Securities Lending, Shorting, and Pricing

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First Version: March 29, 2001
Current Version: September 24, 2001

Abstract

We present a model of asset valuation in which short-selling is achieved by searching for security lenders and by bargaining over the terms of the lending fee. If lendable securities are difficult to locate, then the price of the security is initially elevated, and expected to decline over time. This price decline is to be anticipated, for example, after an initial public offering (IPO), among other cases, and is increasing in the degree of heterogeneity of beliefs of investors about the likely future value of the security. The initial price of a security may be above even the most optimistic buyer's valuation of the security's future dividends, because of the additional prospect of lending fees for owners.

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1 Introduction

The common method of shorting an equity or a fixed-income security is to borrow the security and sell it. Later, one would buy it in order to return it to the lender, profiting by any price decline, net of borrowing fees. In some cases, on which we focus, it may be difficult to locate securities available for lending.

We build a dynamic model of the determination of prices, lending fees, and short interest (the quantity of securities held short). Agents trade because of differences of opinions, and would-be shorters must search for security lenders and bargain over the lending fee.

We provide a closed-form equilibrium solution, including the dynamics of the price, of the lending fees, and of the short interest. The price is elevated by the prospect of future lending fees, and may, in the beginning, be even higher than the valuation of the most optimistic agent.[1] As time passes, the short interest increases and the prospective fees from future loans decrease. The private valuation of the marginal investor simultaneously declines. Both effects imply that the price declines.

As opposed to the rather conflicted literature relating the current level of short interest to expected returns,[2] our model suggests that price declines can be more directly related to *changes* in the short interest over time. For example, a rapid buildup in the short interest signals strong un-met shorting demand, an associated high lending fee, and thus, other things equal, that lower-than-normal expected returns will suffice to incent a given investor to hold the security. As the short interest grows over time, moreover, the long interest grows one for one, so a rapid increase in short interest implies a rapid succession of purchases by less and less optimistic buyers, and associated reductions in price. The price is not reduced one for one with the change of valuation by the marginal investor, however, since this change is anticipated by agents. Further, in judging expected returns, one must take care as to

[1]Harrison and Kreps (1978) and Morris (1996) obtain a similar result, but due to the opportunity to speculate.

[2]See Aitken, Frino, McCorry, and Swan (1998), Asquith and Meulbroek (1996), Brent, Morse, and Stice (1990), Dechow, Hutton, Meulbroek, and Sloan (2000), Figlewski (1981), Figlewski and Webb (1993), MacDonald and Baron (1973), Safieddine and Wilhelm (1996), Seneca (1967), and Woolridge and Dickinson (1994). Senchack and Starks (1993) do consider changes, but only in as much as they take as proxies for the unexpected amount of shorting.

whose probability assessments are considered, because of our assumption of heterogeneous expectations. These points are clarified in the context of our model.

Our theoretical model is consistent with the empirical results of Jones and Lamont (2001), who find that, for NYSE stocks during the period 1926-1933, those that have high lending fees, that is, low rebate rates, tend to have inferior average returns.

Our results suggest explanations for several asset-pricing peculiarities. For instance, our model is consistent with low average returns during the period immediately following an initial public offering (IPO), when the heterogeneity of investors' expectations may be highest, and the quantity of shares available for lending may be relatively low, especially until the expiration of lock-up agreements, which contractually delay insider sales.[3] Likewise, our model is consistent with the observation of Geczy, Musto, and Reed (2001) that lending fees are relatively high immediately after an IPO, and on average decline over time, as the float increases. The presence of special (low-rebate) securities lending, on its own, could account for a substantial adverse impact on conventional measures of IPO returns. From Figure 1 of Geczy, Musto, and Reed (2001), for example, the cumulative effect of low-rebate securities lending for the first six months after an IPO, on average over their sample, amounts to approximately 0.75% of the market value of the underlying equities.[4] This implies that an investor who could be assured of placing purchased shares immediately into lending agreements would be willing to accept an average reduction of approximately 1.5% in annualized expected return over the first six months, when considering an alternative investment of the same type, but not on special. (This assumes that random variation in the rebate special is unsystematic and thus carries no risk premium.) While there are in fact delays in arranging lending agreements, and this is one of the points of our paper, this suggests an impact of securities lending fees on the expected returns demanded of IPOs that is not to be ignored when judging IPO performance in the secondary market.

Although they do not address IPOs specifically, Jones and Lamont (2001) suggest that lending fees are insufficient on their own to account for the low

[3]Evidence for long-run IPO underperformance is provided by Ritter (1991), Loughran and Ritter (1995), and others, but has been questioned by Brav, Geczy, and Gompers (2000) and Eckbo and Norli (2001).

[4]This is based on the information plotted in their figure, and approximate. We do not have the underlying numerical data.

average performance of stocks on special, for their 1926-1933 NYSE data set.

Furthermore, our model may help resolve the puzzling price behavior of certain equity carve-outs. For instance, Lamont and Thaler (2001), Mitchell, Pulvino, and Stafford (2001), and Ofek and Richardson (2001) point to spinoffs in which the stub value (the implied market value of the portion of the parent company that is not spun off) can be initially negative, seemingly inconsistent with limited liability and optimizing behavior by agents. A recent extreme example is the spinoff of Palm by 3Com. We show that extremely small or even negative stub values are implied if two groups of investors hold opposite and complementary views about both the spinoff and the stub, and if lendable shares are sufficiently hard to locate. In accordance with our model's predictions, stub values typically increase over time, while lending fees diminish.

An informal explanation of IPO "underperformance" as a consequence of heterogeneous beliefs and imperfect shorting abilities was suggested by Miller (1977). This effect was modeled by Figlewski (1981), Lintner (1969), and Jarrow (1980) in a static CAPM framework, assuming an exogenously given transactions cost or restriction on shorting.[5] More recently, some static models (D'Avolio (2001) in the context of equities, and Duffie (1996) and Krishnamurthy (2001) for treasuries) have directly considered a market equilibrium for lending fees. By considering behavior over time, our model captures additional intuition for the determination of prices and lending fees. For instance, our model makes it apparent that lending fees (and the price) reflect the expected number of times that a particular share is to be lent, and clarify the roles paid by the float and by the capital available for shorting.

We show that a higher float, other things equal, implies lower lending fees and prices, at all times, since it makes shorting easier and hastens the decrease of the marginal investor's valuation and of the future potential perceived gains from trade.[6] Larger differences of opinions, on the other hand, imply larger (perceived) gains from a lending agreement, whence larger lending fees.

[5]Viswanathan (2001) allows for strategic behavior in such a setting. Williams (1977) developed a dynamic heterogeneous-beliefs model. Chen, Hong, and Stein (2001) provide empirical support based on evidence of dispersion of beliefs. Harrison and Kreps (1978) and Morris (1996) illustrate the speculative opportunities that arise with heterogeneous beliefs and shorting restrictions. Allen, Morris, and Postlewaite (1993) model "bubbles" in the presence of asymmetric information with short-sales constraints.

[6]D'Avolio (2001) finds a similar result, which stems from risk aversion.

We also enrich our basic model by endogenizing the amount of shorting capital, showing that, other things equal, more capital is brought to bear on shorting in settings with greater differences of opinions between optimists and pessimists. Further, we show that settlement lags can have significant impacts on short interest, lending fees, and prices.

Our model does not address the manner in which information revelation is suppressed by short-sale costs or other shorting constraints, as modeled by Diamond and Verrecchia (1987), nor do we consider the potential impact of derivatives trading (which may be viewed as alternatives to shorting, when shorting is costly or constrained), empirically investigated by Jennings and Starks (1986) Skinner (1990), and Senchack and Starks (1993).

The remainder of the paper is organized as follows. The next section provides a description of the institutional features of the markets for securities lending and shorting. Section 3 provides the model, including a characterization of the equilibrium, comparative statics, and a treatment of the impact of delayed settlement. Section 4 applies the model to the behavior of the prices of spinoffs relative to their parent firms, as in the case of Palm and 3Com. Appendix A shows, as a benchmark, that, without frictions, the unique equilibrium lending fee is zero. Proofs are in Appendix B.

2 The Market for Securities Lending

In a typical securities lending transaction, a would-be shorter, such as a hedge fund, would request a "locate" from its broker. The broker might locate the stock in its own inventory, or in the accounts of those of its customers permitting the use of their securities for lending. Failing this, the broker could turn to a custodian bank, or to another potential lender. Natural lenders include institutional investors such as insurance companies, index funds, and pension funds, who tend to have large and long-duration buy-and-hold investments. Brokers may even have exclusive contracts with institutional investors for access to portfolios of securities for lending purposes, as in a recent major exclusive lending deal between CSFB and CalPERS.[7] The broker's search for

[7] On November 3, 2000, CSFB offered the following press release. "Credit Suisse First Boston (CSFB), in the largest deal of its type, announced today that it has been selected by the California Public Employees' Retirement System (CalPERS) to be an exclusive securities lending principal borrower for CalPERS' passively managed Wilshire 2500 and small-cap stock portfolios totaling more than $57 billion in equity assets. In this arrange-

lendable securities might be conducted using an electronic locate system, or by email, fax, or telephone. On May 22, 2001, ten large financial institutions announced the formation of Equilend, an automatic multi-broker lending facility. (Notably, CSFB was not one of the ten initial participating firms.) A *Financial Times* reporter[8] outlining the proposed role of Equilend described traditional methods for brokering shorts as "labor-intensive, because the appropriate shares or securities can take time to locate."

