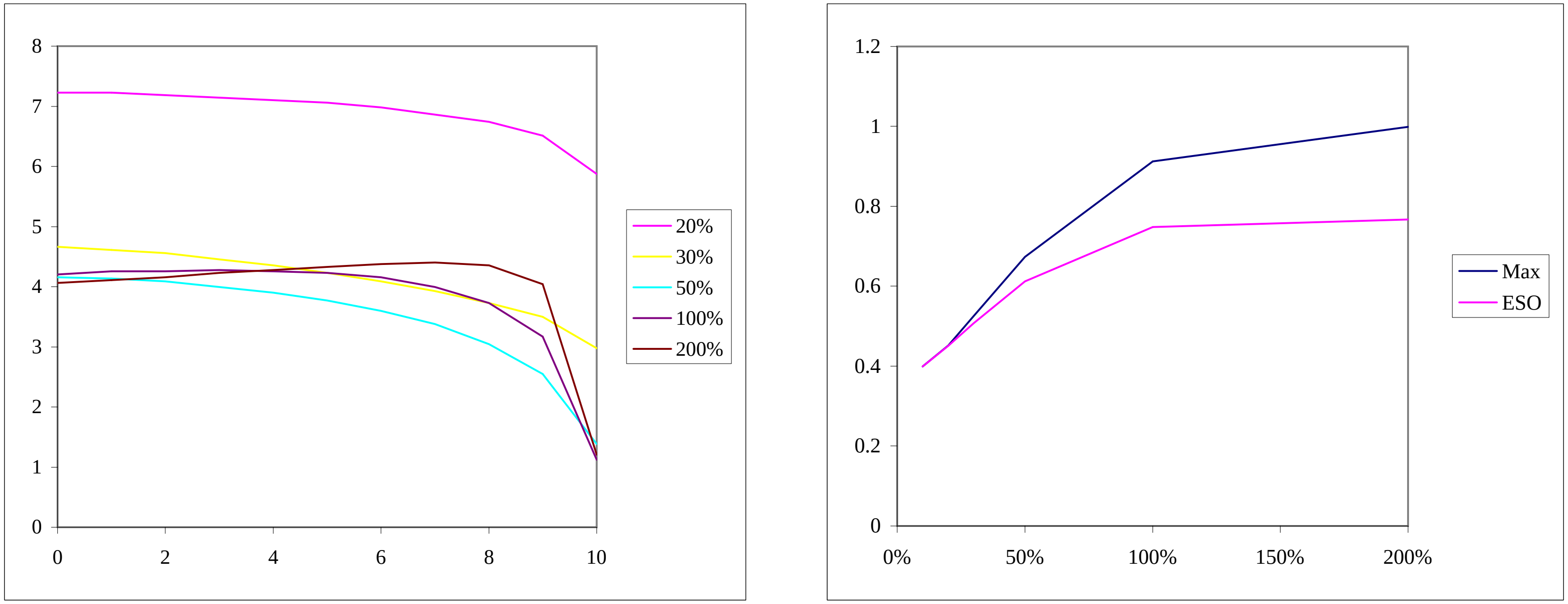


Figure 4: Exercise Boundaries and Option Values for Various Levels of Stock Volatility

