The Underlying Dynamics of Credit Correlations

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Abstract

We propose a hybrid model of portfolio credit risk where the dynamics of the underlying latent variables is governed by a one factor GARCH process. The distinctive feature of such processes is that the long-term aggregate return distributions can substantially deviate from the asymptotic Gaussian limit for very long horizons. We introduce the notion of correlation spectrum as a convenient tool for comparing portfolio credit loss generating models and pricing synthetic CDO tranches. Analyzing alternative specifications of the underlying dynamics, we conclude that the asymmetric models with TARCH volatility specification are the preferred choice for generating significant and persistent credit correlation skews.

1 Introduction

The credit derivatives market, which exceeds $12 trillion according to most recent estimates from the International Swaps and Derivatives Association [2], encompasses a wide range of instruments, from plain vanilla credit default swaps, to credit swaptions, portfolio CDS, and synthetic CDO tranches which are becoming a part of the standard toolkit of credit investors [18]. Together with the growth of the credit derivatives market there has been a great deal of progress in quantitative modeling for both single-name credit derivatives and for structured credit products (see [14] and [25] for a comprehensive review and further references).

The latest advances in credit correlation modeling were in part motivated by the growth and sophistication of the so called correlation trading strategies, namely strategies involving standardized tranches referencing the Dow Jones CDX (US) and iTraxx (Europe) broad market indexes. The synthetic CDO

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market in general, and the standard tranche market in particular, allows investors to take rather specific views on the shape of the credit loss distribution of the underlying diversified collateral portfolio. The investor's views on various slices of this distribution are now well exposed through the pricing of liquid standard tranches, which in turn is expressed through their implied correlations.

More recently, market participants have switched from using implied correlations defined for each tranche to the notion of base correlation which has proven useful because it allowed translation of the pricing function of the set of standard tranches which was a function of two variables (attachment and detachment points) to a one-dimensional pricing function of the base equity tranches which only depends on the detachment point. This mapping is similar to a mapping of the bull spread options with various lower and upper limits onto a sequence of call options with various strikes – with the base equity tranches being analogous to a call option on the survival of the portfolio, and the generic mezzanine tranches being analogous to bull spread options on survival (see [23] for more details). The base correlation framework has become a de-facto industry standard, and historical comparisons of base correlation levels and its skew are frequently used to justify investment decisions.

In this paper we propose a simple model of portfolio credit risk with a one factor GARCH dynamics of loss generating latent variables. Our objectives in designing the model were to give a plausible explanation to the prominent correlation skew observed in synthetic CDO markets, and to investigate which of the properties of the underlying portfolio loss generating models are most relevant for this task. Our conclusions confirm some of the results known to analysts in this field, such as the importance of asymmetry in the loss distribution, and provide a substantially more detailed understanding to the origins of this asymmetry, its dynamics and dependence on term to maturity and other model parameters.

We begin the paper by providing some motivation for the choice of the model type in section 2. In particular, we argue that there are many parallels between modeling of synthetic CDO tranches and modeling of out of money put options on equity indexes. From these analogies it follows that a dynamic model with a richer structure than the standard log-normal Black-Scholes-Merton model must be considered to account for the important features of derivatives traded in the marketplace, most importantly the volatility skew and term structure (for equities market) and the correlation skew (for synthetic CDO market). We conclude that the models of GARCH type have the right properties as candidates for the underlying dynamics describing credit correlations.

Assuming a factor-GARCH model for single-period returns, we derive analytical formulas for skewness and kurtosis of the cumulative return distributions for a variety of specifications of the single period GARCH process. We conclude that for sufficiently long horizons (greater than several months) the effects of stochastic volatility and volatility asymmetry dominate the effects of non-normality of single-period return shocks. We then demonstrate that the empirically estimated parameters of the market factor time series, proxied by the S&P 500 index, do indeed lead to non-Gaussian distribution of cumulative returns for horizons up to 5 or even 10 years.

The connection with credit correlation modeling is made in section 4, where we show that the pairwise lower tail dependence of equity returns and the pairwise default correlation defined in the latent variable framework via the same returns are asymptotically equal as the default threshold (tail threshold) is taken to the lower zero limit. The one factor GARCH model leads to a significantly different dependence of both measures on the risk threshold compared to the previously studied copula models. Both the lower tail dependence and the pairwise default correlations are shown to increase at very low thresholds which is precisely the behavior that would be expected of any model that aims to explain the steep correlation skew growing toward higher attachment points (i.e. lower default thresholds).

In section 5 we lay the groundwork for extending our analysis to portfolio credit risk models by giving a brief introduction to the general copula framework, the pricing methodology for synthetic CDO tranches, and the large homogeneous portfolio approximation which we adopt in the rest of the paper. In section 5.2 we argue that the simple pairwise credit correlation is insufficient for description of the portfolio loss distributions even in the LHP approximation, as it only relates to the second moment of the distribution, the volatility of losses, and does not fully specify the shape of the distribution tails. As a tool for a more complete description of portfolio loss distribution, we introduce the correlation spectrum measure which both simplifies and extends the widely used notion of base correlations to a framework suitable for comparison of various default loss generating models.

In section 6 we compute the correlation spectra for various loss generating models and use them to study the impact of stylized characteristics of market factor dynamics on the portfolio credit risk. First, we show that models with fat tails, such as the static t-copula model, cannot generate upward sloping correlation skew unless the distribution of the market factor is decoupled from the distribution of the idisyncratic returns (as it done, for example, in the double-t copula model), and the latter have thinner tails than the market factor. Among the dynamic loss generating models, we are able to discriminate between the specifications of market factor time series and practically rule out those which do not have an asymmetric volatility process. We demonstrate that the empirical parameters estimated for S&P 500 as a market factor correspond to a substantial credit correlation skew in our methodology, thus confirming that a large portion of the synthetic CDO tranche pricing reflects real risks and not just risk premia. We conclude the section by examining the dependence of the correlation skew on various model parameters such as term to maturity and level of hazard rates – and thereby demonstrate one of the most important advantages of our methodology, in which the correlation skew is not an input but an output of the model and therefore its properties and dependencies can be predicted rather than postulated.

Section 7 presents a brief summary and outline of remaining open questions and possible extensions of our methodology. The Appendices present additional proofs and empirical details.

2 Modelling Credit and Equity Derivatives

There are many analogies between modeling of equity derivatives and modeling of credit derivatives in the latent variable framework. While some of these analogies are superficial, others are intimately related to the structure of the products and the structure of the models used to price them.

The simplest and often cited analogy is between the implied volatility and implied correlation. Quoting the implied volatility of an equity option (together with the level of the underlying stock and the option strike) is equivalent to quoting its price within the standard Black-Scholes-Merton model. In the same fashion quoting the implied correlation (together with spread levels of the reference portfolio and the tranche attachment and detachment levels) is equivalent to quoting the price of a synthetic CDO tranche within the so called Gaussian copula model which has become a de-facto standard in the industry. The Gaussian copula model as applied to portfolio credit risk was introduced by Li [19], and extends similar approaches developed earlier for portfolio market value-at-risk [7], and long-term insurance portfolio loss [11] modeling.

This analogy becomes much deeper and more useful if we focus on the finer details of derivatives pricing. Just as the observation of a non-trivial implied volatility surface reflects deviations from the Black-Scholes model assumptions, the observation of the non-trivial base correlation skew reflects deviations from the Gaussian copula model assumptions. These assumptions are essentially equivalent to those of CreditMetrics model of portfolio loss distribution [4] which, in turn, were derived from an adaptation of Merton's structural model of credit risk [22] with corresponding assumption of log-normality of asset returns. In the Gaussian copula model, the multi-variate probability distribution of times to default is generated as a transformation (with a constant dependence structure) of the multi-variate distribution of asset returns of portfolio constituents. Thus, it stands to reason that either the assumption of the single-factor log-normal distribution of asset returns, or the assumption of the constant dependence structure implied in the Gaussian copula model, or both, are inconsistent with synthetic CDO tranche pricing as reflected by the well established presence of the base correlation skew.

The observation that using the Gaussian copula model is in principle equivalent to using a version of Merton's original model is under-appreciated by many researchers. With this implicit use of Merton's model also come certain well-known drawbacks such as the insufficient probabilities of downside risks for investment grade issuers in the near- and intermediate terms. From the econometric perspective, the main drawbacks of the classic Merton model are its inability to account for a number of well established stylized facts regarding the time series properties of observed equity returns, such as the stochasticity and persistence of volatilities, asymmetry of volatility response to returns with levels that are well beyond what that can be explained by the simple capital structure leverage effect, and the presence of fat tails and other non-Gaussian features.

The adaptation of the copula-based methodology to reduced-form models

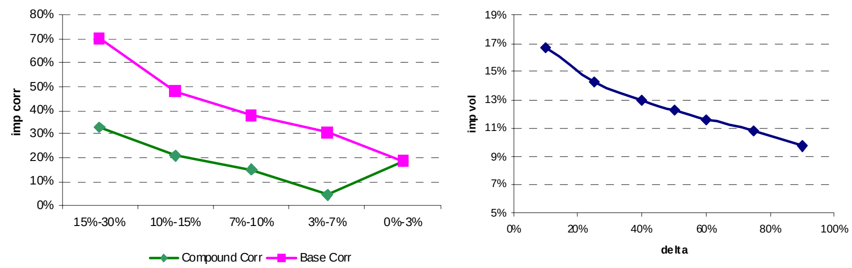

Figure 1: Compounded and base correlation skew for CDX.NA.IG series 3 as of March 2005 (left hand side) and implied volatility skew for S&P 500 index options with 1 year expiry as of March 2005 (right hand side).

of default risk [26], and its re-interpretation in terms of generic latent variable models [12] have opened the possibility to reconcile the parsimony of the copula methodology with more flexible models of single-name credit risk. In particular, one no longer has to explicitly assume that the latent variable driving the generation of default times is log-normally distributed. Among the important steps towards more realistic modeling of the dependence structure of portfolio risks within this hybrid framework are the multi-factor Gaussian copula models [13], the extension to non-Gaussian copulas and in particular to Student-t copulas [20] reflecting the fat-tailed distribution of asset returns, and the explicit modeling of asymmetric latent variable distributions [1].

A lot of intuition about the shape of the base correlation can be gained by simply noting that, given a certain level of underlying index spreads, the higher the attachment point K, the farther out-of-money is the senior tranche (i.e. the tranche which is exposed to losses above K and up to 1). In the case of the equity index options the out-of-money put options are typically priced with a higher level of implied volatility which corresponds to a much fatter downside tails of the implied return distribution. Similarly, the senior synthetic CDO tranches are typically priced at a higher level of base correlation which corresponds to fatter downside tails of default loss distributions (compare the figures in 1, where we have drawn the correlation skew graph in somewhat unusual way, by placing the farther out-of-money senior tranches to the left of x-axis to emphasize the similarity with put options).

Such pricing is commonly attributed to investors' risk aversion to large loss scenarios and correspondingly higher risk premia demanded for securities exposing them to such scenarios. However, we believe that it would be unfair to think of the entire cost premium between various in- and out-of-money tranches as risk premium and that there are real risks which are being compensated by these additional costs, albeit perhaps still accompanied by (relatively smaller) risk premia.

To justify this line of thought let us return for a moment to the case of equity index options and recall that the empirical distribution of returns does indeed exhibit significant downside tails, and that a large part of the implied volatility skew can be explained by the properties of the empirical distribution [5]. Let us list some of the key stylized facts that are known to be relevant for explanation of the equity index option pricing: 1) the fat tails in the return distribution can explain the implied volatility smile; 2) the asymmetry in the return distribution is a necessary ingredient for explaining the implied volatility skew; 3) there exists an implied volatility surface with non-trivial strike and term structure; 4) the term dependence of the volatility surface is determined by the long-run aggregated return distribution characteristics which can be significantly different from those of the short-term (single-period) return distribution; 5) the implied volatility surface has a much more pronounced skew for stock indexes than for individual stocks, reflecting a more important role of the common driving factors compared to idiosyncratic returns in the explanation of the downside risks.

Given the above mentioned analogies between the synthetic CDO tranches and equity index options, it is quite natural to look for similar stylized facts that could explain the shape of the base correlation as a function of detachment level and, potentially, term to maturity, and its key dependencies on the market and model parameters. As we already noted, the standard Gaussian copula framework implicitly relies on the Merton-style structural model for definition of default correlations.

Therefore, if we are to give empirical explanation to the observed base correlation skew we must start by giving an empirical meaning to the variables in this model. Our working hypothesis in this paper will be that the meaning of the "market factor" in the factor copula framework is the same as the market factor used in the equity return modeling. As such, it is often possible to use an observable broad market index such as S&P 500 as a proxy for the economic market factor, with an added convenience that there exists a long historical dataset for its returns and a rich set of equity options data from which one can glean an independent information about their implied return distribution.

This hypothesis is not uncommon in portfolio credit risk modeling – for example, the authors of [20] emphasized the importance of using a fat-tailed distribution of asset returns in the copula framework in part by citing the empirical evidence from equity markets. However most researchers have focused on the single-period return distribution characteristics.

In contrast, we focus in this paper on the long-run cumulative returns, and prove that their distribution is quite distinct from that of the short-term (single-period) returns. As we will show in the rest of this paper, it is the time aggregation properties and the compounding of the asymmetric volatility responses that make it possible to explain the credit correlation skew for 5- or even 10-year horizons. Moreover, a dynamic explanation of the skew such as presented in this paper, allows one to make rather specific predictions for the dependence of this skew on both the term to maturity and on the hazard rates and other model parameters.

3 Time series models of short and long horizon equity returns

In many applications we first specify time series properties of stock returns for high frequency time intervals (daily or weekly) and then derive the distribution of stock prices over longer horizons measured in months or even years. The popular log normal specification that forms the basis of the Black-Scholes-Merton option pricing model assumes constant mean returns and volatilities and iid Gaussian return shocks which leads to the same (log-normal) shape of the distribution of stock prices for all future horizons.

Models with more realistic dynamics can lead to richer distribution of time aggregated returns with fat tails and negative skewness even if we assume Gaussian distributions for the return innovations. In particular, models of GARCH type conform well to the stylized facts regarding both short- and long horizon equity returns. The autoregressive stochastic volatility process [8] captures the essence of volatility persistence and clustering observed in the historical time series. In an extended GARCH framework (see [3] and [9] for a comprehensive review), the non-Gaussian return shocks and the asymmetric response of volatility to return innovations account for a significant amount of the explanatory power in most model specifications, especially with regard to description of long-horizon aggregate returns. The term structure of fat-tailness and skewness of aggregated returns depends on the parametric form chosen for the volatility process [6].

