VALUING CREDIT DEFAULT SWAPS II: MODELING DEFAULT CORRELATIONS

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Abstract

This paper extends the analysis in *Valuing Credit Default Swaps I: No Counterparty Default Risk* to provide a methodology for valuing credit default swaps that takes account of counterparty default risk and allows the payoff to be contingent on defaults by multiple reference entities. It develops a model of default correlations between different corporate or sovereign entities. The model is applied to the valuation of vanilla credit default swaps when the seller may default and to the valuation of basket credit default swaps.

In Hull and White (2000) we explained how a vanilla credit default swap (CDS) can be valued when there is no counterparty default risk. This is a two stage procedure. The first stage is to calculate the risk-neutral probability of default at future times from the yields on bonds issued by the reference entity (or by corporations considered to have the same risk of default as the reference entity). The second stage is to calculate the present value of both the expected future payoff and expected future payments on the credit default swap. Either the value of an existing CDS or the CDS spread for a new deal can then be obtained. In this paper we develop an approach for modeling default correlations so that the analysis can be extended to include counterparty default risk and instruments where the payoff is dependent on defaults by multiple reference entities.

There are two types of models of default risk in the literature: structural models and reduced form models. The inspiration for structural models is provided by Merton (1974). Assume a company has a very simple capital structure where its debt has a face value of D, provides a zero coupon, and matures at time T. Merton shows that the company's equity can be regarded as a European call option on its assets with a strike price of D and maturity T. A default occurs at time T if the option is not exercised. Merton's model has been extended by Black and Cox (1976) and Longstaff and Schwartz (1995), who allow default to take place whenever the asset value falls below a certain level. Zhou (1997) produces an analytic result for the default correlation between two firms under this type of model.

Reduced form models focus on the risk-neutral hazard rate, $h(t)$. This is defined so that $h(t)\Delta t$ is the probability of default between times t and $t + \Delta t$ as seen at time t assuming no earlier defaults. These models can incorporate correlations between defaults by allowing hazard rates to be stochastic and correlated with macroeconomic variables. Examples of research following this approach are Duffie and Singleton (1999) and Lando (1998). Reduced form models are mathematically attractive and reflect the tendency for economic cycles to generate default correlations. They can be made consistent with the risk-neutral probabilities of default backed out from corporate bond prices. Their main disadvantage is that the range of default correlations that can be achieved is limited. Even when there is a perfect correlation between two hazard rates, the corresponding correlation between defaults in any chosen period of time is usually very low. This is liable to be a problem in some circumstances. For example, when two companies operate in the same industry and the same country or when the financial health of one company is for some reason heavily dependent on the financial health of another company, a relatively high default correlation may be warranted.

Jarrow and Yu (1999) provide an interesting way of overcoming this weakness of the reduced form model. If A and B are two related companies, they allow a large jump in the default intensity for company B to take place when there is a default by company A. In this paper we present an alternative approach that is a natural development of the structural models of Merton (1974), Black and Cox (1976), Longstaff and Schwartz (1995) and Zhou (1997). Our model is exactly consistent with the risk-neutral default probabilities backed out from bond prices. The default experience of large numbers of companies can be jointly simulated by sampling from multivariate normal distributions.

We first describe the model and then provide two applications. The first application is to vanilla swaps when there is counterparty default risk. The second is to basket credit default swaps.

1. The Model

As in Hull and White (2000), we find it convenient to explain our model in terms of the *default probability density*, $q(t)$. This is defined so that $q(t)\Delta t$ is the probability of default between times t and $t + \Delta t$ as seen at time zero. We emphasize that $q(t)$ is not the same as the hazard (default intensity) rate, $h(t)$. As mentioned earlier, the latter is defined so that $h(t)\Delta t$ is the probability of default between times t and $t + \Delta t$ as seen at time t, conditional on no earlier default. The two measures are related by

$$q(t) = h(t)e^{-\int_0^t h(\tau)\,d\tau}$$

They provide the same information about the default probability environment.

