Risk Management with Benchmarking*

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Abstract

Portfolio theory must address the fact that in reality, portfolio managers are evaluated relative to a benchmark, and therefore adopt risk management practices to account for the benchmark performance. We capture this risk management consideration by allowing a prespecified shortfall from a target benchmark-linked return, consistent with growing interest in such practice. In a dynamic setting, we demonstrate how a risk averse portfolio manager optimally under- or overperforms a target benchmark under different economic conditions, depending on his attitude towards risk and choice of the benchmark. Investors can therefore achieve their desired gain/loss characteristics for funds under management through an appropriate combined choice of the benchmark and money manager.

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1. Introduction

Portfolio theory, a cornerstone of financial economics, counterfactually assumes that all individuals invest directly in financial markets (Allen (2001)). In practice, most investments are professionally managed. While investors' preferences and/or managerial risk profiles drive investment decisions, investment performance is evaluated ex post. Relative performance evaluation is widespread across many segments of the financial industry, and may arise naturally in the presence of various market frictions, as well as be rooted in behavioral explanations.[1] If it is too costly to assess the causes of a particular performance, an evaluation relative to a visible benchmark may be an optimally economizing compromise.[2] Similarly, inherent cognitive resource constraints naturally lead to adopting heuristic simplifications (Hirshleifer (2001)), and evaluating performance in relative terms may be such a simplification. Due to this scrutiny of performance, professional portfolio managers use risk management practices that account explicitly for benchmark performance. In this paper we study the optimal policies under risk management with benchmarking.

It is well-recognized (Jorion (2000)) that in the presence of benchmarking, any meaningful risk measure should account for the possibility of the investment portfolio underperforming its benchmark, and must consider the portfolio's tracking error (excess return over the benchmark). Such downside risk may be hedged (or insured against) by limiting underperformance via portfolio insurance (Basak (1995), Grossman and Zhou (1996)) for a riskless money market benchmark, or via minimum performance constraints (Teplá (2001)) for a stochastic benchmark. Nevertheless, a serious shortcoming of this (strict) downside hedging with respect to a benchmark is that it may be too costly to implement, giving up considerable upside potential of the benchmark return. Moreover, overperforming the benchmark as a goal is ruled out as it is infeasible. In this paper, we consider a more flexible and affordable risk management framework, where the risk manager is allowed to target overperforming (beating) the benchmark return by a minimum amount, or underperforming by not more than a maximum amount. These targets are feasible since not delivering a target return is allowed with a prespecified shortfall probability. Such a "tracking error constraint" with a potential shortfall is intuitively appealing since risk managers, or those who evaluate their performance, may tolerate various forms of shortfall in order to meet other goals (like beating the stock market in some states). As a result, the use of such a downside risk measure is indeed rapidly spreading in practice, and has also been advocated in the professional literature (RISK (1998, 2000a, 2000b), Jorion (2000, Chapter 17)), beckoning further investigations.

Our primary objective in this paper is to investigate the optimal dynamic behavior of a risk

[1] Relative performance evaluation is performed almost universally, with the only possible exception of the hedge fund industry, where absolute performance is also important: see discussions in, e.g., Fung and Hsieh (1997), p. 276; Chan, Karceski, and Lakonishok (1999), p. 956; the *Economist* (September 1, 2001), pp. 60-61.

[2] The focus on visible investment opportunities in the presence of informational costs is discussed in Merton (1987) and in Shapiro (2001).

manager striving to meet a tracking error constraint, in conjunction with the standard utility maximizing objective. Consistent with the leading benchmarking practice, the risk manager benchmarks the stock market return over his investment horizon. We adopt the familiar Black and Scholes (1973) economy for the financial investment opportunities, and assume the risk manager is guided by constant relative risk aversion preferences. One of the analytical subtleties present in our setting stems from the non-concavity introduced by the tracking error constraint into the risk manager's optimization problem.[3] Throughout the analysis, we compare the optimal behavior of the benchmarking risk manager with that of the downside hedger and the non-risk manager (the latter behavior hereafter referred to as the "normal" policy).

Risk management with benchmarking, when shortfall is allowed, emerges as rich in implications. Absent risk management considerations, a non-risk managing agent's optimal (normal) policy is driven by its sensitivity (given by the agent's risk tolerance) to changing economic conditions (represented by changes in state prices). Under risk management, our analysis identifies economies characterized by the sensitivity of the benchmark relative to a non-risk manager's sensitivity and additionally relative to unity, where the risk manager exhibits distinct patterns of economic behavior in choosing his optimal horizon wealth and trading strategies. In economies where the stock market benchmark reacts less to changes in economic conditions than the more sensitive normal policy, the benchmark beats the normal policy in economic downturns (bad states), but underperforms in upturns (good states). Consequently, downside hedging with respect to the stock market leads a risk manager to maintain the normal-type policy in good states, while matching the allowed underperformance level relative to the stock market in bad states. When shortfall is allowed, the risk manager additionally, optimally chooses in which states to fall short of the benchmark. Here, he identifies the states with the highest state-contingent relative cost of matching the benchmark vs. following the normal policy, so that the benefit from reverting to the normal policy is highest. Indeed, for economies with benchmark sensitivity below unity – the shortfall states are the bad states, above unity – the intermediate states, and when coinciding with normal sensitivity (and being above unity) – the shortfall is in the good states.

In the above economies, where the benchmark is not more sensitive than the normal policy, and when the shortfall states are either good or intermediate, the losses in bad states under both downside hedging and benchmarking are never higher than those without risk management. Despite this being a reassuring outcome, the practical usefulness of our benchmarking risk management framework is underscored by the fact that losses under benchmarking can be *further* reduced relative to those under downside hedging. This is a consequence of the downside hedger necessarily underperforming the stock market in the bad states, while the benchmarker in fact finds it optimal

[3] We note that in our model, the tracking error constraint is taken as given. For an analysis of optimality of risk management based constraints imposed on portfolio managers see Basak, Pavlova, and Shapiro (2001).

to beat the market in those states. As the latter behavior does account for risk aversion, it may be appealing to some investors, as well as merit regulatory consideration.

In economies where the benchmark reacts more to changes in economic conditions than the normal policy, the benchmark beats the normal policy only in good states. This leads the downside hedger to match an allowed underperformance level (insure against large relative losses) in good states, while adopting the normal policy in bad states. This is in sharp contrast to the findings of related work on portfolio insurance and Value-at-Risk-based risk management (e.g., Basak and Shapiro (2001)), where good states are not insured. When the benchmark is at least as sensitive as the normal policy, and when the shortfall states are either intermediate or bad, the gains in good states of both the downside hedger and the benchmarker are never lower than those without risk management. Although that alone may appease some investors, the novelty of our analysis is to show that since the benchmarker is able to overperform the stock market while the downside hedger cannot, the benchmarker's gains can be chosen to be even higher in good states.

An important by-product of our results is in offering guidance to achieve desirable gain/loss characteristics for professionally managed investments, while adhering to the fundamental principles of sound portfolio choice. Furthermore, our analysis can also indicate when a particular gain/loss profile can be obtained under higher Sharpe ratios than those of the benchmark or of non-risk managers (Section 4.2). Controlling gain/loss (as well as other) characteristics may be accomplished by entrusting funds to a money manager with an appropriate risk appetite (inherently his, or delegated to him), thereby choosing the desired normal policy for a given benchmark. Alternatively, for a given managerial risk profile, one can appropriately choose a hybrid benchmark index. The latter is facilitated by our analysis remaining valid for indices composed of a given mix of money market and stock market returns (Section 4.4).

We uncover further properties of the benchmarking risk manager's behavior by studying his optimal pre-horizon wealth and dynamic investments. In economies where the benchmarker chooses to not fall short in bad states, there is always a region of intermediate states where as economic conditions deteriorate, the benchmarker becomes wealthier. The resulting non-monotonicity of the pre-horizon wealth suggests caution in attempting to deduce the state of the economy by observing portfolio wealth alone. We also show that in economies where the risk manager falls short of the benchmark in the intermediate states, a small shift in economic conditions may trigger considerable reaction by the risk manager in the form of changing his dynamic investments and risk exposure, possibly shifting between large leveraged and short positions.[4]

Closely related to our analysis of tracking error are the works of Roll (1992), Brennan (1993), Gómez and Zapatero (2000), and Jorion (2001b), within a static mean-variance framework. Roll

[4]The risk manager will not deviate from the above-outlined optimal policies, because deviations will be detected and penalized ex-post by those who evaluate the risk manager's performance (Section 2.2 elaborates on this point).

(1992) studies the portfolio problem of minimizing the tracking error variance for a given expected tracking error, the so-called TEV criterion. Accordingly, he derives the TEV frontier in the mean-variance tracking error space, and demonstrates that TEV efficient portfolios are not total return mean-variance efficient. Jorion (2001b) complements Roll's analysis by describing TEV-constrained portfolios by an ellipse in the total return mean-variance space. He also emphasizes how TEV criterion may result in the overall portfolio having more risk than the benchmark, and advocates an additional constraint on the volatility of the portfolio's total absolute return. Brennan (1993) and Gómez and Zapatero (2000) additionally study the equilibrium implications of benchmarking and derive a two-beta CAPM, where a new risk factor arises due to benchmarking. Gómez and Zapatero (2000) also provide strong empirical support for their equilibrium model. Working in a dynamic continuous-time setting, Teplá (2001) solves the utility maximization problem of an agent who downside hedges with respect to a stochastic benchmark, which is not necessarily the stock market. Our analysis of downside hedging complements Teplá's in that by choosing the benchmark as the stock market and exploiting the stock market dependence on economic conditions, we provide new results on the behavior of a downside hedging risk manager. Finally, Browne (1999), in a similar continuous-time setting studies a number of objective functions involving a stochastic benchmark, including minimizing the expected time to reach the benchmark, and maximizing the probability of beating the benchmark without underperforming it by a given amount.

The paper is organized as follows. Section 2 describes the economy with benchmarking practices, and provides results for special settings of interest. Section 3 solves the optimization problem of a risk manager who is benchmarking the stock market, and analyzes the resulting portfolio strategies. Section 4 discusses alternative formulations and extensions. Section 5 concludes the paper. Proofs are in the appendix.

2. The Economic Setting

2.1 The Economy

We consider a continuous-time, finite-horizon, $[0, T]$ economy. Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, on which is defined a 1-dimensional Brownian motion w. All stochastic processes are assumed adapted to $\{\mathcal{F}_t; t \in [0, T]\}$, the augmented filtration generated by w. All stated (in)equalities involving random variables hold P-almost surely.

Financial investment opportunities are given by an instantaneously riskless money market account and a risky stock, as in the Black and Scholes (1973) economy. The money market provides a constant interest rate r. The stock price, S, follows a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dw_t,$$

where the stock instantaneous mean return, μ, and standard deviation, σ, are constant. Dynamic market completeness (under no-arbitrage) implies the existence of a unique state price density process, ξ, given by

$$d\xi_t = -r\xi_t dt - \kappa\xi_t dw_t \ ,$$

where $\kappa \equiv (\mu - r)/\sigma$ is the constant market price of risk in the economy. As is well known (e.g., Karatzas and Shreve (1998)), the state price density serves as the driving economic state variable in an agent's dynamic investment problem. The quantity $\xi_T(\omega)$ is interpreted as the Arrow-Debreu price per unit probability P of one unit of wealth in state $\omega \in \Omega$ at time T.

An agent in this economy is endowed at time zero with an initial wealth of W_0. The agent chooses a nonnegative, horizon wealth, W_T, and an investment policy, θ, where θ_t denotes the fraction of wealth invested in the stock at time t. The agent's wealth process W then follows

$$dW_t = [r + \theta_t(\mu - r)]\, W_t dt + \theta_t \sigma W_t dw_t \ .$$

The agent is modeled as deriving utility, u, from horizon wealth. We assume that the agent has constant relative risk aversion (CRRA) preferences, $u(W_T) = \frac{W_T^{1-\gamma}}{1-\gamma}$, $\gamma > 0$.

With no further restrictions or considerations (such as risk management), this *normal* or *non-risk managing* agent, N, chooses the optimal horizon wealth to be (see, e.g., Cox and Huang (1989))

$$W_T^N = I(y^N \xi_T) = \frac{1}{(y^N \xi_T)^{1/\gamma}} \ ,$$

where $I(\cdot)$ denotes the inverse of $u'(\cdot)$, and $y^N > 0$ solves $E[\xi_T I(y^N \xi_T)] = W_0$. As demonstrated in the sequel, an important feature of this horizon wealth is its elasticity with respect to the economic state variable ξ_T, which is a constant given by

$$\frac{\partial W_T^N}{\partial \xi_T} \frac{\xi_T}{W_T^N} = -\frac{1}{\gamma} \ .$$

Henceforth, we refer to the quantity $1/\gamma$ as the *sensitivity of the normal*, optimal horizon policy to economic conditions.

2.2 Benchmarking the Stock Market

Given the importance and prevalence of benchmarking in practice, our objective is to model an agent who manages the relative performance, or tracking error, of his portfolio along with other objectives. Specifically, consistent with industry-wide practices and academic literature, we define the *tracking error* of an agent's horizon wealth relative to a benchmark X as:

$$R_T^W - R_T^X = \frac{1}{T} \ln \frac{W_T}{W_0} - \frac{1}{T} \ln \frac{X_T}{X_0} \ ,$$

where R_T denotes the continuously compounded return over the horizon $[0, T]$. The benchmark X represents the level of a portfolio, or an index, or any economic indicator. To embed benchmarking within a risk management framework, we assume that the risk managing agent wishes to obey the following "tracking error constraint:"

$$P(R_T^W - R_T^X \geq \varepsilon) \geq 1 - \alpha \ . \tag{1}$$

The constraint (1) states that the risk manager maintains his tracking error to be above some prespecified level ε with confidence $1-\alpha$. The case of $\varepsilon > 0$ corresponds to a risk manager aiming to overperform (beat) the return on the benchmark by at least ε, and $\varepsilon < 0$ to a risk manger aiming to not underperform the benchmark return by more than $|\varepsilon|$. The realizations of risk manager's return, R_T^W, below the target return, $R_T^X + \varepsilon$, are those of an unacceptable shortfall, and we refer to α as the *shortfall probability.* That is, the risk manager permits the performance of his portfolio to deteriorate below the target return ($R_T^W < R_T^X + \varepsilon$) with probability α.