The volatility dynamics affects not only the marginal distributions of stock returns but also the distribution of stock co-movements over long horizons or more generally the copula of long horizon returns. The log-normal model implies a Gaussian copula for any time horizon whereas multivariate models with more realistic dynamic properties result in non-Gaussian copulas.

In this paper we focus on two non-Gaussian features of long horizon return copulas: tail dependence and asymmetry. In this section we describe a simple one factor model with TARCH(1,1) dynamics that allows us to incorporate persistence and asymmetry in volatility and correlations and yet is tractable enough to derive qualitative and quantitative results for non-Gaussian properties of long horizon return distributions. We begin by describing the univariate model, and then generalize it to a multi-variate framework with a single factor structure of returns.

3.1 Univariate model: TARCH(1,1)

Let r_t be the log-return of a particular stock or an index such as SP500 from time $t-1$ to time t . $\mathcal{F}_t$ denotes the information set containing realized values of all the relevant variables up to time t. We will use the expectation sign with subscript t to denote the expectation conditional on time t information set: $E_t(.) = E(.|\mathcal{F}_t)$. The time step that we use in the empirical part is 1 day or 1 week. As we already mentioned, predictability of stock returns is negligible over such time horizons and therefore we assume the conditional mean is constant and equal to zero[1]:

$$m_t \equiv E_{t-1}(r_t) = 0 \tag{1}$$

The conditional volatility $\sigma_t^2 \equiv E_{t-1}(r_t^2)$ of r_t in TARCH(1,1) has the autoregressive functional form similar to the standard GARCH(1,1) but with an additional asymmetric term [31]:

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= \omega + \alpha r_{t-1}^2 + \alpha_D r_{t-1}^2 1_{\{r_{t-1} \le 0\}} + \beta \sigma_{t-1}^2 \end{aligned} \tag{2}$$

We assume that $\{\varepsilon_t\}$ are iid with zero mean, unit variance, finite skewness s_ε and kurtosis k_ε.We also assume that $\omega > 0$ and α, α_D, β are non-negative so that the conditional variance σ_t^2 is guaranteed to be positive.

Let us introduce the notations for the moments of ε_t that will be used in some of the formulas below:

$$\begin{aligned} m_\varepsilon &\equiv E\left(\varepsilon_t\right) = 0 \\ v_\varepsilon &\equiv E\left(\varepsilon_t^2\right) = 1 \\ v_\varepsilon^d &\equiv E\left(\varepsilon_t^2 1_{\{\varepsilon_t \le 0\}}\right) \\ s_\varepsilon &\equiv E\left(\varepsilon_t^3\right) \\ s_\varepsilon^d &\equiv E\left(\varepsilon_t^3 1_{\{\varepsilon_t \le 0\}}\right) \\ k_\varepsilon &\equiv E\left(\varepsilon_t^4\right) \\ k_\varepsilon^d &\equiv E\left(\varepsilon_t^4 1_{\{\varepsilon_t \le 0\}}\right) \end{aligned} \tag{3}$$

The persistence of stochastic volatility in the model is governed by the parameter ζ which is calculated as follows[2]:

$$\zeta \equiv E\left(\beta + \alpha \varepsilon_t^2 + \alpha_D \varepsilon_t^2 1_{\{\varepsilon_t \le 0\}}\right) = \beta + \alpha + \alpha_D v_\varepsilon^d \tag{4}$$

If $\zeta \in [0, 1)$ then conditional variance mean-reverts to its unconditional level $\sigma^2 = E\left(\sigma_t^2\right) = \frac{\omega}{1-\zeta}$. The following parameter ξ will also be useful in describing the higher moments of TARCH(1,1) returns and volatilities:

$$\xi \equiv E\left(\beta + \alpha \varepsilon_t^2 + \alpha_D \varepsilon_t^2 1_{\{\varepsilon_t \le 0\}}\right)^2 = \beta^2 + \alpha^2 k_\varepsilon + \alpha_D^2 k_\varepsilon^d + 2\alpha\beta + 2\alpha_D \beta v_\varepsilon^d + 2\alpha\alpha_D k_\varepsilon^d \tag{5}$$

We can rewrite 2 in terms of the increments of the conditional volatility $\Delta\sigma_{t+1}^2 \equiv \sigma_{t+1}^2 - \sigma_t^2$ and the volatility shocks η_t

[1] We will discuss later the "risk-neutralization" of the return process which requires certain drift restrictions in the derivatives pricing context.

[2] Note that for ε_t^2 with symmetric distribution $v_\varepsilon^d = 0.5$.

$$
\begin{aligned}
r_t &= \sigma_t \varepsilon_t \\
\Delta\sigma_{t+1}^2 &= (1-\zeta)\left(\sigma^2-\sigma_t^2\right)+\sigma_t^2\eta_t \\
\eta_t &\equiv \alpha\left(\varepsilon_t^2-1\right)+\alpha_D\left(\varepsilon_t^2 1_{\{\varepsilon_t\leq 0\}}-v_\varepsilon^d\right)
\end{aligned}
\tag{6}
$$

The speed of mean reversion in volatility is $1-\zeta$ and is small when ζ is close to one which is usually true for daily and weekly equity returns – hence the persistence of the stochastic volatility. Using this result, we can estimate the term dependence of the periodic (short-term) returns variance

$$
E_{t-1}\sigma_{t+n}^2 = \sigma^2+\zeta^n\left(\sigma_t^2-\sigma^2\right) \text{ for } n\geq 0 \tag{7}
$$

The TARCH(1,1) volatility shocks η_t are iid, with zero mean and the following variance:

$$
var(\eta_t) = var(\alpha\varepsilon_t^2+\alpha_D\varepsilon_t^2 1_{\{\varepsilon_t\leq 0\}}) = (\alpha+\alpha_D\xi)^2 k_\varepsilon+\alpha_D^2(1-\xi)\xi(k_\varepsilon+1) \tag{8}
$$

Persistent and volatile volatility produces fat tails in the unconditional return distribution even for models with Gaussian shocks. It is easy to see from 6 that conditional volatility of σ_{t+1}^2 is proportional to σ_t^4 and $var(\eta_t)$

$$
var_{t-1}\left(\sigma_{t+1}^2\right) = var_{t-1}\left(\sigma_t^2\eta_t\right) = \sigma_t^4 var(\eta_t) \tag{9}
$$

The correlation of conditional volatility with the return in the previous period depends on the covariance of return and volatility innovations

$$
corr_{t-1}\left(r_t,\sigma_{t+1}^2\right) = corr_{t-1}\left(\varepsilon_t,\eta_t\right) = \frac{\alpha s_\varepsilon+\alpha_D s_\varepsilon^d}{\sqrt{var(\eta_t)}} \tag{10}
$$

The negative correlation of return and volatility shocks, often cited as the "leverage effect"[3], is the main source of the asymmetry in the return distribution. We can see from formula 10 that negative return-volatility correlation can be achieved either through negative skewness of return innovations $s_\varepsilon < 0$, through asymmetry in volatility process $\alpha_D > 0$ or combination of the two. We call these static and dynamic asymmetry, respectively.

In this paper we are interested in the effects of the volatility dynamics on the distribution of long horizon returns. While a closed form solution for the probability density function of TARCH(1,1) aggregated returns is not available, we can still derive some analytical results for its conditional and unconditional moments: volatility, skewness and kurtosis.

[3] Though we note here that the magnitudfe of this "leverage effect" in return time series for stocks of most investment grade issuers far exceeds the amount that would be reasonable based purely on their capital structure leverage.

3.1.1 Volatility

The conditional variance $V_{t,t+T}$ of the normalized log return $R_{t,t+T} = \frac{1}{\sqrt{T}}(\ln S_{t+T} - \ln S_t) = \frac{1}{\sqrt{T}}\sum_{u=t+1}^{t+T} r_u$ encompassing T periods from t to $t+T$ follows directly from the term structure dependence of the periodic return variance 7:

$$V_{t,t+T} = E_t R_{t,t+T}^2 = \frac{1}{T} E_t \left(\sum_{t+1 \le u \le t+T} \sigma_u^2 \right) = \sigma^2 + \left(\sigma_{t+1}^2 - \sigma^2\right) \frac{1}{T} \frac{1-\zeta^T}{1-\zeta} \quad (11)$$

The unconditional variance is therefore the same as for the short-term returns:

$$V_T = E(V_{t,t+T}) = \sigma^2 \quad (12)$$

The deviation of the T-horizon conditional volatility $V_{t,t+T}$ from its unconditional level σ^2 depends on the current deviation of the short horizon volatility $\sigma_{t+1}^2 - \sigma^2$, aggregation horizon T and the level of volatility persistence ζ. See figure 2 for illustration.

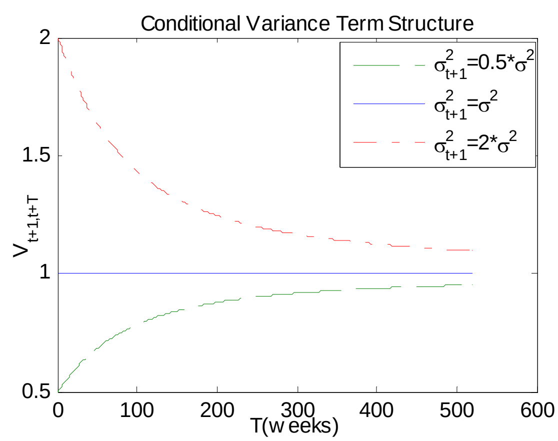

Figure 2: Term structure of conditional variance of time aggregated return $R_{t+1,t+T}$. TARCH(1,1) has persistence coefficient $\zeta = 0.98$ and the followng parametrization: $\sigma^2 = 1$, $\alpha = 0.01$, $\alpha_D = 0.10$, $\beta = 0.92$, $\varepsilon_t \sim N(0,1)$. We plotted volatility term structure for three different initial volatilities: $\sigma^2/2$, $\sigma^2$and $2\sigma^2$

3.1.2 Skewness

Skewness is a convenient measure of return distribution asymmetry. The following proposition gives the formulas for conditional and unconditional third moments of aggregated returns generated by the TARCH(1,1) model.

Proposition 1 *Suppose* $0 \leq \zeta < 1$ *and the return innovations have finite skewness,* s_ε*, and finite "truncated" third moment,* s_ε^d*. Then the conditional third moment of* $R_{t,t+T}$ *has the following representation for TARCH(1,1)*

$$E_t R_{t,t+T}^3 = \frac{1}{T^{3/2}} s_\varepsilon \sum_{u=1}^{T} E_t\left(\sigma_{t+u}^3\right) + \frac{3}{T^{3/2}}\left(\alpha s_\varepsilon + \alpha_D s_\varepsilon^d\right) \sum_{u=1}^{T} \frac{1-\zeta^{T-u}}{1-\zeta} E_t\left(\sigma_{t+u}^3\right) \tag{13}$$

In addition, if $E\sigma_t^3$ *is finite, then unconditional skewness of* $R_{t,t+T}$ *is given by*

$$S_T \equiv \frac{E R_{t,t+T}^3}{E(R_{t,t+T}^2)^{3/2}} = \left[\frac{1}{T^{1/2}} s_\varepsilon + 3\frac{1}{T^{3/2}}\left(\alpha s_\varepsilon + \alpha_D s_\varepsilon^d\right) \frac{T(1-\zeta) - 1 + \zeta^T}{(1-\zeta)^2}\right] E\left(\frac{\sigma_t}{\sigma}\right)^3 \tag{14}$$

Proof. *See appendix A for the details.* ∎

The conditional third moment is a function of the conditional term structure of σ_t^3, term horizon T and volatility parameters. The conditional skewness can be computed using second and third conditional moments derived above. The asymmetry in the return distribution arises from two sources - skewness of return innovations and asymmetry of the volatility process. Note that the second term in the formulas for conditional and unconditional skewness is directly related to the correlation of return and volatility innovations. If return-volatility correlation is zero ($\alpha s_\varepsilon + \alpha_D s_\varepsilon^d = 0$) then $S_T = \frac{1}{T^{1/2}} s_\varepsilon E\left(\frac{\sigma_t}{\sigma}\right)^3$. If return innovations are symmetric then asymmetric volatility drives the asymmetry in the return distribution. In Figure 3 we show conditional and unconditional skewness term structures. For realistic parameters corresponding approximately to parameters of the TARCH(1,1) estimated for weekly SP500 log returns, both conditional and unconditional skewness is negative. It decreases in the medium term, attains the minimum at approximately the 2 year point and then decays to zero as T increases. The skewness term structure conditional on the high/low current volatility is above/below the unconditional skewness.

3.1.3 Kurtosis

The fourth conditional moment, if it exists, describes the fat-tailness of the conditional return distribution and the volatility of return volatility. Formula 9 gives us the conditional volatility of the conditional volatility. The fourth conditional moment of one period return is proportional to the kurtosis of return innovations:

$$E_t r_{t+1}^4 = \sigma_{t+1}^4 k_\varepsilon \tag{15}$$

For symmetric return shocks and symmetric volatility dynamics ($s_\varepsilon = 0$ and $\alpha_D = 0$) kurtosis K_T has a simple representation in terms of the model parameters, according to the following proposition.