a. Risk aversion coefficient = 0.5



b. Risk aversion coefficient = 2

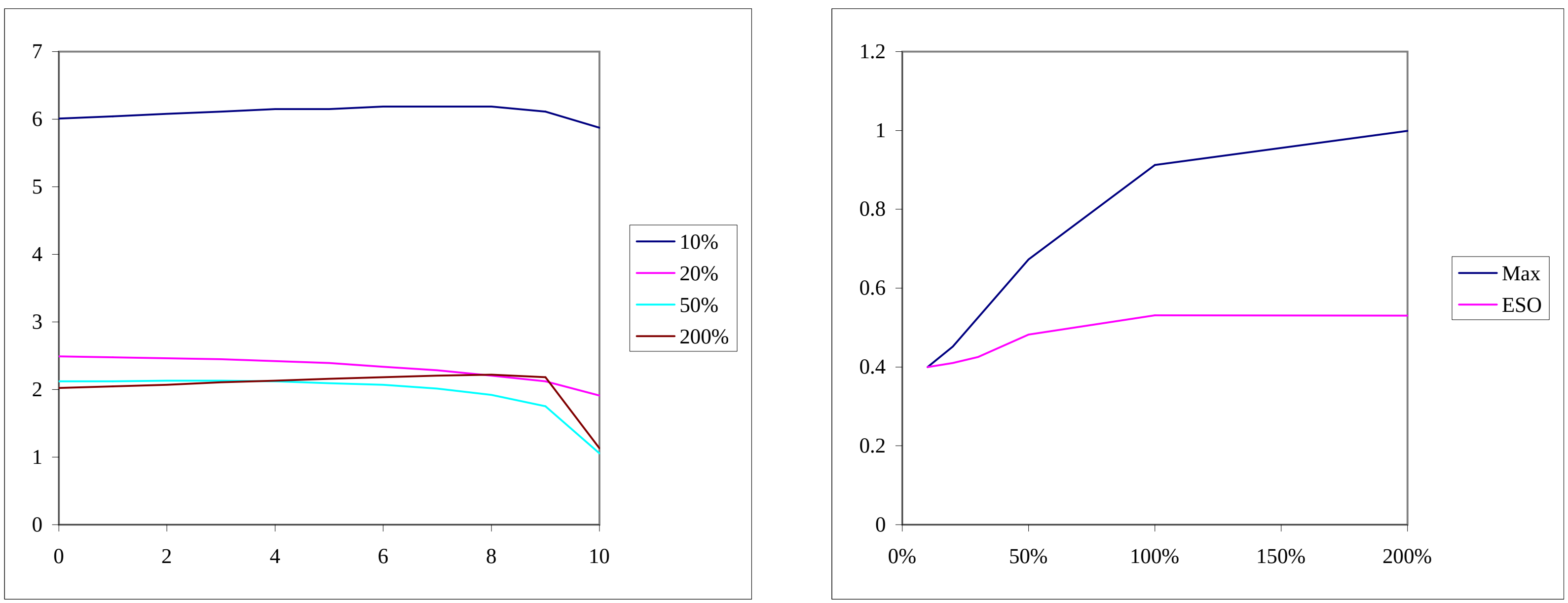
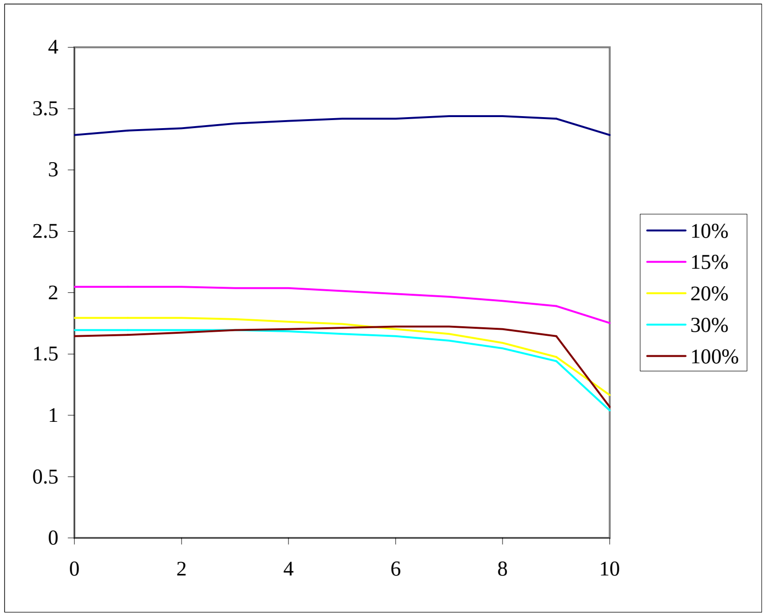
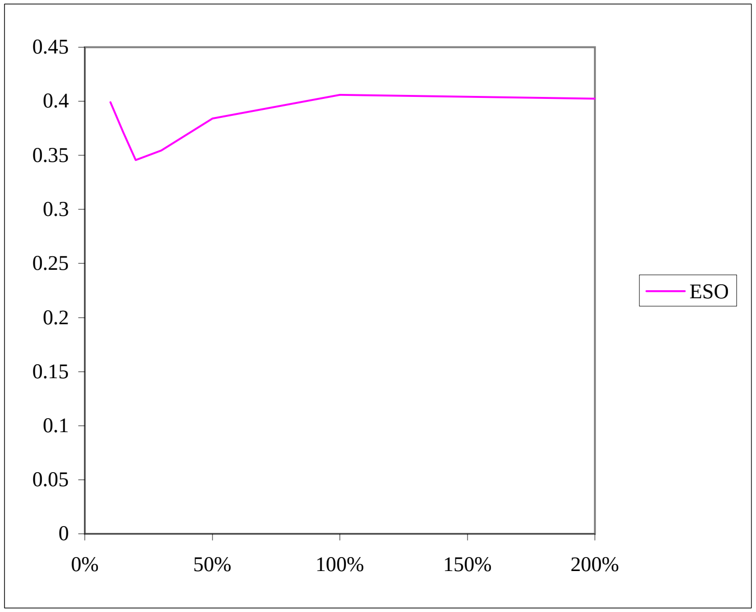


Figure 4 cont'd: Exercise Boundaries and Option Values for Various Levels of Stock Volatility

c. Risk aversion coefficient = 4





d. Risk aversion coefficient = 10