An important justification of our reduced form tracking error constraint stems from the fact that risk-managers' performance in practice is evaluated ex post, i.e., backtested, on a repeated basis, with penalties imposed on those who unacceptably fall short of their target return. Although currently tracking error based risk-management strategies are formally backtested by regulators only relative to a state-independent benchmark (e.g., money market), backtesting relative to other benchmarks is implicitly performed by the economic environment, and ultimately by clients whose money is being managed. Clients, for example, can view the frequency at which the risk manager reports shortfalls over fixed consecutive intervals (weeks, months). Following common practice (e.g., Jorion (2000, 2001a)), this frequency can be translated into an unconditional probability, α, of a shortfall over a given interval, $[0, T]$. We assume that maintaining a prespecified shortfall probability, to avoid penalties, drives the risk manager to optimally follow over $(0, T]$ a dynamic policy designed at the outset ($t = 0$) to finance the horizon wealth, W_T. Penalties may include outflow of funds resulting in lost fees, costs of legal actions, or damaged professional reputation. The potential for backtesting penalties at time T is a leading implicit friction, which we capture in a reduced form by specifying our tracking error constraint (1) only at time 0. It is this friction that discourages the risk manager to deviate over $(0, T]$ from the time-0 constrained optimal policy, as deviations are penalized.[5]

[5]To maintain our focus, we implicitly take the mechanism of backtesting penalties as given. Such a mechanism could arise naturally to promote truthful reporting to avoid capital charges (in a bank-regulatory context) or penalties (in the context of the money-management industry). The mechanism could be implemented over a single period when performance is verifiable state by state, or over consecutive periods where the shortfall probability is deduced via historical exceedence frequency, or via more efficient statistical methods (Jorion (2001a)). Alternatively, competitive pressure could force the risk manager to follow the constrained policy. When money managers compete for client fees, clients can inflict penalties merely by observing competitors' performance, with no need for state verification or for collection of historical data. One could easily envision a setting, for example, with two money managers competing for fees. Given appropriate fee structure, and absent collusion, neither manager would deviate over $(0, T]$ from the optimal policy solving our problem (and satisfying our constraint at $t = 0$) in fear of the competitor being the one more likely to match the benchmark, and consequently the one to attract client fees. Finally, it is also conceivable that the risk manager simply precommits to execute the constrained policy in light of many possible considerations (e.g., to avoid transactions costs in dealing with third parties that may provide custom derivative instruments to implement the policy, to minimize computational costs, or to demonstrate trading skills).

In this paper, we focus on the most common, natural benchmark: the stock market (Section 4.4 extends the analysis to hybrid benchmarks). When the risk manager is benchmarking the performance of the stock market, then $R_T^X = R_T^S = \frac{1}{T} \ln \frac{S_T}{S_0}$. In this case, the constraint (1) leads the risk manager to strive maintaining his horizon wealth above a level given by:

$$X_T = W_0 e^{(R_T^S + \varepsilon)T} = e^{\varepsilon T} \frac{W_0}{S_0} S_T \ . \tag{2}$$

This is the wealth generated by investing the initial endowment at the target return, or equivalently the wealth generated by investing a tracking error adjusted initial wealth, $X_0 = e^{\varepsilon T} W_0$, in the stock market. We refer to X_T as the *horizon benchmark level*, and note that terminal wealth W_T may fall short of the benchmark level with probability α. Although R_T^X is independent of ε (by definition), we incorporate ε into the definition of the benchmark level X_T in (2) to highlight that the risk manager's wealth is determined by targeting stock market performance R_T^S, adjusted for the level of required overperformance ($\varepsilon > 0$), or allowed underperformance ($\varepsilon < 0$). Since in our economic setting S_T is decreasing in ξ_T, so is the benchmark level X_T; as economic conditions deteriorate at the horizon, so does the level of the benchmark. We note that the stock market level being decreasing in the state price density is consistent with all related equilibria studied in the literature [normal pure-exchange economy (Lucas (1978)), portfolio insurance (Basak (1995)), Value-at-Risk (Basak and Shapiro (2001))]. Moreover, an important quantity that is identified in the subsequent analysis is the elasticity of the horizon level of the benchmark with respect to the economic state variable ξ_T, which is a constant given by

$$\frac{\partial X_T}{\partial \xi_T} \frac{\xi_T}{X_T} = -\frac{\sigma}{\kappa} \ .$$

We refer to the quantity σ/κ as the *sensitivity of the benchmark* to economic conditions, and assume $\sigma/\kappa > 0$ without loss of generality (see footnote 8).

Our reduced form tracking error constraint (1), has the convenient property that it nests other cases of interest investigated in the literature. When $\alpha = 1$, it nests the non-risk managing agent, who is not concerned with benchmarking. For the case of the benchmark being the riskless money market account, $R_T^X = r$, the formulation reduces to Value-at-Risk based risk management (Basak and Shapiro (2001)). When $\alpha = 0$ the constraint is a "hard constraint," nesting the case of portfolio insurance (Basak (1995), Grossman and Zhou (1996)) for the money market benchmark, and the case of minimum performance constraint (Teplá (2001)) for the stock market benchmark.

2.3 Benchmarking the Money Market

When the risk manager is benchmarking the performance of the money market, $R_T^X = r$, the horizon benchmark level has no sensitivity to economic conditions, and is constant given by

$$X_T = W_0 e^{(r+\varepsilon)T} \equiv \underline{X} \ .$$

This is the case of a Value-at-Risk risk manager with "floor" $\underline{X}$ studied by Basak and Shapiro (2001). The optimal horizon wealth of the money market benchmarking agent, M, is given by

$$W_T^M = \begin{cases} I(y^M \xi_T) & \textit{if } \ \xi_T < \underline{\xi} \\ \underline{X} & \textit{if } \ \underline{\xi} \le \xi_T < \overline{\xi} \\ I(y^M \xi_T) & \textit{if } \ \overline{\xi} \le \xi_T \ , \end{cases}$$

where $y^M > 0$ solves $E[\xi_T W_T^M] = W_0$, $\underline{\xi} = u'(\underline{X})/y^M$, $\overline{\xi}$ solves $P(\overline{\xi} \le \xi_T) = \alpha$, and is depicted in Figure 1.

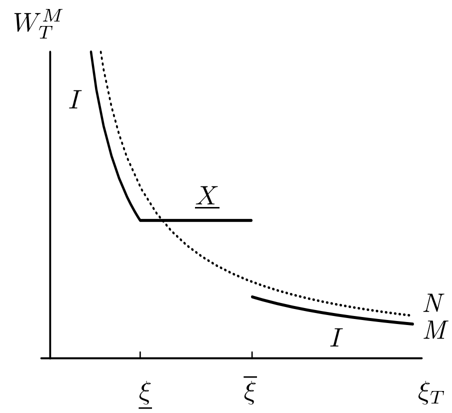


Figure 1: Optimal horizon wealth, W_T^M, of the money market benchmarking risk manager M (solid plot), and of the non-risk manager N, W_T^N (dotted plot).

As evident from Figure 1, the M agent's optimal horizon wealth falls into three distinct regions of economic behavior. In "good states" (the region with the lowest realizations of ξ_T), the M agent behaves like a non-risk manager, following the normal-type policy (for expositional convenience, the function I appears in our figures without the $y\xi_T$ argument, and we refer to it as the normal, non risk managing behavior). In the intermediate states ($\underline{\xi} \le \xi_T \le \overline{\xi}$) the agent matches the money market benchmark level $\underline{X}$ (insures himself against losses). In the "bad states" (the region with the highest realizations of ξ_T), he chooses to fall short of the benchmark, and follow a normal policy (being completely uninsured), since these are the costliest states to match the benchmark. Due to the cost of matching the benchmark in the intermediate states, the agent's gains are lower in good states, and although his losses are lower in some intermediate states, he incurs larger losses than a non-risk manager in bad states. The discontinuity in the optimal horizon wealth at $\overline{\xi}$ arises due to the fact that the bad, uninsured states are simply the worst states up to a probability of exactly α.

2.4 Benchmarking with Downside Hedging

We now consider the case of the tracking error constraint being a hard constraint with $\alpha = 0$. This is a risk management practice with complete downside hedging with respect to a benchmark. We note that such a hard constraint, prohibiting a shortfall, is feasible only if the risk manager gives up some of the benchmark's potential upside gains. In particular, downside hedging requires $\varepsilon < 0$. Consequently, a shortcoming of this approach is that the case of the risk manager being required to outperform the stock market, $\varepsilon > 0$, is ruled out as being infeasible. The case of matching the benchmark return, $\varepsilon = 0$, leads to the trivial policy of investing all wealth in the benchmark.

In the case of the benchmark being the money market, which is not sensitive to economic conditions, the risk manager is a portfolio insurer, *PI* (Basak (1995), Grossman and Zhou (1996)), with "floor" $\underline{X} = W_0 e^{(r+\varepsilon)T}$, where $\varepsilon < 0$ for feasibility. The portfolio insurer's optimal horizon wealth is given by

$$W_T^{PI} = \begin{cases} I(y^{PI}\xi_T) & \text{if } \xi_T < \underline{\xi} \\ \underline{X} & \text{if } \underline{\xi} \leq \xi_T \end{cases},$$

where $y^{PI} > 0$ solves $E[\xi_T W_T^{PI}] = W_0$, and $\underline{\xi} = u'(\underline{X})/y^{PI}$. In the good states, with relatively high horizon wealth, the *PI* behaves like a normal agent. As economic conditions deteriorate, and his wealth hits the money market benchmark level, $\underline{X}$, the *PI* exactly matches the benchmark. Consequently, his gains are lower in the good states, and losses lower in bad states, compared to those without risk management.

The case of the benchmark being the stock market has recently been analyzed by Teplá (2001). Here, we exploit the dependence of the stock market benchmark on economic conditions (level of ξ_T), and provide new results on the optimal behavior of a risk manager, *H*, benchmarking the stock market with downside hedging, in Proposition 1, depicted in Figure 2.

Proposition 1. *The optimal horizon wealth of a risk manager, H, benchmarking the stock market with downside hedging ($\alpha = 0$), and $\varepsilon < 0$, is given by,*
(a) for economies with $\sigma/\kappa < 1/\gamma$:

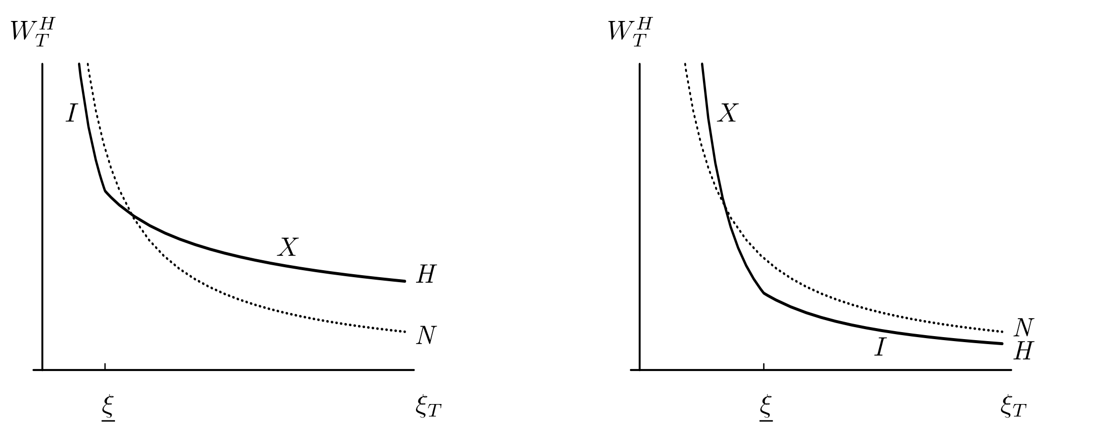
$$W_T^H = \begin{cases} I(y^H\xi_T) & \text{if } \xi_T < \underline{\xi} \\ X_T & \text{if } \underline{\xi} \leq \xi_T \end{cases},$$

(b) for economies with $\sigma/\kappa > 1/\gamma$:

$$W_T^H = \begin{cases} X_T & \text{if } \xi_T < \underline{\xi} \\ I(y^H\xi_T) & \text{if } \underline{\xi} \leq \xi_T \end{cases},$$

(c) for economies with $\sigma/\kappa = 1/\gamma$: $W_T^H = I(y^H\xi_T)$,
where in all economies $y^H > 0$ solves $E[\xi_T W_T^H] = W_0$, $\underline{\xi} = (y^H A^\gamma)^{1/(\gamma\sigma/\kappa - 1)}$, and $A = W_0 \exp[(\varepsilon + (\mu - \sigma^2/2) - (r + \kappa^2/2)\sigma/\kappa)T]$. When $\varepsilon = 0$, then $W_T^H = X_T$, and when $\varepsilon > 0$, downside hedging is not feasible.

In economies where the stock market benchmark is less sensitive to economic conditions than the normal policy, $\sigma/\kappa < 1/\gamma$ (Proposition 1(a), Figure 2(a)), the H agent's optimal behavior is similar to that of an agent benchmarking the money market with downside hedging. When the benchmark reacts less to changes in economic conditions than the normal policy, the stock market benchmark performs worse in good states ($\xi_T < \underline{\xi}$) and better in bad states ($\underline{\xi} \leq \xi_T$), as compared to the normal policy. Consequently, to meet the tracking error constraint (1), the benchmark level X_T (equation (2) with $\varepsilon < 0$) is matched in bad states, while the normal policy $I(y^H \xi_T)$ is adopted in good states. The implication is that (similarly to the *PI*'s policy) gains are lower in good states, and losses are lower in bad states compared to those without risk management.



(a) For economies with benchmark less sensitive than normal policy, $\sigma/\kappa < 1/\gamma$.

(b) For economies with benchmark more sensitive than normal policy, $\sigma/\kappa > 1/\gamma$.

Figure 2: Optimal horizon wealth, W_T^H, of a risk manager, H (Proposition 1), benchmarking the stock market with downside hedging (solid plot), and of the non-risk manager N, W_T^N (dotted plot).

In economies where the stock market benchmark reacts more to changes in economic conditions than a normal policy, $\sigma/\kappa > 1/\gamma$ (Proposition 1(b), Figure 2(b)), the benchmark performs better in good states ($\xi_T < \underline{\xi}$) and worse in bad states ($\underline{\xi} \leq \xi_T$), as compared to the normal policy. Consequently, it is now in the good states that the benchmark level is matched (complete insurance against downside risk), and the bad states is where the normal policy is adopted (and no insurance is undertaken). Interestingly, this is in contrast to the findings of related work on portfolio insurance and Value-at-Risk, where good states are not insured. The implication is that gains are higher in good states, and losses higher in bad states compared to those without risk management.

When the stock market sensitivity equals normal sensitivity (Proposition 1(c)), the benchmark and normal policies respond similarly to economic fluctuations, the normal policy, W_T^N, delivers the stock-market return in all states, and hence $W_T^H = W_T^N$.