Proposition 2 *If the distribution of* ε_t *is symmetric and* $\alpha_D = 0$ *then uncon-*

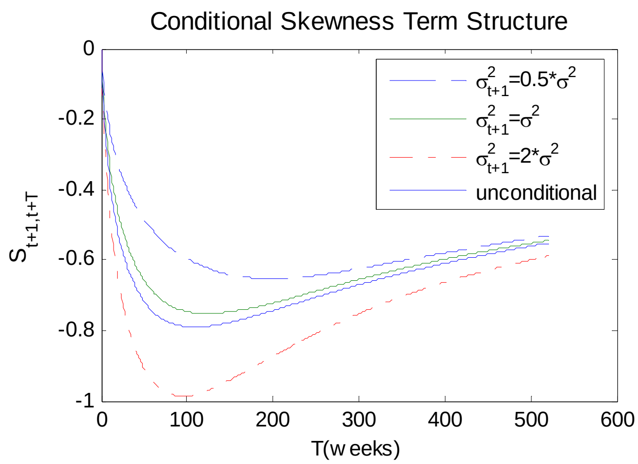

Figure 3: Term structure of conditional skewness of time aggregated return $R_{t+1,t+T}$.TARCH(1,1) has persistence coefficient $\zeta = 0.98$ and the following parametrization: $\sigma^2 = 1$, $\alpha = 0.01$, $\alpha_D = 0.10$, $\beta = 0.92$, $\varepsilon_t \sim N(0,1)$. We plotted unconditional skewness term structure and conditional for three different initial volatilities: $\sigma^2/2$, $\sigma^2$and $2\sigma^2$. The term structure of $E_t\sigma^3_{t+u}$was computed from 10,000 independent simulations.

ditional kurtosis of $R_{t,t+T}$, if exists, is given by the following formula:

$$K_T = 3 + \frac{1}{T}(K_1 - 3) + 6\frac{\gamma_1}{T^2}\frac{T(1-\zeta) - 1 + \zeta^T}{(1-\zeta)^2} \textit{for } T > 1 \qquad (16)$$

$$K_1 = k_\varepsilon \frac{1-\zeta^2}{1-\xi}$$

where k_r and k_ε are the unconditional kurtosis estimates of one period total returns r_t and return innovations ε_t, respectively, ζ is the persistence parameter 4, ξ is the parameter defined earlier in 5, and γ_1 is the first unconditional autocorrelation coefficient of squared returns, defined in A:

$$\gamma_1 \equiv corr\left(r^2_{t-1}, r^2_t\right) = \alpha\left(k_r - 1\right) + \alpha_D\left(k^d_r - v^d_r\right) + \beta k_r/k_\varepsilon$$

Proof. *See appendix A for the details.* ■

3.2 Multivariate model: One factor ARCH with TARCH(1,1) factor volatility dynamics

Let us now turn to a multi-variate model of equity returns for M companies, with a simple dynamic factor structure decomposing the returns into a common (market) and idiosyncratic components. To concentrate on the time dimension of the model we assume a homogeneity of cross-sectional return properties, namely that factor loadings and volatilities of idiosyncratic terms are constant and identical for all stocks. Thus, our homogeneous one factor ARCH model has the following form.

$$\begin{aligned} r_{i,t} &= b\sigma_{m,t}\varepsilon_{m,t} + \sigma\varepsilon_{i,t} \\ \Delta\sigma^2_{m,t+1} &= (1-\rho_m)\left(\sigma^2_m - \sigma^2_{m,t}\right) + \sigma^2_{m,t}\eta_{m,t} \\ \eta_{m,t} &\equiv \alpha\left(\varepsilon^2_{m,t} - 1\right) + \alpha_D\left(\varepsilon^2_{m,t}1_{\{\varepsilon_{m,t}\leq 0\}} - v^d_{m,\varepsilon}\right) \end{aligned} \tag{17}$$

where

- $b \geq 0$ is the constant market factor loading and it is the same for all stocks
- $r_{m,t}$ is the market factor return with zero conditional mean $E_{t-1}(r_{m,t}) = 0$, conditional volatility $\sigma^2_{m,t} \equiv E_{t-1}(r^2_{m,t})$ and unconditional truncated second moment of return innovations $v^d_{m,\varepsilon} \equiv E\left(\varepsilon^2_{m,t}1_{\{\varepsilon_{m,t}\leq 0\}}\right)$
- $\sigma\varepsilon_{i,t}$ are the idiosyncratic return components with constant volatilities σ^2 and zero conditional means $E_{t-1}(\sigma\varepsilon_{i,t}) = 0$
- $\{\varepsilon_{i,t}, \varepsilon_{m,t}\}$ are unit variance iid for each t and all i

In this model conditional volatilities and pairwise conditional correlations of stock returns are time varying and depend only on the volatility dynamics of the market factor.

$$\sigma^2_{i,t} \equiv Var_{t-1}(r^2_{i,t}) = b^2\sigma^2_{m,t} + \sigma^2 \tag{18}$$

$$\rho_{(i,j),t} = \frac{Cov_{t-1}(r_{i,t}, r_{j,t})}{\sqrt{\sigma^2_{i,t}\sigma^2_{j,t}}} = \frac{b^2\sigma^2_{m,t}}{\sigma^2 + b^2\sigma^2_{m,t}} \tag{19}$$

The unconditional correlation between returns of any two stocks is given by

$$\rho_{(i,j)} = \frac{b^2\sigma^2_m}{\sigma^2 + b^2\sigma^2_m} \tag{20}$$

The conditional pairwise correlation $\rho_{(i,j),t}$ is a strictly increasing function of market volatility $\sigma^2_{m,t}$ if $b > 0$ and therefore the persistence and asymmetry of the market volatility $\sigma^2_{m,t+1}$ translates into the persistence and dynamic asymmetry of the stock correlations $\rho_{(i,j),t}$.

Because of the simple linear factor structure and constant market loadings time aggregated equity returns $R_{i,T} = \frac{1}{\sqrt{T}}\sum_{u=1}^{T} r_{i,u}$ also have a one factor representation[4]

$$R_{i,T} = bR_{m,T} + E_{i,T} \tag{21}$$

[4] To simplify the notations we assume that the initial time $t = 0$ and use only subscipt for the time aggregation horizon T.

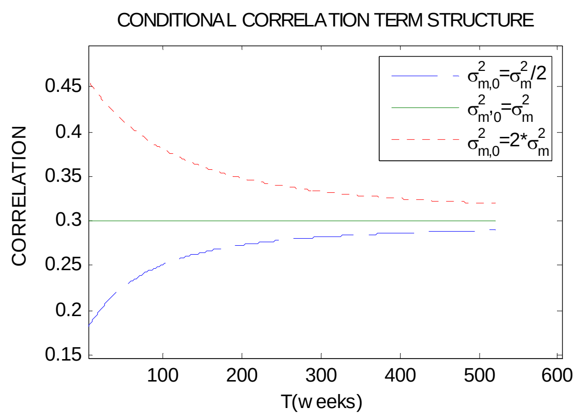

Figure 4: Conditional Correlation Term Structure of time aggregated returns $R_{i,T}$ and $R_{j,T}$. TARCH(1,1) has persistence coefficient $\zeta_m = 0.98$ and the following parametrization: $\sigma_m^2 = 1$, $\alpha_m = 0.01$, $\alpha_{m,D} = 0.10$, $\beta_m = 0.92$, $\varepsilon_{m,t} \sim N(0,1)$. We plotted conditional correlation term structure for three different initial volatilities: $\sigma_m^2/2$, $\sigma_m^2$and $2\sigma_m^2$.

where $R_{m,T} = \frac{1}{\sqrt{T}}\sum_{u=1}^{T} r_{m,u}$ and $E_{i,T} = \frac{1}{\sqrt{T}}\sigma\sum_{u=1}^{T}\varepsilon_{i,u}$ are independent conditional on $\digamma_0$.

The conditional variance, correlation, skewness and kurtosis of aggregated returns $R_{i,T}$ can be easily computed in terms of the corresponding moments of market and idiosyncratic returns.

$$V_{i,T} = E_0\left(R_{i,T}^2\right) = b^2 V_{m,T} + \sigma^2 \tag{22}$$

$$\Gamma_{(i,j),T} = corr(R_{i,T}, R_{j,T}|\digamma_0) = \frac{b^2 V_{m,T}}{b^2 V_{m,T} + \sigma^2} \tag{23}$$

$$S_{i,T} = \frac{E_0\left(R_{i,T}^3\right)}{V_{i,T}^{3/2}} = \Gamma_{(i,j),T}^{3/2} S_{m,T} + \left(1 - \Gamma_{(i,j),T}\right)^{3/2} S_{E,T} \tag{24}$$

$$K_{i,T} = \Gamma_{(i,j),T}^2 K_{m,T} + 6\Gamma_{(i,j),T}\left(1 - \Gamma_{(i,j),T}\right) + \left(1 - \Gamma_{(i,j),T}\right)^2 K_{E,T} \tag{25}$$

We can see from Figure 4 that indeed the term structure of conditional pairwise correlation resembles that of the conditional variance in Figure 2.

3.3 SP500 as a proxy for market return

To provide some empirical context to the theoretical discussion above, let us consider the estimation results of several TARCH(1,1) specifications for SP500 daily and weekly returns. We obtained the daily levels of SP500 from CRSP database. The total number of observations is 10699 and covers the period from 07/02/1962 till 12/31/2004. We constructed daily and weekly log returns and estimated the parameters of TARCH and GARCH models with Gaussian and Student-t shocks for 2 samples - full and post-1990.

Tables 1,2,3 in appendix B show estimated parameters and various data statistics. Note that the Student-t distribution has an additional parameter, degrees of freedom, that adjusts the tails of the error distribution[5]. Since the Gaussian distribution is nested within the Student-t as a limit of large degrees of freedom, and since the estimates of the full unconstrained model result in a relatively small and statistically significant value of the degrees of freedom, we conclude that the data points toward the fat-tailed return shock distribution.

On the other hand, the asymmetric TARCH model is nested within the symmetric GARCH in the limiting case $\alpha_D = 0$. The estimated asymmetric coefficient α_D in the TARCH model is not only non-zero, but significantly higher than the symmetric coefficient α for both complete and post 1990 samples, both daily and weekly frequencies and Gaussian and Student-t shock distributions. Thus, we conclude that the asymmetric volatility is prominently present in the data. The best fit model among those considered is the TARCH(1,1) with Student-t distribution of return innovations. The additional parameters of this model are statistically significant.

In Figure 5 we show the estimate of skewness for overlapping returns of different aggregation horizons measured in days. The full sample shows high negative skewness for one day return because of the 1987 crash. On the post 1990 sample negative skewness rises with aggregation horizon up to 1 year and then slowly decays toward zero. Both samples show significant skewness for horizons of several years. We should note that confidence bounds around skewness curves are quite wide due to the persistence and high volatility of the squared returns and serial correlation of the overlapping observations.

To make sure that asymmetry in volatility is not a result of several extreme negative returns like 1987 crash we provide data statistics and re-estimated parameters of TARCH models for trimmed full and post 1990 samples. The trimming is done by cutting excess volatility in the most extreme 0.05% observations of both positive and negative return. We can see from Tables 1b and 2b that trimming significantly reduced skewness s_r of daily returns but the volatility of daily returns is still significantly asymmetric. Weekly returns are less affected by trimming both in terms of TARCH parameters and unconditional skewness and kurtosis. However, the long-run skewness of aggregate returns remains largely unaffected by the trimming because it is driven mostly by the asymmetry of the volatility and the value of α_D does not change much due to trimming, especially for the model versions with Student-t distributed return shocks.

[5] The Student-T distribution is sometimes critiqued as a model for continously componded returns because the expectation of the exponent of Student-T variable is infinite and therefore expected return over one period is also infinite. In practice (estimation) we can think of Studen-T distribution as being truncated at far enough tails so that the estimation procedure is not changed, while the expectation of the exponent is finite.

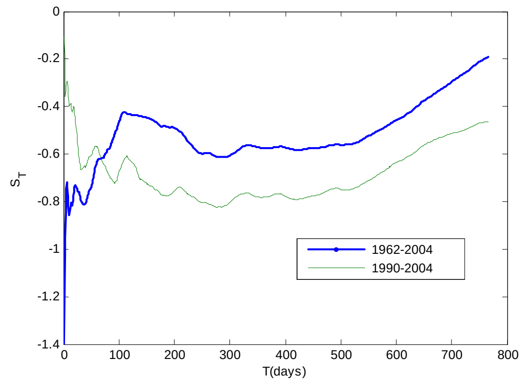

Figure 5: Term structure of skewness for SP500 time aggregated log returns estimated with overlapping samples moments for full and post-1990 data.

4 Modeling Tail Risk and Default Correlation

The dynamic models of aggregate equity returns presented in the previous section can serve as an important ingredient for modeling of tail risks and default correlations. In this paper we are interested in the effects of the return dynamics on the joint distribution of $\mathbf{R}_T = [R_{1,T}, ..., R_{K,T}]'$[6]. Denote

- $F_T(d_i) \equiv P(R_{i,T} \leq d_i | \mathcal{F}_0)$ conditional cdf of aggregate total returns $R_{i,T}$
- $G_T(d_i) \equiv P(E_{i,T} \leq d_i | \mathcal{F}_0)$ conditional cdf of aggregate idiosyncratic returns $E_{i,T}$
- $F_T(\mathbf{d}) \equiv P(\mathbf{R}_T \leq \mathbf{d} | \mathcal{F}_0)$ joint conditional cdf of $\mathbf{R}_T$
- $C_T(\mathbf{u}) \equiv F_T\left(F_T^{-1}(u_1), ..., F_T^{-1}(u_M)\right)$ conditional copula of $\mathbf{R}_T$

Note than the assumption of one factor structure implies that equity returns $\mathbf{R}_T$ are independent conditional on the market return $R_{m,T}$ and therefore $F_T(\mathbf{d})$ can be computed as expectation of the product of conditional cdfs e.g. for unconditional[7] distribution:

$$\bar{F}_T(\mathbf{d}) = E\left(\prod_{i=1}^{M} \bar{P}(R_{i,T} \leq d_i | R_{m,T})\right) = E\left(\prod_{i=1}^{M} \bar{G}_T(d_i - b_i R_{m,T})\right) \tag{26}$$

The tail dependence coefficient and the "default correlation" coefficient are convenient measures of the risk of joint extreme movements for a pair of assets.

[6] the bold letters denote N dimentional vectors e.g. $\mathbf{x} \equiv [x_1, ..., x_N]'$.

[7] for unconditional distributions and copulas we use the same notations but with bar above the corresonding letter, e.g $\bar{F}_T$.

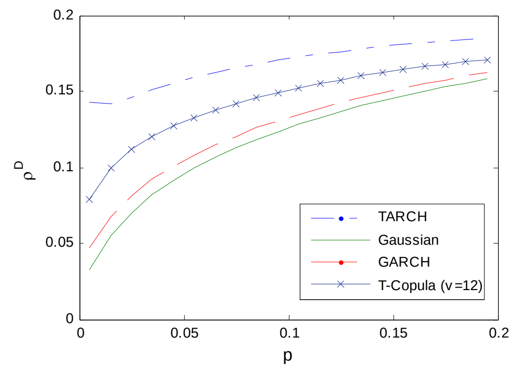

Figure 6: Default correlation as a function of p for TARCH, GARCH,T-Copula and Gaussian models are calculated using 100,000 Monte Carlo simulations. The linear correlation parameter is 0.3 for all 4 models. T-Copula degrees of freedom parameter is 12. TARCH and GARCH market factors correspond to 5 year log returns which are computed based on the returns simulated over weekly intervals. TARCH(GARCH)parameters are $\alpha = 0.01(0.06)$, $\alpha_D= 0.1(0)$, $\beta = 0.92(0.92)$, Gaussian shocks and idiosyncrasies.

