A Model of Credit Risk, Optimal Policies, and Asset Prices*

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This revision: October 2000

*We would like to thank Kose John, Paolo Pasquariello, Rangarajan Sundaram, and the seminar participants at New York University, University of Iowa, and the Wharton School for their helpful comments. We have also benefited greatly from discussions with Anna Pavlova and her extensive comments. All errors are solely our responsibility. Address correspondence to Alex Shapiro, Department of Finance, Stern School of Business, 44 West 4th Street, Suite 9-190, New York, NY 10012-1126.

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Abstract

This paper studies the optimal policies of borrowers (firms or individuals) who may default subject to default costs, and analyzes the asset pricing implications. Borrowers defaulting under adverse economic conditions may, despite incurring default costs, emerge as wealthier than non-borrowers or those who can default costlessly. In many scenarios, borrowers take on less risk exposure than non-borrowers. However, a larger risk exposure by borrowers may occur as well, depending on the structure of default costs and on how debt maturity relates to the planning horizon. In the latter case, borrowers' default policies render binary options useful instruments for lenders in hedging the credit-risk component of their assets. In equilibrium, a lower (higher) risk exposure by borrowers manifests itself in an attenuated (amplified) market volatility and risk premium, but the market value is always higher in economic downturns, and lower in upturns, compared to an economy without the presence of credit risk.

JEL Classifications: G33, G11, G12, C61, D51.

Keywords: Credit Risk, Defaultable Debt, Investments, Asset Pricing, Volatility.

1. Introduction

Corporate and household borrowing has reached record proportions and pace in recent years, to more than triple the size of the U.S. government debt. This dominance of credit-risky debt – on which the borrower has the option to default – is likely to prevail into the future, given the U.S. Treasury policy of cutting on its debt load.[1] From a financial economics perspective, these historic trends raise the need for conceptual frameworks able to link the credit quality of borrowers to underlying economic primitives, as well as able to advance our understanding of the associated optimal policies and asset prices. Moreover, the challenge to better understand borrowers' decisions to default has been recently invigorated by regulators' quest to formally embed models of credit-risk into bank-capital requirements (Basle Committee on Banking Supervision (1999)).

In this paper, we investigate the optimal behavior of a borrower who is allowed to default, and study the ensuing dynamic equilibrium in the presence of this credit risk. We model a borrower (a levered firm or household) within a standard complete-markets continuous-time economy. To maintain as simple a setting as possible, we take as given a zero-coupon debt contract in place, asserting that upon its maturity default may occur. Motivated by observed violations of the strict priority rule, default occurs whenever the fraction of the (levered) assets that can be seized by the lender cannot repay the face value of the debt.[2] The dynamics of the assets are optimally controlled by the borrower. Credit risk then means that in some states of the world the borrower optimally chooses to repay less than the face value, and the debt is thus equivalent to a riskless contract plus a credit-risk component, specifying in which states, and to what extent, the repayment deviates from the face value.

We choose perhaps the most natural imperfection for default to matter economically: default is costly.[3] Upon default, the borrower in our model incurs a fixed cost as well as a cost proportional to the amount of default. Our setting is amenable to analyzing many quantities of interest, and this is facilitated by treating costs in a reduced form, while abstracting from mechanisms that give rise to such costs. Our formulation considers a borrower with an increasing and concave objective function (representing risk averse preferences as a special case), and has the convenient property of nesting the benchmark case of no debt or no default costs (Merton (1971), Cox and Huang (1989)).

We first consider a borrower whose planning horizon coincides with the maturity of the debt.

[1] At the outset of the millennium, out of the \$17.7 trillion in domestic debt (not including the \$7.8 trillion financial sector) only \$3.6 trillion was federal debt (see the 6/9/2000 Federal Reserve's release Z.1 at www.federalreserve.gov), and the Treasury Department resolved to further reduce this debt (www.ustreas.gov/press/releases/ps390.htm). For a concurrent discussion of the data, see the *Wall Street Journal* article "Debtor Nation" on 7/5/2000, p. C1.

[2] For more on departures from the strict priority rule (stating that debtholders' claims must be fully satisfied prior to distributing any value to equityholders), see, e.g., Warner (1977), Franks and Torous (1989), and Eraslan (1999).

[3] Costs of financial distress have been widely documented; see the references in footnote 2, and also, e.g., Altman (1984) and Andrade and Kaplan (1998).

Under general investment opportunities, the borrower's optimal terminal net worth falls into three regions, in which it exhibits distinct economic behavior: no-default, default, and in between, an extended region of default-resistance. In good states, the borrower does not default and the net worth resembles the benchmark policy. In unfavorable, intermediate states, the borrower strives to not default, to avoid default costs, and the net worth is maintained at a default-resistance level determined by the (constant) default boundary. However, in the worst states of the world, resisting default becomes too costly, and the borrower chooses to default. Fixed default costs extend the resistance region, and introduce a wedge between the default-boundary and the optimal wealth upon default. Once fixed costs are incurred, the borrower's behavior across states reverts to a benchmark-type policy, regardless of the amount of default. On the other hand, facing proportional costs "bumps up" the optimal wealth across the default region. Interestingly, upon default the borrower's net worth may indeed be higher than that of a non-borrower, or of a borrower that does not incur default costs.

Under an isoelastic objective function and lognormal state prices, the dynamic investments of a borrower reveal the optimal risk exposure to be lower than in the benchmark, in many scenarios of interest. This is due to the borrower's overall reliance on riskless investments to finance the default-resistance level, as well as to finance the costs imminent upon default. However, with fixed costs present, when the probability of default is high (but not high enough to categorically eliminate solvency), the borrower may take on a larger risk exposure (and more so on approaching the horizon) than in the benchmark. This large risk exposure, driven by the fixed-costs wedge, intends to finance the relatively high level of wealth at the default boundary, should economic conditions turn favorable. The latter behavior, viewed across the state space, translates into the credit-risk component of the debt contract being a portfolio of a put option plus a binary option. Therefore, barring our abstraction from issues of incompleteness and trading costs, our analysis suggests that, beyond the generic usefulness of binary instruments (Ingersoll (2000)), binary options triggered by default events (or by indicative economic fundamentals) may have an economic role in facilitating effective hedging of portfolios exposed to credit-risk.

When the debt matures prior to the planning horizon, the borrower's optimal wealth upon debt-maturity inherits the main features of the case where default coincides with the planning horizon. We obtain additional implications arising from the path-dependent nature of the optimal policy at the planning horizon. For example, the borrower's planning-horizon wealth is shown to be higher if default had occured compared to no default, in the presence of proportional default costs, all else being equal. This is because of the upward-shifting effect that proportional costs have on the borrower's wealth upon default. However, prior to debt-maturity, the risk exposure of a borrower is always lower than in the benchmark. This holds regardless of fixed costs, because when the debt matures prior to the planning horizon, there is no incentive to make large risky bets to avoid the charge of fixed costs, as these costs have no immediate impact on the planning-horizon wealth.

Finally, to investigate the impact on the economy of the prevalence of credit risk, we perform a general-equilibrium analysis. We examine the equilibrium market and its dynamics in a Cox, Ingersoll, and Ross (1985)-type production economy (with a riskless technology), populated by a representative borrower and a representative lender, bound by the zero-coupon debt contract. The lender faces no frictions, and hence behaves as in the benchmark case. We find that prior to debt maturity, in bad states the equilibrium market price is increased in the presence of credit risk, while in good states the market price is decreased. This is because the borrower shifts wealth from good to bad states in striving to meet debt obligations and reduce default costs. Since the presence of default costs induces the borrower to reduce risk exposure in many scenarios, in the examined economy the aggregate investment in risky technologies is reduced as well, while the investment in the riskless technology is increased. The market then becomes less risky, resulting in lower market volatility and risk premium. This is consistent with a related argument in the literature asserting that firms will hedge cash flows in the presence of fixed default costs (see, e.g., Smith and Stulz (1985) and Allen and Santomero (1998)). We complement this argument by a formal equilibrium analysis, and moreover, establish that the volatility is always reduced when default costs are proportional to the amount of default. Somewhat surprisingly, we demonstrate that in the presence of fixed costs, and maturity coinciding with the planning horizon, the opposite may also occur: high-risk investments by borrowers, and hence increased market volatility compared to an economy with no leverage or no default costs.

Our modeling approach relies on an endogenously determined asset-value dynamics, and thus differs considerably from the two major approaches in the asset-pricing literature that deal with credit risk: the "structural" option-based approach, with exogenous asset-value dynamics, stemming from Merton (1974) [and having numerous extension incorporating realistic features such as deviations from the strict priority rule, taxes, or strategic considerations (e.g., Leland (1994), Longstaff and Schwartz (1995), Anderson and Sundaresan (1996))]; and the more recent "reduced form" approach, where default events are specified by an exogenous process (see, e.g., Jarrow and Turnbull (1995), Duffie and Singleton (1999)). Our framework not only allows us to analyze optimal policies and asset prices, but also offers another potentially tractable alternative to pricing various defaultable instruments.

A related line of work examines how equilibrium is affected when borrowers or lenders, with concave optimization, face missing markets or constraints. With different focus, this work emphasizes imperfections and employs economic settings different from ours. Zame (1993) and Dubey, Geanakoplos, and Shubik (1996) study static equilibrium models with utility-penalizing default costs, and demonstrate that market incompleteness provides a role for default in promoting efficiency. Zhang (1997) and Alvarez and Jermann (2000) analyze dynamic models with stochastic income and solvency constraints, in which the possibility to revert to autarky upon default affects the economy, but there is no default in equilibrium. Allen and Gale (2000) and Chang and Sundaresan (2000) (CS) consider models where lenders are restricted from any investment activity, except for initial (welfare-improving) lending, and default occurs in equilibrium. CS's analysis and ours are complementary in that SC also consider a continuous-time setting, but working within an infinite-horizon environment they focus on a perpetual debt contract, upon which the borrower may default and revert to autarky. These models offer many important insights, but are limited to qualitative guidance or must resort to numerical solutions if default indeed occurs in equilibrium.

In order to focus on the ubiquitous imperfection of costs being associated with default, our modeling approach differs from the aforementioned equilibrium models in that we let both borrowers and lenders operate within complete markets, without constraints on investments. Despite the fundamental structure of our setting and the evident realism of the examined imperfection, such an analysis, to our knowledge, has not been performed in the literature. Consequently, unlike the related equilibrium work, we choose to build our general equilibrium analysis upon our closed-form partial-equilibrium results. We thus illustrate how, with borrowers alone affected by the costs of default, the insights gained in partial equilibrium survive in an aggregate setting. Moreover, by employing state prices defined by the given investment opportunities throughout our analysis, we gain not only analytical tractability but also practical implementability, allowing for a variety of extensions, as we illustrate. Our approach can be therefore viewed as filling a gap between the large partial-equilibrium/valuation literature analyzing structural and reduced-form models of credit risk, and the growing literature studying how the presence of borrowing affects general equilibrium.

The paper is organized as follows. Section 2 describes the economic setting. Section 3 solves the optimization problem of a borrower with debt maturing at the planning horizon, and Section 4 analyzes the case of debt maturing prior to the planning horizon. Section 5 provides the equilibrium analysis. Section 6 extends the setting to repeated borrowing, and also shows how our model may be applied to credit-risk management. Section 7 concludes the paper. Proofs are in the appendix.

2. The Economic Setting

2.1 The Economy

We consider a finite-horizon, $[0, T']$, economy with a single consumption good (the numeraire). Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, on which is defined an

N-dimensional Brownian motion $w(t) = (w_1(t), \ldots, w_N(t))^\top$, $t \in [0, T']$. All stochastic processes are assumed adapted to $\{\mathcal{F}_t; t \in [0, T']\}$, the augmented filtration generated by w. All stated (in)equalities involving random variables are understood to hold P-almost surely. In what follows, given our focus is on characterization, we assume all stated processes to be well-defined, without explicitly listing the regularity conditions ensuring this.[4]

We assume there are $N+1$ investment opportunities: one instantaneously riskless and the remainder risky. The vector of instantaneous net returns on the investment opportunities follows the dynamics

$$\begin{pmatrix} r(t)dt \\ \mu(t)dt + \sigma(t)dw(t) \end{pmatrix}, \tag{1}$$

where the interest rate r, the drift coefficients $\mu \equiv (\mu_1, \ldots, \mu_N)^\top$, and the volatility matrix $\sigma \equiv \{\sigma_{ij},\, i = 1, \ldots, N;\, j = 1, \ldots, N\}$ are possibly path-dependent.

Dynamic market completeness (under no-arbitrage) implies the existence of a unique state price density process, ξ, given by

$$d\xi(t) = -\xi(t)[r(t)dt + \kappa(t)^\top dw(t)] \ , \tag{2}$$

where $\kappa(t) \equiv \sigma(t)^{-1}(\mu(t) - r(t)\bar{1})$ is the market price of risk process, and $\bar{1} \equiv (1, \ldots, 1)^\top$. The quantity $\xi(T', \omega)$ is interpreted as the Arrow-Debreu price per unit probability P of one unit of consumption good in state $\omega \in \Omega$ at time T'. Without loss of generality, we set $\xi(0) = 1$.

The economy is populated by a borrower, b, and a lender, ℓ, (bound by a zero-coupon fixed-maturity debt contract described below in Section 2.2), each endowed with an initial wealth of $W_n(0)$, $n = b, \ell$. (Since our focus until Section 5 is on the optimal behavior of a single borrower, we drop, for now, the subscript n.) The borrower (lender) chooses a nonnegative, planning-horizon wealth, $W(T')$, representing terminal net worth, and an investment policy, θ, where $\theta(t) \equiv (\theta_1(t), \ldots, \theta_N(t))^\top$ denotes the vector of fractions of wealth invested in each risky investment opportunity.[5] The wealth process W before (and, when relevant, after) the debt-maturity date then follows

$$dW(t) = W(t)\left[r(t) + \theta(t)^\top(\mu(t) - r(t)\bar{1})\right]dt + W(t)\theta(t)^\top\sigma(t)dw(t) \ . \tag{3}$$

Prior to debt maturity, the total value of the assets, managed by the borrower, V, is endogenously determined, and is given by $V(t) \equiv W(t) + D(t)$, where D is the value of the debt.

[4] See, for example, Karatzas and Shreve (1998) for the required integrability conditions on the quantities to be introduced in this section, as well as the associated Novikov's condition. In the equilibrium constructed in Section 5, these conditions can be shown to be satisfied.

[5] We do not impose constraints, such as short selling, on θ, because for simplicity, we (implicitly) assume the availability of financial instruments to implement investment policies and, if necessary, to circumvent physical constraints on investments. Clearly, the parameters in (1) can be restricted so that particular constraints are never binding, and the solution is unaffected. We also abstract away from considerations of defaultability associated with the given investment opportunities in order to focus on default in the context of a particular contract (Section 2.2).