3. Optimization under Risk Management with Benchmarking

In this section, we solve the optimization problem of a risk manager, who is required to maintain his tracking error relative to the stock market return to be above some prespecified level ε with a given confidence $1-\alpha$, over an investment horizon $[0,T]$.

3.1 Agent's Optimization with Benchmarking the Stock Market

The dynamic optimization problem of the risk manager, B, the benchmarker, who is benchmarking the stock market can be restated as the following static variational problem [using the martingale representation approach (Cox and Huang (1989), Karatzas, Lehoczky, and Shreve (1987))]:

$$\begin{aligned} & \max_{W_T} E[u(W_T)] \\ \textit{subject to} \quad & E[\xi_T W_T] \leq W_0 \ , \\ & P(R_T^W - R_T^X \geq \varepsilon) \geq 1-\alpha \ . \end{aligned} \tag{3}$$

Note that the tracking error constraint complicates the problem not only by introducing nonconcavity into the maximization (as with benchmarking the money market), but also by linking the nature of the nonconcavity to the state-dependent characteristics of the benchmark. Proposition 2 characterizes the optimal solution.[6]

Proposition 2. *The optimal horizon wealth of a risk manager, B, benchmarking the stock market is given by,*

for economies with $\sigma/\kappa < 1/\gamma$:

(a) when $\sigma/\kappa \leq 1$, letting $\overline{\xi}$ satisfy $P(\overline{\xi} \leq \xi_T) = \alpha$, $\underline{\xi} < \overline{\xi}$, we have

$$W_T^B = \begin{cases} I(y^B \xi_T) & \textit{if} \ \ \xi_T < \underline{\xi} \\ X_T & \textit{if} \ \ \underline{\xi} \leq \xi_T < \overline{\xi} \\ I(y^B \xi_T) & \textit{if} \ \ \overline{\xi} \leq \xi_T \ , \end{cases} \tag{4}$$

(b) when $\sigma/\kappa > 1$, letting $\overline{\xi}$, ξ^* satisfy $P(\overline{\xi} \leq \xi_T < \xi^*) = \alpha$, $g(\overline{\xi}) = g(\xi^*)$, $\underline{\xi} < \overline{\xi} < \xi^*$, we have

$$W_T^B = \begin{cases} I(y^B \xi_T) & \textit{if} \ \ \xi_T < \underline{\xi} \\ X_T & \textit{if} \ \ \underline{\xi} \leq \xi_T < \overline{\xi} \\ I(y^B \xi_T) & \textit{if} \ \ \overline{\xi} \leq \xi_T < \xi^* \\ X_T & \textit{if} \ \ \xi^* \leq \xi_T \ , \end{cases} \tag{5}$$

[6] The solution is in closed form, up to the constant Lagrange multiplier, y^B, associated with the budget constraint (3), and in Section 3.2, we will provide explicit numerical solutions for a variety of parameter values. A feasibility condition for a solution is $\min_{\mathcal{A}} E[\xi_T X_T 1_{\{\xi_T \in \mathcal{A}\}}] \leq W_0$, where $\mathcal{A} \equiv \mathcal{R}_+ \setminus [a,b]$, and $0 \leq a < b \leq \infty$ are such that, $P(a \leq \xi_T < b) = \alpha$.

for economies with $\sigma/\kappa > 1/\gamma$:

(c) when $\sigma/\kappa \geq 1$, letting ξ^* satisfy $P(\xi_T < \xi^*) = \alpha$, $\xi^* < \underline{\xi}$, we have

$$W_T^B = \begin{cases} I(y^B\xi_T) & \text{if } \xi_T < \xi^* \\ X_T & \text{if } \xi^* \leq \xi_T < \underline{\xi} \\ I(y^B\xi_T) & \text{if } \underline{\xi} \leq \xi_T \ , \end{cases} \tag{6}$$

(d) when $\sigma/\kappa < 1$, letting $\overline{\xi}$, ξ^* satisfy $P(\overline{\xi} \leq \xi_T < \xi^*) = \alpha$, $g(\overline{\xi}) = g(\xi^*)$, $\overline{\xi} < \xi^* < \underline{\xi}$, we have

$$W_T^B = \begin{cases} X_T & \text{if } \xi_T < \overline{\xi} \\ I(y^B\xi_T) & \text{if } \overline{\xi} \leq \xi_T < \xi^* \\ X_T & \text{if } \xi^* \leq \xi_T < \underline{\xi} \\ I(y^B\xi_T) & \text{if } \underline{\xi} \leq \xi_T \ , \end{cases} \tag{7}$$

for economies with $\sigma/\kappa = 1/\gamma$: if $\varepsilon < 0$, $W_T^B = I(y^B\xi_T)$; if $\varepsilon = 0$, $W_T^B = X_T$; and if $\varepsilon > 0$,

(e) when $\sigma/\kappa > 1$, letting ξ^* satisfy $P(\xi_T < \xi^*) = \alpha$, we have

$$W_T^B = \begin{cases} I(y^B\xi_T) & \text{if } \xi_T < \xi^* \\ X_T & \text{if } \xi^* \leq \xi_T \ , \end{cases} \tag{8}$$

(f) when $\sigma/\kappa < 1$, letting $\overline{\xi}$ satisfy $P(\overline{\xi} \leq \xi_T) = \alpha$, we have

$$W_T^B = \begin{cases} X_T & \text{if } \xi_T < \overline{\xi} \\ I(y^B\xi_T) & \text{if } \overline{\xi} \leq \xi_T \ , \end{cases} \tag{9}$$

(g) when $\sigma/\kappa = 1$, we have $W_T^B = \{I(y^B\xi_T) \text{ or } X_T\}$, with $P(I(y^B\xi_T) < X_T) = \alpha$,

where in all economies $y^B > 0$ solves $E[\xi_T W_T^B] = W_0$, $\overline{\xi}$ and ξ^* denote downward and upward discontinuities in W_T^B, respectively (Figure 3), $\underline{\xi} = (y^B A^\gamma)^{1/(\gamma\sigma/\kappa - 1)}$, A is as in Proposition 1, and $g(\xi) = \left(\gamma(y^B\xi)^{(\gamma-1)/\gamma} - (A/\xi^{\sigma/\kappa})^{1-\gamma}\right)/(1-\gamma) + y^B A \xi^{1-\sigma/\kappa}$. When the tracking error constraint (1) is not binding, then $W_T^B = I(y^B\xi_T)$, $y^B = y^N$.

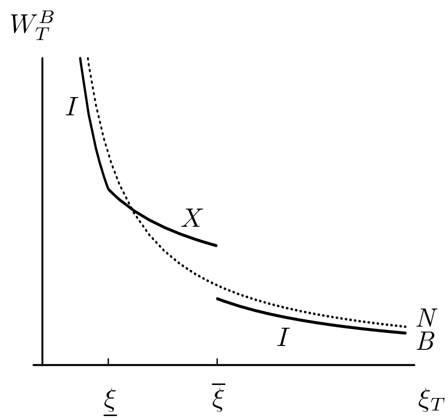
Proposition 2 identifies seven types of economies, each characterized by the sensitivities of the normal policy and the benchmark to changes in the state of the economy. As depicted in Figure 3, the B risk manager follows different policies across these economies, exhibiting either three distinct patterns of economic behavior (economies (a), (c)), four distinct patterns (economies (b), (d)), or two distinct patterns (economies (e), (f)).

In economies where the benchmark is less sensitive than the normal policy, $\sigma/\kappa < 1/\gamma$, downside hedging (when feasible) leads to matching the stock market benchmark in bad states (Proposition 1(a)). When shortfall is allowed, the key difference is that the B risk manager has the option to choose in which "α-fraction" of the states to fall short of the benchmark and revert to a normal-type policy, $I(y^B\xi_T)$. In making his choice, he considers the *state-contingent relative cost* of matching the benchmark vs. following the normal policy; he identifies the states for which the cost of matching the benchmark over and above the normal policy is highest, so that the benefit from reverting to the normal policy is highest in these states.[7] Proposition 2 reveals that the choice depends on whether the benchmark sensitivity is below or above unity. For benchmark sensitivity below unity (economy (a)), $\sigma/\kappa < 1$, the relative cost is highest in bad states, leading the B agent to revert to the normal policy in those states, and to cause the single downward discontinuity at $\bar{\xi}$ (Figure 3(a)).[8] On the other hand, for benchmark sensitivity above unity (economy (b)), $\sigma/\kappa > 1$, it is now the "intermediate-bad" states in which the benchmark is least affordable, leading the B agent to revert to the normal policy in those states, causing the two discontinuities at $\bar{\xi}$ and ξ^* (Figure 2(b)). When the benchmark sensitivity is still above unity, but is further increased to coincide with normal sensitivity (economy (e)), $\sigma/\kappa = 1/\gamma > 1$, it is the good states in which the benchmark is least affordable, provided $\varepsilon > 0$, and the B risk manager beats the stock market return by ε when matching the benchmark level in good states. (When the normal policy and benchmark have the same sensitivity, the normal policy matches the stock market return in all states, and hence for $\varepsilon \leq 0$ the tracking error constraint (1) never binds).

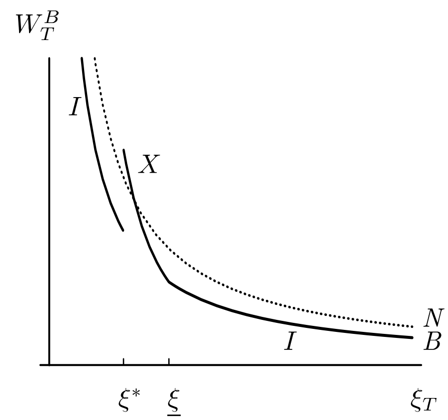
In economies (a), (b), and (e), where the benchmark is not more sensitive than the normal policy, the gains of the B risk manager are lower in good states (as for the M and H risk managers), compared to those without risk management. This is because in good states a normal-type policy is adopted, giving up in these states some gains to provide extra funds required to match the benchmark in other states. In economies (b) and (e), since the benchmark level is matched in bad states, the B agent's losses are lower (as for the H risk manager), while in economy (a), since the normal policy is adopted in bad states, losses are higher (as for the M risk manager), compared to those without risk management. The importance of the B risk manager's ability to match the stock market benchmark level (X_T in (2)) in bad states is underscored by the fact that when he targets overperformance ($\varepsilon > 0$, indeed feasible in bad states due to shortfall in other states), the B risk manager can cut bad-state losses even further. This unique feature of the shortfall approach in economies (b) and (e), allowing risk managers to beat the stock market return (when $\varepsilon > 0$) in bad states, is in sharp contrast to the downside hedging approach forcing risk managers to give up returns (via $\varepsilon < 0$) in these states, and necessarily incur larger losses than those with the shortfall approach. Therefore, for given market parameters (σ, κ), by spotting/enforcing appropriate managerial characteristics (γ) as in Proposition 1(b) or (e), the practice of shortfall-based risk management with benchmarking could be of value to investors, and may also merit a regulatory consideration.

[7] The notion of state-contingent relative cost of the benchmark with respect to the normal policy is formalized by the difference in their respective deflated values, $\xi_T X_T - \xi_T I(y\xi_T)$. As highlighted in the appendix, this difference is largest for high ξ_T in economies (a) and (f), has an interior maximum in (b) and (d), largest for low ξ_T in (c) and (e), and is state independent in (g).

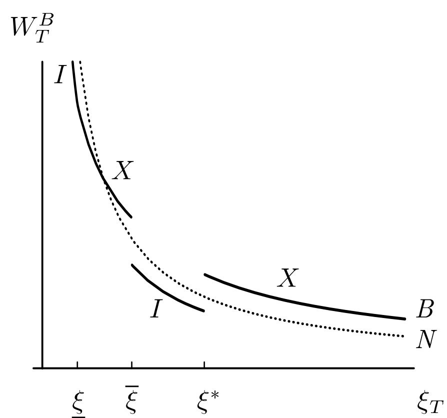
[8] The money market benchmarking agent M's three-region solution (Section 2.3) can now be understood as an example for a benchmark with zero sensitivity. In such a case the benchmark is inherently relatively least affordable in bad states compared to any normal policy, because the latter is adversely affected in bad states for any (risk averse) preferences. It is also evident that the case of negative sensitivity ($\sigma/\kappa < 0$) is captured by the solution in Proposition 2(a). The only difference is that in Figure 3(a), X_T in the intermediate region will be depicted as increasing in ξ_T (and similarly, in the bad states in Figure 2(a)).



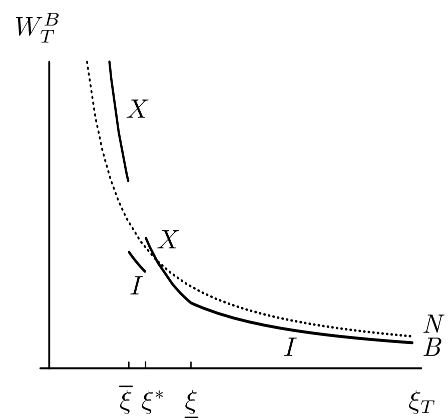
(a) Benchmark less sensitive than normal policy, $\sigma/\kappa < 1/\gamma$, and also $\sigma/\kappa \leq 1$.



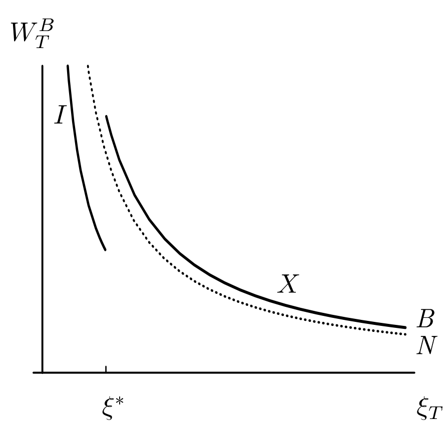
(c) Benchmark more sensitive than normal policy, $\sigma/\kappa > 1/\gamma$, and also $\sigma/\kappa \geq 1$.



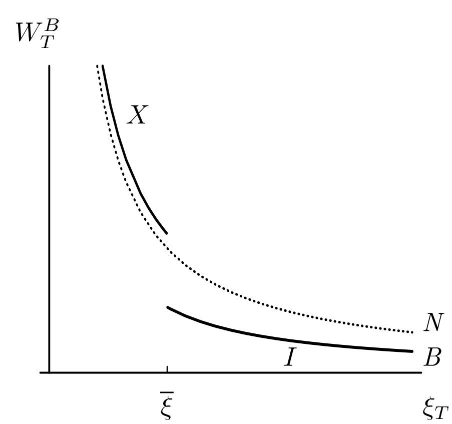
(b) Benchmark less sensitive than normal policy, $\sigma/\kappa < 1/\gamma$, and also $\sigma/\kappa > 1$.