Suppose $R_{i,T}$ and $R_{j,T}$ are the stock returns for companies i and j over the $[0,T]$ time horizon. The coefficient of lower tail dependence and the default correlation coefficient for two random variables with the same continuous marginal cdfs, $F_T(R)$, are defined as

$$\lambda_{i,j}^D = \lim_{p\to+0} P\left(R_{i,T} \le d_p | R_{j,T} \le d_p\right) = \lim_{p\to+0} \frac{C_T(p,p)}{p} \tag{27}$$

$$\rho_{i,j}^D(p) = corr(1_{\{R_{i,T}\le d_p\}}, 1_{\{R_{j,T}\le d_p\}}) = \frac{C_T(p,p) - p^2}{(1-p)p} \tag{28}$$

where p is the probability of crossing the threshold (also interpreted as the default probability), and is related to the latter via the relationship $d_p = F_T^{-1}(p)$. Both measures depend only on the bivariate copula of the two random variables and are asymptotically equal: $\lim_{p\to+0} \rho_{i,j}^D(p) = \lambda_{i,j}^D$.

On Figure 6 we show the default correlation $\rho_{1,2}^D$ as a function of p for 4 different models - TARCH, GARCH, Gaussian and Studen-t copula. For all 4 models the linear correlation of latent returns is set to 0.3. The Gaussian copula is symmetric and have zero tail dependence for both upper and lower tails. We can see on the graph that it also has lowest default correlation for all default probabilities in the range of [0,0.2]. The Student-t copula is also symmetric but has fatter joint tails compared to the Gaussian copula. Its default correlation is above the Gaussian for all p and converges to a positive number (the tail dependence coefficient) as p decreases to zero. TARCH and GARCH are calibrated to have volatility dynamics parameters corresponding approximately to the weekly SP500 returns and the time aggregation horizon is set 5 years. We can see that TARCH has higher default correlation than other 3 models and for very low quantiles is upward sloping. The upturn for the extreme tails is a consequence of the left tail shape of the common factor. The default correlation for very low default probabilities should be close to 1 since the left tail of the factor is fatter than the left tail of the idiosyncratic shocks. The GARCH default correlations are closer to the Gaussian because the 5 year time aggregated market factor is "almost" Gaussian. As we showed in the previous sections both kurtosis and skewness of the market factor converge to zero faster for GARCH than TARCH given the same level of volatility persistence.

On Figure 7 we show default correlation $\rho^D_{1,2}$ as a function of p for the TARCH model but for different aggregation horizons. The default correlations for 1 and 5 year horizons are significantly above the 1 week horizon. The term structure of skewness for this example is shown on Figure 3 in the previous section. We can see from the skewness term structure figures that weekly returns are symmetric whereas time aggregated returns for longer horizons(1 and 5 years) have significant negative skewness which increases the default correlations.

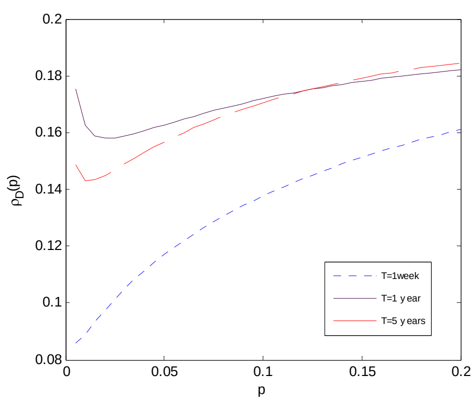

Figure 7: Default correlation as a function of p for the TARCH model for 1 week, 1 year and 5 year time horizons which are calculated based on 100,000 Monte Carlo simulations of weekly returns. The linear correlation is 0.3. TARCH parameters are $\alpha = 0.01$, $\alpha_D = 0.1$, $\beta = 0.92$, Gaussian shocks and idiosyncrasies.

5 Modeling Portfolio Credit Risk

5.1 General Copula Framework

In this section we describe a hybrid semi-dynamic approach to modeling of default correlations in a large homogeneous portfolio of credit exposures. Consider a portfolio of M credit-risky obligors. We start with a static setup with a fixed time horizon $[0, T]$ and to simplify notation skip the time subscript for time dependent variables. At time $t = 0$ all M obligors are assumed to be in non-default state and at time T firm i is in default with probability p_i. We assume we know the individual default probabilities $\mathbf{p} = [p_1, ..., p_M]'$ (either risk-neutral, e.g. inferred from default swap quotes, or actual, e.g. estimated by rating agencies). Let $\tau_i \geq 0$ be the random default time of obligor i and $Y_i = 1_{\{\tau_i \leq T\}}$ the default dummy variable which is equal to 1 if default happened before T and 0 otherwise.

The loss generated by obligor i conditional on its default is denoted as $l_i > 0$. The loss l_i is a product of the total exposure size n_i and percentage losses in case default occurs $1 - \bar{R}_i$ where $\bar{R}_i \in [0, 1]$ is the recovery rate. We also assume that l_i is constant (see [1] for discussion on stochastic recoveries). Portfolio loss L_M at time T is the sum of the individual losses for the defaulted obligors

$$L_M = \sum_{i=1}^{M} l_i 1_{\{\tau_i \leq T\}} = \sum_{i=1}^{M} l_i Y_i \tag{29}$$

The mean loss of the portfolio can be easily calculated in terms of individual default probabilities:

$$E(L_M) = \sum_{i=1}^{M} l_i E(Y_i) = \sum_{i=1}^{M} l_i p_i \tag{30}$$

Risk management and pricing of derivatives contingent on the loss of the credit portfolio, such as CDO tranches, require knowing not only the mean but the whole distribution of portfolio loss with cdf $F_L(x) = P(L_M \leq x)$. Portfolio loss distribution depends on the joint distribution of default indicators $\mathbf{Y} = [Y_1, ..., Y_M]'$ and in a static setup can be conveniently modeled using the latent variables approach [12]. Particularly, to impose structure on the joint distribution of default indicators we assume that there exists a vector of M real-valued random variables $\mathbf{R} = [R_1, ..., R_M]'$ and M dimensional vector of non-random default thresholds $\mathbf{d} = [d_1, ..., d_M]'$ such that

$$Y_i = 1 \Longleftrightarrow R_i \leq d_i \text{ for } i = 1, ..., M \tag{31}$$

Denote $F : \mathbb{R}^M \to [0, 1]$ as a cdf of $\mathbf{R}$ and assume that it is a continuous function with marginal cdf $\{F_i\}_{i=1}^{M}$. For each obligor i the default threshold d_i is calibrated to match the obligor's default probability p_i by inverting the cdf of its aggregate returns $R_i : d_i = F_i^{-1}(p_i)$. According to Sklar's theorem, under the continuity assumption F can be uniquely decomposed into marginal cdfs $\{F_i\}_{i=1}^{M}$ and the M-dimensional copula $C : [0,1]^M \Rightarrow [0,1]$

$$F(\mathbf{d}) = C(F_1(p_i), .., F_M(p_M)) \tag{32}$$

Several popular copula choices are the Gaussian copula model [19], Student-t [20] and Clayton:

- Gaussian copula

$$C^G(\mathbf{p}; \Sigma) = \Phi_\Sigma\left(\Phi^{-1}(p_1), ..., \Phi^{-1}(p_M)\right)$$

- Student-t copula

$$C^T(\mathbf{p}; \Sigma, v) = t_{\Sigma,v}(t_v^{-1}(p_1), ..., t_v^{-1}(p_M))$$

- Clayton

$$C^{Cl}(\mathbf{p}) = \max\left(1 - M + \sum_{i=1}^{M} p_i^{-\beta}\right)^{\beta}$$

The choice of copula C defines the joint distribution of default indicators from which the portfolio loss distribution can be calculated. The number of names in the portfolio can be large and therefore the calibration of the copula parameters can be problematic. To reduce the number of parameters some form of symmetry is usually imposed on the distribution of default indicators. Gordy [15] and Frey and McNeil [12] discuss the mathematics behind the modeling of credit risk in homogeneous groups of obligors and equivalence of the homogeneity assumption to the factor structure of default generating variables. Conditional on the factors, defaults are independent and the conditional joint distribution of default indicators can be easily calculated using the multinomial distribution. To simplify the calculations even more, a large homogenous portfolio (LHP) approximation can be used to approximate the multinomial distribution with a finite number of obligors. Schonbucher and Shubert [26] and Vasicek [29] show that LHP approximation is quite accurate for upper tail of the loss distribution even for mid-sized portfolios of about 100 names. We use symmetric one factor LHP setup in this paper for analytical tractability.

Assumption 1(Symmetric One Factor Model)*: Assume that loss given default $l_i = (1-\bar{R}_i)n_i$ and individual default probabilities p_i are the same for all M names in the portfolio and that the latent variables admit symmetric linear one factor representation:*

$$n_i = n \tag{33a}$$

$$\bar{R}_i = \bar{R} \tag{33b}$$

$$p_i = p \tag{33c}$$

$$R_i = bR_m + \sqrt{1-b^2}E_i \text{ with } 0 \le b \le 1 \tag{33d}$$

where R_m and E_i are independent zero mean, unit variance random variables. $E_i's$ are identically distributed with cdf $G(.)$.

Parameter b defines the pairwise correlation of latent variables which is often referred to as "asset correlation" (this naming reflects the interpretation of latent variables as asset returns in Merton-style structural default models):

$$\rho = b^2 \tag{34a}$$

Suppose we increase the number of names in the portfolio while keeping the total exposure size of the portfolio constant so that $n_i = N/M$. Conditional on R_m the loss of the portfolio contains the mean of independent identically distributed random variables, $L_M = \left(1 - \bar{R}\right) N \frac{1}{M} \sum_{i=1}^{M} 1_{\{R_i \leq d\}}$, which a.s. converges to its conditional expectation as M increases to infinity. We use L without subscript to denote the portfolio loss under LHP assumption.

Proposition 3 ***(LHP Loss)*** *Under Assumption 1*

$$\begin{aligned} L &\equiv \lim_{M \to \infty} \left[\left(1 - \bar{R}\right) N \frac{1}{M} \sum_{i=1}^{M} 1_{\{R_i \leq d\}} \right] = \left(1 - \bar{R}\right) N P \left(R_i \leq d | R_m\right) \qquad (35) \\ &= \left(1 - \bar{R}\right) N G \left(\frac{d - b R_m}{\sqrt{1 - b^2}} \right) \text{ a.s. for any } R_m \in supp\left(G\right) \end{aligned}$$

Proof. see proposition 4.5 in [12] ■

Based on 35 cdf of L can be expressed in terms of the cdf of R_m

$$P\left(L \leq l\right) = P\left(R_m \geq d_1\left(l\right)\right) \tag{36}$$

$$d_1\left(l\right) = \frac{d}{b} - \frac{\sqrt{1 - b^2}}{b} G^{-1} \left(\frac{l}{\left(1 - \bar{R}\right) N} \right) \tag{37}$$

The probability that a diversified portfolio will incur a small loss is high when the probability of market return R_m falling below barrier d_1 is low. In other words a small loss corresponds to the right tail of the market return distribution. The left tail of the market return distribution corresponds to large portfolio losses – the thicker the left tail the more probable is a large loss. The market factor threshold d_1 depends on the single name default barrier d, market factor loadings b and the loss-per-obligor parameters. Note that d_1 is not necessarily monotonic function of b. Only for small losses, such that $l < \left(1 - \bar{R}\right) NG(0)$, it is increasing in b.

For Gaussian copula we have familiar formula for the LHP loss derived by Vasicek [29], where we have substituted the asset correlation parameter in place of the factor loading using the relation 34a:

$$L^G = \left(1 - \bar{R}\right) N\Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}R_m}{\sqrt{1-\rho}}\right) \tag{38}$$

$$P(L \le l) = 1 - \Phi\left(d_1^G(l)\right) \tag{39}$$

$$d_1^G(l) = \frac{\Phi^{-1}(p)}{\sqrt{\rho}} - \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\Phi^{-1}\left(\frac{l}{\left(1-\bar{R}\right)N}\right) \tag{40}$$

5.2 From Loss Distributions to Correlation Spectrum

The mean of the loss distribution is not affected by the choice of copula. The second and higher moments of the loss distribution depend on the copula characteristics. In particular, the variance of the loss can be expressed in terms of bivariate default correlation coefficients, $\rho^D(p)$, defined in Section 4.

$$\begin{aligned} Var(L_M) &= Var\left((1-\bar{R})N\frac{1}{M}\sum_{i=1}^{M} 1_{\{R_i \le d_p\}}\right) \qquad (41)\\ &= (1-\bar{R})^2 N^2 p(1-p)\left(\frac{1}{M} + \frac{M-1}{M}\rho^D(p)\right) \\ Var(L) &= (1-\bar{R})^2 N^2 p(1-p)\rho^D(p) \qquad (42) \end{aligned}$$

where $\rho^D(p) = corr(1_{\{R_i \le d_p\}}, 1_{\{R_j \le d_p\}})$. By comparing the default correlation coefficients, as we did in Section 4, we therefore implicitly compare the impact of the copula choice on the loss variance.