When encountering stocks that are "hard to locate," brokers sometimes cannot "circle" the quantity of lendable shares requested. Brokers may offer "partial fills." Occasionally, a significant amount of time may pass before the necessary stock can be located. (Unfortunately, we do not have data concerning the distribution of time delays for locating lendable stocks.) Factors that are said to be related to the degree of difficulty of locating lendable shares include the capitalization of the issue, the float (the quantity of shares available for trade), whether the stock is included in an index, the stock's liquidity, the degree of concentration of ownership, and the presence of special activity, such as IPOs, mergers, spinoffs, or acquisitions.

Once a security is located, the broker may execute a "pay-for-hold" transaction, compensating the lender for holding the securities until the borrower executes a short sale. This transaction is sometimes called "pre-borrowing." Trades in the stock itself are normally executed in the U.S. within 3 days of the trade. Normally, sell orders that are short sales are marked "short"

ment, CalPERS has given CSFB the exclusive right to borrow the assets held in each of the portfolios for a guaranteed fee. 'The combination of CalPERS and CSFB in this securities lending relationship will give the System's members superior value for their assets while allowing CSFB to continue expanding its Equity Finance franchise,' said Bob Sloan, Managing Director of the global Equity Finance Group at CSFB. 'This places CSFB in a position to further our franchise in the prime brokerage and alternative capital arena,' he continued. 'We are very pleased CalPERS has selected CSFB.' ... eSecLending provided the platform for distributing bidding parameters and guidelines to participating broker/dealers and disseminating bidding results to CalPERS for execution. eSecLending, LLC, (www.eseclending.com), is a new firm offering a web-based auction system for securities lending. The new process is designed to meet the needs of large pension funds, mutual funds and other major investors including online custodians. Burlington, Vt.-based eSecLending serves as the primary developer of the web platform and software, and is responsible for staffing and managing the auction process." (Source: www.csfb.com) The term "portfolio valuation" has apparently been used by brokers for the valuation of such exclusive lending rights. We are not aware of the fee in the CALPERS-CSFB deal.

[8] See "Banks Form Platform for Short-Sellers," by Alex Skorecki, *Financial Times*, May 22, 2001.

for special attention, because they may be executed only on an "uptick," an SEC regulation.

The actual securities-lending transaction, given a locate, can be accomplished on a same-day basis. If conducted through a broker, the broker would typically act as the borrower from the outside lender, and as the lender to the outside borrower. Cash collateral, normally 102% of the market value of the borrowed shares for domestic securities (105% for international securities), is passed from the borrower to the lender in exchange for the shares. The lender "rebates" interest on the collateral at an agreed overnight rate. An overnight rebate rate of r implies a daily interest payment of $r/360$ times the amount of cash collateral. The interest payments may accrue on a daily basis, for month-end settlement. The rebate offered by the broker to its outside borrower would normally be lower than the rebate received by the broker from its outside lender. The extent to which the rebate is below a market rate (such as the federal funds rate in the United States) represents a benefit to the lender over other sources of funding. Occasionally, other securities are used as collateral, rather than cash, in which case an outright lending fee is charged. Only 1% of the security loans by a custodian bank appearing in the database analysed by Geczy, Musto, and Reed (2001) were of this type.

Under SEC Regulation T, retail customers of brokers must, in addition to the cash collateral, post 50% of the market value of the stock in additional collateral, although this additional collateral may be posted in Treasury Bills.[9] Shorting retail customers typically do not receive any interest on their cash collateral, and retail customers whose stocks are being lent typically do not benefit from any lending fee.

In this paper we focus (implicitly) on institutional investors. One of the purposes of this paper is to model and present-value the stream of low-rebate benefits to owners of lendable shares.

Lending agreements are normally on an "open" or "continuing" basis, renewed each day with an adjustment of the cash collateral according to changes in the market price of the stock, and at a newly negotiated rebate

[9]Maintenance margin is 30%, or $5 per share, whichever is greater. Investors may short a stock that they already own, a practice called "shorting against the box," for example in order to create the effective reduction in equity exposure associated with a direct sale, but avoid immediate recognition of capital gains for tax purposes. The additional margin required when shorting against the box is only 5%, according to Brent, Morse, and Stice (1990). Retail customers may in some cases post the 150% in cash collateral with no rebate.

rate. The lender may opt out of a continuing lending arrangement by issuing a "recall notice," in which case the borrower must return the stock. A typical method for the short-seller to return the stock would be to borrow it from another lender. Alternatively, the borrower's broker could issue its own recall notice to another borrower. In some cases, often called "short squeezes," the borrower (or its broker) is unable to locate lendable shares and must be "bought in," that is, must buy the stock outright.[10] If the borrower fails to deliver the security in standard settlement time, the lender itself may buy it, using the cash collateral. The borrower remains responsible for any additional costs to the lender in conducting the buy-in. With a buy-in, the short sale is effectively interrupted. Institutional investors are viewed as preferred lenders, as they tend to hold stock positions over long periods of time, and are relatively unlikely to recall the stock. An unrelated broker, however, would normally be a less desirable lender, as its motives for maintaining a position in the stock over time are relatively uncertain.

During a lending agreement, ownership title (including voting rights and rights to any distributions, including dividends and shares) passes to the borrower.[11] Cash-in-lieu-of-dividend payments are made by the borrower to the lender.

In addition to borrowing for the purpose of profiting from a price decline or to obtain securities to deliver under a prior lending agreement, stocks may also be borrowed in order to hedge an investment (such as an equity derivative or a convertible bond), to gain access to voting rights, or to be the owner of record for dividends, which can be useful for certain accounting or tax reasons, or for dividend-discount reinvestment plan purchases, the benefits of which are documented by Scholes and Wolfson (1989).

Shares are lent in order to obtain the cash collateral as a source of financing, and in order to profit from the associated low rebates, or to meet the terms of an exclusive lending agreement, in return for which the lender received a guaranteed fee, as in the CSFB-CalPERS deal.

[10] A broker might, as a service to a highly valued customer, buy the stock on its own account in order to lend it to the customer.

[11] In Japan, given the Japanese tax treatment of dividends, it is common for the lender to recall the stock prior to dividends, in order to be recognized as the holder of record.

3 Securities Lending and Asset Pricing

This section contains the basic model, based on trade among agents with divergent beliefs about the prospective future value of an asset. Optimists want to buy the asset; pessimists want to sell it short. The key features of this model are: *(i)* an agent can sell shares only if he owns them or has borrowed them, *(ii)* those wishing to borrow or lend must search for each other, and *(iii)* the borrower and lender must negotiate a fee. Our valuation approach is based on Duffie, Gârleanu, and Pedersen (2000), which has a search-and-baragaining structure similar to that earlier used in certain monetary models, particularly Trejos and Wright (1995).

3.1 Model

Our model addresses a hypothetical asset that pays no dividends before a stopping time τ with Poisson arrival intensity[12] γ. At time τ, the present market value V of the future dividends is revealed to all agents.[13] Before this "day of reckoning," no information concerning V is revealed. Of the total amount of shares outstanding, the float (amount of actual shares available for trade) is fixed at F. We could also examine the implications of scheduled changes in the float over time, for example through expiration of lock-up agreements, or through merger or spinoff events.

We assume for simplicity that agents are risk-neutral and show no time preference. A continuum of types of agents, indexed by $\sigma \in [0, 1]$, agree on the probability distribution of the time τ at which V is revealed and on the independence of τ and V. They may, however, have different beliefs about the eventual value, V, of the asset. Specifically, at any time $t < \tau$, an agent of type σ expects the value of the asset to be $V^\sigma = E^\sigma(V)$, where E^σ denotes expectation with respect to the probability measure used by agents of type σ. The agent types are assumed to be ordered so that, without loss of generality, V^σ is strictly increasing in σ. For example, agents of type 1 are the most optimistic; agents of type 0 are the most pessimistic. The masses of the different types of agents is given by a measure μ, in that there is a

[12]We fix a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a filtration $\{\mathcal{F}_t : t \geq 0\}$ of sub-σ-algebras, satisfying the usual conditions as defined by Protter (1990), representing the information commonly available to investors. The stopping time τ is exponentially distributed with mean $1/\gamma$.

[13]That is, V is an $\mathcal{F}_\tau$-measurable random variable.

finite mass $\mu([a,b])$ of agents with a type in an interval[14] $[a,b]$.

As a simplification, we assume that each agent can be long or short at most one share. (This may be viewed as an un-modeled substitute for a risk or credit limit, or for the effect of risk aversion.) Trading of the asset takes place as follows. There is a centralized (Walrasian) market for buying and selling shares. At each time t, shares can be bought and sold instantly at a price P_t. In Section 3.7, we consider the quantitative significance of incorporating a settlement lag, normally 3 days in the United States, and more in most other countries. (Major brokers in Switzerland obtain same-day settlement.)