We assume that the risk-neutral default probability densities for N companies have been estimated using the bootstrap procedure in Hull and White (2000) or some other method. As a first step, we discretize these default probability densities so that defaults can happen only at times t_i $(1 \le i \le n)$. We define $t_0 = 0$ and

$$\delta_i = t_i - t_{i-1}$$

for $1 \le i \le n$. We also define q_{ij} as the risk-neutral probability of default by company j at time t_i $(1 \le i \le n; 1 \le j \le N)$.

The key feature of our model is that there is a variable $X_j(t)$ describing the creditworthiness of company j at time t $(1 \le j \le N)$. We will refer to this variable as the *credit index* for company j. We can think of $X_j(t)$ in a number of ways. In the context of structural models, it can be regarded as some function of the value of the assets of the company j. Alternatively, we can imagine that the usual discrete credit ratings, produced by rating agencies, are replaced by continuous measures and that X_j is some function of the measure for bonds issued by company j.[1]

[1] In their Creditmetrics system, J.P. Morgan (1997) show how a future credit rating can be mapped to a normal distribution and vice versa for the purpose of defining correlations. Our approach carries this idea a little further.

Our objective is to select correlated diffusion processes for the credit indices of the N companies and to determine a default barrier for each company such that the company defaults at time t_i if its credit index is below the default barrier. Without loss of generality we assume that $X_j(0) = 0$ and that the risk-neutral process for $X_j(t)$ is a Wiener process with zero drift and a variance rate of 1.0 per year.

Let K_{ij} be the value of the default barrier for company j at time t_i. Denote the risk-neutral probability density function for $X_j(t_i)$ by f_{ij}. Both K_{ij} and f_{ij} can be determined inductively from the risk-neutral default probabilities, q_{ij}. Based on the process for X_j, $X_j(t_1)$ is normally distributed with a mean of zero and a variance of δ_1. As a result

$$f_{1j}(x) = \frac{1}{\sqrt{2\pi\delta_1}} \exp\left[-\frac{x^2}{2\delta_1}\right] \tag{1}$$

and

$$q_{1j} = N\left(\frac{K_{1j}}{\sqrt{\delta_1}}\right) \tag{2}$$

where N is the cumulative standard normal distribution function. This implies

$$K_{1j} = \sqrt{\delta_1} N^{-1}(q_{1j})$$

For $2 \le i \le n$ we can calculate K_{ij} and f_{ij} from q_{ij}, $K_{i-1,j}$ and $f_{i-1,j}$. The relationship between q_{ij} and K_{ij} is

$$q_{ij} = \int_{K_{i-1,j}}^{\infty} f_{i-1,j}(u) N\left(\frac{K_{ij} - u}{\sqrt{\delta_i}}\right) du \tag{3}$$

Standard numerical methods can be used to set up a procedure for evaluating the integral in this equation for a given value of K_{ij}. An iterative procedure can then be used to find the value of K_{ij} that solves the equation. The density function f_{ij} is

$$f_{ij}(x) = \int_{K_{i-1,j}}^{\infty} f_{i-1,j}(u) \frac{1}{\sqrt{2\pi\delta_i}} \exp\left[-\frac{(x-u)^2}{2\delta_i}\right] du \tag{4}$$

for $x > K_{ij}$. The integral in this equation can be evaluated using standard numerical procedures. The cumulative probability of company j defaulting by time t_i is

$$1 - \int_{K_{ij}}^{\infty} f_{ij}(u) du$$

By increasing the number of default times, this model can be made arbitrarily close to a model where defaults can happen at any time.[2] The default barrier is in general nonhorizontal; that is, in general, K_{ij} is not the same for all i. This introduces some nonstationarity into the default process and is a price that must be paid to make the model consistent with the risk-neutral default probabilities backed out from bond prices. It can be argued that one reason for company j's default barrier being a function of time is that its capital structure is more complicated than the simple capital structure assumed by models such as Merton (1974). We regard the difference between traditional structural models and our model to be analogous to the difference between one-factor equilibrium models of the term structure such as Vasicek (1977) and one-factor no-arbitrage models of the term structure such as Hull and White (1990). The latter are non-stationary in that a function of time is introduced into the drift of the short rate to make the model consistent with an exogenously specified initial term structure. Here we make the default barrier a function of time to make the model consistent with exogenously specified initial default probabilities.