In addition to the variance of L we are also interested in measuring(pricing) the extreme risks - the likelihood of small and large losses. To do that we define the loss tranches which allow us to look at the particular slices of portfolio loss. Let $(K_d, K_u]$ denote a tranche with attachment point K_d and detachment point K_u expressed as fractions of the reference portfolio notional so that $0 \le K_d < K_u \le 1$. The notional of the tranche, $N_{(K_d,K_u]}$, is $N(K_u - K_d)$ where N is the notional of the portfolio. The loss $L_{(K_d,K_u]}$ of the tranche is the fraction of L that falls between K_d and K_u. For simplicity assume that total notional N is normalized to 1.

$$L_{(K_d,K_u]} = f_{(K_d,K_u]}(L) \tag{43}$$

$$f_{(K_d,K_u]}(x) \equiv (x - K_d)^+ - (x - K_u)^+ \tag{44}$$

Tranches with zero attachment point, $(0, K_u]$, and unit detachment point, $(K_d, 1]$, are called equity and senior tranches correspondingly. Loss of any tranche can be decomposed into losses of two equity tranches $L_{(K_d,K_u]} = L_{(0,K_u]} - L_{(0,K_d]}$. Expected loss of the equity tranche $(0, K]$ depends on the portfolio loss distribution and in LHP approximation can be computed using only the distribution of the market factor

$$EL_{(0,K]} = Ef_{(0,K]}(L) = \left(1-\bar{R}\right) E\left[G\left(\frac{d-bR_m}{\sqrt{1-b^2}}\right) 1_{\{R_m \geq d_1(K)\}}\right] + KP\left(R_m < d_1(K)\right) \tag{45}$$

The expectation in (45) can be computed by Monte Carlo simulation or numerical integration if we know G and distribution of R_m (see appendix C). For the Gaussian copula, the integral can be calculated in a closed form.

$$E^G L_{(0,K]} = \left(1-\bar{R}\right) \Phi\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) + K\Phi(d_1) \tag{46}$$

$$d_1 = \frac{1}{\sqrt{\rho}}\Phi^{-1}(p) - \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\Phi^{-1}\left(\frac{K}{1-\bar{R}}\right) \tag{47}$$

Because of its analytical tractability, it is convenient to use the Gaussian copula as a benchmark model when comparing different choices of dependence structure. The asset correlation parameter ρ plays a similar role to the implied volatility for equity options because an equity tranche $(0,K]$ is essentially a call option on the surviving part of the underlying portfolio. Like any long call option position, its value is increasing as a function of the uncertainty of the underlying. The underlying in this case is the distribution of the portfolio losses, and its uncertainty increases with the asset correlation parameter, as can be seen from:

$$Var(L) = EL^2 - \left(1-\bar{R}\right)^2 p^2 = \left(1-\bar{R}\right)^2 \left[\Phi\left(\Phi^{-1}(p), \Phi^{-1}(p); \rho\right) - p^2\right]$$

Note also that a similar dependence on bivariate default correlation ρ^D was derived in 42. Thus, by finding the asset correlation level that in certain sense replicates the results of more complex portfolio loss generating models in the context of a Gaussian copula framework, we can translate the salient features of such models into mutually comparable units. More specifically, we define the correlation spectrum as follows:

Definition 4 *Suppose the loss distribution of a large homogeneous portfolio is generated by a model* $\left\{C, p, \bar{R}\right\}$ *with copula* C*, equal individual default probabilities* p *and recovery rate* $\bar{R}$*. Let* $L_{(0,K]} \in \left[pf_{(0,K]}\left(1-\bar{R}\right),\ f_{(0,K]}\left(\left(1-\bar{R}\right)p\right)\right]$ *be the expected loss of the equity tranche* $(0,K]$*. We define the* ***correlation spectrum*** $\rho(K,p,\bar{R})$ *of the model* $\left\{C, p, \bar{R}\right\}$ *as the correlation parameter of the Gaussian copula that produces the same expected loss* $EL_{(0,K]}$ *for the tranche* $(0,K]$ *for the given horizon* T *and given single-issuer default probability* p

$$\rho(K,p,\bar{R}) \text{ solves } E^G L_{(0,K]}(\rho) = EL_{(0,K]} \text{ for all } K \in [0,1] \tag{48}$$
$$\text{where } EL_{(0,K]} \text{ is expected loss of the tranche}$$
$$(0,K] \text{ generated by model } \left\{C, p, \bar{R}\right\}$$

where $E^G L_{(0,K]}$ *is defined in (46).*

The correlation spectrum as defined above is closely related but not identical to the notion of the base correlation used by many practitioners [23]. The difference is that the base correlation is defined using the prices of the equity tranches, which in turn depend on interest rates, term structure of losses, etc. By contrast, the correlation spectrum is defined without a reference to any market price. It characterizes the portfolio loss generating model, rather than the the supply/demand forces in the market. Therefore we believe it is a more convenient tool for comparing different models, while the base correlation is presumably better for comparing the relative value between actual tranches.

Another important point is that the correlation spectrum depends implicitly on the term to maturity via the cumulative default probability p. However, this is not the only dependence – potentially, the dependence structure characterized by the copula C also exhibits some time dependence when viewed within the context of the Gaussian copula. This statement needs a clarification – the copula C itself is defined in a manner that encompasses all time horizons and therefore cannot depend on any particular horizon. However, when we translate the tranche loss generated with this dependence structure into the simpler Gaussian model the transformation that is required may depend on the horizon. As we will see in subsequent sections, this is indeed the case for the portfolio loss generating models based on GARCH dynamics, which are therefore characterized by a non-trivial term structure of the correlation spectrum.

To ensure that the correlation spectrum is well defined we need to prove that (48) has a unique solution. Let us first prove the following:

Proposition 5 *For the Gaussian copula, the expected loss of an equity tranche is a monotonically decreasing function of ρ and attains its maximum (minimum) when correlation is equal to 0 (1).*

Proof. *Using (46) and the properties of Gaussian distribution*[8] *we derive*

$$
\begin{aligned}
E^G_\rho L_{(0,K]} &\equiv \frac{\partial}{\partial \rho} E^G L_{(0,K]} \qquad (49)\\
&= -\left(1-\bar{R}\right)\left[\frac{1}{2\sqrt{\rho}}\Phi_3\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) + \Phi_2\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right)\frac{\partial}{\partial b}d_1\right]\\
&+ K\phi(d_1)\frac{\partial}{\partial b}d_1\\
&= -\frac{1-\bar{R}}{2\sqrt{\rho}}\phi\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) < 0
\end{aligned}
$$

for any $\rho \in (0,1)$. ■

Therefore, there is a one-to-one mapping between loss distribution and correlation spectrum, and our transformation does not lead to any loss of information. The next proposition shows how to calculate the loss cdf using the correlation spectrum and its slope along the K-dimention.

Proposition 6 *Suppose* $\rho(K, p, \bar{R})$ *is the correlation spectrum for model* $\left\{C, p, \bar{R}\right\}$ *and the probability distribution function of the portfolio loss is a continuous function then the loss cdf can be computed from the correlation spectrum:*

$$P(L \leq K) = P^G(L \leq K) + \rho_K(K, p, \bar{R}) E^G_\rho L_{(0,K]} \qquad (50)$$

where

$$P^G(L \leq K) = 1 - \Phi(d_1) \qquad (51)$$

$$E^G_\rho L_{(0,K]} = \left(1-\bar{R}\right)\frac{1}{2\sqrt{\rho}}\phi\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) \qquad (52)$$

$$d_1 = \frac{1}{\sqrt{\rho}}\Phi^{-1}(p) - \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\Phi^{-1}\left(\frac{K}{1-\bar{R}}\right) \qquad (53)$$

and the functions are evaluated at $\rho = \rho(K, p, \bar{R})$.

---

[8] The following properties of 2 dimentional Gaussian cdf are used in the calculation

$$\Phi_2(x, y; \rho) = \phi(y)\Phi\left(\frac{x - \rho y}{\sqrt{1-\rho^2}}\right)$$

$$\frac{\partial}{\partial \rho}\Phi(x, y; \rho) = \phi(x, y; \rho)$$

where $\phi(.)$ denotes, depending on the number of agruments, pdf of standard Normal distribution and pdf of bivariate Normal with standard Normal marginals and correlation coefficient as a third argument. Numerical subscript denotes the partial derivative with respect to the corresponding argument. First formula is straitforward. The proof of the second can be found in Vasicek([30])

Proof. *first note that the derivative with respect to K of the expected tranche's loss under true copula C is related to the cdf of the loss*

$$\begin{aligned} \frac{d}{dK} EL_{(0,K]} &= \frac{d}{dK} E\left(L - (L-K)_+\right) && (54) \\ &= -\frac{d}{dK} E\left(L-K\right)_+ = E1_{\{L-K\geq 0\}} = 1 - P\left(L \leq K\right) \end{aligned}$$

therefore

$$\begin{aligned} P\left(L \leq K\right) &= 1 - \frac{d}{dK} EL_{(0,K]} && (55) \\ &= 1 - E_K^G L_{(0,K]} - \rho_K(K,p,\bar{R}) E_\rho^G L_{(0,K]} \\ &= 1 - \Phi\left(d_1\right) + \frac{1-\bar{R}}{2\sqrt{\rho}} \phi\left(\Phi^{-1}\left(p\right), -d_1; -\sqrt{\rho}\right) \rho_K(K,p,\bar{R}) \\ &= P^G\left(L \leq K\right) + \left(1-\bar{R}\right) \frac{1}{2\sqrt{\rho}} \phi\left(\Phi^{-1}\left(p\right), -d_1; -\sqrt{\rho}\right) \rho_K(K,p,\bar{R}) \end{aligned}$$

where partial derivative with respect to K is computed as

$$\begin{aligned} E_K^G L_{(0,K]} &\equiv \frac{\partial}{\partial K} E^G L_{(0,K]} && (56) \\ &= -\left(1-\bar{R}\right) \Phi_2\left(\Phi^{-1}\left(p\right), -d_1; -\sqrt{\rho}\right) \frac{\partial}{\partial K} d_1 + K\phi\left(d_1\right) \frac{\partial}{\partial K} d_1 + \Phi\left(d_1\right) \\ &= \Phi\left(d_1\right) \end{aligned}$$

■

We defined the correlation spectrum for the loss distribution with a fixed time horizon. In the next section we illustrate the pricing of portfolio tranche swap contracts and show that the value of the swap depends on the whole term structure of the correlation spectrum up to the maturity of the swap.

5.3 Pricing of Synthetic CDO Tranches

In this section we briefly define the payoff structure of synthetic CDO tranche contracts and their pricing. Consider a synthetic CDO with fixed maturity T written on a synthetic portfolio. The loss $L_{(K_d,K_u]}\left(t\right)$ of the tranche $(K_d, K_u]$ at time $t \leq T$ is a fraction of portfolio loss $L\left(t\right)$ that falls between K_d and K_u.

$$L_{(K_d,K_u]}\left(t\right) = f_{(K_d,K_u]}\left(L(t)\right) \quad (57)$$

The swap contract for a particular tranche is swap of cash flows between the "premium leg" and the "protection leg". The protection buyer agrees to pay a fixed fee s to the protection seller in the proportion to the survived notional of the tranche. The protection seller compensates the tranche losses to the insured until the maturity of the contract.

Since the swap contract is a contingent claim on the portfolio loss it can be priced using the risk-neutral distribution of the portfolio losses. We assume that interest rate risk is not correlated with credit risk and denote by $D(0,t)$ the price at time 0 of a zero coupon bond maturing at time t. The payoff structure of both premium and protection legs is linear in the tranche's loss and therefore to price these legs when the interest rates are not correlated with default risk we only need to know the term structure of expected tranche losses. Introduce the tranche's default probability $P_{(K_d,K_u]}(t)$ as expected fraction of the tranche's notional that is lost due to defaults by time t.

$$P_{(K_d,K_u]}(t) = \frac{E_0\left[L_{(K_d,K_u]}(t)\right]}{N_{(K_d,K_u]}} \tag{58}$$

For simplicity assume that time is continuous. The value of the protection leg at time 0

$$\begin{aligned} V_0^{protection} &= E_0\left(\int_{t=0}^{T} D(0,t)\, dL_{(K_d,K_u]}(t)\right) \\ &= N_{(K_d,K_u]}\int_{t=0}^{T} D(0,t)\, dP_{(K_d,K_u]}(t) \end{aligned} \tag{59}$$

Assuming that protection fee is paid in Δq intervals e.g. quarterly the value of the premium leg at time 0

$$\begin{aligned} V_0^{premium} &= E_0\left(\sum_{q=1}^{T/\Delta q} D(0,q\Delta q)\left[N_{(K_d,K_u]} - L_{(K_d,K_u]}(q\Delta q)\right] s\Delta q\right) \\ &= N_{(K_d,K_u]} s\Delta q \sum_{q=1}^{T/\Delta q} D(0,q\Delta q)\left[1 - P_{(K_d,K_u]}(q\Delta q)\right] \end{aligned} \tag{60}$$

The par spread of the swap contract is the spread that makes the values of protection and premium legs equal. As we already mentioned swap contract cash flows are linear functions of the tranche losses and therefore the values of the both legs depend only on the tranche's expected losses. Because timing of the losses is important when the interest rates are not zero we need the whole term structure of the tranche's expected losses up to the maturity of the swap to price the contract.

The portfolio loss $L(t)$ at time t in the Gaussian copula framework depends on time t only through the single-issuer default probability p_t[9]. As we showed in proposition 2 the correlation spectrum is the equivalent representation of the loss distribution for a fixed time horizon. The dependence of the correlation spectrum $\rho(K, p, \bar{R})$ on t is also achieved through the second argument - single-issuer default probability p. For example the expected tranche loss at time t

[9] The portfolio loss generating copula does not change with T because it corresponds to the copula of time to default distribution which by definition does not depend on T.

is the expected tranche loss of the Gaussian model with correlation parameter $\rho\left(K, p_t, \bar{R}\right)$ and single-issuer default probability p_t:

$$EL_{(0,K]}(t) = E^G L_{(0,K]}\left(\rho\left(K, p_t, \bar{R}\right), p_t\right) \tag{61}$$

The expected tranche loss that happens between t and $t + dt$ can therefore be computed from the correlation spectrum using its level and the slope in the p-dimention:

$$\begin{aligned} dEL_{(0,K]}(t) &= \frac{dE^G L_{(0,K]}(t)}{dp_t} dp_t \\ &= \left(E_p^G L_{(0,K]}(t) + \rho_p\left(K, p_t, \bar{R}\right) E_\rho^G L_{(0,K]}(t)\right) dp_t \end{aligned} \tag{62}$$

where

$$E_\rho^G L_{(0,K]}(t) = -\frac{1-\bar{R}}{2\sqrt{\rho}} \phi\left(\Phi^{-1}(p_t), -d_1; -\sqrt{\rho}\right) \tag{63}$$

$$E_p^G L_{(0,K]}(t) \equiv \frac{\partial}{\partial p} E^G L_{(0,K]}(t) = \left(1-\bar{R}\right) \Phi\left(\frac{-d_1 + \sqrt{\rho}\Phi^{-1}(p_t)}{\sqrt{1-\rho}}\right) \tag{64}$$

$$d_1 = \frac{1}{\sqrt{\rho}} \Phi^{-1}(p_t) - \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \Phi^{-1}\left(\frac{K}{1-\bar{R}}\right) \tag{65}$$

and the functions are evaluated at $\rho = \rho(K, p, \bar{R})$. Thus, both legs of the swap contract can be priced using the correlation spectrum surface, single-issuer default probability term structure and the formulas (58) -(62).