The borrower (lender) maximizes the expected value of $v(W(T'))$. The function $v(\cdot)$ is assumed twice continuously differentiable, strictly increasing, strictly concave, and to satisfy the Inada conditions: $\lim_{x\to 0} v'(x) = \infty$ and $\lim_{x\to\infty} v'(x) = 0$. A concave objective function renders our analysis widely applicable as it allows us: to represent the objective function of any utility-maximizing agent (as in, e.g., Zame (1993), Alvarez and Jermann (2000), Chang and Sundaresan (2000)); to incorporate, in a reduced form, the presence of managerial self-interest (as argued, e.g., by Stulz (1984), Allen and Santomero (1998)) and/or concavified compensation structures (as advocated, e.g., by John and John (1993), John, Saunders, and Senbet (2000)); or to capture a concave non-stochastic investment opportunity faced by an expected-shareholders'-value maximizing firm beyond the modeled horizon (similarly to, e.g., Froot, Scharfstein, and Stein (1993), Froot and Stein (1998)). In the sequel, for expositional convenience, we sometimes emphasize results by adopting for the borrower the interpretation of a levered *firm*, but our results are equally valid for an individual borrower/household.[6]

2.2 Modeling the Debt Contract and the Costs of Default

Our objective is to examine, in as simple a setting as possible, how the possibility of costly default affects optimal policies of the borrower (who controls the dynamics of the assets V), and to study the implication of this optimal behavior for aggregate quantities in the economy of Section 2.1. We would like to capture two observed phenomena associated with defaultable debt contracts. First, upon default, the lender is only able to seize a fraction of the borrower's assets, which is reflected in violations of the strict priority rule. Second, a borrower may default despite having sufficient funds to service the debt. To this end, we assume a given debt contract in place between the borrower and the lender, where the contract structure is specified as follows:

Assumption 1. (Debt Contract) *The payoff of a zero-coupon debt contract with face value F and maturity date $T \leq T'$ is $D(T) = \min\{(1-\beta)V(T), F\}$, where $0 \leq \beta \leq 1$.*

The contract asserts that *default* occurs at the debt-maturity date $T \leq T'$ whenever the face value is not repaid in full, $D(T) = (1-\beta)V(T) < F$, implying solvency for $W(T) \geq \frac{\beta F}{1-\beta}$, and we refer to $\frac{\beta F}{1-\beta}$ as the *default boundary.* The *recovery rate* β captures in reduced form the two aforementioned observed phenomena (see also Remark 1(iii)). One natural way to interpret our debt-contract formulation is to note that any borrower's assets are made up of tangible and intangible components. The fraction of assets seizable by the lender, $(1-\beta)V$, represents then the

[6] We assume the objective function $v(\cdot)$ to satisfy the Inada conditions, which are standard for an individual borrower, to maintain compatability with the benchmark investment-choice model with no debt. Inada conditions are also standard for neo-classical production functions. However, none of our qualitative results rely on this assumption. Increased concavity of $v(\cdot)$ may be mapped to higher risk aversion, more self-interested management, or more pronounced features of the post-horizon investment opportunity.

tangible, collateralizable part, which is the liquidation value of the assets (for any value of V). The intangible, noncollateralizable part, βV, represents borrower-specific intangible assets such as human capital and organizational knowledge base. Prior to debt maturity, the intangible assets, $\beta V(t)$, are an integral part of total assets, and are fully capitalized in our complete markets setting.[7] But according to this interpretation, once $(1-\beta)V(T)$ is seized, the intangible part can no longer be capitalized (yet it is valuable to the borrower, i.e., can be used by the borrower to generate future cash flows). Note that our formulation conforms to the traditional approach (as in Merton (1974)) to model defaultable (discount) debt. In particular, default may occur only at a deterministic date, T, when the debt matures, where this date is fixed to precede, or coincide with, the planning horizon, T'. Section 3 examines the case of $T = T'$, and Section 4 the case of $T < T'$.

It is undisputed that, in general, corporate or personal default is costly (due to impaired business reputation and stigmatization, unfavorable asset-liquidation terms, or other direct expenses). We focus on the following structure of the costs associated with default:

Assumption 2. (Default Costs) *Upon default* $(D(T) < F)$*, the borrower incurs the costs* $C(T) = \phi + \lambda(F - D(T))$*, where* $\phi \geq 0$*,* $\lambda \geq 0$*. Otherwise* $(D(T) = F)$*,* $C(T) = 0$*.*

To capture essential components of default costs, our cost structure combines the two primary types of costs that have been discussed in the literature; we allow for a *fixed-costs* component ϕ (similarly to Chang and Sundaresan (2000)), and for a component proportional to the amount of default $(F - D(T))$, where λ is the *proportional cost* per unit of default.[8] These costs may more generally be interpreted as financial distress costs (that include, e.g., lost business and wasted managerial resources) incurred when the borrower is in danger of defaulting (Bodie and Merton (2000, pp. 429-430)). The financial-distress region (as a function of V) can be defined explicitly by financial ratios within debt covenants (in our setting $\frac{V(T)}{F} < \frac{1}{1-\beta}$), or implicitly by market's perception of distressed financial ratios. From Assumptions 1 and 2 it is clear that the borrower may default and incur costs while having $V(T) > F$. This is consistent, for example, with our tangibles-plus-intangibles interpretation of total assets, where the borrower cannot liquidate the intangibles and hence costs cannot be avoided. More generally, when costs are interpreted as costs

[7] McGrattan and Prescott (2000) estimate that productive intangible assets in the U.S. are valued at roughly 40% of gross national product, which is translated into about 20% of capitalized *aggregate* corporate equity, and conceivably into a larger fraction of market value within some industries.

[8] Zame (1993) and Dubey, Geanakoplos, and Shubik (1996) incorporate proportional costs in units of utility, whereas we model costs in units of the numeraire. Anticipating future results, each of the two types of costs that we employ indeed affects the optimal behavior in a different manner. It can be shown that Assumptions 1 and 2 are equivalent to the cost function, $C(\cdot)$, being $C(W(T)) = [\frac{\beta(\phi+\lambda F)}{\beta+\lambda(1-\beta)} - \frac{\lambda(1-\beta)}{\beta+\lambda(1-\beta)} W(T)] 1_{\{W(T) < \frac{\beta F}{1-\beta} - \phi\}}$, where for $\phi > 0$, $W(T)$ never takes values in the $[\frac{\beta F}{1-\beta} - \phi, \frac{\beta F}{1-\beta})$ interval. (For $(\phi, \beta, F) > 0$, this structure imposes the restriction that $\phi < \frac{\beta F}{1-\beta}$.) Clearly, for $\phi > 0$, the cost function is discontinuous in $W(T)$ and is not convex on $\mathcal{R}$, and hence it is not amenable to a straightforward treatment by existing techniques (such as in Liu (1998)). Even for $\phi = 0$, nonconvexity of $C(\cdot)$ can arise under alternative specifications of costs, for example, as in Remark 1(ii).

of financial distress, these are incurred regardless of debt service (even if face value is repaid), and our debt-contract formulation then captures, in reduced form, scenarios where lenders are only able to seize assets valued less than F (perhaps due to intangibility, but possibly due to other reasons such as bargaining between different stakeholders).[9]

Our formulation nests the benchmark investment model (henceforth B) with no debt (Merton (1971), Cox and Huang (1989)). Specifically, when $F = 0$, there is no debt ($V \equiv W$) and the optimal solution is the B-model wealth, $W^B(T')$. Moreover, when $\beta = 0$, to satisfy the Inada conditions, the borrower never defaults ($V(T) > F$ guaranteeing $W(T') > 0$) and again $W^B(T')$ is optimal. In a third extreme, $\phi = \lambda = 0$, default is costless, and although it may occur it does not impact the borrower, who can thus still finance the optimal policy $W^B(T')$. Therefore, in the latter case, the face value F, and the recovery rate β affect only the value of the debt, and hence V but not W, and leverage (W/V) has no impact on how the borrower's net worth is invested. Given our interest to focus on borrowers' wealth, we collectively refer to the above three cases (although differing in V) as the benchmark.

Remark 1. **(Alternative Modeling of Debt and/or Costs):** Consider (i) Debt payoff given by $D(T) = \min\{(1-\beta)(V(T) - C(T)), F\}$; (ii) Larger default costs for larger asset base: $C(T) = \lambda V(T)$, when $D(T) < F$; (iii) Borrower's recovery rate is given by β_1, while the default region is parameterized by β_2, $0 \leq \beta_2 < \beta_1 \leq 1$: $D(T) = (1-\beta_1)V(T)$ if $V(T) < \frac{F}{1-\beta_2}$, otherwise $D(T) = F$. We adopt the formulation in Assumption 1, instead of (i), to clarify at the outset that the borrower alone bears all costs, and that these costs do not necessarily represent immediate expenses; default affects future business and financing opportunities, so that although $\beta V(T)$ is retained upon default, it cannot be "consumed" entirely. Moreover, our formulation lends itself to more convenient comparisons with the benchmark. We will demonstrate (see Remark 2) that specifying the debt as in (i), employing alternative cost structures as, e.g., in (ii), as well as capturing by two separate parameters the two aforementioned default-related observed phenomena does not qualitatively change the insights gained from our setting.

3. Optimization when Planning-Horizon Default is Allowed

In this section, we solve the optimization problem of a borrower bound by a debt contract maturing at the planning horizon ($T = T'$), where the borrower may choose to default at T, subject to default costs. We then analyze the properties of the solution.

[9] Yet another alternative interpretation of the borrower's defaulting despite having $V(T) > F$, is that the borrower faces various imperfections and costs (such as costs of immediacy), and hence chooses to default, and incur default costs that are still lower than some other (not modeled here) costs that would have been incurred had the borrower attempted to fully repay F.

3.1 Borrower's Optimization

We solve the dynamic optimization problem of the borrower using the martingale representation approach (Cox and Huang (1989), Karatzas, Lehoczky, and Shreve (1987)), which allows the problem to be restated as the following static variational problem:

$$\begin{aligned} & \max_{W(T)} E[v(W(T))] \\ \textit{subject to} \quad & E[\xi(T)(W(T)+C(T))] \leq W(0) \ , \end{aligned} \tag{4}$$

where the costs of default $C(T)$, the terminal net worth $W(T)$, and the associated total-assets value $V(T)$ satisfy Assumptions 1 and 2. The budget constraint states that initial wealth must be sufficient to cover the value of terminal wealth plus potential costs.[10] We note that the optimization problem in (4) is nonstandard, as it is complicated by the nonlinearity and discontinuity in the cost structure, introducing not only nonconcavity into the objective, but also nonconvexity into the budget constraint. Proposition 1 characterizes the optimal solution, assuming it exists.[11]

Proposition 1. *When debt maturity coincides with the borrower's planning horizon* ($T = T'$)*, the borrower's optimal terminal net worth is*

$$W^*(T) = \begin{cases} I(y\xi(T)) & \textit{if } \ \xi(T) < \xi_* \quad : \textit{no-default}, \\ \frac{\beta F}{1-\beta} & \textit{if } \ \xi_* \leq \xi(T) < \xi^* \quad : \textit{default-resistance}, \\ I\left(\frac{\beta y}{\beta+\lambda(1-\beta)}\xi(T)\right) & \textit{if } \ \xi^* \leq \xi(T) \quad : \textit{default}, \end{cases} \tag{5}$$

where $I(\cdot)$ *is the inverse function of* $v'(\cdot)$, $\xi_* \equiv v'(\frac{\beta F}{1-\beta})/y$, *and* $\xi^* \geq \xi_*$, $y \geq 0$ *solve the following system:* $v(I(x\xi^*)) - v(I(y\xi_*)) = x\xi^*(I(x\xi^*) - I(y\xi_*) + \phi)$ *with* $x \equiv \beta y/(\beta + \lambda(1-\beta))$, $E[\xi(T)(W^*(T;y) + C(W^*(T;y)))] = W(0)$.

Consequently:

(i) If $W(0) = W^B(0)$, *then* $y \geq y^B$.

(ii) $W^*(T) \leq W^B(T)$ *under no-default, and for* $\lambda = 0$ *under default.*

However, under default for $\lambda > 0$, *we may have* $W^*(T) > W^B(T)$.

(iii) For $\phi = 0$, $\xi^* = \xi_*(\beta + \lambda(1-\beta))/\beta$.

[10] In fact, it follows that for $t < T$, $W(t) = \frac{1}{\xi(t)}E[\xi(T-)W(T-)|\mathcal{F}_t] = \frac{1}{\xi(t)}E[\xi(T)(W(T)+C(T))|\mathcal{F}_t]$. Consequently, $V(t) \equiv W(t)+D(t)$, and $V(T) \equiv W(T-)+D(T) = W(T)+C(T)+D(T)$. Note that since the debt contract introduces nonconcavity into the objective, our problem appears related to the case where nonconcavity is introduced into a fund manager's objective via a call-option type compensation (see, e.g., Carpenter (2000)). However, because in our formulation the debt contract is accounted for in the budget constraint, in the absence of default costs the benchmark solution is obtained, whereas the fund-manager's problem leads to an all-or-nothing two-region solution.

[11] We prove that if a terminal wealth satisfies (5) then it is the optimal policy for the borrower. We will provide explicit numerical solutions for a variety of parameter values.

Figure 1 depicts the optimal terminal net worth of the borrower and illustrates how it may relate to the B-policy.

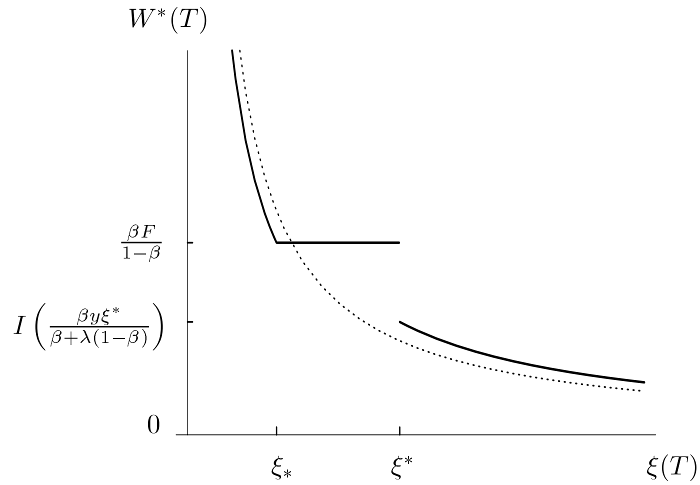


Figure 1: The borrower's time-T optimal wealth, $W^*(T)$ (solid plot), and the time-T B-policy, $W^B(T) = I(y^B\xi(T))$ (dotted plot), when debt maturity coincides with the borrower's planning horizon ($T = T'$).

Figure 1 reveals the borrower to exhibit three distinct patterns of economic behavior, mapped into three regions of the state space: no-default, default, and in between an extended region of default-resistance (or "resistance" for brevity). In the latter, the borrower resists default and the target wealth does not change upon maturity in response to deteriorating economic conditions (represented by increasing $\xi(T)$). In the no-default "good" states (low $\xi(T)$), the borrower behaves like in the B-case, while not defaulting on the debt obligation. However, unfavorable states ($\xi(T)$ above ξ_*) are endogenously classified into two subsets: the default "bad" states ($\xi(T) \geq \xi^*$), in which the borrower defaults, and the default-resistance "intermediate" states ($\xi_* \leq \xi(T) < \xi^*$), in which the wealth level is maintained at the default boundary.[12] Hence the probability of default is endogenously set by the choice of ξ^* to equal the probability mass of the states where $\xi(T) \geq \xi^*$.

The optimal behavior is driven by the undesirability of costly default. The default-resistance region then arises due to the asymmetry of the cost structure across the state space.[13] Specifically,

[12] In the equilibrium analyzed in Section 5, we will show that the no-default "good" states, low price of consumption good $\xi(T)$, are associated with a higher market value than in the default "bad" states, high $\xi(T)$. We reserve the label "no-default" for the region to the left of ξ_*, where $W^*(T)$ is strictly above the default-boundary value, although the borrower does not default in the intermediate region as well.