In equilibrium, the price P_t clears the market. An agent can sell stock, however, if and only if he owns it or has borrowed it. In order to borrow a share, an agent must first find another agent who owns a share that can be lent. Once contact with the lending agent is made, the parties must agree on a borrowing fee before the loan can be executed.

We assume that agents are randomly matched with intensity λ. That is, given a group of agents with mass m, a particular agent finds someone from that group with intensity $\lambda m/2$, and someone from that group finds that agent with intensity $\lambda m/2$, for a total contact intensity for that agent of λm. This assumption is based, informally at this stage, on an application of the law of large numbers for a "continuum" of agents that is typical in models based on random matching. (Using independent matching, this can be formally justified by taking limits as the number n of agents goes to $+\infty$, with equally-likely probability of meeting a particular agent, given a contact. We defer a more careful measure-theoretic treatment of this idealized limiting behavior to other work.) Similarly, agents from a subset of agents of current mass $m_A(t)$ come into contact with agents from a subset of current mass $m_B(t)$ at the total (almost sure) rate $\lambda m_A(t) m_B(t)$. Our model is equivalent to one in which borrowers find other agents with some intensity $\lambda_B < \lambda$, while lenders find other agents with intensity $\lambda_L = \lambda - \lambda_B$. In this sense, borrowers are more effective at searching than are lenders if $\lambda_B > \lambda_L$, but our quantitative results depend only on the total contact intensity $\lambda = \lambda_B + \lambda_L$.

When an agent wishing to short meets an owner of shares, they bargain over the current rate R_t at which borrowing fees are paid. These fees are continually re-negotiated until either side terminates the contract, so that the total fee paid during an open lending agreement between times s and

[14] The only measurability requirement we have is that intervals are measurable.

t is $\int_s^t R_u \, du$. (The integral make senses if the borrowing-fee process R is integrable, which is the case in the equilibria that we analyse.) As there is no time preference, credit risk, or risk aversion, if loans were offered in our model, then the market interest rate would be zero, so we can also view R_t as the "special," that is, the difference between the rebate and the normal short-term interest rate.

For now, we characterize equilibria in which only "pessimists," meaning agents of the lowest-valuation type $\sigma = 0$, can short. In Section 3.6, we show that this restriction is without loss of generality provided there are frictional transactions costs for shorting and a sufficiently large mass of pessimists.

3.2 General Properties of Equilibrium

In this section we derive some general features of the securities lending market and its interplay with the market for the underlying security.

We assume that, at time zero, the short interest is zero, that is, no shorting has yet happened. Because of the Walrasian market for shares, the float is initially allocated to the most optimistic agents, that is, to a mass F of agents whose valuations are at least as high as that of any agent not initially allocated shares.

Over time, pessimists meet lenders, borrow shares, and shortsell. These shares are bought by successively less optimistic agents. At time t, shares are bought by the current "marginal investors," the most optimistic investors who do not already own shares. We are looking for the equilibrium price process, $P(t)$, the equilibrium borrowing fee, $R(t)$, and the equilibrium short interest, $S(t)$ (the total amount of shares held short). We may take the commonly available information at time t to be that generated by prices, rebates, revelation of V, and by the times of borrower-lender contacts.[15]

At time t, the total long interest is $F+S(t)$, and therefore the next buyer's (marginal investor's) type $\sigma(t)$ is well defined as $\sigma(t) = \bar{\sigma}(S(t))$, where[16]

$$\bar{\sigma}(S) = \inf\{\sigma \,:\, \mu([\sigma, 1]) \leq F + S\}. \tag{1}$$

[15]This means that the information set $\mathcal{F}_t$ is that generated by $\{P_s, R_s, V1_{\{\tau \leq s\}}, 1_{\{\phi \leq s\}} : \phi \in \Phi, s \leq t\}$, where Φ is the set of times at which identified pairs of agents make contact. In the equilibria that we examine, each agent cares only about observation of τ, V, and that agent's own contact times.

[16]If the cumulative distribution function of types is strictly increasing and continuous, then the condition $\mu([\bar{\sigma}(S), 1]) = F + S$ uniquely defines $\bar{\sigma}(S)$.

As of time t, the quantity $U(S(t))$ of un-filled shorters, that is, the quantity of pessimists who have not already obtained a short position, is

$$U(S(t)) = \mu(0) - S(t),$$

the total number of pessimists less the short interest.

The rate $S'(t)$ at which the short interest $S(t)$ is building up over time depends on the rate of contact between un-filled shorters and owners of stocks. This total rate of contact is the product of the meeting intensity λ, the mass $U(S(t))$ of unfilled shorters, and the float F. Thus,

$$S'(t) = \lambda F U(S(t)) 1_{\{F+S(t)<\mu((0,1])\}}. \tag{2}$$

The indicator factor $1_{\{F+S(t)<\mu((0,1])\}}$ allows for the cessation of shorting once all optimists already own shares. This ordinary differential equation (2) determines the equilibrium short interest $S(t)$ and, together with (1), the equilibrium allocation of the security.

We model the price $P(t)$ and borrowing fee $R(t)$ that apply in the event that $t < \tau$. The actual price and borrowing fee jump to V and 0, respectively, on date τ. We analyze only equilibria in which $P(t)$ and $R(t)$ are deterministic.

Since, in equilibrium, a lending agreement is not terminated before the day τ of reckoning, the total expected future lending fee L_t from an already-matched borrower to the lender, from any time $t < \tau$ onwards, using the fact that $\mathcal{P}(\tau > u \,|\, \tau > t) = e^{-(u-t)\gamma}$, is

$$L_t = E_t \left(\int_t^\tau R_u \, du \right) = \int_t^\infty R_u e^{-(u-t)\gamma} \, du, \tag{3}$$

where E_t denotes expectation given the information available at time t. (This expectation does not depend on type.) The expected total income, at any time t before τ, associated with eventually lending the stock, once a borrower is located, is

$$\mathcal{L}_t = \int_t^\infty e^{-(s-t)\gamma} e^{-\int_t^s \lambda U(S(u)) \, du} \lambda U(S(s)) L(s) \, ds, \tag{4}$$

using the fact that $e^{-\int_t^s \lambda U(S(u)) \, du} \lambda U(S(s))$ is the conditional density, evaluated at time s, of the first time at which a given owner encounters some un-matched pessimist, given no such contact by time t. The lending deal will be conducted at time s only if V remains un-revealed at that time, explaining the factor $\mathcal{P}(\tau > s \mid \tau > t) = e^{-(s-t)\gamma}$.

At any time $t < \tau$, an (optimistic) agent of type σ who has not already bought the security has an expected benefit from buying at some time $u \geq t$ of

$$E^{\sigma}\left((V + \mathcal{L}_u - P_u)1_{\{u<\tau\}} \;\middle|\; t < \tau\right) = (V^{\sigma} + \mathcal{L}_u - P_u)\, e^{-(u-t)\gamma}. \tag{5}$$

By the definition of the time-t marginal-investor type $\sigma(t)$, it must be optimal for this type to be ready to buy for the first time at time t. For this to be the case, the marginal benefit to this type of waiting, in terms of price reduction net of foregone lending fees, must be equal at time t to the marginal cost of waiting, in terms of the expected rate of loss for this type caused by not having purchased the asset in time to have profited from the expected price change at the day of reckoning, time τ. This expected opportunity-loss rate is the mean arrival rate γ of the revelation of V, multiplied by the mean expected gain $V^{\sigma(t)} + \mathcal{L}(t) - P(t)$ given prior purchase. That is, we must have the first-order condition

$$\frac{d}{dt}[\mathcal{L}(t) - P(t)] = \gamma[V^{\sigma(t)} + \mathcal{L}(t) - P(t)], \tag{6}$$

which, is also obtained from (5) by differentiation with respect to u, evaluating the result at $u = t$ and at $\sigma = \sigma(t)$, and finally setting the result equal to zero. We can treat (6) as a linear ordinary differential equation in $\mathcal{L}(t) - P(t)$, with the solution given by the following result.

Proposition 1 *In any equilibrium, the price and expected lending fee from a prospective fee satisfy*

$$P_t = \mathcal{L}_t + \int_t^{\infty} V^{\sigma(u)} e^{-(u-t)\gamma}\gamma\, du. \tag{7}$$

The price, P, can therefore be viewed as the expected future revenue $\mathcal{L}$ associated with the potential to lend the asset, plus the weighted average of the valuations of future marginal investors, where the weight for type $\sigma(u)$ is the probability density $e^{-(u-t)\gamma}\gamma$ that this investor will be marginal just as V is revealed.

This provides a natural relationship between the price, P, and the expected potential lending fee, $\mathcal{L}$. In order to identify the price and lending fee separately, one must treat the bargaining game between the borrower and the lender.

3.3 Negotiating the Lending Fee

In this section, we model the negotiation of the lending fee. We consider a potential lender and borrower who have made contact and must agree on a lending fee. To determine the relative strength of their bargaining positions, we first determine their outside options.