(d) Benchmark more sensitive than normal policy, $\sigma/\kappa > 1/\gamma$, and also $\sigma/\kappa < 1$.



(e) Benchmark as sensitive as normal policy, $\sigma/\kappa = 1/\gamma$, and also $\sigma/\kappa > 1$.



(f) Benchmark as sensitive as normal policy, $\sigma/\kappa = 1/\gamma$, and also $\sigma/\kappa < 1$.

Figure 3: Optimal horizon wealth, W_T^B, of a risk manager, B (Proposition 2), benchmarking the stock market (solid plot), and of the non-risk manager N, W_T^N (dotted plot).

The behavior of the B risk manager in economies (c), (d), and (f) may be understood analogously to that in economies (a), (b), and (e). In economies where the benchmark reacts more to changes in economic conditions than the normal policy, $\sigma/\kappa > 1/\gamma$, downside hedging (when feasible) leads to matching the stock market benchmark in good states (Proposition 1(b)). The B risk manager, who is allowed a shortfall, reverts to the normal-type policy in the best states (economy (c)) up to a probability of α, when the benchmark sensitivity exceeds unity; these are simply the states with the highest state-contingent relative cost. This shortfall leads to the upward discontinuity of ξ^* in the B agent's optimal horizon wealth (Figure 3(c)). When benchmark sensitivity is below unity (economy (d)), it is now the "intermediate-good" states in which the benchmark is least affordable, leading the B agent to revert to the normal policy in those states, causing the two discontinuities at $\overline{\xi}$ and ξ^* (Figure 3(d)). When the benchmark sensitivity is further decreased to coincide with normal sensitivity (economy (f)), the B agent then reverts to the normal policy in the relatively costliest bad states, provided $\varepsilon > 0$ (for $\varepsilon \leq 0$ the tracking error constraint never binds). Finally note that in economy (g) having unit sensitivities implies that for both the benchmark and normal policies, a change in value in response to a change in state prices is compensated by that same change in state prices, so that in fact relative cost is the same across states. The B risk manager can then choose arbitrarily in which states to underperform the benchmark, as long as those states are of probability α.

In economies (c), (d), and (f), where the benchmark is at least as sensitive as the normal policy, the B risk manager's losses are higher in bad states compared to those without risk management (as for the M and H risk managers). This is due to the fact that the B risk manager, who adopts a normal-type policy in bad states, must reduce his wealth in those states to be able to afford matching the benchmark in other states. In economies (d) and (f), since the benchmark level is matched in good states, the B risk manager's gains are higher (as for the H risk manager), and in economy (c) gains are lower, since the normal policy is adopted (as for the M risk manager), compared to without risk management. However, since beating the stock market return is feasible for the B risk manager, unlike for the H risk manager, the B risk manager's gains in good states in economies (d) and (f) can be chosen to always be higher than those of the H risk manager. This feature of enhanced gains under the shortfall approach is yet another important reason why such risk management technique may be widely adopted, as opposed to other alternatives.

3.2 Properties of the Optimal Solution

Proposition 3 presents explicit expressions for the risk manager's optimal wealth and portfolio strategies before the planning horizon, and also presents new results for the special case of downside hedging.

Proposition 3.

(i) The time-t optimal wealth of the risk manager, B, benchmarking the stock market is given by:

$$\begin{aligned} W_t^B &= \Big[1_{\{a,c,d,f\}} + \mathcal{N}(d(\gamma,\underline{\xi}))1_{\{a,b\}} - \mathcal{N}(d(\gamma,\overline{\xi}))1_{\{a,b,d,f\}} \\ &\qquad + \mathcal{N}(d(\gamma,\xi^*))1_{\{b,c,d,e\}} - \mathcal{N}(d(\gamma,\underline{\xi}))1_{\{c,d\}}\Big] Z(\gamma)(y^B\xi_t)^{-1/\gamma} \\ &\quad + \Big[1_{\{b,e\}} - \mathcal{N}(d(\kappa/\sigma,\underline{\xi}))1_{\{a,b\}} + \mathcal{N}(d(\kappa/\sigma,\overline{\xi}))1_{\{a,b,d,f\}} \\ &\qquad - \mathcal{N}(d(\kappa/\sigma,\xi^*))1_{\{b,c,d,e\}} + \mathcal{N}(d(\kappa/\sigma,\underline{\xi}))1_{\{c,d\}}\Big] Z(\kappa/\sigma)A\xi_t^{-\sigma/\kappa} \ , \end{aligned} \tag{10}$$

where the arguments of the indicator function $1_{\{\}}$ refer to the economies identified in Proposition 2, $\mathcal{N}(\cdot)$ is the standard-normal cumulative distribution function, y^B is as in Proposition 2, and

$$Z(v) \equiv e^{\frac{1-v}{v}\left(r+\frac{||\kappa||^2}{2v}\right)(T-t)} \ , \qquad d(v,x) \equiv \frac{\ln\frac{x}{\xi(t)} + (r + \frac{2-v}{2v}||\kappa||^2)(T-t)}{||\kappa||\sqrt{T-t}} \ .$$

(ii) The fraction of wealth invested in stocks is:

$$\theta_t^B = q_t^B\theta^N \ ,$$

where θ^N, the optimal fraction of wealth invested in stocks under the normal policy, and q_t^B, the exposure relative to the normal policy, are given by

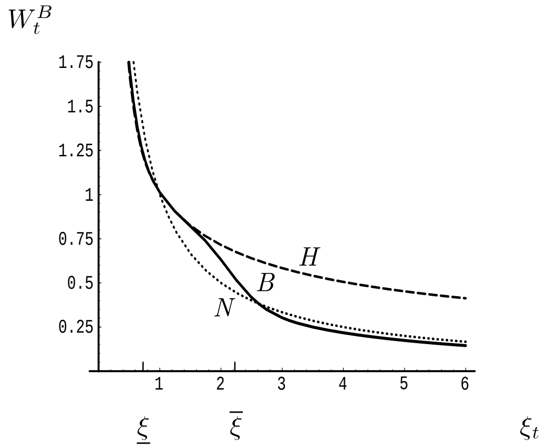
$$\begin{aligned} \theta^N &= (\kappa/\sigma)/\gamma, \\ q_t^B &= 1 + \Big[1_{\{b\}} - \mathcal{N}(d(\kappa/\sigma,\underline{\xi}))1_{\{a,b\}} + \mathcal{N}(d(\kappa/\sigma,\overline{\xi}))1_{\{a,b,d\}} \\ &\qquad - \mathcal{N}(d(\kappa/\sigma,\xi^*))1_{\{b,c,d\}} + \mathcal{N}(d(\kappa/\sigma,\underline{\xi}))1_{\{c,d\}}\Big] (\gamma\sigma/\kappa - 1)\, Z(\kappa/\sigma)A\xi_t^{-\sigma/\kappa}/W_t^B \\ &\quad + \Bigg\{ \Big[\varphi(d(\gamma,\underline{\xi}))1_{\{a,b\}} - \varphi(d(\gamma,\overline{\xi}))1_{\{a,b,d,f\}} + \varphi(d(\gamma,\xi^*))1_{\{b,c,d,e\}} \\ &\qquad - \varphi(d(\gamma,\underline{\xi}))1_{\{c,d\}}\Big] Z(\gamma)(y^B\xi_t)^{-1/\gamma} \\ &\quad - \Big[\varphi(d(\kappa/\sigma,\underline{\xi}))1_{\{a,b\}} - \varphi(d(\kappa/\sigma,\overline{\xi}))1_{\{a,b,d,f\}} + \varphi(d(\kappa/\sigma,\xi^*))1_{\{b,c,d,e\}} \\ &\qquad - \varphi(d(\kappa/\sigma,\underline{\xi}))1_{\{c,d\}}\Big] Z(\kappa/\sigma)A\xi_t^{-\sigma/\kappa}\Bigg\} \ \gamma/(W_t^B||\kappa||\sqrt{T-t}) \ , \end{aligned} \tag{11}$$

where $\varphi(\cdot)$ is the standard-normal probability density function.

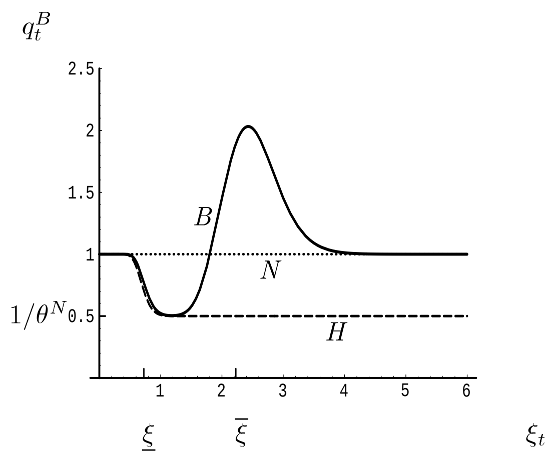
(iii) When $\varepsilon < 0$, $\sigma/\kappa \neq 1/\gamma$, and the risk manager benchmarks the stock market with downside hedging, H, the optimal policies are given by (10) and (11), for $\alpha = 0$, so that in (a) $\overline{\xi} = \infty$, in (c) $\xi^* = 0$, and in (b) and (d) $\overline{\xi} = \xi^*$. When $\overline{\xi} = \infty$, $\sigma/\kappa = 1/\gamma$, (10) and (11) coincide with the normal policy. When $\varepsilon = 0$, X_t is the optimal policy, with relative risk exposure of $1/\theta^N$.

Figures 4 and 5 depict the policies given in Proposition 3.[9] Figure 4 presents the results for economies (a) and (c) when the B risk manager's goal is merely to limit underperformance ($\varepsilon < 0$),

[9] The optimal horizon wealth (Proposition 2) can be expressed as the wealth generated by a normal policy plus an option to exchange this wealth for the horizon benchmark level, plus a short binary option position with exercise range

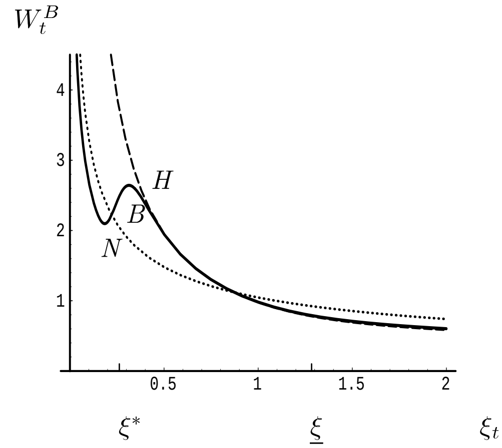


(i) Wealth of B, H and N risk managers

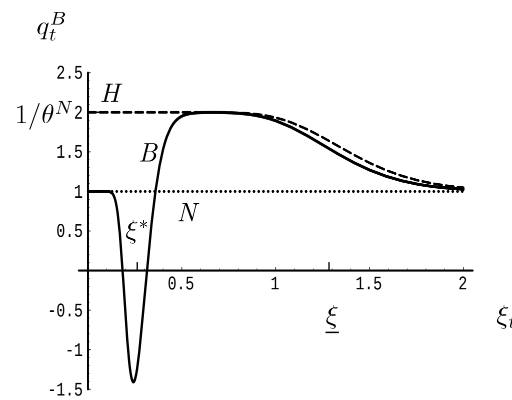


(ii) Relative risk exposure of B, H and N risk managers

Economy (a): Benchmark less sensitive than normal policy, $\sigma/\kappa < 1/\gamma$, and also $\sigma/\kappa \leq 1$.



(i) Wealth of B, H and N risk managers



(ii) Relative risk exposure of B, H and N risk managers

Economy (c): Benchmark more sensitive than normal policy, $\sigma/\kappa > 1/\gamma$, and also $\sigma/\kappa \geq 1$.

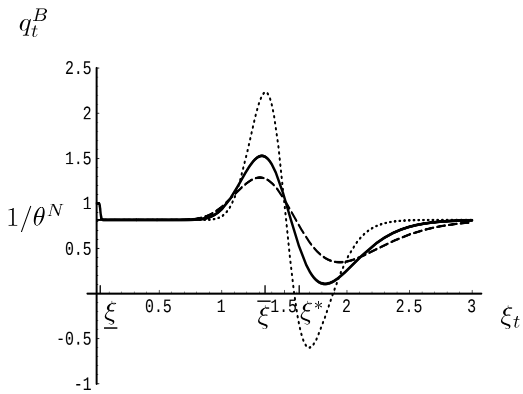
Figure 4: The time-t (i) wealth and (ii) exposure to risky assets relative to the normal policy (Proposition 3), for the risk manager, B, benchmarking the stock market (solid plots), the risk manager, H, with downside hedging (dashed plots), and the non-risk manager, N (dotted plots), in economies (a) and (c) (see Figure 5 for economies (b), (d), (e), or (f)). In both economies, $\alpha = 0.01$, $\varepsilon = -0.025$, $t = 0.8$, $T = 1$, $r = 0.05$, and $W_0 = 1$. In (a), $\sigma/\kappa = 0.5$, $1/\gamma = 1$, $\sigma = 0.2$, then $y^B = 1.15$, $\underline{\xi} = 0.76$, $\overline{\xi} = 2.23$. In (c), $\sigma/\kappa = 1$, $1/\gamma = 0.5$, $\sigma = 0.5$, then $y^B = 1.35$, $\xi^* = 0.26$, $\underline{\xi} = 1.28$.

corresponding to the shortfall region, and payoff given by the shortfall amount. The expression in Proposition 3(i) can be understood as the pre-horizon value of this option package in each economy (for a given state, ξ_t), explaining the appearance of the Black and Scholes (1973)-type terms (due to lognormality), as well as the non-monotonous patterns (due to the binary option) in the figures.