Sample Data

The results in the rest of this paper are based on the data in Table 1. This data shows credit spreads for AAA-, AA-, A-, and BBB-rated bonds. We assume that the recovery rates on all bonds is 30%, the Treasury zero curve is flat at 5% (with semiannual compounding), and that all the bonds pay a 7% coupon semiannually. Although credit ratings are attributes of bonds rather than companies, it will be convenient to refer to the companies issuing the bonds as AAA-, AA-, A-, and BBB-rated companies, respectively.

[2] An alternative model, incorporating the possibility of defaults happening at any time, is one where defaults happen if K_{ij} is reached at any time between t_{i-1} and t_i. There is an analytic expression for the default probability density between times t_{i-1} and t_i conditional on $X_j(t_{i-1})$ and an analytic expression for f_{ij} conditional on $X_j(t_{j-1})$. The model can, therefore, be implemented in a similar way to the model we present. However, it is more complicated because equation (3) is replaced by an equation involving a double integral.

Credit spreads vary through time. The spreads in Table 1 are designed to be representative of those encountered in practice. They are based in part on data in Fons (1994) and Litterman and Iben (1991).

The BBB data is the same as that used in Hull and White (2000) and leads to the default probability density in Table 2. Figure 1 shows the default barrier corresponding to this data. This was constructed by using equations (1) to (4) and assuming that defaults can take place at times 0.05, 0.15, 0.25, ... 9.95. The probability of default at each of the first ten points is 0.00219; at the next ten points it is 0.00242; and so on. The default barrier starts at zero and is initially steeply downward sloping. This shape, which is quite different from that of the default barriers used in traditional structural models, is necessary to capture the probability of default during the first short period of time.[3]

Default Correlations

Define ρ_{jk} as the instantaneous correlation between the credit indices for companies j and k. When j and k are public companies, we can assume (analogously to J.P. Morgan's Creditmetrics) that ρ_{jk} is the correlation between their equity prices. When this is not the case, we can use other proxies. For example, when j is a private company we can replace it by a public company that is in the same industry and geographical region for the purposes of calculating ρ_{jk}. When j is a sovereign entity, we can use the exchange rate for the currency issued by the sovereign entity as a substitute for equity price when the ρ_{jk}'s are calculated.

The default correlation between company j and k for the period between times T_1 and T_2 is usually defined as the correlation between the following two variables:

(a) A variable that equals 1 if company j defaults between times T_1 and T_2 and zero otherwise; and

[3] Duffie and Lando (1997) make the point that a disadvantage of traditional structural models is that the the probability of default during the first short period of time is zero. Our model overcomes this disadvantage.

(b) A variable that equals 1 if company k defaults between times T_1 and T_2 and zero otherwise

Define

$Q_j(T)$: The cumulative probability of a default by company j between times 0 and T

$P_{jk}(T)$: The probability that both company j and company k will default between times 0 and T

$\beta_{jk}(T)$: The default correlation between companies j and k for the period between times 0 and T.