If the lender walks away from the negotiations at time t, he expects (in equilibrium) a present value of lending fees of $\mathcal{L}_t$ from the next borrower, which is thus his outside option value. Similarly, the borrower's outside option value at time t is the expected value associated with finding another lender, which is

$$\mathcal{S}_t \;=\; \int_t^\infty \lambda_t^S e^{-\gamma(s-t)} e^{-\int_t^s \lambda_u^S \, du} (P_s - L_s - V^0)\, ds, \tag{8}$$

where $\lambda_t^S = \lambda F 1_{\{F+S(t)<\mu((0,1])\}}$ is the intensity with which another lender is located.

If the agents agree to transact now, then the lender receives L_t and the borrower get his expected utility from shorting, $P_t - L_t - V^0$. Hence, the gain from trade between these agents is

$$L_t + (P_t - L_t - V^0) - \mathcal{L}_t - \mathcal{S}_t = G_t - \mathcal{S}_t,$$

where we have used Proposition 1, and where

$$G_t = E_t(\Delta V(\tau) \;\big|\; \tau > t) = \int_t^\infty \Delta V_u e^{-(u-t)\gamma} \gamma \, du, \tag{9}$$

where $\Delta V_t = V^{\sigma(t)} - V^0$. Thus, if the lender has a fraction q of the bargaining power,[17] then the equilibrium lending fee for a loan in progress is

$$L_t = \mathcal{L}_t + q(G_t - \mathcal{S}_t). \tag{10}$$

This is an equilibrium outcome of Nash (1950) bargaining, and can be justified by an alternating-offer game with risk of breakdown (Binmore, Rubinstein, and Wolinsky (1986)),[18] or by a simultaneous-offer bargaining game (Kreps (1990)).

[17] The bargaining power q need not be constant over time, but we take it so for simplicity of exposition.

[18] We solve an explicit bargaining game over the total fees paid in expectation, L, but we do not report it here. The numerical results imply a value for q that is almost constant and very close to 0.5.

Solving Equations (4) and (8), using (10), we get the following calculations.

Theorem 2 *Suppose the lender has a fraction q of the bargaining power. Then the expected present value L_t of the lending fee paid by the borrower to a lender already contacted at time t is given by (10), where*

$$\mathcal{S}_t = \int_t^\infty (1 - e^{-\int_t^u (1-q)\lambda_z^S dz})\Delta V_u e^{-(u-t)\gamma}\gamma \, du \tag{11}$$

$$\mathcal{L}_t = \int_t^\infty e^{-\gamma(s-t)}\lambda_s^L q(G_s - \mathcal{S}_s)\, ds, \tag{12}$$

where $\lambda_t^L = \lambda U(S(t))$ is the rate with which a lender finds a borrower. The price P_t is given by Equation (7). Of all equilibria, the one given by $q = 0$ has the lowest lending fees ($L_t = \mathcal{L}_t = 0$) and prices, while the one given by $q = 1$ has the highest possible lending fees (L_t and $\mathcal{L}_t$) and prices.[19]

Since the bargaining, as derived above, takes place over L_t, the literal interpretation of L_t is a lump-sum lending fee paid if the lending arrangment begins at time t, and is to continue until the day of reckoning. This lump-sum payment, however, is consistent with continuous payments R_t. The rate R_t, moreover, is "renegotiation proof," in the sense that a later bargaining over lending fees will lead to no change in the path of R. One obtains R_t by differentiating 4, whence

$$L_t' = -R_t + \gamma L_t, \tag{13}$$

yielding

$$R_t = q(G_t - \mathcal{S}_t)(\lambda_t^L - (1-q)\lambda_t^S) + q\gamma\Delta V_t \tag{14}$$

$$= q\left[(\lambda_t^L + \gamma)(G_t - \mathcal{S}_t) - (G_t - \mathcal{S}_t)'\right] \geq 0\,. \tag{15}$$

3.4 Characterizing the Prices and Lending Fees

In this section, we derive some properties of prices and lending fees that apply in all of the equilibria that we have identified. First, we have some natural time dynamics.

[19]These prices and lending fees are minimal and maximal, respectively, accross all possible equilibria, not just within the class of bargaining equilibria considered here.

Proposition 3 *For $t < \tau$, the expected future borrowing fees $L(t)$ and $\mathcal{L}(t)$, the price, $P(t)$, and the volume of trade, $S'(t)$, are all decreasing in t. The short interest $S(t)$ is increasing in t. As $t \to \infty$, it holds that $\mathcal{L}(t) \to 0$, and P_t approaches the Walrasian price.*[20]

We turn to a characterization based on comparative statics. That is, we compare the equilibrium properties of economies that are distinguished by their parameters. We consider first the dependence of lending fees on the differences of opinions between optimists and pessimists. We say (in the sense of comparative statics) that there is an increased *difference of opinions* between optimists and pessimists if the pessimists' valuation, V^0, decreases and if the cross-sectional distribution of the optimists' valuations increases in the sense of first-order stochastic dominance (FOSD).

Proposition 4 *With any increase in the difference of opinions between optimists and pessimists, there is an increase in the lending fees, L_t and $\mathcal{L}_t$. With any increase (in the sense of FOSD) in the optimists' valuations, holding constant the pessimists' valuation V^0, the asset price P_t increases.*

Increasing the float F has two effects. First, it increases the quantity of agents who can hold the security, both directly, and indirectly by facilitating a more rapid growth in the short interest. Hence, the marginal investor is less optimistic with a larger float. Second, a larger float is associated with a reduction in the expected number of times that a given share will be lent. Both of these effects reduce the price and the expected lending fee, as stated below. This result may partially address the influence of a small float on the initial valuation of IPOs, fixing the fundamentals and the total number of shares outstanding,

Proposition 5 *The price, P_t, and the lending fees, L_t and $\mathcal{L}_t$, are decreasing in the float, F, and increasing in the lender's bargaining power, q.*

Increasing the search intensity, λ, or increasing the initial quantity $\mu(0)$ of would-be shorters, which we may think of as proportional to the amount of capital available for shorting, decreases the valuation of the marginal investor, pushing down the price and the lending fee. At the same time, however, it increases the expected number of times that a given share can be lent, which on its own would increase the lending fee and hence the price.

[20]The Walrasian price is the valuation of the marginal investor given the maximal amount of shorting; see Appendix A.

Proposition 6 *The price, net of expected future lending fees, $P - \mathcal{L}$, decreases with λ and with $\mu(0)$, and increases with γ. The effects of λ, $\mu(0)$, and γ on the price P and the expected lending fee $\mathcal{L}$, separately, however, are ambiguous.*[21]

Precise statements can still be made regarding the behavior of the equilibrium as parameters approach extreme values. Of particular interest is the influence on the price of an increasingly liquid market for borrowing — that is, as the meeting intensity λ goes to infinity.

Proposition 7 *Suppose the lender's bargaining power q is strictly less than 1. Then, as the search intensity λ tends to infinity, for all t, the expected future lending fees L_t (at the time of a loan) and $\mathcal{L}_t$ (if searching for a loan at time t) tend to zero, and the price P_t tends to the Walrasian price W_t.*[22]

The intuition behind this result is the following. If pessimists have some bargaining power ($q < 1$) and there are not enough pessimists to absorb the entire demand for the asset, then, as λ gets large, the reservation values of pessimists approach the total future fees from a lending agreement, because they have a high chance of meeting another owner immediately. Since the quantity of unfilled shorts decreases rapidly toward zero, lenders are not in the same situation. With a large quantity of pessimists, the marginal investor's valuation quickly becomes that of a pessimist, leaving no gains from lending.

If, on the other hand, lenders have perfect bargaining power ($q = 1$), then, as of time 0, the potential expected gain from lending an asset repeatedly until the day of reckoning all accrue to the owner at time 0. This expected gain is the expected number of times that the asset is expected to be lent, multiplied by the difference in valuation between the marginal investor and the pessimist. In the limit as λ gets large, the asset is lent the maximum possible number of times,

$$\beta = \frac{1}{F} \min\{\mu(0), \mu((0,1]) - F\}.$$

We can summarize this case as follows.

[21] By an "ambiguous" effect, we mean that there are parameters for which the effect is positive, and other parameters for which the effect is negative.

[22] This is the price that obtains when there exist Walrasian markets for both trading and borrowing the security; see Appendix A.

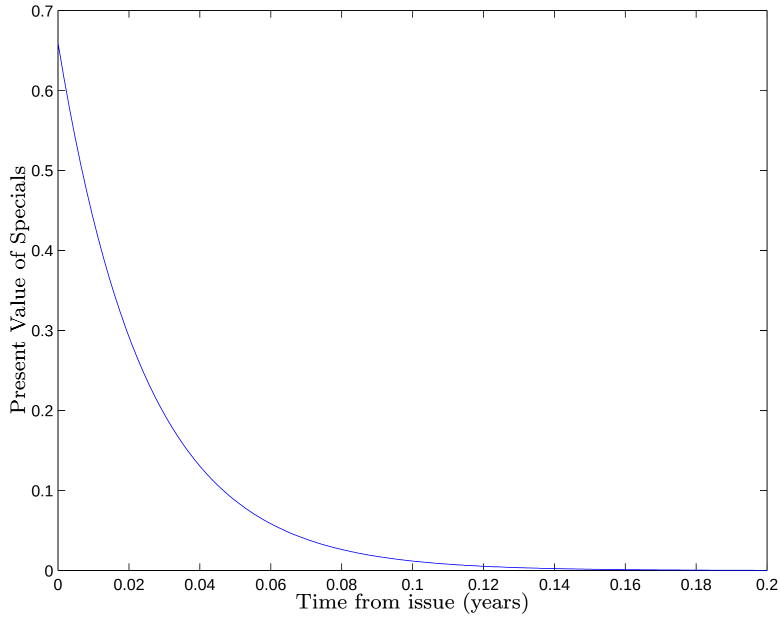

Figure 1: The expected total lending fees, $\mathcal{L}_t$.