[13] The separation of the state space into three regions, with a discontinuity of the optimal policy across states, is also obtained by Basak and Shapiro (2000) in a different context – a risk management analysis. However, the optimal policy in (5) is distinctly different from theirs, and unless additional parametric restrictions are imposed (e.g., as in Section 6.2), the policy in general will not comply with a particular risk management requirement. Default-resistance over an extended region, in which a borrower neither defaults nor increases the net worth, is analogous to the behavior of agents facing other types of nonlinearity in their cost/price structure. Examples include: an agent facing a securities

striving to not default in states where without default costs it would have been optimal to default, the borrower attempts to maintain over some of these states the minimum wealth level that avoids triggering default costs. This level must then correspond to the value of the default boundary, and the flat, constant-wealth shape arises because in our setting the default boundary is state-independent. However, when the default-resistance value is too costly to maintain, recognizing that default is allowed, the borrower chooses to default. Default is chosen in the worst states, as in these states it is most expensive to finance the state-independent default-boundary wealth. To compensate for the wealth level in the default-resistance states and for the costs incurred upon default, the wealth across the no-default region must be decreased (property (*ii*) in Proposition 1), although it maintains the B-like structure.

Fixed costs, ϕ, contribute to the borrower's incentives to extend the resistance region, and are the sole cause for the discontinuity of the net worth $W^*(T)$ in the transition into the default region. However, once default occurs, fixed costs are incurred regardless of the amount of default, and it is optimal to revert to the B-like policy. Consequently, only the proportional costs parameter, λ, affects the shape of $W^*(T)$ in the default region. The lower is $W^*(T)$ (and hence $V^*(T)$) in the default region, the lower is the debt payment, leading to larger proportional costs.[14] To counteract this, the borrower aims at a higher wealth upon default. Therefore, the optimal policy differs from the B-policy by being *positively* related to the proportional-costs parameter λ. This explains the somewhat unexpected feature of the optimal policy stated in property (*ii*) and depicted in Figure 1, where $\lambda > 0$ is such that a levered firm defaulting at a time of economic downturn, despite its suffering default costs, fairs better than an otherwise equal unlevered firm or a firm facing costless default. A distinct feature of the solution is that the discontinuity in Figure 1 is larger than the fixed costs, ϕ. This is due to there being two effects of fixed costs when the debt maturity coincides with the planning horizon. The first is the direct effect of fixed costs on wealth. The second is the indirect effect arising due to the concave objective over wealth at debt maturity. This latter effect overextends the resistance region introducing upon default an additional discontinuity over and above ϕ.

Inspection of Figure 1 allows us to summarize the dependence of the solution on the parameters F, β, λ, and ϕ. As the face value, F, or the recovery rate, β, increase, so does the default boundary. Then region boundaries, ξ_* and ξ^*, decrease, but so that the resistance region shrinks. (Indeed, in the limit of $\beta = 1$, the default region extends over all states, and maximal costs are

market with proportional transaction costs who exhibits an extended region where he does not rebalance his portfolio (Davis and Norman (1990)); an agent facing a different interest rate for borrowing versus lending who exhibits an extended region over which he neither borrows nor lends (Cvitanić and Karatzas (1992)); an agent facing an import quota over a period of time who exhibits an extended region of no trade (Basak and Pavlova (2000)).

[14] Linking $V(T)$ to the underlying primitives, it is easy to verify (see proof of Proposition 1, and Corollary 1 next) that in the default region $V(T)$ is positively related to $W^*(T)$: $V^*(T) = \left(I\left(\frac{\beta y}{\beta+\lambda(1-\beta)}\xi(T)\right) + \phi + \lambda F\right)\frac{1}{\beta+\lambda(1-\beta)}$.

incurred even though $D(0) = 0$.) This, along with decreasing $W^*(T)$ in the default and no-default regions, allows the borrower to meet the higher default-resistance level. For high enough F or β the wealth in the default region falls *below* the benchmark $W^B(T)$. As the proportional costs parameter, λ, increases, the borrower acts to decrease the probability of default, and at the same time to raise the wealth in the default region to minimize the burden of proportional costs. Accordingly, the resistance region expands in both directions, and the wealth in the shrinking no-default region is decreased, thereby financing the increased level at the bad states. A higher λ also increases the curvature of the policy upon default, rendering it more variable across states. An increase in fixed costs, ϕ, similarly extends the resistance region, achieving the goal of lowering the default probability and hence decreasing the deadweight of fixed default costs. However, being insensitive to the magnitude of default, increased ϕ induces a decreased level of wealth in the shrinking no-default *and* default regions. When ϕ increases high enough relative to λ, the wealth in the default region falls *below* the benchmark value. At the other extreme, when ϕ vanishes, so does the discontinuity in $W^*(T)$.

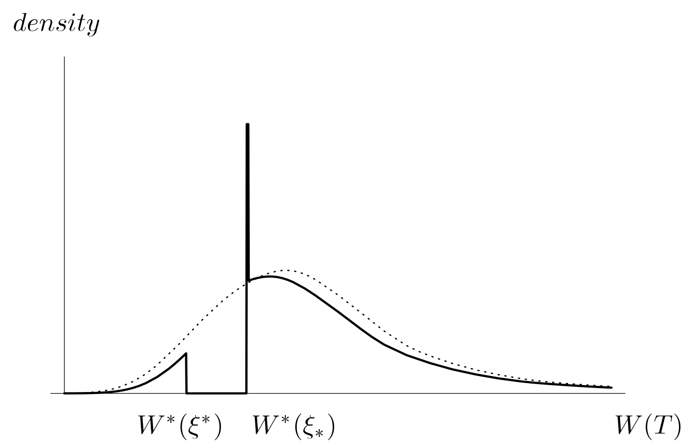


Figure 2: The probability density function of the borrower's time-T optimal wealth (solid plot), and the B-policy (dashed plot), when debt maturity coincides with the borrower's planning horizon ($T = T'$).

Figure 2 depicts the shape of the probability density function corresponding to the terminal wealth policies in Figure 1. There is a probability mass build up in the borrower's terminal wealth, at the default boundary $W^*(\xi_*) = \frac{\beta F}{1-\beta}$. The borrower then has a discontinuity, with no states having wealth between $W^*(\xi_*)$ and $W^*(\xi^*) = I\left(\frac{\beta y \xi^*}{\beta+\lambda(1-\beta)}\right)$. Note that relative to the benchmark, the depicted distribution in the default region is shifted to the right, meaning more wealth with higher probability, as in Figure 1. On the other hand, when fixed costs dominate, the default-region tail shrinks, while shifting to the left relative to the benchmark, similarly to the left-shifted no-default-region tail, whereas the probability mass build up at the default boundary increases.

Corollary 1 elaborates upon the borrower's optimal capital structure, when debt maturity coincides with the planning horizon, describing the equity component (cum default costs), and the debt liability.

Corollary 1. *When debt maturity coincides with the borrower's planning horizon* ($T = T'$),
(i) the borrower's optimal terminal wealth cum costs is given by

$$W^*(T;y) + C^*(T;y) = W^B(T;y) + \max\{I(y\xi_*) - W^B(T;y), 0\} - \Big(\max\{I(x\xi^*) - W^B(T;x), 0\} + (I(y\xi_*) - \phi - I(x\xi^*))1_{\{\xi^* \le \xi(T)\}}\Big) \frac{\beta}{\beta + \lambda(1-\beta)},$$

(ii) the optimal debt payout policy is given by

$$D^*(T) = F - \Big(\max\{I(x\xi^*) - W^B(T;x), 0\} + (I(y\xi_*) - \phi - I(x\xi^*))1_{\{\xi^* \le \xi(T)\}}\Big) \frac{1-\beta}{\beta + \lambda(1-\beta)},$$

where $W^B(T;s) = I(s\xi(T))$*, and* y*,* x*,* ξ_**,* ξ^* *are as in Proposition 1. Moreover, as* $\xi(T) \to \infty$*,* $D^*(T) = \frac{(1-\beta)(\phi + \lambda F)}{\beta + \lambda(1-\beta)}$*. The default-region boundary,* ξ^**, may lie above or below the benchmark costless-default-region boundary,* $\xi^B \equiv v'(\frac{\beta F}{1-\beta})/y^B$*.*

In Corollary 1(i), the equity component, cum costs, takes the form of the B-wealth plus a put option thereon, plus a short position in a package that includes a put and a "binary" option. The long put position guarantees the default-boundary value in the resistance region, while the short package is structured to guarantee the funds necessary to cover the default costs. The binary-option component arises because of the aforementioned additional discontinuity in the wealth upon transition into the default region, arising due to the indirect effect of fixed cost.

Figure 3 describes the payoff of the debt contract across the state space.

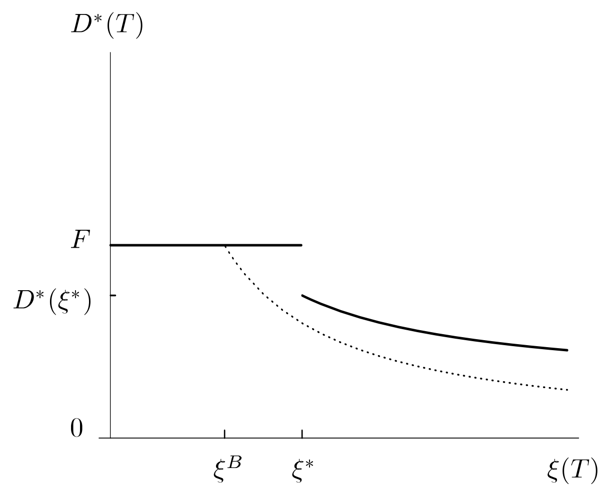


Figure 3: The time-T debt payoff with costly default (solid plot), and with costless default (dotted plot), when debt maturity coincides with the borrower's planning horizon ($T = T'$).

In the presence of fixed default costs incurred at the planning horizon, Corollary 1(ii) illustrates that the debt credit-risk component, although being a put option when expressed in terms of $V(T)$, $\max\{F-(1-\beta)V(T),0\}$, is in fact a portfolio of options when analyzed across the state space; the credit-risk component combines a put option *and* a binary option (the latter accounting for the discontinuity at ξ^* in Figure 3). This portfolio of options enters into the debt contract due to the debt's structural dependence on the assets value, $V(T)$, and hence on the terminal net worth. Note that since $V(T)$ must include funds to cover default costs (unlike in the B-case), $D^*(T)$ will be higher than the B-value for $\xi(T)$ large enough, even though the default-region boundary in the costless-default benchmark, ξ^B, may be higher than ξ^* for some parameter values. Therefore, at the most adverse states, lenders recover in our setting a larger fraction of the face value from borrowers that incur default costs, than from borrowers that default costlessly.

Remark 2. (The Solution with Alternative Modeling of Debt and/or Costs): The optimal policies corresponding to the formulations (i), (ii), and (iii) in Remark 1 are
(i) for debt payoff given by $D(T)=\min\{(1-\beta)(V(T)-C(T)),F\}$,

$$W^{(i)}(T)=\begin{cases} I(y\xi(T)) & : \textit{no-default}, \\ \frac{\beta F}{1-\beta}+\phi & : \textit{default-resistance}, \\ I\left(\frac{y(\beta-\lambda(1-\beta))}{\beta}\xi(T)\right) & : \textit{default}, \end{cases}$$

(ii) for defaults costs given by $C(T)=\lambda V(T)$, when $D(T)<F$,

$$W^{(ii)}(T)=\begin{cases} I(y\xi(T)) & : \textit{no-default}, \\ \frac{\beta F}{1-\beta} & : \textit{default-resistance}, \\ I\left(\frac{y\beta}{\beta-\lambda}\xi(T)\right) & : \textit{default}, \end{cases}$$

(iii) when $D(T)=(1-\beta_1)V(T)$ if $V(T)<\frac{F}{1-\beta_2}$, otherwise $D(T)=F$, where $0\le\beta_2<\beta_1\le 1$,

$$W^{(iii)}(T)=\begin{cases} I(y\xi(T)) & : \textit{no-default}, \\ \frac{\beta_2 F}{1-\beta_2} & : \textit{default-resistance}, \\ I\left(\frac{y\beta_1}{\beta_1+\lambda(1-\beta_1)}\xi(T)\right) & : \textit{default}, \end{cases}$$

where region boundaries are specified analogously to the specification in Proposition 1, and are omitted here for brevity. Consistent with the economic rationale underlying $W^*(T)$, $W^{(i)}(T)$, $W^{(ii)}(T)$, and $W^{(iii)}(T)$ inherit the shape depicted in Figure 1. However, the default boundary in (i) is cost-dependent, the wealth in (ii) *always* falls below the B-wealth upon default (as the lower is the net worth, the lower are the costs), and the default boundary in (iii) does not depend on the recovery rate β_1. Also note that the debt payoff in (i) will vanish at the most adverse states.

3.2 Further Properties of the Borrower's Optimal Policy

To perform a detailed analysis of the optimal behavior of a borrower, we specialize the setting to an isoelastic objective function, $v(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and to log-normal state prices with constant interest rate and market price of risk. Proposition 2 presents explicit expressions for the borrower's optimal wealth and investment policy before the planning horizon/debt-maturity.