It follows that

$$\beta_{jk}(T) = \frac{P_{jk}(T) - Q_j(T)Q_k(T)}{\sqrt{[Q_j(T) - Q_j(T)^2][Q_k(T) - Q_k(T)^2]}} \tag{5}$$

To calculate the default correlation, $\beta_{jk}(T)$, from the credit index correlation, ρ_{jk}, we can simulate the credit indices for companies j and k to calculate $P_{jk}(T)$ and equation (5) can then be used to obtain $\beta_{jk}(T)$. Table 3 shows the results of doing this for AAA, AA, A, and BBB companies. It illustrates that $\beta_{jk}(T)$ depends on T and is less than ρ_{jk}. For a given value of ρ_{jk}, $\beta_{jk}(T)$ increases as the credit quality of j and k decrease. These results are similar to those produced by Zhou (1997) for his model, which is a particular case of the one we propose.

2. Calculation of CDS Spreads with Counterparty Credit Risk

In Hull and White (2000) we explained how to value a CDS with a notional principal of $1 when there is no counterparty default risk. Here we extend the analysis to allow for the possibility of a counterparty default. As in Hull and White (2000), we assume that default events, Treasury interest rates, and recovery rates are mutually independent. We also assume that a bondholder's claim in the event of a default equals the face value of the bond plus accrued interest. Define

T: Life of credit default swap

$\hat{R}$: Expected recovery rate on reference obligation in the event of a default[4]

$\theta(t)\Delta t$: Risk-neutral probability of default by reference entity between times t and $t+\Delta t$ and no earlier default by counterparty

$\phi(t)\Delta t$: Risk-neutral probability of default by counterparty between times t and $t+\Delta t$ and no earlier default by reference entity

$u(t)$: Present value of payments at the rate of $1 per year on payment dates between time zero and time t

$e(t)$: Present value of an accrual payment at time t equal to $t-t^*$ dollars where t^* is the payment date immediately preceding time t.

$v(t)$: Present value of $1 received at time t

w: Total payments per year made by CDS buyer per $1 of notional principal

s: Value of w that causes the credit default swap to have a value of zero. This is referred to as the CDS spread.

π: The risk-neutral probability of no default by either counterparty or reference entity during the life of the credit default swap

$A(t)$: Accrued interest on the reference obligation at time t as a percent of face value.

[4] Theoretically this should be the expected recovery rate in a risk-neutral world. In practice recovery rates are usually assumed to be the same in a risk-neutral world and the real world.

The payments cease if there is a default by the reference entity or a default by the counterparty. If there is a default by the reference entity, there is a final accrual payment. In the event of a default by the counterparty, we assume no final accrual payment. The present value of the payments is therefore

$$w\int_0^T [\theta(t)u(t) + \theta(t)e(t) + \phi(t)u(t)]\, dt + w\pi u(T)$$

If a credit event occurs at time t, the expected value of the reference obligation as a percent of its face value is $[1 + A(t)]\hat{R}$. The expected payoff from the CDS per \$1 of face value is therefore

$$1 - [1 + A(t)]\hat{R} = 1 - \hat{R} - A(t)\hat{R}$$

The present value of the expected payoff is

$$\int_0^T [1 - \hat{R} - A(t)\hat{R}]\theta(t)v(t)\, dt$$

and the value of the credit default swap to the buyer is

$$\int_0^T [1 - \hat{R} - A(t)\hat{R}]\theta(t)v(t)\, dt - w\int_0^T [\theta(t)u(t) + \theta(t)e(t) + \phi(t)u(t)]\, dt + w\pi u(T) \tag{6}$$

It follows that the CDS spread is

$$s = \frac{\int_0^T [1 - \hat{R} - A(t)\hat{R}]\theta(t)v(t)\, dt}{\int_0^T [\theta(t)u(t) + \theta(t)e(t) + \phi(t)u(t)]\, dt + \pi u(T)} \tag{7}$$

The CDS spread can be calculated by evaluating both the numerator and denominator in equation (7) using Monte Carlo simulation.[5] The credit index for both the reference