Proposition 8 *If the lender has all bargaining power* ($q = 1$), *then*

$$\begin{aligned} \lim_{\lambda\to\infty} \mathcal{L}_0 &= \beta(W_0 - V^0) \\ \lim_{\lambda\to\infty} P_0 &= W_0 + \beta(W_0 - V^0). \end{aligned}$$

Furthermore, for $t > 0$, *in the limit as* $\lambda \to \infty$, *we have* $\mathcal{L}_t = 0$ *and* $P_t = W_t$.

3.5 Numerical Example

We illustrate with an example. We assume that there is a mass $\mu(0) = 0.25$ of pessimistic agents whose personal valuation of the asset is $V^0 = 100$, while the valuations of the other agents, of total mass 2, are uniformly distributed between V^0 and $V^1 = 110$. The total supply of the asset is $F = 0.04$, the meeting intensity of agents is $\lambda = 1000$, and the intensity of arrival of the reckoning day is $\gamma = 0.5$. The gains from trade are split evenly (that is, $q = 0.5$).

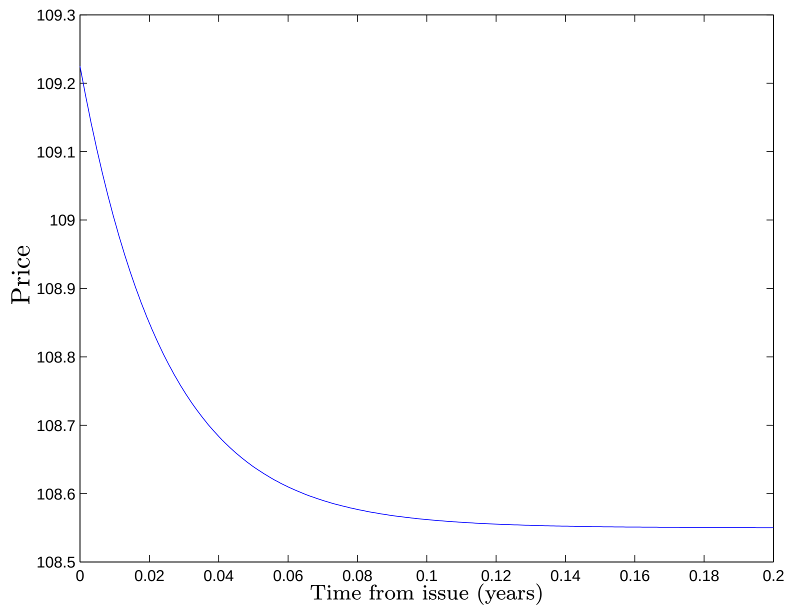

Figure 2: The dependence of the asset price on time.

With this specification, the equilibrium can be solved explicitly. Figure 1 shows the expected present value of fees that an owner receives by (eventually) lending (that is, $\mathcal{L}$). For the parameters chosen, this represents a price premium at time 0 of approximately 0.6% associated with the opportunity to lend the asset. (The maximal premium, for the case in which lenders have all of the bargaining power, is about 50%.) This lending-fee price premium decreases rapidly over time, to practically 0 within two months.

The price itself, depicted in Figure 2, drops over time as the expected future value of lending opportunities declines, and as the expected marginal valuation at the day of reckoning also declines with the introduction of more and more shares from shortselling, "burning through" the pool of un-invested optimists. The price decline is mostly due to the decline of the expected lending fee.

Figure 3 illustrates the gradual build-up of the short interest, as a proportion of the float. Finally, Figure 4 plots the price at time 0 as a function of λ. One can notice both the non-monotonicity of the price as well as its eventual convergence to its smallest possible value, the Walrasian one.

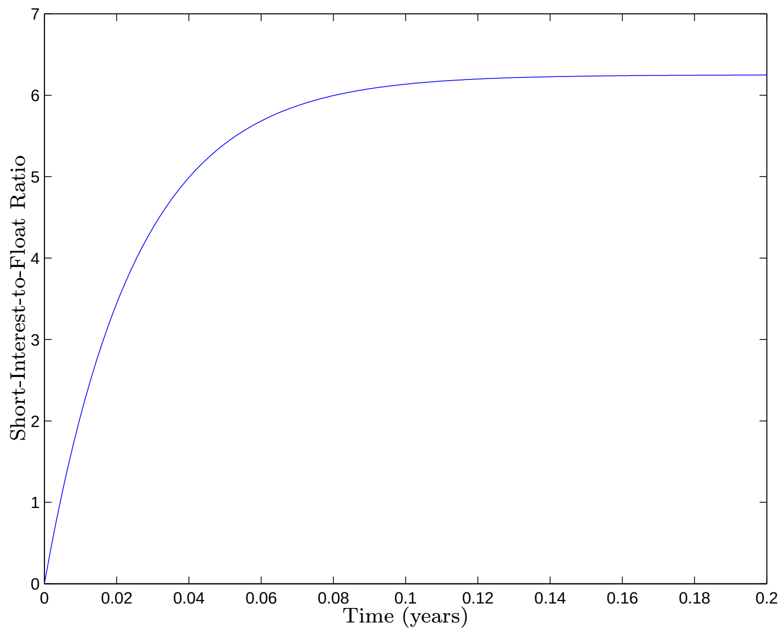

Figure 3: The dependence of the short interest on time.

3.6 The Equilibrium Amount of Shorting Capital

Now we consider the endogenous determination of the amount of capital that is made available for shorting. We assume that there is some fixed frictional cost, c, of shorting (in addition to the lending fees). Only agents who have incurred this cost are in a position to search for opportunities to borrow shares. The value $\mathcal{S}_t$ to a pessimist of acquiring this ability, in the equilibrium of Section 3.3, is characterized as follows.

Proposition 9 *At any time $t < \tau$, the expected value, $\mathcal{S}_t$, to a pessimist associated with the opportunity to be a shorter (i) decreases with t, (ii) is decreasing and continuous in the quantity, $\mu(0)$, of pessimists, and (iii) tends to 0 as $\mu(0)$ increases to infinity. If the marginal-type function $\bar{\sigma}(\,\cdot\,)$ defined by (1) is strictly decreasing[23] at 0, then $\mathcal{S}_0$ is strictly decreasing in $\mu(0)$.*

[23] This means that there exists no $\epsilon > 0$ such that $\bar{\sigma}(\epsilon) = \lim_{S \to 0} \bar{\sigma}(S)$. If this is not the case, then we obtain the weaker result that $\mathcal{S}_0$ is constant for $\mu(0)$ in a set $[0, \bar{\mu}]$ with $\bar{\mu} > 0$, and strictly decreasing for $\mu(0) > \bar{\mu}$. (For this, it suffices that the distribution of types has a positive density.)

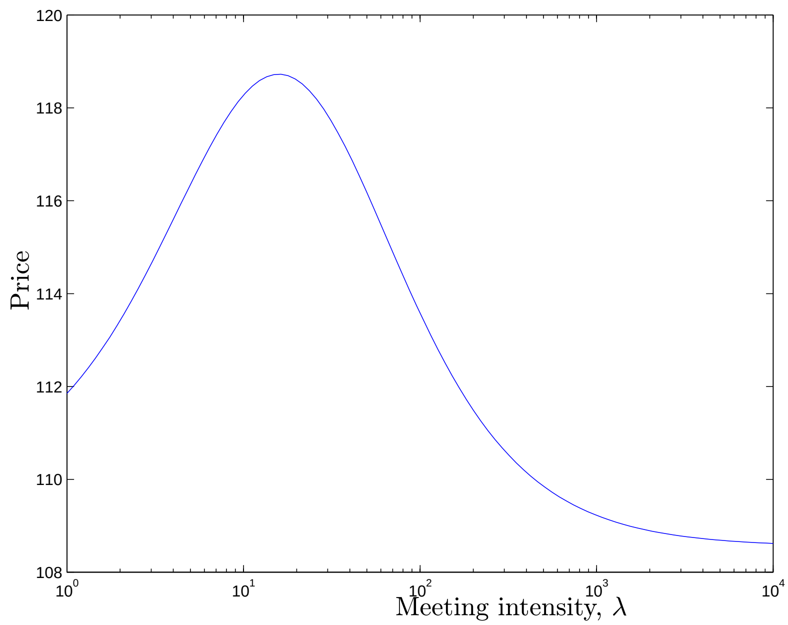

Figure 4: Dependence of the price on the borrower-lender search intensity.