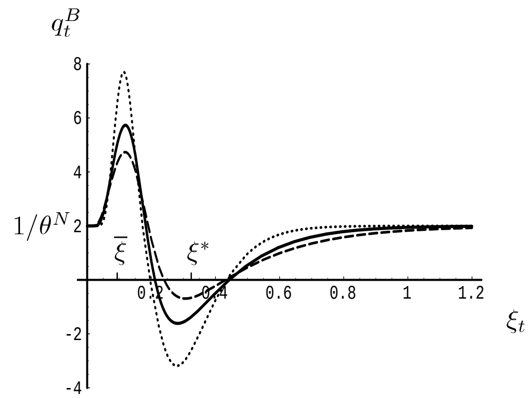
allowing us to simultaneously study the policies of the risk manager, H, who downside hedges. In economy (a), where the benchmark is less sensitive than the normal policy, both the B and H risk managers match the horizon benchmark level in intermediate states, with the former choosing to fall short in bad states. Thus, both managers' pre-horizon wealth behaves similarly to that of a non-risk manager, N, in good states (with the H policy giving up some of the upside), tending to the current value of the benchmark in intermediate states. In bad states, the H risk manager's pre-horizon wealth continues to track the less sensitive benchmark, whereas the B risk manager's wealth reverts back to resemble that of the N agent. Similarly, the risk exposure, relative to that without risk management, for both risk managers resembles the normal policy in good states, and as ξ_t increases, this risk exposure decreases towards $1/\theta^N < 1$, the relative risk exposure required to replicate the benchmark. In bad states, the H risk manager remains invested in the benchmark. The B risk manager, however, increases his exposure back up to, then above, and finally back down to, the normal policy level as ξ_t continues to increase. In states near $\overline{\xi}$, there is a fair chance that the B risk manager will be able to match the benchmark, but only if he takes a large stock position and the economy does not experience a downturn ($\xi_T < \overline{\xi}$).

The behavior of pre-horizon wealth and risk exposure in economy (c) is explained similarly, after allowing for the fact that the B risk manager falls short of the horizon benchmark level in good states. In the region of ξ^*, the B risk manager reduces, rather than increases, his stock market exposure, possibly even taking a short position, to allow him to increase his wealth and match the benchmark if economic conditions turn out to be not very favorable ($\xi_T > \xi^*$). A noteworthy feature of economy (c) is that due to the upward discontinuity at ξ^* of the horizon policy, over a region of the state space, the pre-horizon wealth of the B risk manager increases, rather than decreases for deteriorating economic conditions. Therefore, contrary to standard results, where optimal wealth suffers as economic conditions deteriorate, here we see the opposite feature. Under shortfall-based risk management, over a region of the state space risk managers (and their clients) become wealthier as the economy worsens. A byproduct of this behavior is that the same level of wealth may be observed under three different economic scenarios (e.g., consider the $W_t^B = 2.5$ level obtained for three different values of ξ_t in Figure 4 for economy (c)), suggesting caution in attempting to deduce the state of the economy by observing portfolio wealth alone. This feature of increasing wealth can be observed under wide range of economic primitives, as it is prevalent around ξ^* in economies (b), (d), and (e) as well (for brevity not depicted in the figures).

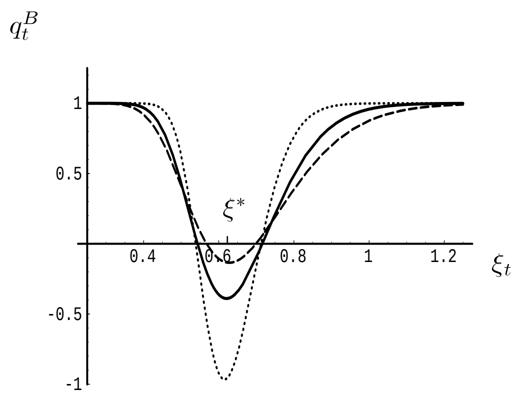
Economy (b) differs from economy (a), and economy (d) from economy (c), in that the B risk manager falls short of the benchmark in intermediately bad or in intermediately good states, respectively. At time t, in both (b) and (d), the chances of a shortfall are higher in the region around $\overline{\xi}$ to ξ^*, rather than for either very high or low values of ξ_t. Therefore, as in Figure 5, around the lower end of this region, the B risk manager takes a large stock position allowing him to meet the



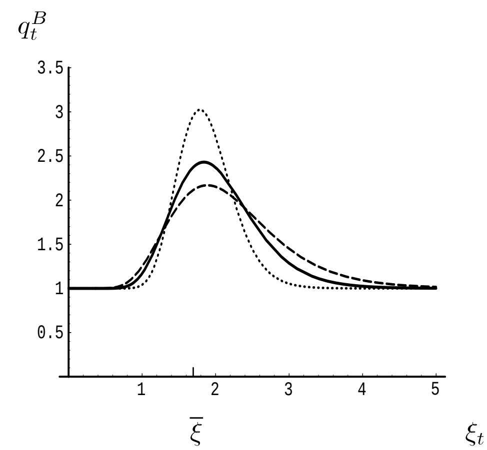
Economy (b): Benchmark less sensitive than normal policy, $\sigma/\kappa < 1/\gamma$, and also $\sigma/\kappa > 1$.



Economy (d): Benchmark more sensitive than normal policy, $\sigma/\kappa > 1/\gamma$, and also $\sigma/\kappa < 1$.



Economy (e): Benchmark as sensitive as normal policy, $\sigma/\kappa = 1/\gamma$, and also $\sigma/\kappa > 1$.

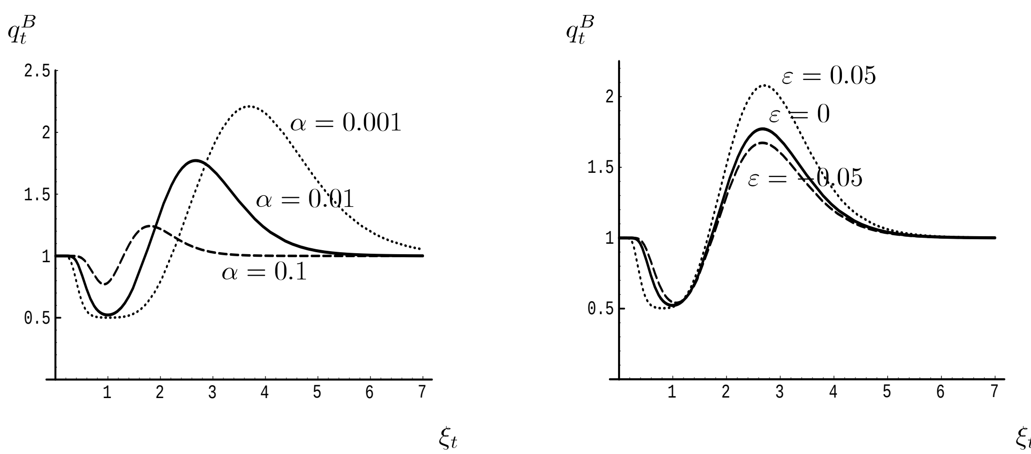


Economy (f): Benchmark as sensitive as normal policy, $\sigma/\kappa = 1/\gamma$, and also $\sigma/\kappa < 1$.

Figure 5: The time-t exposure to risky assets relative to the normal policy, for the risk manager, B, benchmarking the stock market (Proposition 3) for economies (b), (d), (e) and (f). In all economies $T = 1$, $r = 0.05$, $W_0 = 1$, $\alpha = 0.05$, and $\varepsilon = 0.03$. The solid, dashed, and dotted plots represent $t = 0.5$, $t = 0.25$, and $t = 0.75$, respectively. In (b), $\sigma/\kappa = 1.36$, $1/\gamma = 1.67$, $\sigma = 0.33$, then $y^B = 1.92$, $\underline{\xi} = 0.03$, $\overline{\xi} = 1.35$, $\xi^* = 1.62$. In (d), $\sigma/\kappa = 0.8$, $1/\gamma = 0.40$, $\sigma = 0.45$, then $y^B = 6.99$, $\overline{\xi} = 0.09$, $\xi^* = 0.32$, $\underline{\xi} = 8.22$. In (e), $\sigma/\kappa = 1/\gamma = 1.25$, $\sigma = 0.3$, then $y^B = 1.80$, $\xi^* = 0.62$. In (f), $\sigma/\kappa = 1/\gamma = 0.5$, $\sigma = 0.2$, then $y^B = 2.37$, $\overline{\xi} = 1.69$.

benchmark if the economy does not experience a downturn ($\xi_T < \overline{\xi}$, as in economy (a)), around the upper end, the B risk manager reduces his stock market exposure allowing him to match the benchmark if the economy does not prosper ($\xi_T > \xi^*$, as in economy (c)). In the worst states in economy (b), and the best states in economy (d), the B risk manager's risk exposure tends to $1/\theta^N$. An interesting implication of the optimal policy in economies (b) and (d) is that one can observe considerable shifts in portfolio composition, possibly shifting from leveraged to short positions, and vice versa, upon relatively minor changes in economic conditions (as captured by changes in ξ_t). Hence, if shortfall-based risk management is indeed explicitly or implicitly being followed by institutional investors, our results suggest a potential explanation to the puzzling, but yet observed phenomena, where seemingly small arrivals of news regarding fundamentals may at times indeed carry no considerable reaction from market participants, but at other times cause significant portfolio rebalancing. Clearly, in economies (b) and (d) (as in the other economies), the nature of the risk management practice (α, ε) as well as the actual state of the economy (ξ_t) determines how pronounced the impact of external news is.

In economy (e), the B risk manager falls short of the horizon benchmark level in good states, and in economy (f) in bad states. In both economies, the relative risk exposure tends to unity for states in which the B risk manager follows the normal policy, as well as for states in which he matches the benchmark (because the replicating portfolio is given by the normal policy when $\sigma/\kappa = 1/\gamma$). In economy (e), the B risk manager reduces his relative stock market exposure in the region of ξ^* (as in (b), (c), and (d)), and in economy (f) he increases his relative exposure in the region of $\overline{\xi}$ (as in (a), (b), and (d)) to allow him match the benchmark or fall short depending on how the economy fares. Figure 5 also displays the dependence of the relative risk exposures in economies (b), (d), (e) and (f) on time. In all economies (including (a) and (c), not displayed), decreasing the time-to-horizon causes the B risk manager to deviate further from the normal policy in the region in which chances of shortfall are highest, amplifying in that region the swings in portfolio positions in response to news.



(i) the effect of α (ii) the effect of ε

Figure 6: The benchmarking risk manager's, B, relative risk exposure for varying levels of (i) $\alpha \in \{0.001, 0.01, 0.1\}$, and (ii) $\varepsilon \in \{-0.05, 0, 0.05\}$ in economy (a). The solid plots represent the following parameter values: $\sigma/\kappa = 0.5$, $1/\gamma = 1$, $\alpha = 0.01$, $\varepsilon = 0$, $r = 0.05$, $\sigma = 0.2$, and $W_0 = 1$. Then $y^B = 1.28$, $\underline{\xi} = 0.55$, and $\overline{\xi} = 2.23$.

Sensitivities to the shortfall probability, α, and the extent of over/under-performance, ε, are depicted in Figure 6 for economy (a), and are suggestive of the results across other economies. The B risk manager deviates further from the normal policy as α decreases and as ε increases, in each case reflecting the greater influence of the tracking error constraint. The effect is most pronounced in the region of maximum exposure around $\overline{\xi}$, as around $\underline{\xi}$, the risk exposure is bounded below by $1/\theta^N$ ($= 0.5$ for the parameters in the figure). The maximum exposure for decreasing α occurs for higher values of ξ_t, whereas the maximum exposure for increasing ε occurs at about the same value, since in the former case the shortfall region is shrinking, whereas in the latter it is fixed. Therefore, if simultaneous large portfolio swings by numerous risk managers are considered undesirable (either because of price stability concerns, or transaction costs considerations), it may be desirable to not encourage all risk managers to adopt the same confidence level, but rather promote diversity in the choice of α (unlike the current uniform choice of $\alpha = 0.01$ for VaR users in the banking sector).

4. Alternative Formulations and Extensions

4.1 Benchmarking with Limited Expected Relative Losses

We have so far considered the most basic shortfall approach, captured by the quantile-based tracking error constraint (1), which focuses on the shortfall probability α of not meeting the target return $R_T^X + \varepsilon$. An alternative approach is to limit both the probability and magnitude of the shortfall, and the simplest way to achieve that is to adopt an expectations-based constraint that limits the losses relative to the horizon benchmark level in (2):

$$E\left[\xi_T\left(e^{(R_T^X+\varepsilon)T} - e^{R_T^W T}\right) 1_{\{R_T^W - R_T^X < \varepsilon\}}\right] \leq \beta\ . \tag{12}$$

Proposition 4 presents the optimal policy of a risk manager, L, benchmarking the stock market ($R_T^X = R_T^S$) subject to limited expected relative losses (LERL) as in (12).[10]

Proposition 4. *The optimal horizon wealth of a risk manager, L, benchmarking the stock market subject to limited expected relative losses is given by,*
(a) for economies with $\sigma/\kappa < 1/\gamma$:

$$W_T^L = \begin{cases} I(z_1\xi_T) & \text{if } \ \xi_T < \underline{\xi} \\ X_T & \text{if } \ \underline{\xi} \leq \xi_T < \overline{\xi} \\ I((z_1 - z_2)\xi_T) & \text{if } \ \overline{\xi} \leq \xi_T\ , \end{cases}$$

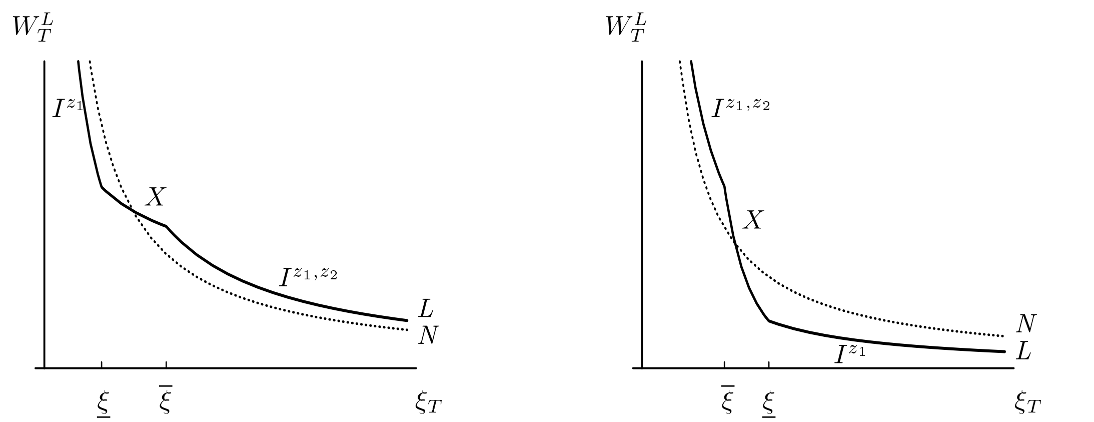
[10] The constraint in (12) is cast in units of wealth, because as we saw in Section 3, terminal wealth is in fact the relevant choice variable. Another expectations-based constraint to consider is $E\left[\left((R_T^X + \varepsilon) - R_T^W\right) 1_{\{R_T^W - R_T^X < \varepsilon\}}\right] \leq \eta$. Although this constraint is less tractable to analyze, our insights can be extended to this case as well.