Proposition 2. *When debt maturity coincides with the borrower's planning horizon ($T = T'$), assume $v(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and r and κ are constant. Then:*
(i) The borrower's optimal wealth before the debt-maturity date is given by

$$\begin{aligned} W^*(t) &= \frac{X(T-t)}{(y\xi(t))^{\frac{1}{\gamma}}} + \left[\frac{\beta F}{1-\beta} e^{-r(T-t)} \mathcal{N}(-d_2(\xi_*)) - \frac{X(T-t)\mathcal{N}(-d_1(\xi_*))}{(y\xi(t))^{\frac{1}{\gamma}}}\right] \\ &\quad - \left[\left(\frac{\beta F}{1-\beta} - \phi\right) e^{-r(T-t)} \mathcal{N}(-d_2(\xi^*)) - \frac{X(T-t)\mathcal{N}(-d_1(\xi^*))}{(\beta y\xi(t)/(\beta+\lambda(1-\beta)))^{\frac{1}{\gamma}}}\right] \frac{\beta}{\beta+\lambda(1-\beta)}, \end{aligned} \tag{6}$$

where $t < T$, $\mathcal{N}(\cdot)$ is the standard-normal cumulative distribution function,

$$\begin{aligned} \ln X(s) &\equiv \frac{1-\gamma}{\gamma}\left(r + \frac{||\kappa||^2}{2}\right) s + \left(\frac{1-\gamma}{\gamma}\right)^2 \frac{||\kappa||^2}{2} s \ , \\ d_2(x) &\equiv \frac{\ln\frac{x}{\xi(t)} + (r - \frac{||\kappa||^2}{2})(T-t)}{||\kappa||\sqrt{T-t}} \ , \\ d_1(x) &\equiv d_2(x) + \frac{1}{\gamma}||\kappa||\sqrt{T-t} \ , \\ \xi_* &= \frac{1}{y}\left(\frac{1-\beta}{\beta F}\right)^{\gamma} , \end{aligned} \tag{7}$$

y and ξ^* solve $\gamma\left(\frac{\beta+\lambda(1-\beta)}{\beta y\xi^*}\right)^{\frac{1-\gamma}{\gamma}} + \beta y\xi^* \frac{(1-\gamma)(\beta F-\phi(1-\beta))}{(1-\beta)(\beta+\lambda(1-\beta))} = \left(\frac{\beta F}{1-\beta}\right)^{1-\gamma}$, *and* $W^*(0; y) = W(0)$.
(ii) The fraction of wealth invested in the risky investment opportunities is

$$\theta^*(t) = m^*(t)\theta^B(t) \ ,$$

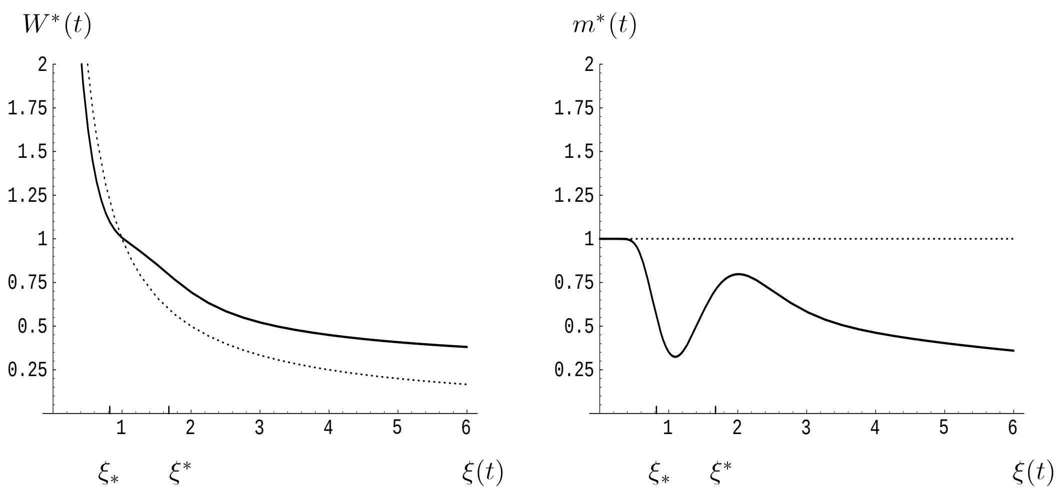
where the B-value, θ^B, and the exposure to risky investments relative to the benchmark, m^*, are

$$\begin{aligned} \theta^B(t) &= \frac{1}{\gamma}[\sigma(t)^\top]^{-1}\kappa, \\ m^*(t) &= 1 - \left[\frac{\beta F}{1-\beta}\mathcal{N}(-d_2(\xi_*)) - \frac{\beta}{\beta+\lambda(1-\beta)}\left(\frac{\beta F}{1-\beta} - \phi\right)\mathcal{N}(-d_2(\xi^*)) \right. \\ &\quad \left. - \frac{\gamma\beta}{\beta+\lambda(1-\beta)}\left(\frac{\beta F}{1-\beta} - \phi - \left(\frac{\beta+\lambda(1-\beta)}{\beta y\xi^*}\right)^{\frac{1}{\gamma}}\right)\frac{\varphi(d_2(\xi^*))}{||\kappa||\sqrt{T-t}}\right] \frac{e^{-r(T-t)}}{W^*(t)}, \end{aligned} \tag{8}$$

respectively, and $\varphi(\cdot)$ is the standard-normal probability distribution function.
(iii) The exposure to risky investments relative to the benchmark is bounded below: $m^*(t) \geq 0$. Under no fixed costs, $\phi = 0$, $m^*(t) \leq 1$. However, for $\phi > 0$, we may have $m^*(t) > 1$.

The option-based interpretation in Corollary 1(i) clarifies the expression of the time-t optimal wealth in (6). The first term takes the form of the optimal B-wealth, the second and third terms represent the cost of a Black and Scholes (1973)-type put option on the B-wealth with strike price $\frac{\beta F}{1-\beta}$, the fourth and fifth terms are the proceeds from shorting a portfolio of a put plus a binary option. Consequently, when the fraction invested in the risky investments is expressed as a multiple of the B-policy, the three square-bracketed terms in (8) correspond, respectively, to the positions in the long put and the short options portfolio.

Figure 4 plots the borrower's optimal time-t wealth and risk exposure, and compares these with the B-case. Figure 4(a) reveals that the pre-horizon borrower's wealth behaves similarly to the benchmark in all states, while being lower in the good states and higher in the bad. In the intermediate region, borrower's wealth exhibits concavity in $\xi(t)$ and it is easy to visualize how this concavity will increase as time approaches the horizon, and tend to the discontinuous shape in Figure 1. In these intermediate states the borrower begins to accumulate wealth to guarantee the resistance level, whereas in the bad states the borrower starts to allocate funds to cover the almost imminent default costs, rendering $W^*(t)$ bounded away from zero. The shift of wealth into the intermediate and bad states is feasible due the decreased wealth in the good states.



(a) Wealth vs. the B-case (b) Risk Exposure vs. the B-case

Figure 4: The (a) time-t wealth and (b) time-t risk exposure relative to the benchmark (dotted plot), when debt maturity coincides with the borrower's planning horizon ($T = T'$). The parameters used are: $\gamma = 1$, $F = 1$, $\beta = 0.5$, $\phi = 0.1$, $\lambda = 0.2$, $W(0) = 1$, $r = 0.05$, $||\kappa|| = 0.4$, $T = 1$, $t = 0.5$. Then, the time-T region boundaries and the time-0 debt value, respectively, are $\xi_* = 0.82$, $\xi^* = 1.68$, $D^*(0) = 0.91$.

Figure 4(b) illustrates the typical shape of the borrower's optimal investment policy, characterized by four segments in the $\xi(t)$ space. First, in the good states, with default being unlikely, the benchmark behavior prevails. Second, in the relatively cheap unfavorable states, the borrower increases the fraction of wealth in the riskless investment aiming to secure the default-resistance level. Third, as $\xi(t)$ rises further, the borrower's risk exposure begins to rise as well, tending back towards the B-policy, but in the case of Figure 4(b) not surpassing it. The fourth segment occurs when $\xi(t)$ is high enough to deter the borrower from further risk taking, and the optimal policy gradually shifts towards a totally riskless position. It is straightforward to verify, using (6)-(8), that the humped shape in Figure 4(b) survives for all parameter values. Formally, this nonmonotonic behavior across the state-space is linked to the replication of the options described in Corollary 1(i). Intuitively, the investment policy is driven by the combined need to finance the default-resistance region as well as the funds required to cover default-costs in the default region. The hump in Figure 4(b) then arises when $\xi(t)$ is in the proximity of ξ^*, because it is only a risky position, sensitive to economic fluctuations, that can facilitate the financing of the two distinct wealth levels over near-by states. Clearly, when $\xi(t)$ is already very high, then default is very likely, it is too costly to bet on a favorable realization of a large risky investment, and a borrower favors riskless investments that, although are unlikely to lead to solvency, would nevertheless cover the costs of default.

Figure 5 displays a sensitivity analysis of $m^*(t)$ to F, β, ϕ, λ, and time. Increasing F or β raises the default boundary, and has the qualitatively similar effect of increasing the likelihood of default and shrinking the resistance region, and therefore in Figures 5(a) and (b) the humped deviation from the benchmark is reduced to a smaller region of states. When default costs (ϕ or λ) increase, the threat of costly default exerts more influence, extending the resistance region, and hence in Figures 5(c) and (d) the humped deviation from the benchmark spreads to a larger region of states.

Figures 5(d) and 5(e) illustrate the somewhat surprising result, stated in Proposition 2(iii), that indeed for some parameter values, and in particular as the time-to-horizon decreases, the exposure of a borrower to risky investments *increases* in some states, compared to the benchmark. The increase occurs across the region of states that straddles ξ^*. Then, conditions are such, that in these states large investment in risky projects is the only strategy allowing to avoid the penalties of default by reaching the resistance level, should economic conditions turn favorable, although leading to default if the state of the economy slightly deteriorates. Interestingly, there is justification for such an aggressive behavior only when the presence of the fixed-costs wedge is coupled with the debt maturing at the planning horizon. Otherwise, $\phi = 0$ eliminates the sharp disparity of wealth around ξ^*; and $T < T'$, which we study next, removes the urgency of the highly levered bets, even if $\phi > 0$. The analysis therefore illustrates, that overall, in many scenarios of interest (and under most of the examined parameter space), the optimal policy of a levered firm is in fact the one of a perennial lower risk exposure relative to the benchmark.

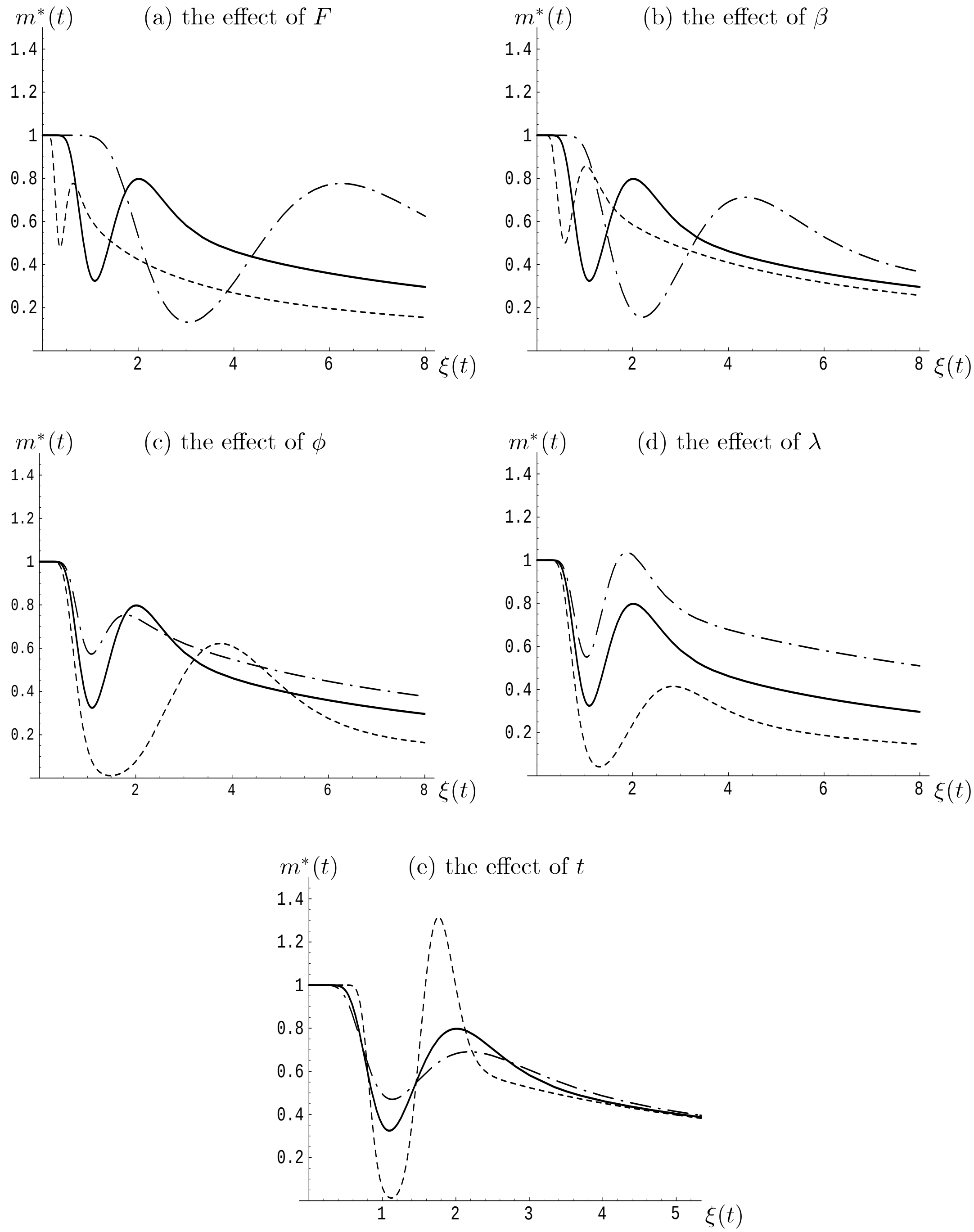


Figure 5: The effect of the debt-contract parameters (F, β), the default-costs parameters (ϕ, λ), and the effect of time (t), on the borrower's risk exposure relative to the B-case, when debt maturity coincides with the borrower's planning horizon ($T = T'$). The solid line in all charts represents the following case: $\gamma = 1$, $F = 1$, $\beta = 0.5$, $\phi = 0.1$, $\lambda = 0.2$, $W(0) = 1$, $r = 0.05$, $||\kappa|| = 0.4$, $T = 1$, $t = 0.5$. Then, $\xi_* = 0.82$, $\xi^* = 1.68$, $\xi^B = 1$. (a) The dashed plot is for $F = 2$, the dot-dashed for $F = 0.5$. (b) The dashed plot is for $\beta = 0.6$, the dot-dashed for $\beta = 0.4$. (c) The dashed plot is for $\phi = 0.4$, the dot-dashed for $\phi = 0.025$. (d) The dashed plot is for $\lambda = 1.0$, the dot-dashed for $\lambda = 0.01$. (e) The dashed plot is for $t = 0.9$, the dot-dashed for $t = 0.1$.

4. Optimization when Pre-Horizon Default is Allowed

In this section, we study the optimization problem of a borrower with a debt contract maturing prior to the planning horizon: $T < T'$. This setting is of obvious general interest, but it is also particularly relevant for those firms and individual borrowers that borrow more heavily in the early stages of their life cycle, decreasing and eliminating debt as they mature. Although simplifying the life cycle to a dichotomy of "with" and "without" debt, the setting in this section will provide us with additional new insights regarding the economic forces interacting at the time of default.

4.1 Borrower's Optimization

Using the martingale representation approach, the dynamic optimization problem of the borrower is restated as the following static variational problem:

$$\begin{aligned} &\max_{W(T),W(T')} E[v(W(T'))] \\ \textit{subject to} \quad & E[\xi(T)(W(T)+C(T))] \leq W(0), \\ & E[\xi(T')W(T')|\mathcal{F}_T] \leq \xi(T)W(T), \end{aligned}$$

where default costs, $C(T)$, and net worth upon debt maturity, $W(T)$, satisfy Assumptions 1 and 2. The budget constraint is broken into two components to clarify the impact of possible default; the first component states that initial wealth must be sufficient to cover potential default costs upon maturity, and the second component is as in the B-case.

While we retain all the main features of default coinciding with the planning horizon, in order to highlight the implications unique to the model with pre-horizon default, we henceforth assume isoelastic objective and log-normal state prices.