This result provides for a determinate endogenous level of capital for shorting, in the following sense. Suppose there is an unlimited pool of pessimists that consider the opportunity to incur the "entry cost" c. At time 0, given the properties stated by the proposition and provided that c is strictly less than the benefit level $\mathcal{S}_0$ associated with no pessimists ($\mu(0) = 0$), there is a unique quantity $\mu(0)$ of pessimists that actively seek short positions with the property that the benefit $\mathcal{S}_0$ precisely justifies the cost c. In equilibrium (of the entry game that we do not formally model here), this quantity $\mu(0)$ of pessimists enters, and is indifferent to doing so, because $\mathcal{S}_0 = c$. As the benefit $\mathcal{S}_t$ decreases with t, those entering at time 0 correctly anticipate that there is no subsequent entry. With this equilibrium entry of pessimists, no optimist would short, as the expected profit to an optimist from shorting is smaller than that of a pessimist, who in equilibrium is indifferent to entering.

Thus, as anticipated with our initial model, only type-0 agents would choose to shortsell under these conditions. The expected profit from shorting depends on the differences of opinions, as stated below.

Proposition 10 *The equilibrium amount, $\mu(0)$, of capital available for short-*

ing increases with an increase in the difference of opinions between optimists and pessimists.

The quantity $\mu(0)$ of those actively seeking short positions depends ambiguously on other model parameters. For instance, a decrease in the float F leads to a higher valuation by the marginal investor, making shorting more attractive, but also makes it harder to find shares to short.

If there were a limited pool of pessimists, agents of different valuation types would acquire the ability to short. Of these, relatively more optimistic shorters would wait until the price declines sufficiently to justify closing their short positions and forming long positions instead. We have avoided the calculation of an equilibrium for this, more complicated, situation.

3.7 Delayed Settlement

We have so far assumed instantaneous settlement of trades. In many markets, however, settlement occurs with a lag. In most U.S. equity markets, for instance, settlement is "$T + 3$," meaning 3 days after the transaction date, while the market for securities lending is normally based on same-day settlement. Thus, the "spot market" for equities is, in effect, actually a 3-day forward market. It follows that if X sells a share to Y today, then Y could not begin lending the share until 3 days from now. Settlement lags reduce the rate at which short interest can build up.

In this section, we present a simple extension of our model that captures the notion of delayed settlement. We denote the settlement lag by θ. We assume for simplicity that when a share is sold, but has not yet been delivered, it is not available for lending to anyone. In the example above, this means that, during the settlement period, neither X nor Y may not lend the share sold to Y.

With a settlement lag, the number of shares (potentially) available for lending is a time-varying process, which we denote by A. The basic ordinary differential equation (2) determining the short interest $S(t)$ is now replaced by the system of equations[24]

$$S'(t) = \lambda A(t) U(S(t)) 1_{\{F+S(t)<\mu((0,1])\}} \tag{16}$$

$$A'(t) = -S'(t) + S'(t-\theta), \tag{17}$$

[24] At $t = 0$, the derivatives involved are derivatives from the right.

the second of which reflects the fact that the quantity $A(t)$ shares available for shorting is reduced by the current volume of shorting and increased by newly delivered shares (those that were borrowed and sold θ units of time ago).

With delayed settlement, our model captures a sense in which a security can become harder to locate. A shorter locates a share with intensity $\lambda A(t)$, which is low if the number, $A(t)$, of shares available for shorting is low. This happens, for instance, if there is a large number of agents who want to short, or if there is a long settlement period. (The model can, indeed, produce cyclical variation in the number of shares available for shorting.)

Proposition 11 *An increase in the settlement lag θ causes a reduction in the short interest $S(t)$ for all t.*

The expected lending fee for a prospective loan, for an agent buying a security at time t with a settlement delay of θ, is $e^{-\gamma\theta}\mathcal{L}_{t+\theta}$, where $\mathcal{L}$ is defined by (4). As a consequence, the price is

$$P_t = e^{-\gamma\theta}\mathcal{L}_{t+\theta} + \int_t^\infty V^{\sigma(u)} e^{-(u-t)\gamma}\gamma\, du, \qquad (18)$$

modifying Proposition 1. One can separately compute the price and the lending fee in light of delayed settlement, solving the bargaining game with an arbitrary sharing of bargaining power.[25]

We demonstrate the quantitive effect of delayed settlement by extending the example considered previously. We assume a settlement lag of $\theta = 0.01$ (approximately 2.5 working days). Figure 5 shows the price, with both instantaneous and delayed settlement. For the parameters chosen, delayed settlement results in a higher present value of the opportunity to lending. In fact, the initial price is even higher than the private valuation of the most optimistic agent.

[25] With delayed settlement, the gain from trade is $G_t - \mathcal{S}_t - (\mathcal{L}_t - e^{-\gamma\theta}\mathcal{L}_{t+\theta})$. Thus,

$$L_t = \mathcal{L}_t + q(G_t - \mathcal{S}_t - (\mathcal{L}_t - e^{-\gamma\theta}\mathcal{L}_{t+\theta})),$$

and $\mathcal{L}$ and $\mathcal{S}$ satsify the differential equations:

$$\begin{aligned} \mathcal{S}_t' &= -(1-q)\lambda_t^S\left(G_t - S_t - (\mathcal{L}_t - e^{-\gamma\theta}\mathcal{L}_{t+\theta})\right) + \gamma\mathcal{S}_t \\ \mathcal{L}_t' &= -q\lambda_t^L\left(G_t - \mathcal{S}_t - (\mathcal{L}_t - e^{-\gamma\theta}\mathcal{L}_{t+\theta})\right) + \gamma\mathcal{L}_t, \end{aligned}$$

where $\lambda_t^L = \lambda U(S(t))$ and $\lambda_t^S = \lambda A_t$.

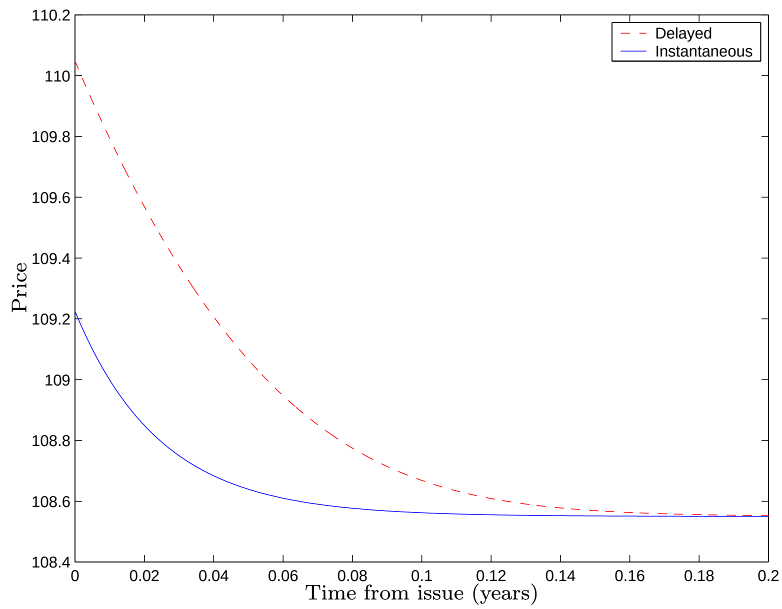

Figure 5: Price with both delayed and instantaneous settlement

4 Example: Equity Carve-Outs

The "strange" behavior of the prices of certain equity carve-outs has received recent attention (Lamont and Thaler (2001), Mitchell, Pulvino, and Stafford (2001), Ofek and Richardson (2001)). For instance, 3Com, which owned Palm, made an initial public offering of 5% of Palm shares on March 2, 2000, and promised to later distribute the remaining Palm shares to 3Com shareholders (conditional on IRS approval). The "stub" value of 3Com, its total market valuation net of the market value of its holdings of Palm, became *negative* shortly after the Palm issue! This is, at least superficially, at odds with the absence of arbitrage and the limited liability of equities. We now illustrate how our modeling approach allows for the possibility of negative stub values, with no arbitrage and with optimizing agents.