[5] For computational efficiency, the control variate technique and antithetic variable technique described in Boyle (1977) are used. The control variate technique involves using Monte Carlo simulation to calculate the numerator and denominator in equation (7) for both the situation where there is counterparty default risk and the situation where there is no counterparty default risk. The paths sampled for the reference entity's assets are the same in both cases. This provides an estimate of the incremental effect of counterparty defaults on the numerator and denominator. It can be combined with very accurate estimates of the numerator and denominator for the no-counterpart-default case that are produced using the approach in Hull and White (2000).

entity and the counterparty must be simulated. If the reference entity defaults first (that is, the credit index for the reference entity falls below its default barrier before the credit index for the counterparty does so), payments continue up to the time of default with a final accrual payment and there is a payoff. If the counterparty defaults first (that is, the credit index for the counterparty falls below its default barrier before the credit index for the reference entity does so), payments continue up to the time of the default with no final accrual payment and no payoff. If neither the counterparty nor the reference entity default (that is, neither credit index reaches its barrier), payments continue for the life of the credit default swap and there is no payoff.

In Hull and White (2000) we considered a CDS swap where

a) The life of the contract is five years

b) The buyer makers semiannual payments

c) The reference entity is a BBB, as defined by Tables 1 and 2

d) The reference obligation lasts five years, pays a 10% coupon, and has a 30% expected recovery rate.

We showed that in the absence of counterparty default risk the CDS spread is 1.944%. Table 4 shows the spread for the same CDS when entered into with AAA, AA, A, and BBB counterparties. When the credit index correlation between the counterparty and the reference entity is zero, the impact of counterparty default risk is very small. But as the correlation increases and the credit quality of the counterparty declines, counterparty default risk has a bigger effect.

When the counterparty defaults, one option open to the purchaser of the CDS is to enter into new contract with a new counterparty to reinstate the default protection for the rest of the life of the original contract. If there is no correlation between the reference entity and the counterparty, the probability of the reference entity defaulting is unaffected by a counterparty default. If forward credit spreads are similar to spot credit spreads, the analysis in Hull and White (2000) shows that the CDS spread for the new contract should be similar to that for the original contract. This explains why the impact of counterparty default is small in the zero correlation case in Table 4.[6] The payoffs that are not realized by the CDS purchaser because of counterparty defaults are almost exactly offset by the payments that the purchaser does not have to make subsequent to a counterparty default.

When the correlation between the counterparty and the reference entity is positive, a default by the counterparty is likely to be accompanied by a below average credit index for the reference entity. If the buyer enters into a replacement contract with a new counterparty for the remaining life of the CDS, the payment will on average be higher than that in the original contract. Table 4 shows that this is reflected in the CDS spread.

Analytic Approximation When Default Correlations Are Known

Counterparty default risk reduces both the present value of the expected payoffs from a CDS and the present value of the purchaser's expected payments. To provide some insights into this, we now present a very simple analytic approximation for the change in the CDS spread when there is counterparty default risk. The approximation can be used when the default correlation between the reference entity and the counterparty has already been estimated—either directly from default data or in some other way. Define:

Q_r: The probability of default by the reference entity during time T

Q_c: The probability of default by the counterparty during time T

P_{rc}: The joint probability of default by the counterparty and the reference entity during time T. (This can be calculated from Q_r, Q_c, and the default correlation using equation (5).)

g: The proportional reduction in the present value of the expected payoff on the

[6] As an experiment, we tested the impact of counterparty default risk in the zero correlation case when the spreads over Treasuries for one-, two-, three-, four-, and five-year bonds issued by the reference entity are 100, 200, 300, 400, and 500 basis points, respectively. In this case forward credit spreads are much greater than spot spreads and the impact of counterparty default risk is much greater than in Table 4. It ranges from 4 basis points for a AAA counterparty to 10 basis points for a BBB counterparty.

CDS arising from counterparty defaults.

h: The proportional reduction in the present value of expected payments on the CDS arising from counterparty defaults.

$\hat{s}$: The CDS spread assuming no counterparty default risk. This can be calculated as described in Hull and White (2000).