Proposition 3. *When debt maturity is prior to the borrower's planning horizon* ($T < T'$)*, assume* $v(W) = \frac{W^{1-\gamma}}{1-\gamma}$*,* $\gamma > 0$*, and* r *and* κ *are constant. Then, the borrower's optimal planning-horizon wealth is*

$$W^{\star}(T') = I(z(T)\xi(T')) = \frac{1}{(z(T)\xi(T'))^{\frac{1}{\gamma}}}.$$

The borrower's optimal wealth upon debt maturity and the Lagrange multiplier $z(T)$ *are*

$$W^\star(T) = \begin{cases} E\left[\frac{\xi(T')}{\xi(T)} I(z(T)\xi(T')) \middle| \mathcal{F}_T\right] = \frac{X(T'-T)}{(z(T)\xi(T))^{\frac{1}{\gamma}}} \quad \text{and} \quad z(T) = y \\ \qquad\qquad \text{if } \xi(T) < \xi_\star \qquad : \text{no-default}, \\ \\ \frac{\beta F}{1-\beta} \qquad \text{and} \qquad z(T) = \frac{\xi_\star}{\xi(T)} y \qquad \text{if } \xi_\star \le \xi(T) < \xi^\star \quad : \text{default-resistance}, \\ \\ E\left[\frac{\xi(T')}{\xi(T)} I(z(T)\xi(T')) \middle| \mathcal{F}_T\right] = \frac{X(T'-T)}{(z(T)\xi(T))^{\frac{1}{\gamma}}} \quad \text{and} \quad z(T) = \frac{\beta y}{\beta + \lambda(1-\beta)} \\ \qquad\qquad \text{if } \xi^\star \le \xi(T) \qquad : \text{default}, \end{cases}$$

where $\xi_\star \equiv \frac{1}{y}\left(\frac{1-\beta}{\beta F} X(T'-T)\right)^\gamma$, $\xi^\star \equiv \xi_\star \frac{\beta+\lambda(1-\beta)}{\beta}\left(\frac{\beta F}{\beta F - \phi(1-\beta)}\right)^\gamma$, $X(s)$ *is as in (7), and* $y \ge 0$ *solves the budget constraint* $E[\xi(T)(W^\star(T;y) + C(W^\star(T;y)))] = W(0)$.

Consequently:

(i) If $W(0) = W^B(0)$, *then* $y \ge y^B$.

(ii) $W^\star(T) \le W^B(T) = \frac{X(T'-T)}{(y^B\xi(T))^{\frac{1}{\gamma}}}$, *under no-default, and for* $\lambda = 0$ *under default.*

However, under default for $\lambda > 0$, *we may have* $W^\star(T) > W^B(T)$.

(iii) The optimal planning-horizon policies:

$W^\star(T')$ *post-no-default,* $W^\star(T')$ *post-default-resistance, and* $W^\star(T')$ *post-default – after the realization of either* $\xi(T) < \xi_\star$, $\xi_\star \le \xi(T) < \xi^\star$, *or* $\xi^\star \le \xi(T)$, *respectively – satisfy:*

(a) $W^\star(T'; \xi^\star \le \xi(T)) > W^\star(T'; \xi(T) < \xi_\star)$ *for* $\lambda > 0$, *holding with equality for* $\lambda = 0$.

(b) $W^\star(T'; \xi_\star \le \xi(T) < \xi^\star) > W^\star(T'; \xi(T) < \xi_\star)$ *for* $\phi > 0$ *or* $\lambda > 0$.

(c) $W^\star(T'; \xi_\star \le \xi(T) < \xi^\star) > W^\star(T'; \xi^\star \le \xi(T))$ *for* $\phi > 0$ *and* $\lambda = 0$, *with the inequality reversed for* $\phi = 0$ *and* $\lambda > 0$.

When $\phi > 0$ *and* $\lambda > 0$, $W^\star(T'; \xi_\star \le \xi(T) < \xi^\star, \xi(T) \to \xi^\star) > W^\star(T'; \xi^\star \le \xi(T))$, *with the inequality reversed for* $\xi(T) \to \xi_\star$.

Proposition 3 (and properties (i)-(ii)) reveals that upon maturity the three-region structure, and the emerging behavior within each region resemble those in Propositions 1 and Figure 1, with the wealth upon maturity now being the present value of the planning-horizon wealth in the no-default and default regions. However, we now clearly see how the three regions are formed. The no-default region is set by the choice of $\xi_\star$, and then to set the other two regions, the proportional costs parameter, λ, and the fixed costs, ϕ, enter separately into two multiplicative terms that determine $\xi^\star$ in relation to $\xi_\star$. So the structure of the default costs explicitly determines how aggressive the borrower is in avoiding default by extending the default-resistance region. The larger is λ, or ϕ, the larger is the default-resistance region relative to the no-default and default regions. Moreover, due to the adverse impact of fixed-costs, hitting the borrower for the slightest amount of default, $\xi^\star/\xi_\star$ increases the more concave is the borrower's objective function.

Focusing on the optimal behavior in the no-default region vs. the default region, property $(iii)(a)$ states that despite paying proportional default costs, the planning-horizon wealth, $W^\star(T')$, is higher post-default compared to post-no-default.[15] This is true regardless of whether under default the deflated wealth, $\xi(T)W^\star(T)$, is above or below the deflated wealth under no-default. The wealth upon debt-maturity, $W^\star(T)$, deviates from the B-structure in the default region only when $\lambda > 0$, and, as with $T = T'$, is bumped up to reduce the proportional default costs. Hence the path-independence of the B-solution no longer holds, and for a given $\xi(T')$ the post-default wealth exceeds the post-no-default wealth. $W^\star(T')$ post-default is the same as post-no-default in the case of fixed costs only ($\phi > 0,\ \lambda = 0$) because there are no proportional costs to modify the B-like structure of the optimal policy upon maturity. Therefore, the resulting $W^\star(T')$ inherits the B-like path independence in both the no-default and default regions.

Examining the optimal planning-horizon wealth post-default-resistance, property $(iii)(b)$ shows it to be higher than the post-no-default wealth and, in the case of fixed-costs only ($\phi > 0,\ \lambda = 0$), property $(iii)(c)$ shows it to be higher than the post-default wealth. This is because the extra funds allocated to reach the default boundary at maturity are subsequently used to finance the post-default-resistance wealth. However, $W^\star(T')$ post-default-resistance does not exceed the post-default wealth in the case of proportional costs only ($\phi = 0,\ \lambda > 0$) because of the bumped up $W^\star(T)$ across the default region. When $\phi > 0$ and $\lambda > 0$, the default-resistance region is further stretched to the right (via increased $\xi^\star$). Then, there are resistance states with $\xi(T)$ close enough to $\xi^\star$ in which the deflated wealth at T is exceptionally high, resulting in a $W^\star(T')$ higher than the value achieved post-default, for a given $\xi(T')$.

Corollary 2 presents the borrower's optimal capital structure upon debt-maturity:

Corollary 2. *When debt maturity is prior to the borrower's planning horizon ($T < T'$), assume $v(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and r and κ are constant. Then,*
(i) the borrower's optimal wealth cum costs upon debt-maturity is given by

$$W^\star(T;y) + C^\star(T;y) = W^B(T;y) + \max\left\{X(T'-T)I(y\xi_\star) - W^B(T;y), 0\right\} - \max\left\{X(T'-T)I(x\xi^\star) - W^B(T;x), 0\right\}\tfrac{\beta}{\beta+\lambda(1-\beta)},$$

(ii) the optimal debt payout policy is given by

$$D^\star(T) = F - \max\left\{X(T'-T)I(x\xi^\star) - W^B(T;x), 0\right\}\tfrac{1-\beta}{\beta+\lambda(1-\beta)},$$

[15] Property (iii) compares the planning-horizon wealth, $W^\star(T')$, for a given $\xi(T')$, across scenarios of arriving to the given $\xi(T')$ via three alternative regions in the $\xi(T)$ space. These are not welfare comparisons, but rather a description of the optimal-policy's path dependence, highlighting the different impact of fixed and proportional default costs. The results provide sharp implications for a firm's size based on its credit history, and these implications are testable for an appropriately constructed sample of firms.

where $W^B(T;s) = X(T'-T)I(s\xi(T))$, $I(s) = s^{-\frac{1}{\gamma}}$, $X(s)$ *is as in (7), and* $\xi_\star$, $\xi^\star$, y *are as in Proposition 3,* x *as in Proposition 1.*

The expressions in Corollary 2(i)-(ii) are similar to those in Corollary 1, and arise due to the same arguments as in Section 3. The major difference here is that put options are the only instruments embedded within the optimal policies. The reason being that when debt maturity precedes the planning horizon ($T < T'$), the fixed default costs upon debt maturity do not immediately affect the concave objective over the planning-horizon wealth. The ability to spread the impact of fixed costs over the planning-horizon states removes the urgency to avoid fixed costs upon maturity, and hence undermines the rationale for over-extending the default-resistance region. So, when $T < T'$, the fixed costs only have a direct effect on wealth, reducing it by ϕ. Therefore, unlike in the case of $T = T'$, there is no need for investment strategies (implemented by binary options) designed to finance upon maturity a larger than ϕ net-worth discontinuity. As a result, when $T < T'$, the debt credit-risk component, analyzed across the time-T state space, is entirely captured by a put option, which is in-the-money when $\xi^\star < \xi(T)$.

4.2 Further Properties of the Borrower's Optimal Policy

Proposition 4 presents explicit expressions for the borrower's optimal wealth and investment policy before the debt-maturity date.

Proposition 4. *When debt maturity is prior to the borrower's planning horizon ($T < T'$), assume* $v(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, *and* r *and* κ *are constant. For* $t < T$:
(i) The borrower's optimal wealth before the debt-maturity date is given by

$$\begin{aligned} W^\star(t) \; = \; & \frac{X(T'-t)}{(y\xi(t))^{\frac{1}{\gamma}}} + \left[\frac{\beta F}{1-\beta} e^{-r(T-t)} \mathcal{N}(-d_2(\xi_\star)) - \frac{X(T'-t)\mathcal{N}(-d_1(\xi_\star))}{(y\xi(t))^{\frac{1}{\gamma}}}\right] \\ & - \left[\left(\frac{\beta F}{1-\beta} - \phi\right) e^{-r(T-t)} \mathcal{N}(-d_2(\xi^\star)) - \frac{X(T'-t)\mathcal{N}(-d_1(\xi^\star))}{(\beta y\xi(t)/(\beta+\lambda(1-\beta)))^{\frac{1}{\gamma}}}\right] \frac{\beta}{\beta+\lambda(1-\beta)}, \end{aligned} \tag{9}$$

where $X(s)$ *is as in (7),* $\xi_\star$, $\xi^\star$, *and* y *are as in Proposition 3.*
(ii) The fraction of wealth invested in the risky investment opportunities is

$$\theta^\star(t) = m^\star(t)\theta^B(t) \; ,$$

where the exposure to risky investments relative to the benchmark, $m^\star$, *is*

$$m^\star(t) = 1 - \left[\frac{\beta F}{1-\beta}\mathcal{N}(-d_2(\xi_\star)) - \frac{\beta}{\beta+\lambda(1-\beta)}\left(\frac{\beta F}{1-\beta} - \phi\right)\mathcal{N}(-d_2(\xi^\star))\right] \frac{e^{-r(T-t)}}{W^\star(t)}. \tag{10}$$

(iii) The exposure to risky investments is bounded below and above: $0 \le m^\star(t) \le 1$.

Since Corollary 2 illustrated that the separation of the maturity and planning-horizon dates eliminated the need for aggressive risky betting prior to T, $W^\star(t)$ in (9), although similar to (6), does not include a binary component. In (9), $\xi_\star$ and $\xi^\star$ are set so that $W^\star$ is composed from a first term in the form of the B-wealth, plus a put option thereon with strike $\frac{X(T'-T)}{(y\xi_\star)^{\frac{1}{\gamma}}} = \frac{\beta F}{1-\beta}$, and $\left(\frac{\beta}{\beta+\lambda(1-\beta)}\right)^{\frac{\gamma-1}{\gamma}}$ units of a short put thereon with strike $\frac{X(T'-T)}{(y\xi^\star)^{\frac{1}{\gamma}}} = \left(\frac{\beta F}{1-\beta} - \phi\right)\left(\frac{\beta}{\beta+\lambda(1-\beta)}\right)^{\frac{1}{\gamma}}$. This portfolio of options guarantees the default-resistance wealth as well as the funds needed to cover default costs, and hence increases the fraction of wealth invested in the riskless investment. This results in a typical shape for $m^\star$ as in Figure 4(b), for all $t < T$ and all parameter values, meaning that levered firms, facing pre-horizon costly default, unambiguously reduce their risk exposure relative to unlevered firms or firms facing no default costs.

5. Equilibrium in the Presence of Credit Risk

Given the prevalence of defaultable debt in the economy, it is of interest to evaluate its impact on asset prices at an aggregate level. In this section, we provide a simple general-equilibrium production model in which the partial equilibrium behavior of the borrower persists, and affects aggregate quantities including market value and dynamics. It is not our intention to provide the most general setting where most pertinent quantities are endogenously determined.

5.1 The Equilibrium Setting

Under costly default, we have illustrated that in many scenarios of interest a levered firm invests a higher fraction of its net worth in riskless investments than does an unlevered firm. Moreover, in striving to meet its debt obligations, the levered firm optimally "shifts" wealth from the good to bad states of the world. This motivates us to consider a production economy in which both the supply of riskless investments and the aggregate consumption/wealth are endogenous. Hence, unlike in a pure-exchange environment, aggregate consumption/wealth may actually be postponed or shifted and, unlike the case of a fixed (zero) supply bond, aggregate nonzero holdings in a riskless investment are allowed.

Accordingly, within the framework of Section 2, we adopt a variation on the Cox, Ingersoll, and Ross (1985) continuous-time production economy. In this economy, the investment opportunities available to both a representative borrower, b, and a representative lender, ℓ, are constant-returns-to-scale production technologies, using the single consumption good as their only input and producing the consumption good as output. The production technologies have perfectly elastic supplies, and net returns given by (1), where the (exogenously specified) parameters r, μ, and σ are assumed constant.[16]

The initial net worth of the representative borrower, $W_b(0)$, and the representative lender, $W_\ell(0)$, is exogenously specified in units of the consumption good. For tractability, we specialize to the borrower and the lender having an isoelastic objective function, $v_n(W_n) = \frac{W_n^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, $n = b, \ell$. The optimization problem of the borrower is solved in Sections 3 and 4. The dynamic optimization problem of the lender may be restated as the following variational problem:

$$\max_{W_\ell(T')} E\left[\frac{W_\ell(T')^{1-\gamma}}{1-\gamma}\right] \quad \textit{subject to} \quad E\left[\xi(T')W_\ell(T')\right] \leq W_\ell(0). \tag{11}$$

The lender's optimal planning-horizon wealth is given by

$$W_\ell(T') = \frac{W_\ell(0)}{X(T')\xi(T')^{\frac{1}{\gamma}}}, \tag{12}$$

where $X(s)$ is as in equation (7). Consequently, the lender's time-t optimal wealth and fractions of wealth in risky-technology investments are given by

$$W_\ell(t) = \frac{W_\ell(0)}{X(t)\xi(t)^{\frac{1}{\gamma}}}, \tag{13}$$

$$\theta_\ell(t) = \frac{1}{\gamma}\left(\sigma^\top\right)^{-1}\kappa. \tag{14}$$

We note that the lender's optimization problem and its solution are identical to those of a benchmark investor in an economy with no debt or no default costs. This is because in our complete-markets environment, the lender is capable of "undoing" the effects of the debt contract, perfectly hedging its credit risk component.

Equilibrium in our production economy requires the borrower and the lender to be acting optimally, and for all wealth to be invested in the production technologies. Our goal is to compare equilibrium in the presence of credit risk with equilibrium in the benchmark economy with no debt or default costs. In particular, we focus on pertinent quantities before the debt-maturity date, T, since as we have illustrated in Section 4, the borrower reverts back to a benchmark policy after debt maturity, and hence the ensuing equilibrium resembles that in the benchmark economy.