6 Comparing Portfolio Loss Generating Models

In section 4 we demonstrated that dynamic models such as GARCH and TARCH can produce significant pairwise default correlation even for very low default thresholds. Thus, one can hope that these models should also be able to capture the important aspects of multi-variate default losses in a diversified portfolio setting. But in order to discriminate between these models and to understand which of their characteristics are the most important from a credit modeling perspective, one must have a good measure that makes such comparisons not only possible but hopefully apparent and intuitive. The market standard measure is the base correlation. However, this measure is best suited for comparison of pricing of similar tranches rather than comparison of different models. In particular, base correlation implicitly depends on the level and term structure of interest rates, as well as conventions such as coupon payment frequency, up-front pricing, etc.

Our goal in this paper is not so much to price a specific set of tranches under given market conditions as to provide a general framework for judging the versatility of various dynamic portfolio credit risk models. All such models, whether defined via dynamic multivariate returns model like in this paper or in various versions of the static copula framework ([19], [12], [26], [20], [13], [1]), can be characterized by the full term structure of loss distributions. Thus, without loss of generality, we can refer to all models of credit risk as loss generating models, with an implicit assumption that any two models that produce identical loss distributions for all terms to maturity are considered to be equivalent. The correlation spectrum, introduced in section 5.2, conveniently transforms specific choice of a loss generating model into a two dimensional surface $\rho(K, T)$ of the Gaussian copula correlation parameter, with the main dimensions being the loss threshold (detachment level) K and the term to maturity T. All other inputs such as the recovery rate R, the term structure of (static) hazard rates h, the level of linear asset correlation ζ, the Student-t degrees of freedom ν, various GARCH model coefficients, etc. – are considered as model parameters upon which the two-dimensional correlation spectrum itself depends. Note that in the previous sections we have expressed the correlation spectrum as a function of detachment level and the underlying portfolio's cumulative expected default probability p rather than the term to maturity T. Given our assumption of the static term structure of the hazard rates h these two formulations are equivalent. In this section we prefer to emphasize the dependence on maturity horizon in order to facilitate the comparison with base correlation models and also to analyze the dependence on the level of hazard rates separately from the term to maturity dimension.

In this framework, we can compare various dynamic and static loss generating models by comparing their correlation spectra, as well as the characteristic dependencies of the correlation spectra on changes of model parameters. Of course, the correlation spectrum of a static Gaussian copula model [19] is a flat surface with constant correlation across both detachment level K and term to maturity T. Any deviation from a flat surface is therefore an indication of a non-trivial loss generating model, and we can judge which features of the model are the important ones by examining how strong a deviation from flatness do they lead to.

6.1 Models with static dependence structure

Let us begin with the analysis of one of the popular static loss generation models. On Figure 8 we show the correlation spectrum computed for the Student-t copula with linear correlation $\rho = 0.3$ and $\nu = 12$ degrees of freedom. Student-t copula is in the same elliptic family as the Gaussian copula but has non-zero tail dependence governed by the degrees of freedom parameter. As a model of single-period asset returns the Student-t distribution has been shown to provide a significantly better fit to observations than the standard normal [20].

However, from the Figure 8 we can see that the static Student-t copula does not generate a notable skew in the direction of detachment level K, and in fact generates a mild downward sloping skew for very short terms, which is contrary to what is observed in the market. The main reason for this is the rigid structure

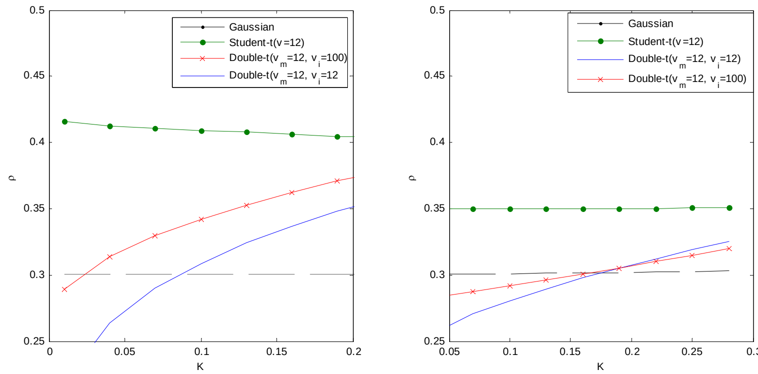

Figure 8: Correlation spectrum slices corresponding to 1-year (left chart) and 5-year (right chart) horizons with default probabilities 0.02 and $1-(1-0.02)^5 = 0.0961$ for Gaussian, Student-t Copula with 12 degrees of freedom, and Double-t Copulas with (v_m =12, v_i =100) and (v_m =12, v_i =12) degrees of freedom. Linear correlation is 0.3 for all copulas.

of this model, with the tails of the idiosyncratic returns tied closely to the tails of the market factor. This can be explicitly seen in the derivation of the Student-t copula as a mixture model [27], and leads to a joint survival of all issuers when the χ^2-distributed mixing variable takes on a large value (almost) regardless of the realization of either the market factor or the idiosyncratic returns. Instead of producing a varying degree of correlation depending on the default threshold, the Student-t copula model simply produces a higher overall level of correlation.

On the other hand, the more flexible double-t copula model [17] produces a significantly upward sloping skew, as can be seen from Figure 8. The main feature of the double-t copula that is responsible for the skew is the cleaner separation between the common factor and idiosyncratic returns – there is no longer a single mixing variable which ties the two sources of risk together. The presence of a fat-tailed market factor while the idiosyncratic returns can in fact get efficiently diversified in the LHP framework naturally leads to an upward sloping correlation skew. Recall that the higher values of the detachment level K correspond to farther downside tails of the market factor where it dominates the idisyncratic returns, resulting in a higher effective correlation.

Furthermore, by making the fully independent idiosyncratic returns more fat tailed one achieves a steeper skew – compare the two examples of the double-t copula, with the degrees of freedom of the idiosyncratic returns set to 100 (i.e. nearly Gaussian case) and to 12 (i.e. strongly fat-tailed case), respectively. Finally, we observe that the slope of the correlation skew gets flatter as the time horizon grows. All of these features will have their close counterparts in the dynamic models which we will consider next.

6.2 Multi-Period (Dynamic) Loss Generating Model

Let us now turn to loss generating models based on latent variables with multi-period dynamics. We have concluded in the previous section that a clean separation of the market factor and the idiosyncratic returns appears to be a prerequisite for producing an upward sloping correlation skew. Fortunately, the dynamic multi-variate models which we considered in section 3 all have this property, both for single-period and for aggregated returns.

On Figure 9 we show the correlation spectrum computed for a loss generating model based on GARCH dynamics with Gaussian residuals, with a linear correlation set to $\rho = 0.3$, and GARCH model parameters taken from the weekly SP500 estimates in Appendix B.

As we can see, this model does exhibit a visible deviation from the flat correlation spectrum for short maturities. However, as we already noted in section 3, the distribution of aggregate returns for the symmetric GARCH model quickly converges to normal. Therefore, it is not surprising to see that the correlation spectrum also flattens out fairly quickly and becomes virtually indistinguishable from a Gaussian copula for maturities beyond 5 years. Thus, we conclude that the symmetric GARCH model with Gaussian residuals is inadequate for description of liquid tranche markets where one routinely observes steep correlation skews at maturities as long as 7 and 10 years.

Based on the empirical results of 3.2 we know that a GARCH model with Student-t residuals provides a better fit to historical time series of equity returns. A natural question is whether allowing for such volatility dynamics can lead to a persistent correlation skew commensurate with the levels observed in synthetic CDO markets.

The results of section 3.2 suggest that the additional kurtosis of the single-period returns represented by the Student-t residuals does not matter very much for aggregate return distributions at sufficiently long time horizons. Indeed, Figure 10 shows that the GARCH model with Student-t residuals exhibits a correlation skew that is quite a bit steeper at the short maturities, yet is almost as flat and featureless at the long maturities as its non-fat-tailed counterpart – there is a small amount of skew at 10 years, but it is too small compared to the steepness observed in the liquid tranche markets. Thus, we conclude that one has to focus on the dynamic features of the market factor process in order to achieve the desired correlation skew effect.

Our next candidates are the TARCH models with either Gaussian or Student-t return innovations. We have seen in 3 that the asymmetric volatility dynamics of these models leads to a much more persistent skewness and kurtosis of aggregated equity returns that actually grow rather than decay at very short horizons, and survive for as long as 10 years for the range of parameters corresponding to the post-1990 sample of SP500 weekly log-returns. Hence, our hypothesis is that a latent variable model with TARCH market fynamics might be capable of producing a non-trivial credit correlation skew for up to 10 year maturity.

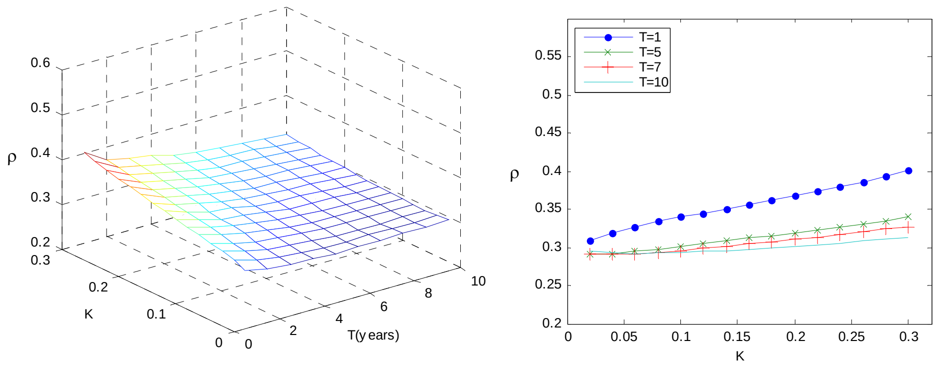

Figure 9: Correlation spectrum for GARCH model (α=0.045, β=0.948) with Gaussian shocks and the slices of the correlation spectrum for 1, 3, 5 and 7 year maturities.

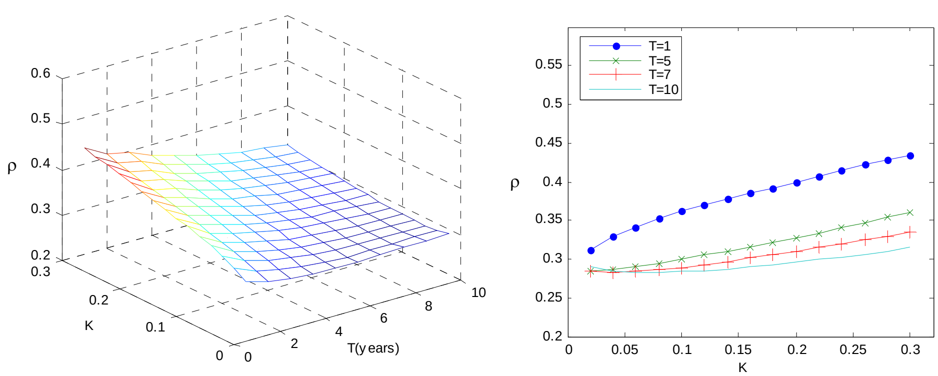

Figure 10: Correlation spectrum for GARCH model (α=0.045, β=0.948) with Student-t shocks (v=8.3) and the slices of the correlation spectrum for 1, 3, 5 and 7 year maturities.

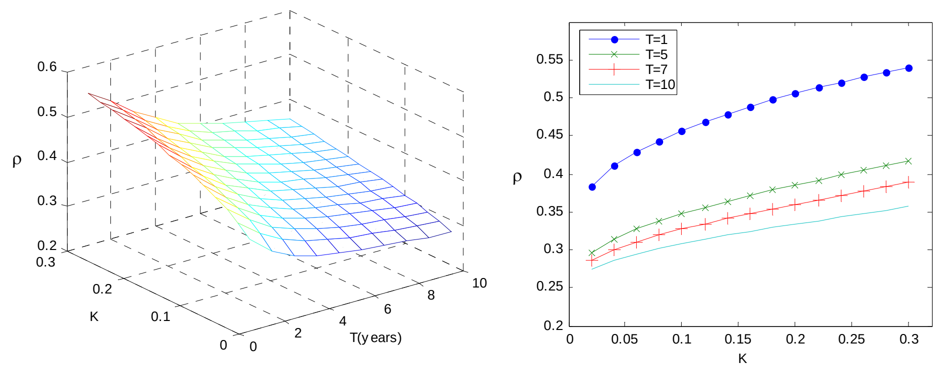

Figure 11: Correlation spectrum for TARCH model (α=0.004, α_D=0.094, β=0.927) with Gaussian shocks and the slices of the correlation spectrum for 1, 3, 5 and 7 year maturities.

The Figures 11 and 12 show the correlation spectra for the TARCH-based loss generating models. The most immediate observation is that both versions of the model produce a rather persistent correlation skew. Although the correlation spectrum surface flattens out with growing term to maturity, the steepness of the skew is still quite significant even at 10 years. Just as in the case of the symmetric GARCH model, the fat-tailed residuals lead only to marginal steepening of the correlation spectrum compared to the case with Gaussian residuals.

Contrast these properties of the dynamic GARCH-based models with the features of the static double-t copula. Upon a closer inspection of Figures 8 and 11 we can see that the TARCH model with Gaussian shocks and Gaussian idiosyncracies produces a slightly steeper 5-year correlation skew than the double-t copula, even when the latter is taken with fat-tailed idiosyncracies. When we turn on the Student-t return residuals for the market factor dynamics (see Figure 12) the differences in the 5-year skew become quite siginificant. Thus, we conclude that the sources of persistent correlation skew are, in order of their importance: 1) the independence of market factor and idiosyncratic returns, 2) the asymmetry of market factor aggregate return distribution, 3) fat tails in the market factor return dynamics, 4) fat tails in the idiosyncratic return distribution.