(b) for economies with $\sigma/\kappa > 1/\gamma$:

$$W_T^L = \begin{cases} I((z_1 - z_2)\xi_T) & \text{if } \ \xi_T < \overline{\xi} \\ X_T & \text{if } \ \overline{\xi} \leq \xi_T < \underline{\xi} \\ I(z_1 \xi_T) & \text{if } \ \underline{\xi} \leq \xi_T \ , \end{cases}$$

(c) for economies with $\sigma/\kappa = 1/\gamma$: $W_T^L = I((z_1 - z_2)\xi_T)$ coincides with W_T^N,

where in all economies $z_1 > z_2 \geq 0$ solve $E[\xi_T W_T^L] = W_0$ with (12) holding with equality, $\underline{\xi} = (z_1^\gamma)^{1/(\gamma\sigma/\kappa - 1)}$, $\underline{\xi} = ((z_1 - z_2)^\gamma)^{1/(\gamma\sigma/\kappa - 1)}$, and A is as in Proposition 1.



(a) For economies with benchmark less sensitive than normal policy, $\sigma/\kappa < 1/\gamma$.

(b) For economies with benchmark more sensitive than normal policy, $\sigma/\kappa > 1/\gamma$.

Figure 7: Optimal horizon wealth, W_T^L, of a risk manager, L (Proposition 4), benchmarking the stock market with LERL (solid plot), and of the non-risk manager N, W_T^N (dotted plot).

Figure 7(a) highlights the result in Proposition 4(a), that in economies where the stock market is less sensitive than the normal policy, the LERL approach guarantees lower losses in bad states than those without risk management. Clearly, a similar conclusion will arise in the context of benchmarking the money market with zero sensitivity (see the limited expected losses (LEL) analysis in Basak and Shapiro (2001)). However, since it is most cost effective to fall short of the benchmark in bad states, the risk management with LERL is less desirable for those interested to beat the stock market in bad states. It is the quantile shortfall approach that can allow overperformance of the market in bad states (when investments are entrusted to managers possessing appropriate characteristics as in Proposition 2(b)(e)).

Figure 7(b) depicts the result in Proposition 4(b), that in economies where the stock market is more sensitive than the normal policy, the LERL approach leads to larger losses in bad states, compared to those without risk management. This is similar to the outcome under the quantile-based approach. However, the quantile shortfall approach offers the flexibility (under appropriate characteristics) to beat the stock market in good states, an option unavailable to the L risk manager. Therefore, contrary to the case of benchmarking the money market, the expectations based risk measure is not unambiguously more desirable than the quantile measure on a gain/loss basis. Nevertheless, the continuous nature of the L policy eliminates the need for taking large short or leveraged positions prior to the horizon, and that may be an attractive feature from a regulatory perspective due to potential implications on market volatility in some states of the world.

4.2 Performance Measures

Section 3 illustrated how imposing a tracking error constraint on the risk manager can lead to a rich diversity of gain/loss profiles for funds under management. While these gain/loss profiles provide the most complete characterization of investment behavior, in practice investors may want to obtain, or only have access to, certain "summary" measures of performance. The primary measure of performance reported by various investment information providers is total return, while volatility (standard deviation of returns) and Sharpe ratio (risk-adjusted risk premium) are also commonly reported (see, e.g., www.fidelity.com, www.morningstar.com). Moreover, returns are typically reported relative to a benchmark; for example Morningstar, a leading information provider, reports the amount by which a given fund over- or underperformed its primary index (the S&P500 for stock-oriented funds) during a calendar year. Given how widely the above measures are reported and the fact that they are often used to compare and rank investment performance, an important question is whether a tracking error constraint will lead a risk manager to follow policies with higher expected returns and Sharpe ratios than would obtain without risk management or by investing directly in the benchmark.

To address this issue, we focus initially for simplicity on economies (e) and (f), in which the benchmark is as sensitive as the normal policy and the non-risk manager optimally invests all funds under management in the benchmark. Consequently, performance measure comparisons relative to the normal policy are equivalent to those relative to the benchmark.[11] Table 1 presents the ex-ante expected return, volatility, and Sharpe ratio of the B risk manager for varying levels of the shortfall probability α, and the target overperformance return ε, expressed as percentages of the corresponding normal policy or benchmark values.[12] The parameter values are chosen to capture

[11] In the other economies, the analysis is complicated somewhat by the fact that the normal policy differs from the benchmark. In economies (a) and (c), depending on the choice of parameter values, the expected return of the normal policy can be greater than, less than, or equal to the expected benchmark return. In economies (b) and (d), the expected return of the normal policy is always less than that of the benchmark. At the same time, the Sharpe ratio of the normal policy is always less than that of the benchmark in economies (a) and (b), and greater in economies (c) and (d).

[12] Given a realized return, $R_T^W \equiv \frac{1}{T}\ln(W_T^B/W_0)$, where W_T^B for a given economy is as in Proposition 2, the expected return, $E[R_T^W]$, and variance, $Var[R_T^W]$, over $[0,T]$ are calculated as outlined in the Appendix. The Sharpe ratio is defined as $(E[R_T^W]-r)/\sqrt{Var[R_T^W]}$. The expected return, variance, and Sharpe ratio for the non-risk manager and the benchmark are defined and computed analogously. We also note that in economy (e) relative risk aversion has

reasonably realistic combinations, and thereby provide empirically relevant assessments of the ex-ante performance measures of the B risk manager's strategies.

In economies (e) and (f), the expected return for the B risk manager is a weighted average of the expected target return, $E[R_T^X] + \varepsilon$, and the expected shortfall return, with weights $(1 - \alpha)$ and α, respectively (see Appendix). The expected shortfall return (the expected return of the normal type policy the B risk manager reverts to in the shortfall states) is less than or equal to the expected return of the benchmark in economies (e) and (f), and decreases as the shortfall probability decreases or as the overperformance level increases. Inspection of the results in Table 1 establishes that, in both economies, the effect of the higher return attained by the B risk manager in states in which he matches the target return can outweigh the lower return in the shortfall states. Hence, attaining a higher expected return than without risk management, or by investing directly in the benchmark, is possible, except when the tracking error constraint is difficult to meet (a small shortfall probability combined with a high target overperformance return).

	Expected Return			Volatility			Sharpe Ratio		
$\alpha =$	2.5%	5%	10%	2.5%	5%	10%	2.5%	5%	10%
Economy (e)									
$\varepsilon = 0.5\%$	100.2	100.4	100.5	96.1	96.4	96.8	104.7	105.3	105.0
$\varepsilon = 1\%$	98.6	100.1	100.6	93.3	93.1	93.7	101.8	108.0	109.1
$\varepsilon = 2\%$	84.6	96.9	99.9	101.0	88.5	88.1	43.5	100.1	113.3
Economy (f)									
$\varepsilon = 0.5\%$	101.4	101.3	101.1	103.1	102.7	102.4	99.0	99.3	99.4
$\varepsilon = 1\%$	102.5	102.4	102.2	107.1	105.8	105.0	96.9	98.0	98.5
$\varepsilon = 2\%$	103.4	104.2	104.1	119.3	113.5	110.7	88.1	93.8	96.1

Table 1: The ex-ante expected return, volatility, and Sharpe ratio in economies (e) and (f), calculated for varying levels of shortfall probability, α, and target overperformance return, ε, expressed as percentages of the corresponding normal policy (and benchmark) values. The fixed parameter values are $r = 0.05$, $\sigma = 0.25$, $T = 1$, $W_0 = 1$, and $\gamma = 0.8$ in economy (e), $\gamma = 2$ in economy (f).

Whereas expected returns are higher for most scenarios across the two economies, after adjusting for risk a different pattern emerges. As illustrated in Table 1, the ex-ante Sharpe ratio of the B risk manager exceeds that of the non-risk manager (and the benchmark) for almost all the reported values of both the shortfall probability and the target overperformance return in economy (e), but always lies below it in economy (f). This difference in behavior of Sharpe ratios is driven by

to be less than 1. We disregard values below 0.5 as these lead to negative Sharpe ratios for the stock market. In economy (f), relative risk aversion has to lie above 1. Results in Table 1 are representative for values of risk aversion $\gamma \in [0.5, 1)$ in economy (e), and $\gamma > 1$ in economy (f).

the difference in risk exposures of the B risk manager across the two economies. As discussed in Section 3.2, in economy (f), the B risk manager's risk exposure is always greater than or equal to the normal policy level and deviates further from this level, resulting in increased volatility, as α decreases or ε increases. On the other hand, in economy (e), the B risk manager's risk exposure is always less than or equal to the normal level. As α decreases, or as ε increases, the deviation in economy (e) of the risk exposure from the normal level initially results in lower overall volatility, contributing to a higher Sharpe ratio. However, as α continues to decrease or ε to increase, the increased deviation from the normal level results in increasingly large short positions which increase overall volatility back up to and then above the normal level. Correspondingly, the B risk manager's Sharpe ratio falls below that of the non-risk manager and that of the benchmark when the tracking error constraint becomes extremely difficult to meet. Notwithstanding, note that in economy (e), the effect of the reduced volatility can result in a risk manager's Sharpe ratio being higher than that without risk management, even when the corresponding expected return is lower.

	Expected Return			Volatility			Sharpe Ratio		
$\alpha =$	2.5%	5%	10%	2.5%	5%	10%	2.5%	5%	10%
Values relative to the normal policy									
$\varepsilon =$ -2%	101.2	100.9	100.6	80.8	86.4	92.6	126.6	117.9	109.3
$\varepsilon =$ 0%	100.2	100.2	100.2	76.7	83.7	91.1	130.7	120.0	110.1
$\varepsilon =$ 2%	93.6	98.2	99.1	83.6	82.0	89.9	104.7	117.7	109.3
Values relative to the benchmark									
$\varepsilon =$ -2%	101.2	100.9	100.6	134.7	143.9	154.3	76.0	70.7	65.6
$\varepsilon =$ 0%	100.2	100.2	100.2	127.9	139.4	151.8	78.4	72.0	66.1
$\varepsilon =$ 2%	93.6	98.2	99.1	139.3	136.6	149.9	62.8	70.6	65.6

Table 2: The ex-ante expected return, volatility, and Sharpe ratio in economy (a), calculated for varying levels of shortfall probability, α, and target overperformance return, ε, expressed as percentages of the corresponding normal policy and benchmark values. The fixed parameter values are $r = 0.05$, $\sigma = 0.25$, $T = 1$, and $W_0 = 1$. Note that $1/\gamma + \sigma/\kappa = 2$, for the expected returns of the benchmark and the normal policy to be equal, implying $0.5 < \gamma \leq 1$. We use a representative value of $\gamma = 0.8$ in the Table.

In the other economies ((a) through (d)), the normal policy differs from the benchmark, nonetheless similar results obtain. For example, Table 2 reports the ex-ante expected return, volatility and Sharpe ratios for varying levels of α and ε in economy (a). The upper and lower rows of the table report these values relative to the normal policy, and the benchmark, respectively. The parameter values are chosen so that the expected returns for the normal policy and the benchmark are the same. However, in economy (a) the volatility of the normal policy is greater than that of the benchmark, whereas the Sharpe ratio of the normal policy is less than that of the benchmark (because in (a), $\sigma/\kappa < 1/\gamma$, and the non-risk manager is always leveraged; $\theta^N = (\kappa/\sigma)/\gamma > 1$). As in economies (e) and (f), the expected return for the B risk manager exceeds that of the non-risk manager and the benchmark, for many reasonable combinations of parameter values. However, adjusting for risk, the Sharpe ratio of the B risk manager exceeds that of the non-risk manager for almost all values of α and ε, but lies below the benchmark Sharpe ratio. (This can once again be explained by the difference in risk exposure and can be shown to be robust to the relationship between the expected returns of the normal policy and the benchmark.) In economy (c), the opposite relationship obtains: the Sharpe ratio of the B risk manager exceeds that of the benchmark for almost all values of α and ε, but lies below the Sharpe ratio of the normal policy.

Based on comprehensive numerical analysis, our results indicate that for a given economy an appropriate combined choice of α and ε can lead to higher expected returns than without risk management or than that of the benchmark. In considering the risk of achieving a given return, the Sharpe ratio of the B risk manager can exceed that of the non-risk manager in certain economies, but lies below it in others. Moreover, when the normal policy differs from the benchmark, the Sharpe ratio of the B risk manager can exceed either the normal policy or the benchmark, but not both. Thus, if investors or fund managers require high risk-adjusted measures of returns, such as the Sharpe ratio, they should ensure that, given the basis for comparison (either the normal policy or the benchmark), the characteristics of the economy (i.e., benchmark sensitivity, and managerial risk profile) are such that it is indeed possible for the B risk manager's Sharpe ratio to exceed the reference Sharpe ratio.

4.3 Multiple Sources of Uncertainty

When there are multiple sources of uncertainty in the economy, our results regarding benchmarking the stock market remain the same provided stock market fluctuations are driven be the "aggregate"/"systematic" uncertainty, as captured by the state price density process ξ. Although benchmarking the stock market is the most common practice, if one is interested in benchmarking some sector of the market, that sector will in general not be driven solely by the "systematic" uncertainty, but will also be affected by "specific"/"idiosyncratic" uncertainty. Notwithstanding, our insights are still applicable.

Consider, for example, an economy where uncertainty is generated by two Brownian motions (w_1, w_2), and where financial investment opportunities are given by the money market account, and two risky stocks (S, Q), each with a price following a geometric Brownian motion. Further assume that the risk manager has logarithmic preferences ($\gamma = 1$), and benchmarks the performance of the first stock ($R_T^X = R_T^S$), with allowed shortfall probability α. Without loss of generality, normalize $W_0 = S_0$, and suppose that an exactly matched performance is desired ($\varepsilon = 0$) so that $X_T = S_T$. In this case, one can show that the optimal policy of a risk manager benchmarking the stock S is given by:

$$W_T^B = \begin{cases} I(y^B\xi_T) = 1/(y^B\xi_T) & \text{if } \ \xi_T < 1/(y^B S_T) & \text{(I)} \\ S_T & \text{if } \ 1/(y^B S_T) \leq \xi_T < c/S_T & \text{(II)} \\ I(y^B\xi_T) = 1/(y^B\xi_T) & \text{if } \ c/S_T \leq \xi_T \ , & \text{(III)} \end{cases}$$

where c satisfies $P(\xi_T S_T \geq c) = \alpha$, and y^B is determined by the budget constraint. Although the primitive sources of uncertainty are the two Brownian motions, this two-dimensional state space can be equivalently represented in terms of S_T and ξ_T. The optimal policy, W_T^B, exhibits three distinct patterns of behavior over three regions of the (S_T, ξ_T) state space, where region (III) is the shortfall region (in which $W_T^B < S_T$). However, it is the correlation between S_T and ξ_T that will determine the location of each region within the (S_T, ξ_T) plane.