5.2 Equilibrium Market Price, Volatility, and Risk Premium

The price of the *market* portfolio, W_M, is defined as the aggregate wealth invested in the production technologies, which equals the sum of the borrower's and lender's net worth:

$$W_M(t) \equiv \sum_{i=0}^{N}\left(\theta_{bi}(t)W_b(t) + \theta_{\ell i}(t)W_\ell(t)\right) = W_b(t) + W_\ell(t).$$

[16] For some recent applications using this type of a production model, with one technology being riskless, see, for example, Obstfeld (1994), Basak (2000), Dumas and Uppal (2000). In contrast, the Cox, Ingersoll, and Ross (1985) model has one riskless bond in zero net supply and no riskless production technology.

The equilibrium market-price dynamics can be represented by

$$dW_M(t) = W_M(t)\left[\mu_M(t)dt + \sum_{j=1}^{N}\sigma_{M,j}(t)dw_j(t)\right],$$

where μ_M is the *market drift* and $||\sigma_M(t)|| = \sqrt{\sum_{j=1}^{N}\sigma_{M,j}(t)^2}$ is the *market volatility*. Proposition 5 presents the equilibrium market price and contrasts it with the benchmark economy:

Proposition 5. *The equilibrium market price in a benchmark economy is*

$$W_M^B(t) = \frac{W_b(0) + W_\ell(0)}{X(t)\xi(t)^{\frac{1}{\gamma}}}.$$

When default is costly, the equilibrium market price before debt maturity is given by

$$W_M(t) = \frac{W_b(0) + W_\ell(0) - Z(0)}{X(t)\xi(t)^{\frac{1}{\gamma}}} + Z(t), \tag{15}$$

where $t < T$, $Z(t) > 0$ is given by

$$\begin{aligned} Z(t) \;=\; & \left[\frac{\beta F}{1-\beta}e^{-r(T-t)}\mathcal{N}(-d_2(\underline{\xi})) - \frac{(W_b(0) - Z(0))\mathcal{N}(-d_1(\underline{\xi}))}{X(t)\xi(t)^{\frac{1}{\gamma}}}\right] \\ & - \left[\left(\frac{\beta F}{1-\beta} - \phi\right)e^{-r(T-t)}\mathcal{N}(-d_2(\overline{\xi})) - \frac{(W_b(0) - Z(0))\mathcal{N}(-d_1(\overline{\xi}))}{X(t)(\beta\xi(t)/(\beta+\lambda(1-\beta)))^{\frac{1}{\gamma}}}\right]\frac{\beta}{\beta+\lambda(1-\beta)}, \end{aligned} \tag{16}$$

$Z(0)$ solves (16) at $t = 0$, $\underline{\xi} \in \{\xi_*, \xi_\star\}$, $\overline{\xi} \in \{\xi^*, \xi^\star\}$, $(\xi_*, \xi^*, d_1(x), d_2(x))$ are as in Proposition 1, and $(\xi_\star, \xi^\star)$ as in Proposition 3, with $y_b = (X(\overline{T})/W_b(0) - Z(0))^\gamma$, $\overline{T} \in \{T, T'\}$.
Consequently, (i) $W_M(t) < W_M^B(t)$ for $\xi(t) \to 0$; (ii) $W_M(t) > W_M^B(t)$ for $\xi(t) \to \infty$.

Proposition 5 shows the market price in the presence of costly default to equal that in the B-case with a reduced (by $Z(0)$) initial level, plus a positive stochastic term, $Z(t)$, reflecting the option package replicated by the borrower. In the bad states of the world (high $\xi(t)$), the market price is increased by the presence of credit risk, while in the good states the market price is decreased. This is because in the bad states, the borrower is investing mostly in the riskless technology, so that to insure the default-resistance wealth, as well as the funds needed to cover default costs. The borrower's desire, before maturity, for more wealth in the bad states is thus pushing up the market level relative to the benchmark. Since at the outset the borrower has effectively used up some funds to pay for the "insurance policy" providing the wealth at the bad states, the borrower then accumulates less wealth in the good states, and hence the market price level is decreased. That is, the borrower shifts wealth from good states where it would well-exceed its debt obligations, to bad states where its wealth upon debt maturity is expected to fall closer to the default boundary, and so the market is higher than in the B-case at economic downturns and lower at upturns.

Proposition 6 presents the market-return dynamics and contrasts it with the benchmark economy with no leverage or default costs:

Proposition 6. *The equilibrium market volatility and risk premium in a benchmark economy are*

$$||\sigma_M^B(t)|| = \frac{1}{\gamma}||\kappa||, \quad \mu_M^B(t) - r = \frac{1}{\gamma}||\kappa||^2.$$

When default is costly, the equilibrium market volatility and risk premium before debt maturity are given by

$$||\sigma_M(t)|| = \frac{1}{\gamma}\left(1 - \frac{\overline{W}_b(t)}{W_M(t)}Y(t)\right)||\kappa||, \quad \mu_M(t) - r = \frac{1}{\gamma}\left(1 - \frac{\overline{W}_b(t)}{W_M(t)}Y(t)\right)||\kappa||^2,$$

where

$$\begin{aligned} Y(t) \quad = \quad & \frac{e^{-r(T-t)}}{\overline{W}_b(t)}\left[\frac{\beta F}{1-\beta}\mathcal{N}(-d_2(\underline{\xi})) - \frac{\beta}{\beta+\lambda(1-\beta)}\left(\frac{\beta F}{1-\beta} - \phi\right)\mathcal{N}(-d_2(\overline{\xi}))\right. \\ & \left. -1_{\{T=T'\}}\frac{\gamma\beta}{\beta+\lambda(1-\beta)}\left(\frac{\beta F}{1-\beta} - \phi - \frac{W_b(0)-Z(0)}{X(T)}\left(\frac{\beta+\lambda(1-\beta)}{\beta\xi^*}\right)^{\frac{1}{\gamma}}\right)\frac{\varphi(d_2(\xi^*))}{||\kappa||\sqrt{T-t}}\right], \end{aligned}$$

$Z(0)$ solves (16) at $t=0$, $\underline{\xi} \in \{\xi_*, \xi_\star\}$, $\overline{\xi} \in \{\xi^*, \xi^\star\}$, $\overline{W}_b(t) \in \{W_b^*(t), W_b^\star(t)\}$, $(\xi_*, \xi^*, d_2(x), W_b^*(t))$ are as in Proposition 1, and $(\xi_\star, \xi^\star, W_b^\star(t))$ as in Proposition 3, with $y_b = (X(\overline{T})/W_b(0) - Z(0))^\gamma$, $\overline{T} \in \{T, T'\}$.

Consequently, in the economy where debt maturity coincides with the borrower's planning horizon but there are no fixed default costs ($T = T', \phi = 0$), or in the economy where debt maturity is prior to the borrower's planning horizon ($T < T'$), we have $Y(t) \in (0,1)$ so that

(i) $||\sigma_M(t)|| < ||\sigma_M^B(t)||$, (ii) $\mu_M(t) - r < \mu_M^B(t) - r$.

However, when debt maturity coincides with the borrower's planning horizon and there are fixed default costs ($T = T', \phi > 0$), we may have $||\sigma_M(t)|| > ||\sigma_M^B(t)||$, $\mu_M(t) - r > \mu_M^B(t) - r$.

Proposition 6 states that the equilibrium market volatility and risk premium are reduced in many scenarios by the presence of credit risk. This is because, as seen in Propositions 2 and 4, a borrower in these scenarios has a lower demand for risky investment opportunities than in the B-case. Hence, within this production economy, the aggregate investment in the risky production technologies is reduced compared with the investment in the riskless technology, and so the market becomes less risky, as reflected by the lower market volatility and risk premium. This volatility result is consistent with a related argument regarding the role of fixed default costs in inducing firms to engage in cash-flow hedging practices (e.g., Smith and Stulz (1985) and Allen and Santomero (1998)). However, as demonstrated after Proposition 2 in Section 3.2, in the presence of fixed default costs, when planning-horizon default is allowed, a levered firm may indeed demand more in the risky technologies than it does in the B-case (e.g., when approaching the debt-maturity date). Therefore, in this case the "cash-flow hedging" argument can no longer be straightforwardly extrapolated, and such a case in fact leads to an increase in market volatility (and risk premium) compared to an economy without leverage or default costs.

6. Extensions and Applications

6.1 Extension to Repeated Borrowing

Our analysis so far focused, for clarity, on a single debt contract. However, our setting readily lends itself to dealing with multiple debt contracts, cross-sectionally or intertemporally. To highlight the intertemporal dimension, consider for example the case of repeated borrowing, where as a debt contract matures at time T, the borrower enters into a new contract with face value F', maturity T', and recovery rate β'. Default in this setting is allowed both pre- and at the planning horizon.[17] The payoff and the associated default costs of the second debt contract are (following the structure in Section 2.2): $D(T') = \min\{(1-\beta')V(T'), F'\}$, $C(T') = \{\phi' + \lambda'(F' - D(T'))\}1_{\{D(T')<F'\}}$, respectively, where $0 \le \beta' \le 1$, $\phi' \ge 0$, $\lambda' \ge 0$. The borrower's optimization problem is reduced then to solving the following problem:

$$\begin{aligned} &\max_{W(T),W(T')} E[v(W(T'))] \\ \textit{subject to} \quad &E[\xi(T)(W(T)+C(T))] \le W(0), \\ &E[\xi(T')(W(T')+C(T'))|\mathcal{F}_T] \le \xi(T)W(T). \end{aligned}$$

Proposition 7 characterizes the optimal solution (where we use the hat superscript, $\hat{}$, to distinguish the endogenous quantities here from those in the previous sections).

Proposition 7. *When the first debt contract matures at time T, and the second debt contract matures at the planning horizon T', assume $v(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and r and κ are constant. Then,*

[17] We refer to this setting as "repeated borrowing" for expositional purposes, but within our paradigms the identifying feature of a contract is its maturity. In deriving the optimal policies of interest, this setting is then observationally equivalent to entering into the first contract at any time prior to T, and into the second contract at any time prior to T' (only V is affected by how we account for the "timing" of borrowing). Our model therefore allows to study optimal policies in conjunction with the formation of term structures of defaultable bonds, or equivalently in conjunction with the pricing of coupon bonds (where the borrower may default on each coupon payment). We leave these issues for future explorations.

the time-T' optimal wealth of the borrower is

$$\hat{W}(T') = \begin{cases} \dfrac{1}{(z(T)\xi(T'))^{\frac{1}{\gamma}}} & \text{if } \xi(T') < \hat{\xi}_*(T) \quad : \text{no-default}, \\ \dfrac{\beta' F'}{1-\beta'} & \text{if } \hat{\xi}_*(T) \le \xi(T') < \hat{\xi}^*(T) \quad : \text{default-resistance}, \\ \left(\dfrac{\beta' + \lambda'(1-\beta')}{\beta' z(T)\xi(T')}\right)^{\frac{1}{\gamma}} & \text{if } \hat{\xi}^*(T) \le \xi(T') \quad : \text{default}, \end{cases}$$

where $\hat{\xi}_*(T) = \zeta_*/z(T)$, $\hat{\xi}^*(T) = \zeta^*/z(T)$. The time-$T$ optimal wealth and the Lagrange multiplier $z(T)$ are

$$\hat{W}(T) = \begin{cases} E\left[\dfrac{\xi(T')}{\xi(T)}(\hat{W}(T') + C(T'))\Big|\mathcal{F}_T\right] = \dfrac{G(z(T))}{(z(T)\xi(T))^{\frac{1}{\gamma}}} + H(z(T)) \;\; \text{and} \;\; z(T) = y \\ \qquad\qquad \text{if } \xi(T) < \hat{\xi}_\star \quad : \text{no-default}, \\ \dfrac{\beta F}{1-\beta} \quad \text{and } z(T) \text{ solves} \quad \dfrac{G(z(T))}{(z(T)\xi(T))^{\frac{1}{\gamma}}} + H(z(T)) = \dfrac{\beta F}{1-\beta} \\ \qquad\qquad \text{if } \hat{\xi}_\star \le \xi(T) < \hat{\xi}^\star \quad : \text{default-resistance}, \\ E\left[\dfrac{\xi(T')}{\xi(T)}(\hat{W}(T') + C(T'))\Big|\mathcal{F}_T\right] = \dfrac{G(z(T))}{(z(T)\xi(T))^{\frac{1}{\gamma}}} + H(z(T)) \;\; \text{and} \;\; z(T) = \frac{\beta y}{\beta+\lambda(1-\beta)} \\ \qquad\qquad \text{if } \hat{\xi}^\star \le \xi(T) \quad : \text{default}, \end{cases}$$

where the constants ζ_*, ζ^*, $\hat{\xi}_\star$, $\hat{\xi}^\star$, and the functions $G(\cdot) > 0$, $H(\cdot) > 0$ are given in the appendix. $y \ge 0$ solves the budget constraint $E[\xi(T)(\hat{W}(T;y) + C(\hat{W}(T;y)))] = W(0)$.
Consequently:
(i) If $W(0) = W^B(0)$, then $y \ge y^B$.
(ii) $\hat{W}(T') = W^*(T')$ for either $F = 0$, $\beta = 0$, or $\phi = \lambda = 0$,
where $W^*(T')$ is as in Proposition 1 (with T' replacing T).
(iii) $\hat{W}(T) = W^\star(T)$ for either $F' = 0$, $\beta' = 0$, or $\phi' = \lambda' = 0$,
where $W^\star(T)$ is as in Proposition 3.

Proposition 7 (properties (*ii*)-(*iii*)) asserts that the optimal planning-horizon wealth, $\hat{W}(T')$, coincides with the optimal policy of Proposition 1, or that the time-T wealth, $\hat{W}(T)$, coincides with the optimal policy of Proposition 3, when some parameter values vanish so that only one of the contracts impacts the borrower. However, for general parameter values, although both $\hat{W}(T)$ and $\hat{W}(T')$ are structured similarly to their counterparts in the single-contract cases, each features a notable difference.

The optimal policy upon maturity of the first debt contract, $\hat{W}(T)$, is modified to account for the default costs that may be incurred at time T', on top of those as of time T considered in

Proposition 3. The reason being that from time 0 the borrower is conscious of future borrowing. Therefore, the choice of region boundaries in the $\xi(T)$ space, and of the wealth within each region, is affected not only by the desire for a balance between the costs of resisting default and the costs incurred upon default at time T, but also by the desire for a similar balance with respect to time T'. Consequently, even at the most adverse states at time T ($\xi(T) \to \infty$), $\hat{W}(T)$ is maintained above a floor ($H > 0$), thereby enabling the borrower to finance the default-resistance region and the costs of default at the planning horizon T', irrespective of the magnitude of default at time T. The borrower's wealth at the most adverse states at time T is thus always higher than in the cases of no borrowing or of costless-planning-horizon default.

Examining the planning-horizon optimal policy, $\hat{W}(T')$, reveals that while retaining the three-region structure of Proposition 1, the boundaries of the regions in the $\xi(T')$ space, $\hat{\xi}_*(T)$ and $\hat{\xi}^*(T)$, are now path-dependent, and are being identified by whether the outcome at time T is no-default, default-resistance, or default. The behavior after T is driven by similar arguments to those outlined in the context of Proposition 3. In particular, the optimal policy post default or post no-default is not sensitive to the particular realization of $\xi(T)$. Moreover, the fact that conditional on $\xi(T')$ the borrower is never worse off post default, compared to post no-default, is translated with repeated borrowing not only to the time-T' wealth level, but also to the time-T' region boundaries – as, for example, is illustrated by the larger no-default region post time-T default, $\hat{\xi}_*(T) = \frac{\beta+\lambda(1-\beta)}{\beta y}\left(\frac{1-\beta'}{\beta' F'}\right)^{\gamma}$, compared to post time-$T$ no-default, $\hat{\xi}_*(T) = \frac{1}{y}\left(\frac{1-\beta'}{\beta' F'}\right)^{\gamma}$.