Counterparty default risk changes the CDS spread from $\hat{s}$ to s where

$$s = \hat{s}\frac{1-g}{1-h} \tag{8}$$

The probability of a counterparty default during the life of the CDS conditional on the reference entity defaulting during the life of the CDS is P_{rc}/Q_r. We assume that there is a 0.5 chance that the counterparty default occurs before the reference entity defaults and a 0.5 chance that it occurs after the reference entity defaults. Ignoring discounting effects this implies

$$g = 0.5\frac{P_{rc}}{Q_r} \tag{9}$$

When the counterparty defaults, the payments made by the purchaser of the CDS may be less than they would be in the no-counterparty-default case. There is a probability of $Q_c - P_{rc}$ that the counterparty defaults and the reference entity does not default. We assume that, when this happens, the payments made by the CDS purchaser are half the average payments in the no-counterparty-default case. There is a probability of P_{rc} that both the counterparty and the reference entity will default. As before we assume that there is a 50% chance that the counterparty default occurs first. We also assume that, when both default with the counterparty defaulting first, the payments made by the purchaser are one third less than in the no-counterparty-default case.[7]. This leads to

$$h = \frac{Q_c - P_{rc}}{2} + \frac{P_{rc}}{6} = \frac{Q_c}{2} - \frac{P_{rc}}{3} \tag{10}$$

[7] This assumption comes from the observation that, when defaults are equally likely at all times for both the counterparty and the reference entity and there is no default correlation, the average time between the two defaults is one third of the life of the CDS.

Equations (8), (9), and (10) suggest the following analytic approximation for the CDS spread is

$$s = \hat{s}\frac{1 - 0.5P_{rc}/Q_r}{1 - Q_c/2 + P_{rc}/3} \tag{11}$$

This result incorporates many courageous assumptions and approximations. It assumes that the probability of default by the reference entity is constant through the life of the CDS; it assumes that the probability of default by the counterparty is constant throughout the life of the CDS; it does not consider the impact of discounting effects on g and h; it does not consider the impact of correlation on the relative timing of defaults by the reference entity and the counterparty; it ignores payment accrual issues; and so on. A much more complex analytic approximation would be required to deal with some of these points.

As a test of equation (11) we used the default correlations in Table 3 to estimate the numbers in Table 4. The results are shown in Table 5. For the range of situations considered, the approximation appears to work reasonably well when the correlation is not too large. For example, when the credit index correlation is 0.4 or less, the analytic approximation is accurate to within one basis point. However, we stress that equation (11), and similar more complicated analytic approximations, can be used only when default correlations have already been estimated in some way.

3. Basket Credit Default Swaps

In a basket credit default swap a number of different reference entities and reference obligations are specified. The buyer makes payments in the usual way. The first reference entity to default triggers a payoff, either in cash or by physical delivery. As in the case of a regular CDS, the payoff typically equals $1 - R - A(t)R$ where R and $A(t)$ are the recovery rate and the accrued interest on the reference obligation for the defaulting reference entity. There are then no further payments or payoffs. As in the case of a vanilla credit default swap, a final accrual payment is usually required when there is a default.

When there is zero correlation between the reference entities and no counterparty default risk, a similar approach to Hull and White (2000) can be used to value a CDS or calculate the CDS spread. If $Q_{r,j}(t)$ $(1 \le j \le N)$ is the cumulative probability of the jth reference entity defaulting by time t, the probability of the first default happening between times t_1 and t_2 is

$$\prod_{j=1}^{N}[1 - Q_{r,j}(t_1)] - \prod_{j=1}^{N}[1 - Q_{r,j}(t_2)]$$

When the correlation between reference entities is non-zero, it is necessary to use a model such as the one we have introduced in this paper to value a basket credit default swap. We redefine variables as follows:

$\theta(t)\Delta t$: Risk-neutral probability of the first default by a reference entity happening between t and $t + \Delta t$ and no earlier default by the counterparty.