When the benchmark, S_T, represents a dominant sector within the economy, it is driven mainly by ξ_T. Then the risk manager's problem becomes effectively one-dimensional, and the solution resembles one of the three-regions policies in Proposition 2(a)(c) (depicted in Figure 3(a)(c)), depending on the underlying parameters ($\gamma = 1$ means that we are either in economies (a) or (c)). As the correlation between S_T and the ξ_T weakens, S_T can take many values upon a given realization of ξ_T. Still, whether the optimal policy follows the benchmark or the normal behavior is determined, as in the one-dimensional case, by considering the relative sensitivities, and the state-contingent relative costs of the two types of behavior.

If S_T has low sensitivity with respect to ξ_T, it will tend to not rise significantly in good states (low ξ_T) and to not decrease significantly in bad states (high ξ_T). Then, it is the condition for region (I) that will hold in good states (because the product $\xi_T S_T$ is low there), and the condition for region (III) that will hold in bad states (because $\xi_T S_T$ is high there). Consequently, the location of the shortfall region in the (S_T, ξ_T) plane will be where ξ_T is high and where S_T is not too low. If, on the other hand, S_T is highly sensitive with respect to ξ_T, it will tend to rise significantly in good states (low ξ_T) and decrease significantly in bad states (high ξ_T). Then, it is the condition for region (III) that will hold in good states (because $\xi_T S_T$ is now high there), and the condition for region (I) that will hold in bad states (because $\xi_T S_T$ is now low there). Therefore, the location of the shortfall region in the (S_T, ξ_T) plane will now be where S_T is high and ξ_T is not too low.

4.4 Hybrid Benchmarks and Other Extensions

Section 4.3 examined one particular benchmark that is different from the aggregate stock market, and clearly many other alternatives may be of interest. Focusing on what is actually used in practice, there appears to be considerable interest in hybrid composite benchmarks. These benchmarks combine both money market and stock market returns in a hypothetical unmanaged combination. Such hybrid returns are reported to the public by leading financial institutions, to be used as a reference in evaluating performance of managed funds. The direct analog of a return on such a benchmark in our setting is given by $R_T^X = \beta r + (1-\beta) R_T^S$, where β is the weighting of the money market return, and $1-\beta$ of the stock market return. The horizon level for the hybrid benchmark is given by

$$X_T = W_0 e^{(\beta r + \varepsilon)T} \left(\frac{S_T}{S_0} \right)^{1-\beta} .$$

Clearly, for $\beta = 1$ and $\beta = 0$, we obtain the money market and stock market benchmarks studied in Sections 2 and 3, respectively. Moreover, our analysis in the previous sections goes through using this hybrid level, and using the corresponding sensitivity $(1-\beta)\sigma/\kappa$ (instead of the stock market sensitivity, σ/κ). The applicability of our analysis for the hybrid benchmark offers important flexibility in the benchmark choice. By choosing the appropriate benchmark (via the choice of β) one can lead a risk manager, with a given risk profile, to follow a particularly desirable policy out of those presented in Proposition 2/Figure 3.

Another alternative benchmark to consider in combining both money market and stock market exposure is a constant-mix benchmark. This dynamically managed benchmark continuously maintains a weight δ in the money market, and $1-\delta$ in the stock market, with its dynamics given by

$$dX_t = \delta r X_t dt + (1-\delta)(X_t/S_t) dS_t = [\delta r + (1-\delta)\mu] X_t dt + [(1-\delta)\sigma] X_t dw_t .$$

The horizon benchmark level, which follows from this dynamics, is given by

$$X_T = W_0 e^{(\Delta + \varepsilon)T} \left(\frac{S_T}{S_0} \right)^{1-\delta} ,$$

where $\Delta = \delta(r + (1-\delta)\sigma^2/2)$. For $\delta = 1$ and $\delta = 0$, we obtain the money market and stock market benchmarks, respectively. Our analysis in the previous sections applies to the constant-mix benchmark as well, by using its horizon level and its sensitivity, $(1-\delta)\sigma/\kappa$, to replace those of the stock market. Therefore, the appropriate choice of a constant-mix benchmark (via the choice of δ), also offers the desirable flexibility to direct a risk manager's behavior to be one of those presented in Proposition 2/Figure 3.

Section 4.1 discussed one particular expectations-based risk management approach, whereas another alternative could be to focus on a second moment. Limiting the volatility of the tracking error is also possible in our setting (although less tractable). However, since this limits the upside potential as well, we do not pursue this direction in our analysis. Furthermore, working with more general price dynamics, our insights are easy to apply, and it is clear how additional patterns of behavior may arise depending on the characteristics of the pertinent benchmark. The shortfall states will still be chosen using our general principle of reverting to the normal policy when the benchmark is least affordable.

5. Conclusion

Advances in portfolio theory must account for the institutional features of the asset management industry. In this paper, we focus on an important aspect characterizing this industry – performance evaluation relative to a benchmark, which in turn leads to risk management practices that account for benchmarking. A rigorous understanding of this aspect is in its infancy in the academic literature, not the least reason for which is the analytical difficulty of the problem. We approach the issue in the most natural way, mirroring risk management with benchmarking by combining a tracking error constraint and a utility maximizing behavior. This turns out to be a fruitful combination, as not only it provides a rich set of theoretical results, but as it also paves the way for investors, as well as regulators, to control gain/loss characteristics of money managers. Moreover, on top of offering guidance as to when risk managers can over- or underperform the stock market, while accounting for the risk return tradeoff, we can also indicate when this can be achieved under higher Sharpe ratios than those of the stock market or of non-risk managers. Although we explore several extensions of our setting, it still remains of interest to perform an equilibrium analysis in the presence of benchmarking, along the lines of Basak and Shapiro (2001).

We maintain the view that resolution of uncertainty, price changes, and the resulting trading activity are occuring in practice more frequently (continuously in our model) than the (ex-post) evaluation of performance. The evaluation can be over consecutive periods (as we implicitly assume by focusing on one representative period), or over overlapping periods. However, as long as it is performed periodically (as opposed to continuously), and only a prespecified shortfall is tolerated before penalties are imposed, then the fundamental insights of our model are still applicable, and risk managers will follow optimal policies along the lines suggested in our analysis. Methodologically, this imposes a different structure on the problem than is typically assumed in the literature with continuously imposed portfolio constraints (Cvitanić and Karatzas (1992), Detemple and Murthy (1997)), and our framework can be thus of use to analyze other challenging issues. In particular, benchmarking is of relevance beyond the scope of professional money managers, and there is room to study its implications in other institutional settings, such as that of a pension fund manager who is interested to limit a shortfall relative to future liabilities, which are affected by uncertain retirement patterns.

Appendix: Proofs

Proof of Proposition 1. See proof of Proposition 2 for $\alpha = 0$. ∎

Proof of Proposition 2. When the constraint is binding, the optimality of the solutions in (4)-(9) for each of the economies (a)-(e), as well as the optimality of the indifference solution in (g), is most conveniently proved case by case, for the associated ranges of the benchmark and normal sensitivities. The logic of the proof in each economy, for an agent benchmarking the stock market, is reminiscent of that in the case of benchmarking the state-independent money market (Basak and Shapiro (2001)), in that the proof adapts the convex-duality approach (see Karatzas and Shreve (1998)) to a non-concave problem, and in that it shows sufficiency for optimality of the stated solution. However, the technical nature of the proof here is considerably different, as Lemmas 1 and 2 below deal with the intricate state dependency of the problem introduced by the stochastic stock market benchmark. Since economy (b) is a case with an optimal policy of four distinct regions and two discontinuities across the state space, it represents, to the best of our knowledge, a notably different case compared to any known analysis in the literature, and hence we first focus on the proof in this economy. We then show how the proof proceeds for the other six economies in a similar manner. To save notation, we suppress below the superscript B on the Lagrange multiplier y.

Lemma 1. *For $1 < \sigma/\kappa < 1/\gamma$, and $\overline{\xi}$, ξ^* satisfying $g(\overline{\xi}) = g(\xi^*)$ and $\underline{\xi} < \overline{\xi} < \xi^*$, we have $g(\xi) < g(\overline{\xi})$ for $\underline{\xi} < \xi < \overline{\xi}$ or $\xi^* < \xi$, and $g(\xi) > g(\overline{\xi})$ for $\overline{\xi} < \xi < \xi^*$.*

Proof: Note that $g(\underline{\xi}) = 0$, and since $\gamma - 1 < 0$ and $1 - \sigma/\kappa < 0$, we obtain $\lim_{\xi\to\infty} g(\xi) = 0$. $\frac{\partial g(\xi)}{\partial \xi} = \xi^{-\sigma/\kappa} f(\xi)$, where

$$f(\xi) = -y^{(\gamma-1)/\gamma}\xi^{(\sigma/\kappa-1/\gamma)} + (\sigma/\kappa)A^{1-\gamma}\xi^{\gamma(\sigma/\kappa-1/\gamma)} + yA(1-\sigma/\kappa).$$

We also have $f(\underline{\xi}) = 0$, and because $\sigma/\kappa - 1/\gamma < 0$, we obtain $\lim_{\xi\to\infty} f(\xi) = yA(1-\sigma/\kappa) < 0$. It is thus left to show $\exists! \tilde{\xi} > \underline{\xi}$ such that $f(\xi) > 0$ for $\underline{\xi} < \xi < \tilde{\xi}$, and $f(\xi) < 0$ for $\tilde{\xi} < \xi$, as this will establish the desired properties of g over $(\underline{\xi}, \infty)$. To that end, it is immediate to verify that $\frac{\partial f(\xi)}{\partial \xi} > 0$ if, and only if, $\xi < \check{\xi}$, where $\check{\xi} = ((1/\gamma)/(\sigma/\kappa))^{1/((1-\gamma)(1/\gamma-\sigma/\kappa))}\underline{\xi} > \underline{\xi}$, with the latter inequality holding because $\check{\xi}$ multiplies $\underline{\xi}$ by a constant greater than unity raised to a positive power. The continuity of f, while decreasing over $(\check{\xi}, \infty)$ towards its negative limit for $\xi \to \infty$, guarantees the uniqueness of $\tilde{\xi}$, as required. ∎

Lemma 2. *For $1 < \sigma/\kappa < 1/\gamma$, let $W(\xi) = (y\xi)^{-1/\gamma}1_{\{\xi<\underline{\xi},\ or\ \overline{\xi}\leq\xi<\xi^*\}} + A\xi^{-\sigma/\kappa}1_{\{\underline{\xi}\leq\xi<\overline{\xi},\ or\ \xi^*\leq\xi\}}$, $x = g(\overline{\xi})$, and $h(W,\xi) = u(W) - y\xi W + x1_{\{W\geq A\xi^{-\sigma/\kappa}\}}$. Then, $\forall \xi \geq 0$, $W(\xi) = \arg\max_W h(W,\xi)$.*

Proof: For a given $\xi \geq 0$, $h(W,\xi)$ is not concave in W. However, its local maxima are attained at $I(y\xi) = (y\xi)^{-1/\gamma}$ or at $A\xi^{-\sigma/\kappa}$. To find the global maximizer, we compare the value of h at these two candidate points. When $\xi < \underline{\xi}$, then $(y\xi)^{-1/\gamma} > A\xi^{-\sigma/\kappa}$, and hence $h((y\xi)^{-1/\gamma},\xi) > h(A\xi^{-\sigma/\kappa},\xi)$, so that $(y\xi)^{-1/\gamma}$ is the global maximizer. When $\underline{\xi} \leq \xi$, then $(y\xi)^{-1/\gamma} < A\xi^{-\sigma/\kappa}$, and from the definitions of $g(\cdot)$, $h(\cdot)$ and x we get

$$h((y\xi)^{-1/\gamma},\xi) - h(A\xi^{-\sigma/\kappa},\xi) = g(\xi) - x\ .$$

From Lemma 1, $g(\xi) < x$ for $\underline{\xi} \leq \xi < \overline{\xi}$ or $\xi^* \leq \xi$, and the global maximizer in these regions is $A\xi^{-\sigma/\kappa}$. On the other hand, for $\overline{\xi} \leq \xi < \xi^*$, $g(\xi) > x$, and $(y\xi)^{-1/\gamma}$ is the global maximizer. ∎

The benchmark horizon level in (2) satisfies

$$X_T = W_0 e^{\varepsilon T} S_T/S_0 = W_0 e^{(\varepsilon+\mu-\sigma^2/2)T+\sigma w_T} = A\xi_T^{-\sigma/\kappa}\ , \tag{A1}$$

where the second and third equalities follow from the terminal values of S_T and ξ_T, respectively, as implied by their geometric Brownian motion dynamics. Next, let W_T^B be as in (5), and let W_T be any candidate optimal solution for economy (b), satisfying the tracking error constraint and the static budget constraint in (3). We then have

$$\begin{aligned}
& E[u(W_T^B)] - E[u(W_T)] \\
= \ & E[u(W_T^B)] - yW_0 + x(1-\alpha) - (E[u(W_T)] - yW_0 + x(1-\alpha)) \\
\geq \ & E[u(W_T^B)] - E[y\xi_T W_T^B] + E[x1_{\{W_T^B \geq X_T\}}] - (E[u(W_T)] - E[y\xi_T W_T] + E[x1_{\{W_T \geq X_T\}}]) \geq 0,
\end{aligned}$$

where the first inequality follows from the budget constraint and the tracking error constraint holding with equality for W_T^B, and holding with equality or inequality for W_T. The second inequality follows from Lemma 1, after substituting (A1) in (5), with $y = y^B$. This establishes the optimality of W_T^B in (5) for economy (b).

From Lemma 1, it is evident that in economy (b) there are unique values of $\overline{\xi}$ and ξ^* satisfying $P(\overline{\xi} \leq \xi_T < \xi^*) = \alpha$. For any other values $\overline{\xi}_\circ$ and $\xi^*_\circ$ with $g(\overline{\xi}_\circ) = g(\xi^*_\circ)$, given the established properties of g in economy (b), we have either $\underline{\xi} < \overline{\xi} < \overline{\xi}_\circ < \xi^*_\circ < \xi^*$, or $\underline{\xi} < \overline{\xi}_\circ < \overline{\xi} < \xi^* < \xi^*_\circ$, and consequently $P(\overline{\xi}_\circ \leq \xi_T < \xi^*_\circ) \neq \alpha$. Lemma 1 further implies that as $\alpha \to 0$ in economy (b), we have $\overline{\xi} \to \xi^*$, and we obtain the solution in Proposition 1(a).