6.2 Managing Credit Risk

The credit risk associated with a debt issue, can be managed by the appropriate choice of the debt-contract parameters. Specifically, focusing for illustration on a parameter easily adjustable in practice, a firm, or its creditors, can choose the face value of the debt, F, all else being equal, so that to fix a prespecified probability of default α, which may be necessary, e.g., to maintain a desirable credit rating. Moreover, from a rating agency's perspective, our model helps to identify those levered firms (as characterized by F, β, ϕ, λ) that can meet a given default probability required for a target rating.[18]

We now return to the framework of Section 4, where a single debt contract matures prior to the planning horizon. Proposition 8 characterizes the debt face value required to maintain a desired probability of default under the optimal policy (assuming it exists):

[18] The rationale for credit ratings and their precise meaning and determinants are beyond the scope of our discussion. For more on the rating process and the relation between ratings and default rates see, e.g., Caouette, Altman, and Narayanan (1998; Chapter 6).

Proposition 8. *With a single debt contract maturing prior to the borrower's planning horizon* $(T < T')$, *assume* $v(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, *and* r *and* κ *are constant. Then, the borrower's optimal policy results in a default probability* α, *when the face value of the debt contract is set to*

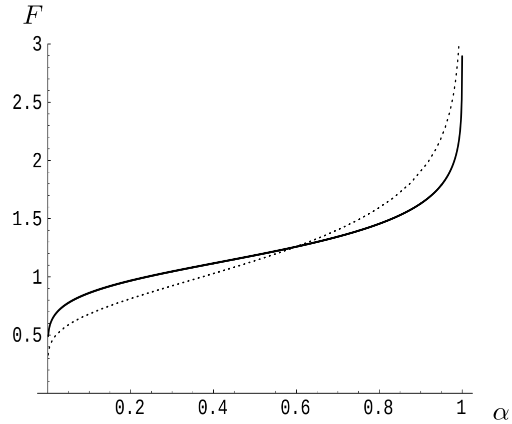
$$F(\alpha) = \frac{1-\beta}{\beta}\left[X(T'-T)\left(\frac{\beta+\lambda(1-\beta)}{\beta y \xi^\star(\alpha)}\right)^{\frac{1}{\gamma}} + \phi\right], \tag{17}$$

where $\xi^\star(\alpha) = e^{-\mathcal{N}^{-1}(\alpha)||\kappa||\sqrt{T}-(r+||\kappa||^2/2)T}$ *is the value of* $\xi^\star$ *for which* $P(D^\star(T) < F|\mathcal{F}_0) = \alpha$, $D^\star(T)$ *is as in Corollary 2,* $\mathcal{N}^{-1}(\cdot)$ *is the inverse of the standard normal cumulative distribution function,* y *solves the time-*0 *budget constraint* $W^\star(0; y, \xi_\star(\alpha; y)) = W(0)$, $W^\star(\cdot)$ *is as in (9), and* $\xi_\star(\alpha; y) = \frac{\beta\xi^\star(\alpha)}{\beta+\lambda(1-\beta)}\left[1 + \frac{\phi}{X(T'-T)}\left(\frac{\beta y \xi^\star(\alpha)}{\beta+\lambda(1-\beta)}\right)^{\frac{1}{\gamma}}\right]^{-\gamma}$, *with* $X(s)$ *as in (7).*

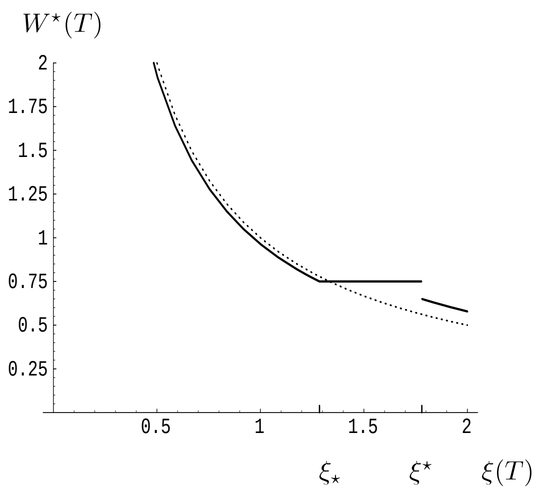
Figure 6(a) summarizes the comparative-statics analysis of the probability of default with respect to the face value. By depicting the correspondence between the face value and the probability of default, we can clearly see which debt contract will comply with a required range of default probabilities. It is interesting to note that at the relatively low levels of leverage, associated with the lower end of default probabilities, a firm facing default costs is less likely to default on a given debt contract than a firm facing costless default. And vice versa at the higher levels of leverage. Clearly, bearing the costs of default disciplines the levered firm to better service its debt, striving to avoid costly default, and then $\xi^\star > \xi^B$. This behavior is illustrated by Figure 6(b) (as well as by Figure 1).[19] However, as shown in Figure 6(c), with a higher debt face value, resisting default becomes much more costly. This extends the default region, which in turn acts to increase the burden of default costs, further weakening the firm's ability to support the default-resistance wealth. Overall, for a given debt contract associated with the higher end of default probabilities, a levered firm facing default costs is more likely to default – despite the disciplinary impact of default costs – than a firm facing no default costs, and then $\xi^\star < \xi^B$.

Proposition 8 may be also useful to borrowers in the context of more formal risk-management practices. In particular, it is evident from Proposition 3, and Figure 1, that a borrower's time-T value-at-risk (VaR) at the $\alpha \times 100\%$ significance level, $VaR(\alpha)$, is given by $W(0) - \frac{\beta F(\alpha)}{1-\beta}$ (see, e.g., Jorion (1997) for more on VaR). Therefore, those who track their VaR, for example at the $\alpha = 0.01$ level (possibly due to regulatory requirements), can enter into a debt contract with a face value $F(0.01)$, using (17), thereby guaranteeing to lose over $[0, T]$ not more than $W(0) - \frac{\beta F(0.01)}{1-\beta}$, with probability 0.99.

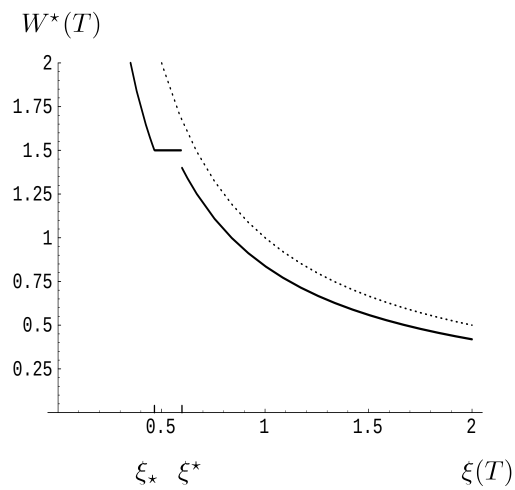
[19] As discussed in Section 4.1, the discontinuity in Figures 6(b)-(c) at $\xi^\star$ is attenuated relative to Figure 1, because when $T < T'$, the discontinuity arises solely due to the actual charge of fixed costs (with no over-extension of the resistance region to avoid this charge, as is the case when $T = T'$). Hence the gap is exactly equal to ϕ.



(a) The probability of default (α) and the debt face value (F)



(b) Wealth vs. the B-case when $F = 0.75$



(c) Wealth vs. the B-case when $F = 1.5$

Figure 6: When debt maturity is prior to the borrower's planning horizon ($T < T'$), in (a), for a given probability of default $\alpha \in [0, 1)$, the associated face value of the debt contract is plotted when default is costly (solid plot) and when default is costless (dotted plot). In (b) and (c), the time-T wealth when default is costly (solid plot) and when default is costless (dotted plot) are plotted for a given value of $\xi(T)$. The parameters used are: $\gamma = 1$, $\beta = 0.5$, $\phi = 0.1$, $\lambda = 0.2$, $W(0) = 1$, $r = 0.05$, $||\kappa|| = 0.4$, $T = 1$, $T' = 2$. Then, the time-T region boundaries and the time-0 debt value, respectively, in (b) where *default boundary* $= F = 0.75$, are $\xi^B = 1.33$, $\xi_\star = 1.28$, $\xi^\star = 1.78$, $D^\star(0) = 0.71$, and in (c) where *default boundary* $= F = 1.5$, are $\xi^B = 0.66$, $\xi_\star = 0.47$, $\xi^\star = 0.60$, $D^\star(0) = 0.99$.

7. Conclusion

We study the optimal decision of borrowers (firms or households) to default on their debt in the presence of default costs, and analyze the associated investment policies and implications for market dynamics. We adopt a complete-markets setting with a general structure of uncertainty, where default matters economically because of the costs inflicted upon a defaulting borrower, and find the borrower's optimal policies to be distinctly different from those of a non-borrower or those who can default costlessly. In doing so, we highlight the different impact of various types of costs, and demonstrate analytically how, depending on the type of costs, a borrower's net worth and risk exposure may be higher or lower than the benchmark levels. We also investigate how these results translate into market price, volatility, and risk premium effects in a production economy.

Borrowers, in our setting, control the dynamics of their assets value, and the credit-risk component of their debt depends on the borrowers' characteristics, as well as on the realization of their investments. Focusing on borrowers who have the option to default is a first and necessary step to understand markets in the presence of credit risk. To maintain the focus on the aspects of optimal default, we model the lenders as facing no frictions. Lenders can implement their optimal benchmark policies by perfectly hedging their credit-risk exposure. In future work, we intend to explore settings where the presence of credit risk also affects the optimal investment policies of lenders.

Furthermore, we maintain the focus on borrowers' optimal policies at the cost of only briefly touching upon the various aspects related to the risky debt itself. However, we do illustrate the applicability of our setting to studying debt in a quite general stochastic environment, and a natural direction for future research is to adopt an environment with an empirically supported dynamics of the riskless short rate and/or the market price of risk. One can then examine the implications of our model for default probabilities, default premia, expected recovery ratios, and for the interaction between hedging against default and hedging against shifts in investment opportunities. These may be performed in the context of a single pure-discount debt contract, or in the presence of multiple contracts, thereby introducing term-structure issues into the analysis. Additional features such as coupons, callability, protective covenants, and taxes can be incorporated as well. Although not a trivial task, exploring these directions in our setting may be rewarding in offering new guidance for investment policies and for pricing, as well as in providing new explanations for observed empirical regularities in the fixed income and equity markets.

Appendix: Proofs

Proof of Proposition 1. When either $F = 0$, $\beta = 0$, or $\phi = \lambda = 0$, then by their definition, $\xi^* = \xi_* \leq \infty$, and $W^*(T) = I(y\xi(T))$, which is optimal following the standard arguments as in the benchmark case. Otherwise, let $g(x\xi) = I(x\xi) - \left(v(I(x\xi)) - v\left(\frac{\beta F}{1-\beta}\right)\right)/(x\xi) - \frac{\beta F}{1-\beta} + \phi$, where $x \equiv \beta y/(\beta + \lambda(1-\beta))$, and note that ξ^* is defined as a solution to $g(x\xi) = 0$. Also note that $g(x\xi_*(\beta+\lambda(1-\beta))/\beta) = \phi$, and $\frac{\partial g(x\xi)}{\partial(x\xi)} < 0$ if, and only if, $\xi > \xi_*(\beta+\lambda(1-\beta))/\beta$. Hence, there exists a unique ξ^* such that $g(x\xi^*) = 0$ and $\xi^* \geq \xi_*(\beta+\lambda(1-\beta))/\beta \geq \xi_*$, where the first inequality holds with equality for $\phi = 0$, yielding property *(iii)*. The remainder of the proof is for the case of $\xi^* > \xi_*$. Since Assumption 1 is formulated in terms of $V(T)$, and since the mapping between $V(T)$ and $W(T)$ is one-to-one ($W(T) = V(T) - F$, if $(1-\beta)V(T) \geq F$; $W(T) = (\beta + \lambda(1-\beta))V(T) - (\phi + \lambda F)$, if $(1-\beta)V(T) < F$), we found it convenient to present the proof using $V(T)$ as the choice variable. We thus need to show that

$$V^*(T) = (I(y\xi(T)) + F)1_{\{\xi(T)<\xi_*\}} + \frac{F}{1-\beta}1_{\{\xi_* \leq \xi(T) < \xi^*\}} + \frac{I(x\xi(T)) + \phi + \lambda F}{\beta + \lambda(1-\beta)}1_{\{\xi^* \leq \xi(T)\}} \quad \text{(A1)}$$

maximizes $E[v(V(T) - D(V(T)) - C(V(T)))]$ subject to $E[\xi(T)(V(T) - D(V(T)))] \leq W(0)$, where $D(V)$ and $C(V)$ satisfy Assumptions 1 and 2, respectively. The proof adapts the common convex-duality approach (see, e.g., Karatzas and Shreve (1998)) to incorporate kinks and discontinuities within the objective and the budget constraint.

Lemma 1. *Pointwise, for all* $\xi(T)$*,*

$$V^*(T) = \arg\max_V\{v(V - D(V) - C(V)) - y\xi(T)(V - D(V))\},$$

where $V^*(T)$ *is given in (A1),* $D(V) = \min\{(1-\beta)V, F\}$*, and* $C(V) = \{\phi+\lambda(F-D(V))\}1_{\{D(V)<F\}}$.