6.3 Dependence on Model Parameters

As explained in the beginning of this section, we consider the correlation spectrum surface as an embodiment of the particular loss generating model. Each such model contains various parameters some of which are empirically estimated (e.g. the degrees of freedom of the Student-t distribution) and some of which are calibrated to a particular problem at hand (e.g. the level of hazard rates and ex-

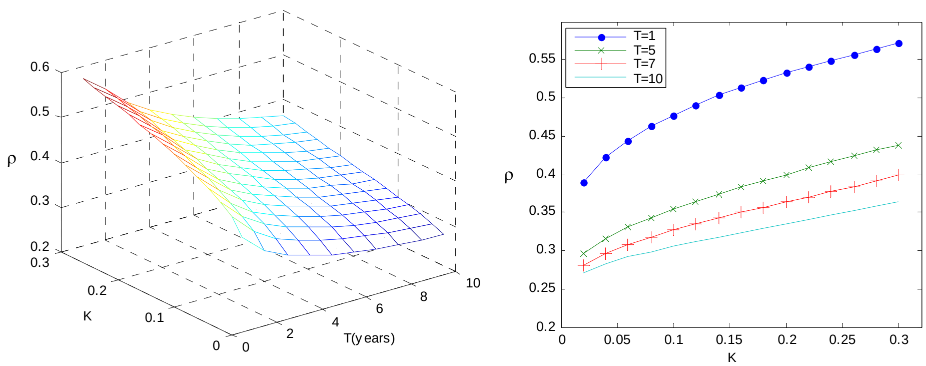

Figure 12: Correlation spectrum for TARCH model (α=0.004, α_D=0.094, β=0.927) with Student-t shocks(v=8.3) and the slices of the correlation spectrum for 1, 3, 5 and 7 year maturities.

pected default probabilities for the collateral portfolio underlying the synthetic CDO tranches under question). While the empirically estimated parameters are not likely to change, the calibrated ones will do so quite frequently as the market conditions change.

In particular, the implied hazard rates can and do change quite significantly even for investment grade credit portfolios. Therefore, the analysis of the dependence of the correlation spectrum on the level of hazard rates has not only an academic relevance as a matter of investigation of the model's range of applicability, but also a practical importance due to reliance of many practitioners on the base correlation methodology which normally takes the correlation skew as an exogenous input and does not incorporate correlation skew adjustments as the market spreads and implied hazard rates change. By contrast, the dynamic multi-period models introduced in this paper produce the correlation spectrum as an output of the model, and therefore can give a specific prediction regarding the way the correlation skew is supposed to change when the model parameters move.

As an example of such predictive behavior of the model consider the correlation spectrum dependence on the hazard rates depicted in Figure 13, where we have shown a particular maturity slice, the 5-year skew, as a function of hazard rates. From the visual comparison of Figures 13 and 11 it appears that the dependence of a correlation skew for a fixed term to maturity but varying level of hazard rates is very similar to the dependence of the correlation spectrum on the term to maturity. The similarity is natural, as the first order effect is the dependence on the level of the cumulative default probability which depends on the product of $h \cdot T$ rather than on the hazard rate or the term to maturity separately. For each level of this product, we get a specific level of the default threshold in the latent variable credit risk model. The higher this threshold,

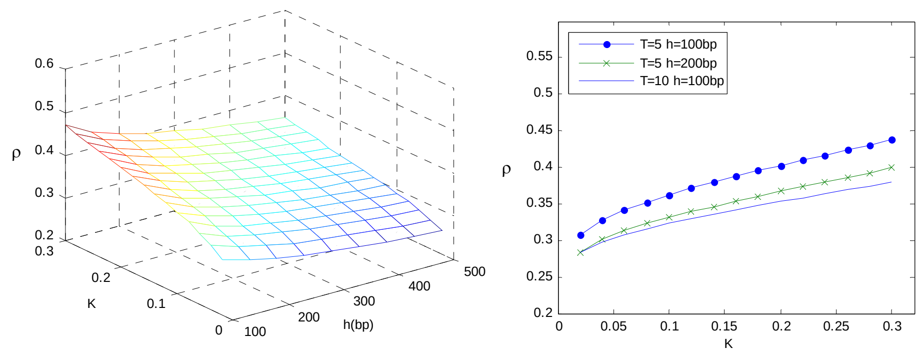

Figure 13: Left figure : the dependence of the 5-year correlation skew on the level of the hazard rates. Right figure shows 3 correlation spectrum slices: 1) T=5-yr h=100bp, 2) T=10-yr h=100bp, 3) T=5-yr h=200bp

the closer is the sampled region to the center of the latent variables distribution and the less it is affected by the tail risk – thus leading to a lower level of the credit correlation.

However, there is a second order effect which makes these two dependencies somewhat different. It is related to the shape of the distribution of aggregate returns for the market factor. Assuming that the parameters of the GARCH process are the same in both cases, we can deduce that the dependence on the term to maturity with fixed hazard rate should exhibit a faster flattening of the correlation spectrum than the dependence on the hazard rate with fixed term to maturity because the increasing aggregation horizon for market factor returns leads to gradual convergence of its distribution towards normal and, as a consequence, to progressively flatter correlation skew.

To make the visual comparison easier, we note that the effect of flattening of the skew while going from a 5-year horizon to the 10-year horizon must be compared against the flattening of the skew while going from 100bp hazard rate to 200bp hazard rate. The right hand side Figure in 13 contains a direct comparison of these three particular slices of the correlation spectrum surface and confirms the intuition put forward above.

7 Summary and Conclusions

In this paper we have introduced and studied a new class of credit correlation models defined via a dynamic portfolio loss generation process within a latent variable approach where the latent variable follows a factor-ARCH with asymmetric TARCH volatility dynamics. We have shown this model to be superior to alternative simpler characterizations of the time series processes including symmetric GARCH volatility dynamics with Gaussian or Student-t residuals when it comes to ability to produce a significant and persistent correlation skew commensurate with the levels observed in the liquid synthetic CDO tranche markets.

To build the foundation for our model, we have studied the time aggregation properties of the multivariate dynamic models of equity returns. We showed that the dynamics of equity return volatilities and correlations leads to significant departures from the Gaussian distribution even for horizons measured in several years. The asymmetry appears to "survive aggregation" longer than fat tails do based on the parameters estimated from the real data. The main source of skewness and kurtosis of the return distribution for long horizons is the dynamic asymmetry of volatility response to return shocks or so-called leverage effect.

We introduced the notion of the correlation spectrum as a tool for comparing the loss generating models, whether defined via a single-period (static) copula, or via multi-period (dynamic) latent-variable framework, and for simple and consistent approach to non-parametric pricing of CDO tranches. We showed that for a portfolio loss distributions with smooth pdf the loss distribution can be easily reconstructed from the correlation spectrum using its level and slope along the K-dimention.

Importantly, in our dynamic loss generating model framework, the correlation spectrum is not only explained, but predicted – based on empirical parameters of the TARCH process and the parameters describing the reference credit portfolio. The model also predicts a specific sensitivity of the correlation spectrum to changes in various such parameters, including the hazard rate. The structural inability of the static models to incorporate the changes in the base correlation have been at the heart of the recent difficulties faced by these models during the synthetic CDO market dislocation in April/May of 2005. While our model is not likely to have given all the answers in such turbulent market conditions either, its ability to accommodate the changes in the correlation skew could help the practitioners get a better handle on the fast moving markets.

One of the possible directions for generalization of our model is to move from a single market factor to a multi-factor framework. The well-documented importance of both macro and industry factors for explanation of equity returns suggests that the same factors could be instrumental in getting a more accurate model of credit correlations as well.

Whether in a single factor or a multi-factor setting, many of our conclusions reflect the limitations of the large homogeneous portfolio approximation which we have adopted in this paper. In particular, it is clear that even deterministic but heterogenous idiosyncrasies, market factor loadings and hazard rates could lead to significant changes in portfolio loss distribution and consequently to the correlation spectrum of the model. An extension of our model to such heterogenous case is possible, although the computational efforts will increase very significantly. Still, the promising features demonstrated by our approach even in the LHP approximation suggest that despite the computational difficulties, such extensions might be a worthy effort. In particular, the explicit modeling of the heterogeneous reference portfolio would have been absolutely necessary if one were to attempt to explain the tranche pricing during significant market dislocations.

Another important simplification which we have made when discussing the results of our model with regard to the correlation spectra is that we have only considered the unconditional return distributions and have not explored the effects of the initial shocks to either TARCH returns of volatility. From the perspective of a credit investor this means that we have described the "equilibrium" (in a loose sense of that word) state of the tranche market, but not the effects related to the relaxation towards the equilibrium. One could expect to find interesting results in this line of research, which would be that much more relevant given the credit market's propensity to undergo unexpected short-term dislocations as we have witnessed several times over the past couple of years.

Among the more practical questions that remain for future investigation are the calculation of the deltas or hedge ratios of synthetic CDO tranches within our framework, defining relative value measures for tranches reflecting the model's ability to produce the "fair" or "predicted" correlation spectrum.

References

[1] L. Anderson, J. Sidenius (2005): "Extensions to Gaussian Copula: Random Recovery and Random Factor Loadings", Journal of Credit Risk, vol. 1, no. 1, p. 29

[2] International Swaps and Derivatives Association (2005): "ISDA Mid-Year 2005 Market Survey", press release published on September 28, 2005.

[3] Bollerslev, T., R.F. Engle, and D. Nelson (1994): "ARCH Models", in The Handbook of Econometrics, vol. IV, eds. D.F. McFadden and R.F. Engle III. North Holland.

[4] G. Gupton, C. Finger, M. Bhatia (1997): "CreditMetrics: Technical Document", RiskMetrics Group.

[5] Day, T. E., and C. M., Lewis (1992): "Stock Market Volatility and the Informational Content of Stock Index Options." Journal of Econometrics 52, p. 267-287.

[6] Drost, F. C. and T. E. Nijman (1993): Temporal aggregation of GARCH processes.Econometrica 61, p. 909–927.

[7] P. Embrechts, F. Lindskog and A. McNeil (2001): "Modeling dependence with copulas and applications to risk management", Working paper, ETH, Zurich

[8] Engle, R.F. (1982): Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom in‡ation. Econometrica 50, p. 987-1006.

[9] Engle, R. F. (2002): "New Frontiers for ARCH Models", Journal of Applied Econometrics, 17, 425–446 (2002).

[10] Engle, R. F. (2002): "Dynamic Conditional Correlation – A Simple Class of Multivariate GARCH Models", Journal of Business and Economic Statistics

[11] E. W. Frees and E. A. Valdez (1998): "Understanding Relationships Using Copulas", North American Actuarial Journal, vol. 2, no. 1, p. 1

[12] R. Frey and A. McNeil, "Dependent defaults in models of portfolio credit risk", Journal of Risk (forthcoming),

[13] Laurent, J.-P., & J. Gregory, 2003, Basket Default Swaps, CDOs and Factor Copulas, ISFA Actuarial School, University of Lyon, working paper available on www.defaultrisk.com

[14] J. Gregory(editor), "Credit Derivatives: The Definitive Guide", Risk Books in association with Application Networks, September 25, 2003

[15] Gordy, M. (2000): "A comparative anatomy of credit risk models," *Journal of Banking and Finance*, 24, 119–149.

[16] He, C., and T. Teräsvirta (1999a): "Properties of moments of a family of GARCH processes," Journal of Econometrics, 92, 173–192

[17] Hull, J., & A. White (2004): Valuation of a CDO and an nth to Default CDS without Monte Carlo Simulation, Journal of Derivatives, 2, 8-23.

[18] D. O'Kane, M. Naldi, S. Ganapati, A. Berd, C. Pedersen, L. Schloegl, R. Mashal, "Lehman Brothers Guide to Exotic Credit Derivatives" Risk, special supplement, October 2003

[19] D. X. Li, "On Default Correlations: a Copula Function Approach", Journal of Fixed Income, vol. (2000)

[20] Mashal, R., Naldi, M. and Zeevi, A. (2003) Extreme Events and Multiname Credit Derivatives. In Credit Derivatives The Definitive Guide, Risk Waters Group, Great Britain, 313-338

[21] Greenberg A., Mashal, R., Naldi, M. and Schlogel, L. (2004) Tuning Correlation and Tail Risk to the Market Prices of Liquid Tranches. Quantitative Credit Research Quarterly, Lehman Brothers, March 3-12.

[22] R. Merton, "On pricing of corporate debt: the risk structure of interest rates", Journal of Finance, vol. 29 (1974) p. 449

[23] L. McGinty, E. Beinstein, R. Ahluwalia, M. Watts, "Credit Correlation: A Guide", JPMorgan Credit Derivatives Strategy, March 2004

[24] Nelsen, R. B. 1999. An Introduction to Copulas. Vol. 139 of Lecture Notes in Statistics, Springer, Berlin.

[25] P. J. Shonbucher, "Credit Derivatives Pricing Models: Models, Pricing and Implementation", J. Wiley & Sons (2003)

[26] Schonbucher, P. J. and D. Schubert. 2001. Copula-dependent default risk in intensity models. Working Paper, Department of Statistics, Bonn University.

[27] Dominic O'Kane and Lutz Schloegl (2003): "A Note on the Large Homogeneous Portfolio Approximation with the Student-t Copula", Finance and Stochastics, Volume 9, No 4, p 577 - 584.

[28] Sklar, A. 1959. Fonctions de r epartition a n dimensions et leurs marges. Publications de l'Institut de Statistique de l'Universit e de Paris, 8:229-231.

[29] O. Vasicek, "Loan Portfolio Value", Risk Magazine, December 2002, pp 160-162

[30] O. Vasicek, "A Series Expansion for the Bivariate Normal Integral", working paper(1998), KMV

[31] Zakoïan, J.-M. (1994). "Threshold Heteroskedastic Models". Journal of Economic Dynamics and Control 18, 931-955.