$\phi(t)\Delta t$: Risk-neutral probability of the counterparty defaulting between times t and $t+\Delta t$ and no earlier default by any of the reference entities.

π: The risk-neutral probability of no default by the counterparty or any of the reference entities during the life of the CDS swap.

$\hat{R}$: The expected recovery rate on a relevant reference obligation after first default

$A(t)$: Expected accrued interest as a percent of notional principal on relevant reference obligation, conditional on the first default happening at time t

Equations (6) and (7) then apply.

A basket CDS spread is calculated by evaluating both the numerator and denominator in equation (7) using Monte Carlo Simulation. The credit index for all reference entities and the counterparty must be simulated. If a reference entity defaults first (that is, the credit index for a reference entity falls below its default barrier before the credit index for the counterparty does so), payments continue up to the time of default with a final accrual payment and there is a payoff. If the counterparty defaults first (that is, the credit index for the counterparty falls below its default barrier before the credit index of any of the reference entities does so), payments continue up to the time of the default with no final accrual payment and no payoff. If the creditworthiness indices for the counterparty and all reference entities stay above their respective default boundaries, payments continue for the life of the basket credit default swap and there is no payoff.

Table 6 shows results for a five-year basket credit default swap with semiannual payments where the counterparty is default-free. All reference entities are the BBB-rated companies and the correlations between all pairs of reference entities are assumed to be the same. All reference obligations are assumed to be five-year bonds with 10% coupons and a 30% expected recovery rate. The table shows that the basket CDS spread increases as the number of reference entities increases and decreases as the correlation increases. The spread decreases as the expected recovery rate decreases. Furthermore the dependence of the spread on the recovery rate becomes more pronounced as the number of reference entities increases. For example, when there are 50 reference entities and the correlation is zero, the spread is 7,799 basis points when the expected recovery rate is 0.1, 7,317 basis points when the expected recovery rate is 0.3, and 6,527 basis points when the expected recovery rate is 0.5.

4. Conclusions

In this paper we have introduced a flexible tool for modeling default correlations. We assume the creditworthiness of a companies can be defined by a credit indices. These indices start at zero and follow correlated Wiener processes. When the credit index of a company reaches a barrier, a default occurs. The barrier is chosen so that the model is exactly consistent with the default probabilities backed out from bond prices.

We have used the model to investigate the impact of counterparty default risk on the value of a vanilla CDS swap. This impact is small when the correlation between the counterparty and the reference entity is zero. It increases as the correlation increases and the creditworthiness of the counterparty declines. We have also used the model to estimate basket CDS spreads. We find that these spreads increase as the number of reference entities in the basket increases and decrease as the correlation between them increases. They also increase somewhat as the expected recovery rate decreases.

The model as developed here can be used to value any credit derivative when the payoff depends on defaults by one or more companies. The model can be extended to incorporate multiple barriers so that it can accommodate instruments that provide payoffs in the event of credit rating changes.

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Table 1
Spreads in Basis Points Between Corporate Bond Yields and Treasury Bond Yields*

Maturity	Credit Rating AAA	AA	A	BBB
1	50	70	100	160
2	52	72	105	170
3	54	74	110	180
4	56	76	115	190
5	58	78	120	200
10	62	82	130	220

*We assume Treasury zero rates are flat at 5% with semiannual compounding, corporate bonds pay semiannual coupons at the rate of 7% per annum, the expected recovery rate is 30%, and the bondholders claim in the event of a default is the face value of the bond plus accrued interest.