Proof: The function $f(V) \equiv v(V - D(V) - C(V)) - y\xi(T)(V - D(V))$
$= [v(V-F) - y\xi(T)(V-F)]1_{\{(1-\beta)V \geq F\}} + [v((\beta+\lambda(1-\beta))V - (\phi+\lambda F)) - y\xi(T)\beta V]1_{\{(1-\beta)V<F\}}$, is not concave in V, but can only exhibit local maxima at $I(y\xi(T)) + F$, if $I(y\xi(T)) + F > \frac{F}{1-\beta}$, at $\frac{I(x\xi(T))+\phi+\lambda F}{\beta+\lambda(1-\beta)}$, if $\frac{I(x\xi(T))+\phi+\lambda F}{\beta+\lambda(1-\beta)} < \frac{F}{1-\beta}$, or at $\frac{F}{1-\beta}$. To find the global maximum, we compare the value of these three local maxima. When $\xi(T) < \xi_*$, we have $I(y\xi(T)) + F > \frac{F}{1-\beta}$ and

$$f\left(I(y\xi(T)) + F\right) > f\left(\frac{F}{1-\beta}\right) \geq \sup_{V<\frac{F}{1-\beta}} f(V), \quad \text{(A2)}$$

where the supremum is obtained when $v((\beta + \lambda(1-\beta))V - (\phi + \lambda F)) - y\xi(T)\beta V$ is evaluated at $V = \frac{F}{1-\beta} < \frac{I(x\xi(T))+\phi+\lambda F}{\beta+\lambda(1-\beta)}$, with the inequality in (A2) holding as equality for $\phi = 0$. So $I(y\xi(T)) + F$ is the global maximizer. When $\xi(T) \geq \xi^*$, we have $I(x\xi^*) \leq \frac{\beta F}{1-\beta}$ since $\xi^* \geq$

$\xi_*(\beta+\lambda(1-\beta))/\beta$, and this implies a tighter lower bound when $\phi > 0$:

$$I(x\xi^*) = \frac{\left(v(I(x\xi^*)) - v\left(\frac{\beta F}{1-\beta}\right)\right)}{x\xi^*} + \frac{\beta F}{1-\beta} - \phi \leq \frac{\beta F}{1-\beta} - \phi. \tag{A3}$$

Therefore, $\frac{I(x\xi(T))+\phi+\lambda F}{\beta+\lambda(1-\beta)} \leq \frac{F}{1-\beta}$, where the equality holds only when $\xi(T) = \xi^*$ with $\phi = 0$. Since now $f\left(\frac{F}{1-\beta}\right) = \sup_{V > \frac{F}{1-\beta}} f(V)$, and also

$$f\left(\frac{I(x\xi(T)) + \phi + \lambda F}{\beta + \lambda(1-\beta)}\right) - f\left(\frac{F}{1-\beta}\right) = -g(x\xi(T))x\xi(T) \geq 0, \tag{A4}$$

where the equality holds only for $\xi(T) = \xi^*$, we have $\frac{I(x\xi(T))+\phi+\lambda F}{\beta+\lambda(1-\beta)}$ as the global maximizer. When $\xi_* \leq \xi(T) < \xi^*$, we have $\frac{I(x\xi(T))+\phi+\lambda F}{\beta+\lambda(1-\beta)} < \frac{F}{1-\beta}$ only when $\phi > 0$ with $\xi(T) > v'\left(\frac{\beta F}{1-\beta} - \phi\right)/x$. Since $0 < g(x\xi) < \phi$ in that range (because $v'\left(\frac{\beta F}{1-\beta} - \phi\right)/x > \xi_*(\beta+\lambda(1-\beta))/\beta$), the inequality in (A4) is reversed, and so $\frac{F}{1-\beta}$ is the global maximizer. ■

Let $V(T)$ be any candidate optimal solution satisfying the static budget constraint (4). We have

$$\begin{aligned}
& E[v(V^*(T) - D(V^*(T)) - C(V^*(T)))] - E[v(V(T) - D(V(T)) - C(V(T)))] \\
= \; & E[v(V^*(T) - D(V^*(T)) - C(V^*(T)))] - E[v(V(T) - D(V(T)) - C(V(T)))] \\
& -yW(0) + yW(0) \\
\geq \; & E[v(V^*(T) - D(V^*(T)) - C(V^*(T)))] - E[y\xi(T)(V^*(T) - D(V^*(T)))] \\
& -E[v(V(T) - D(V(T)) - C(V(T)))] + E[y\xi(T)(V(T) - D(V(T)))] \geq 0,
\end{aligned}$$

where the former inequality follows from the static budget constraint holding with equality for $V^*(T)$, while holding with inequality for $V(T)$. The latter inequality follows from Lemma 1. This establishes the optimality of $V^*(T)$, or equivalently of $W^*(T)$. Then, from (5), it is clear that $W^*(T;y) \geq W^B(T;y)$, and except when equal to $\frac{\beta F}{1-\beta}$, $W^*(T;y)$ is decreasing in y. Hence, to allow the static budget constraint hold with equality, we must have $y \geq y^B$, which establishes property (i). Finally, since the benchmark policy (Cox and Huang (1989)) is $W^B(T) = I(y^B\xi(T))$, and $I'(\cdot) < 0$, property (i) yields the first inequality stated in property (ii), whereas the parameter values used in Figure 4 illustrate the second inequality for $\lambda > 0$. ■

Proof of Corollary 1. From (5), and the expression for $C(W^*(T))$ in footnote 8, we have that

$$\begin{aligned}
W^*(T) + C(W^*(T)) \; = \; & I(y\xi(T))1_{\{\xi(T)<\xi_*\}} + \frac{\beta F}{1-\beta}1_{\{\xi_* \leq \xi(T) < \xi^*\}} \\
& + \left(I\left(\frac{\beta y\xi(T)}{\beta+\lambda(1-\beta)}\right) + \phi + \lambda F\right)\frac{\beta}{\beta+\lambda(1-\beta)}1_{\{\xi^* \leq \xi(T)\}}.
\end{aligned} \tag{A5}$$

Rearranging (A5), and using the structure of x, ξ_*, and $W^B(T)$, yields the expression in property (i). By Assumption 1, $D^*(V^*(T)) = F - \max\{F - (1-\beta)V^*(T), 0\}$, and $V^*(T)$ is given in (A1), which yields the expression in property (ii). As $\xi(T) \to \infty$, the lower bound on $D^*(T)$, is immediate to verify. Finally, we note that in Figure 4(a) ξ^* is higher than $\xi^B = 1$, whereas in Figure 5(a) when $F = 2$, and in Figure 5(b) when $\beta = 0.6$, ξ^* equals 0.51 and 0.82, respectively – both lower than $\xi^B = 1$, thereby serving as examples for the last assertion in the corollary. ■

Proof of Proposition 2.

(i) From (2) and (3), Itô's lemma implies that $\xi(t)W^*(t)$ is a martingale:

$$W^*(t) = E\left[\frac{\xi(T-)}{\xi(t)}W^*(T-)\middle|\mathcal{F}_t\right] = E\left[\frac{\xi(T)}{\xi(t)}(W^*(T) + C(W^*(T)))\middle|\mathcal{F}_t\right], \tag{A6}$$

where the second inequality follows from the definition of W^* over $[0, T)$ as being the borrower's equity-cum-costs, financed by initial endowment $W(0)$, and $W^*(T)$ representing the time-T net worth after accounting for costs. When r and κ are constant, conditional on $\mathcal{F}_t$, $\ln \xi(T)$ is normally distributed with mean $\ln \xi(t) - (r + \frac{||\kappa||^2}{2})(T - t)$ and variance $||\kappa||^2(T - t)$. Substituting (A5) into (A6), using $I(x) = x^{-\frac{1}{\gamma}}$, and evaluating the conditional expectations over each of the three regions of $\xi(T)$ yields (6). The equation defining ξ^* is the counterpart of the corresponding equation in Proposition 1 for the case of an isoelastic objective function. When $\gamma = 1$, the equation solved by $y\xi^*$ becomes $\ln\left(\frac{\beta+\lambda(1-\beta)}{\beta y\xi^*}\right) + \beta y\xi^* \frac{\beta F-\phi(1-\beta)}{(1-\beta)(\beta+\lambda(1-\beta))} = 1 + \ln \frac{\beta F}{1-\beta}$, and we use it in Figures 4 and 5.

(ii) Applying Itô's lemma to (6), we get

$$\begin{aligned}\sigma_{W^*}(t) &= \left(\frac{X(T-t)}{(y\xi(t))^{\frac{1}{\gamma}}} - \frac{X(T-t)\mathcal{N}(-d_1(\xi_*))}{(y\xi(t))^{\frac{1}{\gamma}}} + \frac{X(T-t)\mathcal{N}(-d_1(\xi^*))}{(\beta y\xi(t)/(\beta+\lambda(1-\beta))^{\frac{1}{\gamma}}}\frac{\beta}{\beta+\lambda(1-\beta)}\right)\frac{\kappa}{\gamma}\\ &+ \frac{\beta}{\beta+\lambda(1-\beta)}\left(\frac{\beta F}{1-\beta} - \phi - \left(\frac{\beta+\lambda(1-\beta)}{\beta y\xi^*}\right)^{\frac{1}{\gamma}}\right)\frac{\varphi(d_2(\xi^*))e^{-r(T-t)}}{||\kappa||\sqrt{T-t}}.\end{aligned}$$

From (3), $\sigma_{W^*}(t)$ must equal $\sigma(t)^\top\theta^*(t)W^*(t)$. Using the well-known value of θ^B, we obtain

$$\begin{aligned}m^*(t) &= \left(\frac{X(T-t)}{(y\xi(t))^{\frac{1}{\gamma}}} - \frac{X(T-t)\mathcal{N}(-d_1(\xi_*))}{(y\xi(t))^{\frac{1}{\gamma}}} + \frac{X(T-t)\mathcal{N}(-d_1(\xi^*))}{(\beta y\xi(t)/(\beta+\lambda(1-\beta)))^{\frac{1}{\gamma}}}\frac{\beta}{\beta+\lambda(1-\beta)}\right)\frac{1}{W^*(t)}\\ &+ \frac{\gamma\beta}{\beta+\lambda(1-\beta)}\left(\frac{\beta F}{1-\beta} - \phi - \left(\frac{\beta+\lambda(1-\beta)}{\beta y\xi^*}\right)^{\frac{1}{\gamma}}\right)\frac{\varphi(d_2(\xi^*))}{||\kappa||\sqrt{T-t}}\frac{e^{-r(T-t)}}{W^*(t)}.\end{aligned} \tag{A7}$$

Rearranging (A7) yields (8).

(iii) m^* in (A7) equals a sum of two terms. From (A5) and (6) we have $W^*(t) \geq 0$, and since $\mathcal{N}(-d_1(\xi_*)) \leq 1$, the first term of the sum in (A7) is nonnegative. Noting that $\mathcal{N}(-d_2(\xi_*)) \geq \mathcal{N}(-d_2(\xi^*))$, inspection of (6) reveals that the first term of the sum in (A7) is less than, or equal to 1. The inequality in (A3) implies that the second term of the sum in (A7) contributes a nonnegative value to m^*, which establishes that $m^* \geq 0$. The second term in the sum vanishes for $\phi = 0$, as then $\xi^* = \xi_*(\beta + \lambda(1 - \beta))/\beta$, and so for $\phi = 0$ we have $m^* \leq 1$. Finally, when $\phi > 0$, the dot-dashed plot in Figure 5(d) and the dashed plot in Figure 5(e) provide evidence for the last assertion in (*iii*). ■

Proof of Proposition 3. To show that $W^\star(T)$ and $W^\star(T')$ are the optimal solution to the borrower's optimization problem when $T < T'$ is a straightforward extension of the proof of Proposition 1, and is therefore omitted. Property (*i*) is analogous to property (*i*) in proposition 1, and the inequalities in properties (*ii*)-(*iii*) are immediate to verify. ■

Proof of Corollary 2. The proof is similar to the proof of Corollary 1, and is therefore omitted. ■

Proof of Proposition 4. The proof is as of Proposition 2, with ξ_* and ξ^* replaced appropriately by $\xi_\star$ and $\xi^\star$. ■

Proof of Proposition 5. Summing over the borrower's and lender's time-t optimal wealth, (6)/(9) and (13), substituting for $y_b = (X(\overline{T})/W_b(0) - Z(0))^\gamma$ and algebraically manipulating yields (15)-(16). As $\xi(t) \to 0$, $Z(t) \to 0$, $W_M(t) \to \frac{W_b(0)+W_\ell(0)-Z(0)}{X(t)\xi(t)^{\frac{1}{\gamma}}}$ yielding property (*i*). As $\xi(t) \to \infty$, $W_M^B(t) \to 0$, $W_M(t) \to Z(t)$, yielding property (*ii*). ■

Proof of Proposition 6. Applying Itô's Lemma to (15)-(16) yields the expressions for $||\sigma_M(t)||$ and $\mu_M(t) - r$. When default is costly, for $(T = T', \phi = 0)$ in (8) we have $m^*(t) \in (0,1)$, and for $T < T'$ in (10) we have $m^\star(t) \in (0,1)$. In these cases, we then have $Y(t) \in (0,1)$, and noting that $\overline{W}_b(t) \le W_M(t)$ yields properties (*i*) and (*ii*). For $(T = T', \phi > 0)$ in (10), we may have $m^\star(t) > 1$ and hence $Y(t) < 0$, confirming the last assertion in the proposition. ■

Proof of Proposition 7. The proof is a straightforward combination of the proofs of Propositions 1 and 3. For brevity, we therefore only provide here the expressions for the parameters and functions used in stating the proposition: $\zeta_* = ((1-\beta')/(\beta' F'))^\gamma$, ζ^* solves

$$\gamma' \left(\tfrac{\beta'+\lambda'(1-\beta')}{\beta'\zeta^*}\right)^{\frac{1-\gamma}{\gamma}} + \tfrac{(1-\gamma)(\beta' F' - \phi'(1-\beta'))}{(1-\beta')(\beta'+\lambda'(1-\beta'))}\beta'\zeta^* = \left(\frac{\beta' F'}{1-\beta'}\right)^{1-\gamma}, \text{ when } \gamma \neq 1, \text{ or}$$

$$\ln\left(\tfrac{\beta'+\lambda'(1-\beta')}{\beta'\zeta^*}\right) + \tfrac{\beta' F' - \phi'(1-\beta')}{(1-\beta')(\beta'+\lambda'(1-\beta'))}\beta'\zeta^* = 1 + \ln\frac{\beta' F'}{1-\beta'}, \text{ when } \gamma = 1,$$

and

$$\hat{\xi}_\star = \frac{1}{y}\left(\frac{G(y)}{\beta F/(1-\beta) - H(y)}\right)^\gamma,$$

$$\hat{\xi}^\star = \frac{\beta + \lambda(1-\beta)}{\beta y}\left(\frac{G(\beta y/(\beta+\lambda(1-\beta)))}{\beta F/(1-\beta) - \phi - H(\beta y/(\beta+\lambda(1-\beta)))}\right)^\gamma,$$

$$
\begin{aligned}
G(x) &\equiv X(T'-T)\left(1-\mathcal{N}\left(-\hat{d}_1\left(\zeta_*/x\right)\right)+\left(\frac{\beta'}{\beta'+\lambda'(1-\beta')}\right)^{\frac{\gamma-1}{\gamma}}\mathcal{N}\left(-\hat{d}_1\left(\zeta^*/x\right)\right)\right),\\
H(x) &\equiv e^{-r(T'-T)}\left(\frac{\beta'F'}{1-\beta'}\mathcal{N}\left(-\hat{d}_2\left(\zeta_*/x\right)\right)-\left(\frac{\beta'F'}{1-\beta'}-\phi'\right)\frac{\beta'}{\beta'+\lambda'(1-\beta')}\mathcal{N}\left(-\hat{d}_2\left(\zeta^*/x\right)\right)\right),\\
\hat{d}_2(x) &\equiv \frac{\ln\frac{x}{\xi(T)}+(r-\frac{||\kappa||^2}{2})(T'-T)}{||\kappa||\sqrt{T'-T}},\\
\hat{d}_1(x) &\equiv \hat{d}_2(x)+\frac{1}{\gamma}||\kappa||\sqrt{T'-T}. \qquad \blacksquare
\end{aligned}
$$

Proof of Proposition 8. Setting $P(D^\star(T)<F|\mathcal{F}_0)=\mathcal{N}\left(\frac{\ln\frac{1}{\xi^\star}-(r+\frac{||\kappa||^2}{2})T}{||\kappa||\sqrt{T}}\right)$ equal to α yields the expression for $\xi^\star(\alpha)$. From Proposition 3, $\xi_\star \equiv \frac{1}{y}\left(\frac{1-\beta}{\beta F}X(T'-T)\right)^\gamma$ must satisfy $\xi_\star = \xi^\star(\alpha)\frac{\beta}{\beta+\lambda(1-\beta)}\left(\frac{\beta F-\phi(1-\beta)}{\beta F}\right)^\gamma$, which allows to express the face value as in (17). $\blacksquare$

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