We adopt the following stylized model. Firm A consists of two subsidiaries, B and C. The holding firm A and its subsidiary B are traded on an exchange. We assume for simplicity that each investor can buy or shortsell at most one share of A and of B, and that there exist a unit mass of shares of both A and B, and the same floats for A and B. Hence, we can use the

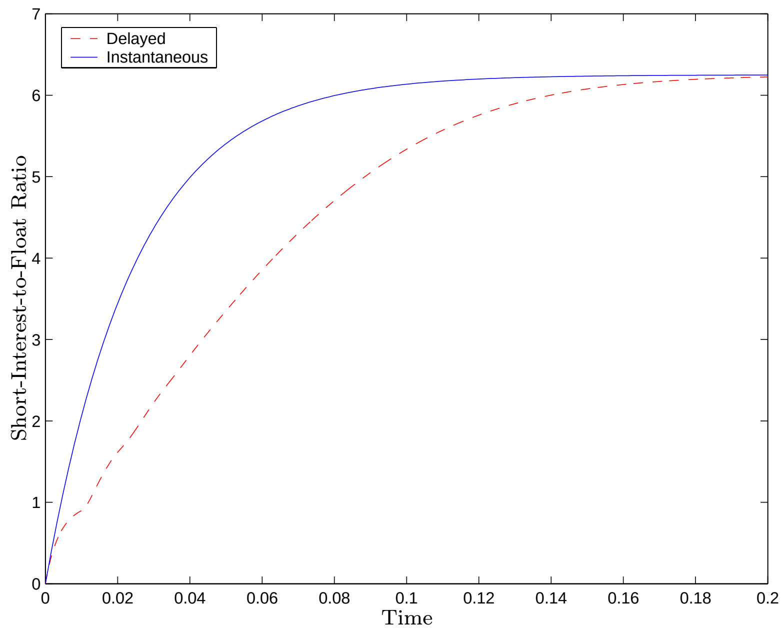

Figure 6: Short interest with both delayed and instantaneous settlement

model of Section 3 both for A and for B. The personal valuations of A, B, and C of an agent of type σ are denoted by $V^{A,\sigma}$, $V^{B,\sigma}$, and $V^{C,\sigma}$, respectively, and are assumed to be non-negative and satisfy $V^{A,\sigma} = V^{B,\sigma} + V^{C,\sigma}$. We are interested in the price p^A of A, the price p^B of B, and the stub value, $p^A - p^B$.

Suppose, first, that all agents agree on the valuation of C, in that $V^{C,\sigma} = V^C$ for all σ. In this case, if the two markets work identically, meaning that λ and q for A are the same as the corresponding parameters for B, then the stub value is, naturally, $p^A - p^B = V^C \geq 0$.

A negative stub value can arise, however, under certain (rather special) circumstances. For instance, suppose there are two groups of agents, one of which is optimistic about B and pessimistic about C, relative to the other group. For a numerical example, suppose the investors of Group 1 have expected valuations of B and C of 85 and 15, respectively. Group-2 investors have expected valuations of B and C of 70 and 30, respectively. All agents thus agree on a valuation of 100 for the holding firm A, which is its equilibrium price. Group-1 agents buy the shares of B; Group-2 agents attempt to

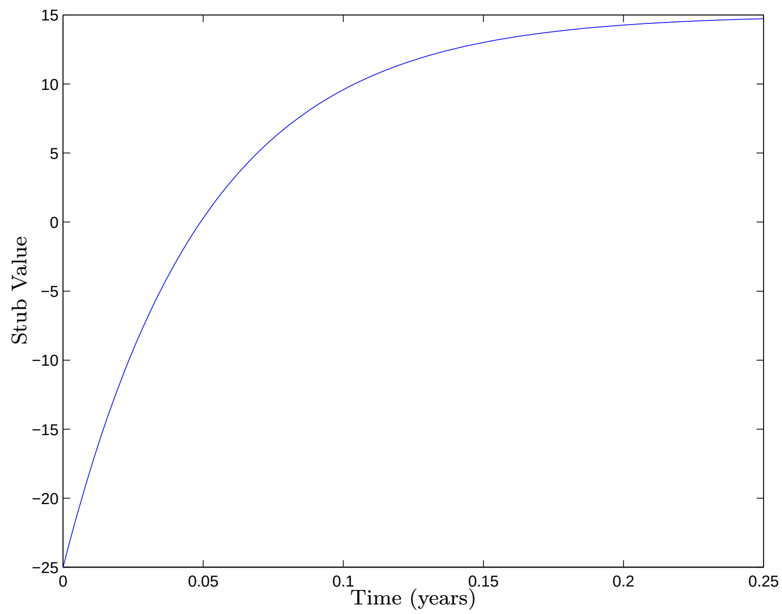

Figure 7: The stub value, initially negative, becomes positive.

short the shares of B. With a contact intensity of $\lambda = 40$, $F = 0.5$, masses 20 of Group-1 investors and 10 of Group-2 investors, an arrival intensity of $\gamma = 5$ for the time at which the valuations of B and C are revealed, and equal borrower-lender bargaining power ($q = 0.5$), our results imply that the market price of B at time 0 is 125.

Figure 7 shows that, as the short interest builds up, the stub value increases and eventually turns positive. This pattern is consistent with the empirical observations of Lamont and Thaler (2001) and Mitchell, Pulvino, and Stafford (2001).

This example shows that a negative stub value is not necessarily inconsistent with optimizing behavior by all agents, contrary to the argument for irrationality proposed by Lamont and Thaler (2001), who write: "It is always true that *someone* has to own the shares issued by the firm; not all buyers can lend their shares." In our model, an inability of would-be borrowers and lenders of stock to instantly locate each other implies that rational agents may indeed pay a price that is inflated by lending fees *and not lend right away*. Whether lending fees, in practice, are large enough to justify observed prices is difficult to assess directly, because of the stylized nature of our model and because of difficulties in measuring agents' private valuations. In the case of the spinoff of Palm by 3Com, Lamont and Thaler (2001) note that, after the first day of trading of Palm (when locating lendable shares of Palm would presumably have been comparatively difficult), the implied stub value of 3Com was minus $60.78 per share! D'Avolio (2001) reports that the annualized lending fee for Palm stock was on the order of 40%. (Similar rates for Palm are noted by Mitchell, Pulvino, and Stafford (2001).) The float, only 3% of outstanding shares, was relatively small. These facts point to a shortage of immediately lendable shares relative to the likely amount of shorting capital. In another case involving a negative stub value, for Stratos Lightwave, Reed (2001) reports an annualized lending fee of approximately 45%.

5 Concluding Remarks

While our results point to price declines during periods over which short interest is building up, we have not directly modeled the associated impact on expected returns, as that would call for identifying a reference set of probability assessments. As far as optimists are concerned, these price declines are acceptable in light of lending fees and the likelihood that they assign to eventual positive price jumps. A case for "inferior expected returns" during the period immediately following IPOs, for example, is more easily made with our model if one adopts the probability assessments of pessimists, or if one simply assumes that optimists are irrationally optimistic.

The model we have presented could be adapted to address more specifically such issues as *(i)* the behavior of prices of equities that have recently had an initial public offering, *(ii)* repo specials and the associated valuation of government bonds, and *(iii)* "bubbles." For these purposes, one might consider extensions of our model that incorporate partial information revelation over time, the updating (and perhaps convergence) of beliefs, disagreement among agents over the implications of new pieces of information, hedging motives for shorting, and fluctuations in the float and in the ease with which agents of various types are located.

A Appendix: Walras Equilibrium

In this section we derive, as a benchmark, the prices and lending fees in an economy with a shorting-through-securities-lending institution, but with *no search frictions.* To be constistent with Section 3, we assume that agents can be long or short at most one share, and that only type-0 agents can short. (If everyone can shortsell, the analysis is analogous.) The equilibrium price in this model is called the "Walrasian price."

At any time, an agent can instantly buy or shortsell shares, and can also lend or borrow shares. In order to shortsell x shares, an agent must borrow at least x shares. To lend x shares, an agent must own at least x shares.

Let $x(\sigma)$ be the (signed) number of shares owned by an agent of type σ. Then, in a Walrasian equilibrium with a positive lending fee, a type-σ agent is lending (borrowing, if negative) $x(\sigma)$ shares. This is because an agent with a long position optimally lends all his shares, and an agent with a short position optimally borrows just the number of shares he needs. Hence, equilibrium in the securities market implies market clearing, in that $\int x(\sigma)\mu(d\sigma) = F$. Equilibrium, and thus market clearing, in the lending market implies that $\int x(\sigma)\mu(d\sigma) = 0$. Thus, in the absence of frictions, there is no equilibrium with positive lending fees if the float is positive.

Consequently, the Walrasian price at any time t is the valuation of the marginal investor at that time, as characterized in the following proposition.

Proposition 12 *Suppose that the float is positive ($F > 0$). Then, at any time t, the unique Walrasian lending fee is zero, and the essentially*[26] *unique Walrasian price is*

$$W_t = \begin{cases} V^0 & \text{if } F + \mu(0) > \mu((0,1]) \text{ and } t < \tau \\ V^{\bar{\sigma}(\mu(0))} & \text{if } F + \mu(0) < \mu((0,1]) \text{ and } t < \tau \\ V & \text{if } t \geq \tau\,. \end{cases} \tag{A.1}$$

B Appendix: Proofs

The proof of Proposition 1 is in the body of the text.

[26] If the marginal investor, $\bar{\sigma}(\mu(0))$, is not unique then there is an interval of equilibrium prices; see (1). The set of values for $\mu(0)$ with this property has Lebesgue measure zero.

Proof of Theorem 2:
Since $\mathcal{S}$ solves the linear ODE

$$\begin{aligned}\dot{\mathcal{S}}_t &= -\lambda_t^S(P_t - L_t - V^0) + (\gamma + \lambda_t^S)\mathcal{S}_t \\ &= -\lambda^S(1-q)G_t + (\gamma + (1-q)\lambda_t^S)\mathcal{S}_t,\end{aligned}$$

we have

$$\mathcal{S}_t = \int_t^\infty e^{-\gamma(s-t)} e^{(1-q)\int_t^s \lambda_z^S dz}(1-q)\lambda^S G_s\, ds, \tag{B.1}$$

which is equal to (11) (integration by parts).