Table 2
Default Probability Density for a BBB-rated Company

Time of Default (yrs)	Default Probability Density
0–1	0.0219
1–2	0.0242
2–3	0.0264
3–4	0.0285
4–5	0.0305
5–10	0.0279

Table 3
Default Correlation of a BBB-Rated Company with a Second Company as a Function of the Credit Index Correlation, the Second Company's Credit Rating and the Length of the Time Period

Time Period (yrs)	Credit Index Correlation	Credit Rating AAA	AA	A	BBB
2	0.0	0.00	0.00	0.00	0.00
	0.2	0.03	0.04	0.04	0.05
	0.4	0.09	0.10	0.11	0.12
	0.6	0.19	0.21	0.22	0.24
	0.8	0.35	0.37	0.40	0.43
5	0.0	0.00	0.00	0.00	0.00
	0.2	0.06	0.06	0.07	0.08
	0.4	0.14	0.15	0.16	0.18
	0.6	0.24	0.26	0.29	0.31
	0.8	0.39	0.42	0.47	0.50
10	0.0	0.00	0.00	0.00	0.00
	0.2	0.08	0.08	0.10	0.10
	0.4	0.17	0.18	0.21	0.22
	0.6	0.28	0.30	0.34	0.36
	0.8	0.41	0.45	0.51	0.55

Table 4
Credit Default Swap Spreads in Basis Points for Different Counterparties and Different Correlations Between the Credit Index of the Counterparty and the Credit Index of the Reference Entity.*

Credit Index Correlation	Counterparty Credit Rating AAA	AA	A	BBB
0.0	194.3	194.3	194.3	194.1
0.2	191.3	190.4	188.7	186.0
0.4	187.4	185.2	181.3	174.6
0.6	182.3	178.3	171.1	158.7
0.8	177.1	170.5	156.8	134.1

*The CDS Spread for a Default-Free Counterparty is 194.4 bps. The contract is on a BBB-rated reference entity, lasts for five years, and requires semiannual payments. The reference obligation is a five-year bond paying a 10% coupon with a 30% expected recovery rate. Standard errors of estimates are less than 0.2 basis points.

Table 5
Estimates of the CDS Spreads in Table 4 Using the Analytic Approximation in Equation (11)

Credit Index Correlation	AAA	AA	A	BBB
0.0	194.0	193.9	193.7	193.2
0.2	191.0	190.2	188.3	185.6
0.4	186.7	184.8	181.3	175.8
0.6	181.0	177.7	171.7	163.2
0.8	173.5	168.1	158.5	145.3

Table 6
Basket Credit Default Swap Spreads
When Reference Entities are all BBBs*

Expected Rec. Rate	Credit Index Correlation	Number of Reference Entities 1	2	5	10
0.1	0.0	196	390	959	1877
	0.2	196	376	848	1492
	0.4	196	357	730	1174
	0.6	196	332	604	888
	0.8	196	296	460	608
0.3	0.0	194	386	946	1842
	0.2	194	371	826	1441
	0.4	194	351	707	1122
	0.6	194	325	582	844
	0.8	194	289	444	580
0.5	0.0	192	380	925	1779
	0.2	192	363	794	1366
	0.4	192	342	672	1050
	0.6	192	315	551	786
	0.8	192	280	420	542

*Life of CDS is 5 years. There is no counterparty default risk. The credit index correlation for all pairs of reference entities is the same. Standard errors of estimates are less than 1 basis point.

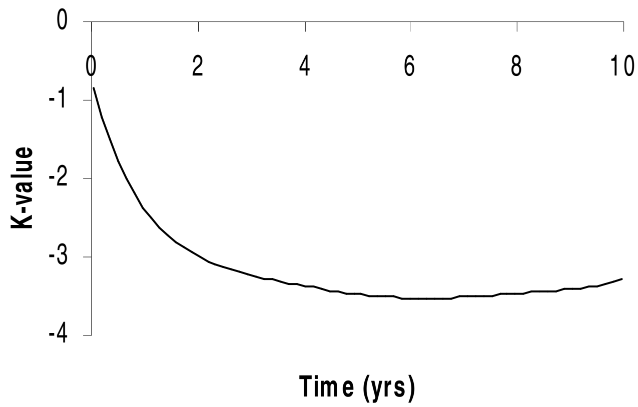

Figure 1

Default Boundary for Data in Table 2