For the remaining economies, the proof follows similar steps, where Lemma 1 is modified to establish that in economy (a) $g(\xi) > g(\overline{\xi})$ for $\underline{\xi} < \overline{\xi} < \xi$; (c) $g(\xi) > g(\xi^*)$ for $\xi < \xi^* < \underline{\xi}$; (d) $g(\xi) > g(\overline{\xi})$ for $\overline{\xi} < \xi < \xi^* < \underline{\xi}$; (e) $g(\xi) > g(\xi^*)$ for $\xi < \xi^*$; (f) $g(\xi) > g(\overline{\xi})$ for $\overline{\xi} < \xi$; and in economy (g) $g(\xi)$ is constant. Lemma 2 then proceeds to verify for a given economy which one of the two candidate solutions is the global maximizer within each region of the state space (and to show that either candidate can be used in economy (g)).

Finally, we note that the state contingent relative cost (see footnote 7) equals $A\xi^{1-\sigma/\kappa} - y^{-1/\gamma}\xi^{1-1/\gamma}$, and is highest in economy (b) at $\hat{\xi} = \left(\frac{1-1/\gamma}{1-\sigma/\kappa}\right)^{1/(1/\gamma-\sigma/\kappa)} \underline{\xi}$. Similarly, an interior solution is obtained in (d), but in (a) and (f) the relative cost is largest for high ξ_T, while in (c) and (e) for low ξ_T. In (g), relative cost equals $A - y^{B^{-1/\gamma}} > 0$, where the inequality is because $A = e^{\varepsilon T} y^{N^{-1/\gamma}}$, $\varepsilon > 0$, and also $y^B > y^N$ for the static budget constraint to hold with equality. ■

Proof of Proposition 3. (i) Using the dynamics of the state price density process and agent's wealth, Itô's Lemma implies that $\xi_t W_t^B$ is a martingale:

$$W_t^B = E\left[\xi_T W_T^B \middle| \mathcal{F}_t\right] / \xi_t \ . \tag{A2}$$

When r and κ are constant, conditional on $\mathcal{F}_t$, $\ln \xi_T$ is normally distributed with variance $||\kappa||^2(T-t)$ and mean $\ln \xi_t - (r + \frac{||\kappa||^2}{2})(T-t)$. For each economy, substituting the expression for W_T^B in Proposition 2 into (A2), and evaluating the expectation over the relevant regions of ξ_T yields (10).
(ii) For each economy, applying Itô's Lemma to (10), results in an expression for σ_t^B, the diffusion term of W_t^B. The expression for θ_t^B follows from the fact that, from the agent's wealth process, σ_t^B must equal $\sigma_t \theta_t^B W_t^B$. Normalizing θ_t^B by the well-known quantity θ^N yields q_t^B.
(iii) For completeness, we present here the solution for the H risk manager, obtained in economy (a) for $\overline{\xi} = \infty$ and (b) for $\overline{\xi} = \xi^*$, when $\varepsilon < 0$:

$$\begin{aligned} W_t^H &= \mathcal{N}(d(\gamma, \underline{\xi})) Z(\gamma) (y^H \xi_t)^{-1/\gamma} + \left[1 - \mathcal{N}(d(\kappa/\sigma, \underline{\xi}))\right] Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa} \ , \\ q_t^H &= 1 + \left[1 - \mathcal{N}(d(\kappa/\sigma, \underline{\xi}))\right] (\gamma\sigma/\kappa - 1) Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa} / W_t^H \\ &\quad + \left(\varphi(d(\gamma, \underline{\xi})) Z(\gamma) (y^H \xi_t)^{-1/\gamma} - \varphi(d(\kappa/\sigma, \underline{\xi})) Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa}\right) \ \gamma / (W_t^H ||\kappa|| \sqrt{T-t}) \ , \end{aligned}$$

and, in economy (c) for $\xi^* = 0$, and in economy (d) for $\overline{\xi} = \xi^*$, when $\varepsilon < 0$:

$$\begin{aligned} W_t^H &= \left[1 - \mathcal{N}(d(\gamma, \underline{\xi}))\right] Z(\gamma) (y^H \xi_t)^{-1/\gamma} + \mathcal{N}(d(\kappa/\sigma, \underline{\xi})) Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa} \ , \\ q_t^H &= 1 + \mathcal{N}(d(\kappa/\sigma, \underline{\xi})) (\gamma\sigma/\kappa - 1) Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa} / W_t^H \\ &\quad + \left(-\varphi(d(\gamma, \underline{\xi})) Z(\gamma) (y^H \xi_t)^{-1/\gamma} + \varphi(d(\kappa/\sigma, \underline{\xi})) Z(\kappa/\sigma) A \xi_t^{-\sigma/\kappa}\right) \ \gamma / (W_t^H ||\kappa|| \sqrt{T-t}) \ . \end{aligned}$$

In all cases, y^H is as in Proposition 1. ■

Proof of Proposition 4. The proof is analogous to the proof of Proposition 2, with the appropriate counterparts of Lemmas 1 and 2, and is therefore omitted. ■

Derivation of Expected Returns and Variances (Section 4.2)

The expected return for the B risk manager is given by:

$$
\begin{aligned}
E[R_T^W] \;=\;& \Big[1_{\{a,c,d,f\}} + \mathcal{N}(d(1,\underline{\xi}))1_{\{a,b\}} - \mathcal{N}(d(1,\overline{\xi}))1_{\{a,b,d,f\}} \\
&\quad + \mathcal{N}(d(1,\xi^*))1_{\{b,c,d,e\}} - \mathcal{N}(d(1,\underline{\xi}))1_{\{c,d\}}\Big] (\ln(y^B\xi_0)^{-1/\gamma} + (r+k^2/2)T/\gamma)/T \\
&+ \Big[1_{\{b,e\}} - \mathcal{N}(d(1,\underline{\xi}))1_{\{a,b\}} + \mathcal{N}(d(1,\overline{\xi}))1_{\{a,b,d,f\}} \\
&\quad - \mathcal{N}(d(1,\xi^*))1_{\{b,c,d,e\}} + \mathcal{N}(d(1,\underline{\xi}))1_{\{c,d\}}\Big] (\ln(A\xi_0^{-\sigma/\kappa}) + (r+k^2/2)T\sigma/\kappa)/T \\
&+ \Big[\varphi(d(1,\underline{\xi}))1_{\{a,b\}} - \varphi(d(1,\overline{\xi}))1_{\{a,b,d\}} \\
&\quad + \varphi(d(1,\xi^*))1_{\{b,c,d\}} - \varphi(d(1,\underline{\xi}))1_{\{c,d\}}\Big] (\kappa/\gamma - \sigma)/\sqrt{T} - \ln W_0/T, \qquad \text{(A3)}
\end{aligned}
$$

where the notation is as in Proposition 3.

To derive (A3), note from the definition of R_T^W, that $E[R_T^W] = (E[\ln W_T^B] - \ln W_0)/T$. Also note that $\ln(y^B\xi_T)^{-1/\gamma} = \ln(y^B)^{-1/\gamma} - (1/\gamma)\ln\xi_T$, and $\ln A\xi_T^{-\sigma/\kappa} = \ln A - (\sigma/\kappa)\ln\xi_T$, where at time 0, $\ln\xi_T$ is distributed normally with mean and standard deviation $\ln\xi_0 - (r+\kappa^2/2)T$ and $\kappa\sqrt{T}$, respectively. Therefore, for each economy, using the fact that if $x \sim \text{Normal}(a,b)$, $E[x1_{\{x<k\}}] = a\mathcal{N}((k-a)/b) - b\varphi((k-a)/b)$, and $E[x1_{\{x>k\}}] = a[1-\mathcal{N}((k-a)/b)] + b\varphi((k-a)/b)$, evaluating the conditional expectations over the relevant regions of ξ_T yields (A3).

In economies (e) and (f), the expected return can be written as:

$$E[R_T^W] = (1-\alpha)(E[R_T^X] + \varepsilon) + \alpha(E[R_T^N] - (\ln(y^B/y^N))/(\gamma T)) \qquad \text{(A4)}$$

where $E[R_T^X]$ and $E[R_T^N]$ denote the expected return of the benchmark and the expected return of the non-risk manager, respectively. Specifically, from the definition of expected return, we have:

$$
\begin{aligned}
E[R_T^N] &= E[\ln(y^N\xi_T)^{-1/\gamma}]/T - \ln W_0/T = (\ln(y^N\xi_0)^{-1/\gamma} + (r+\kappa^2/2)T/\gamma)/T - \ln W_0/T \\
E[R_T^X] &= E[\ln A\xi_T^{-\sigma/\kappa}]/T - \ln X_0/T = (\ln A\xi_0^{-\sigma/\kappa} + (r+\kappa^2/2)T\sigma/\kappa)/T - \varepsilon - \ln W_0/T.
\end{aligned}
$$

The result for economies (e) and (f) in (A4) then follows directly from (A3), using the fact that $\mathcal{N}(d_1(1,\xi^*)) = \alpha$ in economy (e) and $1 - \mathcal{N}(d_1(1,\overline{\xi})) = \alpha$ in economy (f).

Given the definition of R_T^W, the variance is calculated as follows:

$$
\begin{aligned}
Var[R_T^W] &= E[(\ln W_T^B)^2]/T^2 - E^2[\ln W_T^B]/T^2 \\
&= E[(\ln W_T^B)^2]/T^2 - (E[R_T^W]\,T + \ln W_0)^2/T^2. \qquad \text{(A5)}
\end{aligned}
$$

Let $x \equiv \ln(y^B\xi_T)^{-1/\gamma}$, then x is distributed normally with mean and standard deviation $\mu_x \equiv \ln(y^B\xi_0)^{-1/\gamma} + \frac{1}{\gamma}(r+\kappa^2/2)T$ and $s_x \equiv (\kappa/\gamma)\sqrt{T}$, respectively. Similarly, let $z \equiv \ln A\xi_T^{-\sigma/\kappa}$, then z is distributed normally with mean and standard deviation $\mu_z \equiv \ln A\xi_0^{-\sigma/\kappa} + \frac{\sigma}{\kappa}(r+\kappa^2/2)T$ and

$s_z \equiv \sigma\sqrt{T}$, respectively. To evaluate the term $E[(\ln W_T^B)^2]$ in (A5), we evaluate the truncated expectation of x^2 and z^2. For ξ_T truncated from above we have:

$$\begin{aligned}
E[x^2\ 1_{\{\xi_T<a\}}] &= s_x^2 E\left[((x-\mu_x)/s_x)^2\ 1_{\{\xi_T<a\}}\right] + 2\mu_x E[x\ 1_{\{\xi_T<a\}}] - \mu_x^2 E[1_{\{\xi_T<a\}}] \quad \text{(A6)}\\
E[z^2\ 1_{\{\xi_T<a\}}] &= s_z^2 E\left[((z-\mu_z)/s_z)^2\ 1_{\{\xi_T<a\}}\right] + 2\mu_z E[z\ 1_{\{\xi_T<a\}}] - \mu_z^2 E[1_{\{\xi_T<a\}}]. \quad \text{(A7)}
\end{aligned}$$

The second and third terms on the right-hand side of (A6) and (A7) can be evaluated straightforwardly. The first term on the right-hand side in each case involves the truncated expectation of a chi-squared variable with 1 degree of freedom (as both $(x-\mu_x)/s_x$ and $(z-\mu_z)/s_z$ are standard normal). For these terms, substituting in μ_x, s_x, μ_z and s_z, we have:

$$\begin{aligned}
E\left[((x-\mu_x)/s_x)^2\ 1_{\{\xi_T<a\}}\right] &= E\left[\left((-\ln\xi_T/\xi_0-(r+\kappa^2/2)T)/\kappa\sqrt{T}\right)^2\ 1_{\{\xi_T<a\}}\right] = E\left[V 1_{\{v>-\ln a\}}\right]\\
E\left[((z-\mu_z)/s_z)^2\ 1_{\{\xi_T<a\}}\right] &= E\left[\left((-\ln\xi_T/\xi_0-(r+\kappa^2/2)T)/\kappa\sqrt{T}\right)^2\ 1_{\{\xi_T<a\}}\right] = E\left[V 1_{\{v>-\ln a\}}\right],
\end{aligned}$$

where $V \equiv (v-m)^2/s^2$ is distributed chi-squared with 1 degree of freedom, and $v \equiv -\ln\xi_T$, $m \equiv -\ln\xi_0 + (r+\kappa^2/2)T$ and $s \equiv \kappa\sqrt{T}$. Letting $K \equiv (-\ln a - m)^2/s^2$, we have after a series of algebraic manipulations:

$$E[V\ 1_{\{v>-\ln a\}}] = \begin{cases} \frac{1}{2} + \frac{1}{2}\mathrm{CDF}_{\chi_1^2}(K) - \frac{K^{1/2}e^{-K/2}}{\sqrt{2\pi}} & \text{if } -\ln a < m \\ \frac{1}{2} - \frac{1}{2}\mathrm{CDF}_{\chi_1^2}(K) + \frac{K^{1/2}e^{-K/2}}{\sqrt{2\pi}} & \text{if } -\ln a > m. \end{cases} \quad \text{(A8)}$$

Employing a similar approach for ξ_T truncated from below yields:

$$\begin{aligned}
E[x^2\ 1_{\{\xi_T>a\}}] &= s_x^2 E\left[V\ 1_{\{v<-\ln a\}}\right] + 2\mu_x E[x\ 1_{\{\xi_T>a\}}] - \mu_x^2 E[1_{\{\xi_T>a\}}]\\
E[z^2\ 1_{\{\xi_T>a\}}] &= s_z^2 E\left[V\ 1_{\{v<-\ln a\}}\right] + 2\mu_z E[z\ 1_{\{\xi_T>a\}}] - \mu_z^2 E[1_{\{\xi_T>a\}}]
\end{aligned}$$

where

$$E[V\ 1_{\{v<-\ln a\}}] = \begin{cases} \frac{1}{2} - \frac{1}{2}\mathrm{CDF}_{\chi_1^2}(K) + \frac{K^{1/2}e^{-K/2}}{\sqrt{2\pi}} & \text{if } -\ln a < m \\ \frac{1}{2} + \frac{1}{2}\mathrm{CDF}_{\chi_1^2}(K) - \frac{K^{1/2}e^{-K/2}}{\sqrt{2\pi}} & \text{if } -\ln a > m. \end{cases} \quad \text{(A9)}$$

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