Using (4) and (10) we see that $\mathcal{L}$ solves

$$\dot{\mathcal{L}}_t = -\lambda_t^L q(G_t - \mathcal{S}_t) + \gamma\mathcal{L}_t, \tag{B.2}$$

with the solution given by (12).

The optimist will never accept a negative lending fee, whence 0 (obtained when $q = 0$) is the minimal lending fee. To see that $q = 1$ yields the maximal fees, note that $P_t - V^0 \geq L_t$ is a necessary condition for the pessimist to be willing to borrow, a condition which implies that $L_t - \mathcal{L}_t \geq G_t$, whence

$$\dot{\mathcal{L}}_t \geq -\lambda_t^L G_t + \gamma\mathcal{L}_t.$$

Now apply Gronwall's inequality to infer that

$$\mathcal{L}_t \leq \int_t^\infty \int_t^s \lambda_u^L du\ \Delta V_s e^{-\gamma(s-t)}\gamma ds,$$

which is the expression for $\mathcal{L}_t$ corresponding to $q = 1$. □

Proof of Proposition 3:
We first show that $G(t) - \mathcal{S}(t)$ is decreasing in t. This follows from the fact that ΔV_t is decreasing in t and from

$$\begin{aligned}G_t - \mathcal{S}_t &= \int_t^\infty e^{-(1-q)\int_t^u \lambda_z^S dz}\Delta V_u e^{-(u-t)\gamma}\gamma\, du &\text{(B.3)}\\ &= \int_0^\infty e^{-(1-q)\lambda F u}\Delta V_{t+u} e^{-\gamma u}\gamma\, du, &\text{(B.4)}\end{aligned}$$

which is seen to be true because $1_{\{F+S(t)<\mu((0,1])\}} = 0$ implies that $\Delta V_t = 0$. Using this and similar arguments one shows that $\mathcal{L}(t)$, $L(t)$, and $P(t)$ decrease in t.

It is clear from (2) that $S(t)$ is increasing and that $S'(t)$ is decreasing. □

Proof of Proposition 4:
If there is an increase in the difference of opinion between optimists and pessimists, then ΔV_t increases for all t. Hence, from (B.4), $G_t - \mathcal{S}_t$ increases. This implies that L and $\mathcal{L}$ increase.

If the difference of opinions increases, keeping V^0 constant, then $V^{\sigma(t)}$ increases for all t. Then, Proposition 1 shows that P_t increases. □

Proof of Proposition 5:
Increasing the float increases S_t for all t, which decreases $V^{\sigma(t)}$, ΔV_t, and λ_t^L. Hence, using (B.4), $G_t - \mathcal{S}_t$ decreases. Using (12), we see that $\mathcal{L}_t$ decreases. Finally, P_t is seen to decrease using Proposition 1.

It is obvious that $G_t - \mathcal{S}_t$ increases in q, and all the statements about the impact of q follow immedtiately. □

Proof of Proposition 6:
Increasing λ and $\mu(0)$ increases S_t for all t, which decreases $V^{\sigma(t)}$. Hence, the results follow from (7). □

Proof of Proposition 7:
From (B.4), it follows by dominated convergence that $G_t - \mathcal{S}_t \to 0$ as $\lambda \to \infty$, for all t. Note now that

$$\mathcal{L}_0 \leq (G_0 - \mathcal{S}_0) \int_0^\infty \lambda_t^L dt,$$

where the integral is the expected number of times a given asset is lent, which is finite (in fact, it equals β as defined in the text). The statements about L_t and P_t follow immediately. □

Proof of Proposition 8:
When $q = 1$ we have $\mathcal{S} = 0$, and the limit of G_t is ΔV_∞. All the statements are immediate. □

Proof of Proposition 9:
It is clear from (11) that $\mathcal{S}_t$ decreases in t. We next show that $\mathcal{S}_t$ approaches zero as $\mu(0)$ approaches infinity. For $\mu(0) > \mu((0,1]) - F$, we let $T(\mu(0)) = \inf\{t : F + S(t) = \mu((0,1])\}$. From (2), we see that $T(\mu(0)) \to 0$ as $\mu(0) \to$

∞. Now, since $\Delta V_t = 0$ for $t \geq T(\mu(0))$, we see from (11) that $\mathcal{S}_t \to 0$ as $\mu(0) \to \infty$ for all $t \geq 0$.

Inspection of (11) reveals that ΔV_u is the only term depending on $\mu(0)$. We note that $V^{\sigma(u)} = V^{\bar{\sigma}(S(u))}$ is a decreasing and right-continuous function of $S(u)$ which, in turn, is an increasing and continuous function of u and $\mu(0)$. Let $k(u) = (1 - e^{-(1-q)\lambda F u})e^{-u\gamma}\gamma$, f be the function defined by $V^{\sigma(u)} = f(F+S(u))$, and $g(u,\mu(0)) = F+S(u,\mu(0))$, using obvious notation from this point to indicate dependence on $\mu(0)$. Let h be defined by $h(u) = g(u,\mu_0)$. For any $T > 0$ (sufficiently large) and $\Delta\mu$ (sufficiently small) let u_1 and u_2 satisfy by $h(u_1) = A\Delta\mu$, respectively $h(T-u_2) = T - A\Delta\mu$, where A is the modulus of continuity of $g(u,\mu)$ as a function of μ, uniformly in u. Note that

$$\begin{aligned}
&\mathcal{S}(0,\mu_0) - \mathcal{S}(0,\mu_0+\Delta\mu)\\
&\quad= \int_0^\infty k(u)\left(f(g(u,\mu_0)) - f(g(u,\mu_0+\Delta\mu))\right)du\\
&\quad\leq \int_0^\infty k(u)\left(f(g(u,\mu_0)) - f(g(u,\mu_0)+A\Delta\mu)\right)du\\
&\quad= \int_0^{u_1} k(u)f(h(u))du +\\
&\qquad \int_{u_1}^T f(h(u))\left(k(u) - k(h^{-1}(h(u)-A\Delta\mu))\right)du -\\
&\qquad \int_{T-u_2}^T k(u)f(h(u)+A\Delta\mu)du +\\
&\qquad \int_T^\infty k(u)\left(f(g(u,\mu_0)) - f(g(u,\mu_0+\Delta\mu))\right)du.
\end{aligned}$$

(The inequality owes to the monotonicity of f.)
Choosing T large enough will make the last term arbitrarily small. A small enough $\Delta\mu$ combined with continuity makes the other three terms arbitrarily small. Consequently, $\mathcal{S}(0)$ is continuous in $\mu(0)$, and so is $\mathcal{S}(t)$, by essentially the same proof.

Given that ΔV_u is right-continuous in $\mu(0)$, $\mathcal{S}(0)$ is strictly decreasing in $\mu(0)$ if and only if there exists u for which ΔV_u decreases strictly in $\mu(0)$. Note first that, if $\Delta V_u(\mu_0)$ decreases in $\mu(0)$, then, for any $\mu_0' > \mu_0$, defining u' so that $\Delta V_u(\mu_0) = \Delta V_{u'}(\mu_0')$, implies that $\Delta V_{u'}(\mu_0')$ increases in $\mu(0)$ at μ_0'. Second, note that, if $\bar{\sigma}$ decreases strictly around 0, then ΔV_u decreases around $\mu(0) = 0$. $\square$

Proof of Proposition 10:
Increasing the difference of opinions increases ΔV_t for all t, which increases $\mathcal{S}$, as is seen from (11). $\square$

Proof of Proposition 11:
It suffices to consider only times t for which $F+S_t < \mu((0,1])$. Note from (16) and (17) that

$$S'(t) = \lambda(F + S(t-\theta) - S(t))(\mu(0) - S(t)),$$

under the assumptions that $S_t = 0$ for all $t \leq 0$ and $A_0 = F$.

Let now $\theta_1 < \theta_2$ and note that $S_{\theta_1}(t) = S_{\theta_2}(t)$ for $t \leq \theta_1$, and that $S'_{\theta_1}(\theta_1) = S'_{\theta_2}(\theta_1)$ and $S''_{\theta_1}(\theta_1) > S''_{\theta_2}(\theta_1)$, where S'' denotes the second derivative from the right (that is, the derivative from the right of S'). The latter two relations imply that, for $t > \theta_1$ sufficiently close to θ_1, $S_{\theta_1}(t) > S_{\theta_2}(t)$. Assume, in order to get a contradiction, that the (closed) set $\{t > \theta_1 : S_{\theta_1}(t) = S_{\theta_2}(t)\}$ is not empty and let t_0 be its smallest element. Then, $S'_{\theta_1}(t_0) > S'_{\theta_2}(t_0)$, since $S_{\theta_1}(t-\theta_1) > S_{\theta_1}(t-\theta_2) \geq S_{\theta_2}(t-\theta_2)$, contradicting the definition of t_0. $\square$

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