A Kurtosis and Skewness of Aggregated TARCH Returns

In this notes we analyze kurtosis and skewness of aggregated returns $R_T = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} r_t$ when r_t is assumed to follow TARCH(1,1) process

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= (1-\zeta)\,\sigma^2 + \alpha r_{t-1}^2 + \alpha_D r_{t-1}^2 1_{\{r_{t-1}\leq 0\}} + \beta \sigma_{t-1}^2 \end{aligned}$$

where returns innovations ε_t are assumed to be iid, have zero mean and unit variance. We are interested in variance, skewness and kurtosis of time aggregated returns. To make sure that those moments are finite we need corresponding moments of the return innovations to be finite. Particulaly, we assume that ε_t has finite kurtosis. Let us introduce the following notations for the central and truncated moments of ε_t

$$\begin{aligned} m_\varepsilon &\equiv E(\varepsilon_t) = 0 \\ v_\varepsilon &\equiv E\left(\varepsilon_t^2\right) = 1 \\ v_\varepsilon^d &\equiv E\left(\varepsilon_t^2 1_{\{\varepsilon_t \leq 0\}}\right) \\ s_\varepsilon &\equiv E\left(\varepsilon_t^3\right) \\ s_\varepsilon^d &\equiv E\left(\varepsilon_t^3 1_{\{\varepsilon_t \leq 0\}}\right) \\ k_\varepsilon &\equiv E\left(\varepsilon_t^4\right) \\ k_\varepsilon^d &\equiv E\left(\varepsilon_t^4 1_{\{\varepsilon_t \leq 0\}}\right) \end{aligned}$$

Lemma 7 *The following recursions hold for TARCH(1,1) model*

$$\begin{aligned} cov_{t-1}\left(r_t^k, r_{t+u}^2\right) &= \rho cov_{t-1}\left(r_t^k, r_{t+u-1}^2\right) \text{ for } u>1 \\ cov_{t-1}\left(r_t^k r_{t+1}^2\right) &= \alpha var_{t-1}\left(r_t^{k+2}\right) + \alpha_D var_{t-1}\left(r_t^{k+2} 1_{\{r_t \leq 0\}}\right) \end{aligned}$$

Proof.

$$\begin{aligned} &cov_{t-1}\left(r_t^k, r_{t+u}^2\right) = cov_{t-1}\left(r_t^k\left[(1-\zeta)\,\sigma^2 + \alpha r_{t+u-1}^2 + \alpha_D r_{t+u-1}^2 1_{\{r_{t+u-1}\leq 0\}} + \beta\sigma_{t+u-1}^2\right]\right) \\ &= 0 + \alpha cov_{t-1}\left(r_t^k, r_{t+u-1}^2\right) + \alpha_D cov_{t-1}\left(r_t^k, r_{t+u-1}^2 1_{\{r_{t+u-1}\leq 0\}}\right) + \beta cov_{t-1}\left(r_t^k, \sigma_{t+u-1}^2\right) \end{aligned}$$

if u>1 then

$$\begin{aligned} cov_{t-1}\left(r_t^k, r_{t+u-1}^2 1_{\{r_{t+u-1}\leq 0\}}\right) &= v_\varepsilon^d cov_{t-1}\left(r_t^k, r_{t+u-1}^2\right) \\ cov_{t-1}\left(r_t^k, \sigma_{t+u-1}^2\right) &= cov_{t-1}\left(r_t^k, r_{t+u-1}^2\right) \end{aligned}$$

If u=1 then

$$cov_{t-1}\left(r_t^k, \sigma_{t+u-1}^2\right) = 0$$

■

Proposition 8 *Suppose $0 \leq \zeta < 1$ and the return innovations have finite skewness, s_ε, and finite "truncated" third moment, s_ε^d, then conditional third moment of T-period aggregate return $R_{t,t+T}$ has the following representation for TARCH(1,1)*

$$E_t R_{t,t+T}^3 = \frac{1}{T^{3/2}} s_\varepsilon \sum_{u=1}^{T} E_t\left(\sigma_{t+u}^3\right) + \frac{3}{T^{3/2}}\left(\alpha s_\varepsilon + \alpha_D s_\varepsilon^d\right) \sum_{u=1}^{T} \frac{1-\zeta^{T-u}}{1-\zeta} E_t\left(\sigma_{t+u}^3\right)$$

In addition if $E\sigma_t^3$ is finite then unconditional skewness of $R_{t,t+T}$ is given by

$$S_T \equiv \frac{ER_{t,t+T}^3}{E(R_{t,t+T}^2)^{3/2}} = \left[\frac{1}{T^{1/2}} s_\varepsilon + 3\frac{1}{T^{3/2}}\left(\alpha s_\varepsilon + \alpha_D s_\varepsilon^d\right)\frac{T(1-\zeta)-1+\rho^T}{(1-\zeta)^2}\right] E\left(\frac{\sigma_t}{\sigma}\right)^3$$

Proof. *Using Lemma 7 we have*

$$\begin{aligned}
E_t\left(\sum_{u=t+1}^{t+T} r_u\right)^3 &= E_t\left(\sum_{t+1\leq t_1\leq t_2\leq t_3\leq t+T} r_{t_1} r_{t_2} r_{t_3}\right) \\
&= \sum_{u=1}^{T} E_t r_{t+u}^3 + \sum_{t+1\leq t_1<t_2\leq t+T} 3E_t\left(r_{t_1} r_{t_2}^2\right) \\
&= \sum_{u=1}^{T} E_t\left(r_{t+u}^3\right) + 3\sum_{t+1\leq t_1<t_2\leq t+T} \zeta^{t_2-t_1-1}\left(\alpha E_t\left(r_{t_1}^3\right) + \alpha_D E_t\left(r_{t_1}^3 1_{\{r_{t_1}\leq 0\}}\right)\right) \\
&= \sum_{u=1}^{T} E_t\left(r_{t+u}^3\right) + 3\sum_{u=1}^{T}\frac{1-\zeta^{T-u}}{1-\zeta}\left(\alpha E_t\left(r_{t+u}^3\right) + \alpha_D E_t\left(r_{t+u}^3 1_{\{r_{t+u}\leq 0\}}\right)\right) \\
&= s_\varepsilon \sum_{u=1}^{T} E_t\left(\sigma_{t+u}^3\right) + \left(\alpha s_\varepsilon + \alpha_D s_\varepsilon^d\right)\sum_{u=1}^{T}\frac{1-\zeta^{T-u}}{1-\zeta} E_t\left(\sigma_{t+u}^3\right)
\end{aligned}$$

Using the law of iterated expectations

$$E\left(\sum_{u=t+1}^{t+T} r_u\right)^3 = E\left(E_t\left(\sum_{u=t+1}^{t+T} r_u\right)^3\right) = \left[T s_\varepsilon + 3\left(\alpha s_\varepsilon + \alpha_D s_\varepsilon^d\right)\frac{T(1-\zeta)-1+\zeta^T}{(1-\zeta)^2}\right] E\left(\sigma_t\right)^3$$

S_T is then computed using the simple formula for the unconditional variance $E(R_{t,t+T}^2) = \sigma^2$. ■

To derive unconditional kurtosis we define the following unconditional autocorrelations

$$\begin{aligned}
\gamma_n &= \gamma_{-n} = corr(r_{t-n}^2, r_t^2) \\
\varphi_n &= corr(r_{t-n}, r_t^2) \text{ for } n \geq 1 \\
\psi_{i,j} &\equiv E\left(r_{t-i} r_{t-j} r_t^2\right) \text{ for } 1 \leq j < i
\end{aligned}$$

Lemma 9 *γ_n, φ_n and $\psi_{i,j}$ decay exponentially as n and $i-j$ increase*

$$\gamma_n = \zeta\gamma_{n-1} = \zeta^{n-1}\gamma_1 \text{ for } n \geq 1$$
$$\varphi_n = \zeta\varphi_{n-1} = \zeta^{n-1}\varphi_1 \text{ for } n \geq 1$$
$$\psi_{i,j} = \zeta\psi_{i-1,j-1} = \zeta^{j-1}\psi_{i-j+1,1} \text{ for } 1 \leq j < i$$

where γ_1, φ_1 and $\psi_{k,1}$ are given by

$$\gamma_1 = \alpha\left(k_r - 1\right) + \alpha_D\left(k_r^d - v_r^d\right) + \beta k_r/k_\varepsilon$$
$$\varphi_1 = \alpha s_r + \alpha_D s_r^d$$
$$\psi_{k,1} = \alpha E\left(r_{t-k+1}r_t^3\right) + \alpha_D E\left(r_{t-k+1}r_t^3 1_{\{r_t\leq 0\}}\right)$$

with $v_\varepsilon^d = \frac{E\left(r_t^2 1_{\{r_t\leq 0\}}\right)}{Er_t^2}$, $s_r = \frac{E\left(r_t^3\right)}{\left(Er_t^2\right)^{3/2}}$, $s_r^d = \frac{E\left(r_t^3 1_{\{r_T<0\}}\right)}{\left(Er_t^2\right)^{3/2}}$, $k_r = \frac{E\left(r_t^4\right)}{\left(Er_t^2\right)^2}$ *and* $k_r^d = \frac{E\left(r_t^4 1_{\{r_t\leq 0\}}\right)}{\left(Er_t^2\right)^2}$.

Proposition 10 *If*

$\zeta \equiv E\left(\beta + \alpha\varepsilon_t^2 + \alpha_D\varepsilon_t^2 1_{\{\varepsilon_t\leq 0\}}\right) = \beta + \alpha + \alpha_D v_\varepsilon^d < 1$

$\xi \equiv E\left(\beta + \alpha\varepsilon_t^2 + \alpha_D\varepsilon_t^2 1_{\{\varepsilon_t\leq 0\}}\right)^2 = \beta^2 + \alpha^2 k_\varepsilon + \alpha_D^2 k_\varepsilon^d + 2\alpha\beta + 2\alpha_D\beta v_\varepsilon^d + 2\alpha\alpha_D k_\varepsilon^d < 1$

then unconditional kurtosis of r_t, K_1, is finite and

$$K_1 \equiv \frac{Er_t^4}{\left(Er_t^2\right)^2} = k_\varepsilon\frac{1-\zeta^2}{1-\xi}$$

Proof. *If the 4th moment of r_t exists then the following equation must hold*

$$\begin{aligned}
Er_t^4 &= E\left(\varepsilon_t^4\right)E\left(\sigma_t^4\right)\\
&= k_\varepsilon E\left(\left(1-\zeta\right)\sigma^2 + \alpha r_{t-1}^2 + \alpha_D r_{t-1}^2 1_{\{r_{t-1}\leq 0\}} + \beta\sigma_{t-1}^2\right)^2\\
&= k_\varepsilon(\left(1-\zeta\right)^2\sigma^4 + 2\left(1-\zeta\right)\sigma^2 E\left(\alpha r_{t-1}^2 + \alpha_D r_{t-1}^2 1_{\{r_{t-1}\leq 0\}} + \beta\sigma_{t-1}^2\right)\\
&+ \left(\alpha r_{t-1}^2 + \alpha_D r_{t-1}^2 1_{\{r_{t-1}\leq 0\}} + \beta\sigma_{t-1}^2\right)^2)\\
&= k_\varepsilon\left(\left(1-\zeta\right)^2\sigma^4 + 2\left(1-\zeta\right)\rho\sigma^4 + \xi E\sigma_{t-1}^4\right)
\end{aligned}$$

Therefore Er_t^4 nessecerily solves

$$Er_t^4 = k_\varepsilon\left(1-\zeta^2\right)\sigma^4 + \xi Er_t^4$$

■

Proposition 11 *If the distibution of ε_t is symmetric and $\alpha_D = 0$ then unconditional kurtosis of R_T, if exists, is given by the following formula:*

$$K_T = 3 + \frac{1}{T}(K_1 - 3) + 6\frac{\gamma_1}{T^2}\frac{T(1-\zeta) - 1 + \zeta^T}{(1-\zeta)^2} \text{for } T > 1 \tag{66}$$
$$K_1 = k_\varepsilon\frac{1-\zeta^2}{1-\xi} \tag{67}$$

where k_ε *is unconditional kurtosis of* ε_t *and*

$$\xi \equiv E\left(\beta + \alpha\varepsilon_t^2 + \alpha_D \varepsilon_t^2 1_{\{\varepsilon_t \le 0\}}\right)^2 = \beta^2 + \alpha^2 k_\varepsilon + \alpha_D^2 k_\varepsilon^d + 2\alpha\beta + 2\alpha_D \beta v_\varepsilon^d + 2\alpha\alpha_D k_\varepsilon^d.$$

$$\gamma_1 \equiv corr\left(r_{t-1}^2, r_t^2\right) = \alpha\left(k_r - 1\right) + \alpha_D\left(k_r^d - v_r^d\right) + \beta k_r / k_\varepsilon$$

Proof.

$$\begin{aligned} E\left(\sum_{u=t+1}^{t+T} r_u\right)^4 &= \sum_{u=1}^{T} E\left(r_{t+u}^4\right) + 6 \sum_{t+1 \le t_1 < t_2 \le t+T} E\left(r_{t_1}^2 r_{t_2}^2\right) \\ &= \sum_{u=1}^{T} E\left(r_{t+u}^4\right) + 6 \sum_{t+1 \le t_1 < t_2 \le t+T} \left[cov\left(r_{t_1}^2, r_{t_2}^2\right) + E\left(r_{t_1}^2\right) E\left(r_{t_2}^2\right)\right] \\ &= TE\left(r_t^4\right) + 6\frac{T(T-1)}{2} E\left(r_t^2\right)^2 + 6cov\left(r_{t-1}^2, r_t^2\right) \sum_{t+1 \le t_1 < t_2 \le t+T} \zeta^{t_2 - t_1 - 1} \\ &= TE\left(r_t^4\right) + 6\frac{T(T-1)}{2} E\left(r_t^2\right)^2 + 6cov\left(r_{t-1}^2, r_t^2\right) \frac{T(1-\zeta) - 1 + \zeta^T}{(1-\zeta)^2} \end{aligned}$$

substituting the derived 4th moment into the definition of the kurtosis $K_T = E\left(\sum_{u=t+1}^{t+T} r_u\right)^4 / E\left(r_t^2\right)^2$ *completes the proof.* ■

B Estimation Results for SP500

Table 1 SP500 moments.

Sample period	Daily				Weekly			
	s_r	s_r^d	k_r	v_r^d	s_r	s_r^d	k_r	v_r^d
1962-2004	-1.40	-2.43	39.83	0.53	-0.55	-1.35	7.01	0.55
1990-2004	-0.11	-1.14	6.67	0.51	-0.64	-1.36	6.10	0.56
SP500 moments(After trimming 0.1% of extreme positive and negative returns)								
1962-2004	0.05	-1.03	5.95	0.50	-0.39	-1.18	5.26	0.54
1990-2004	0.04	-1.01	5.56	0.50	-0.64	-1.36	6.10	0.56

Table 2
Estimated parameters of GARCH(1,1)/TARCH(1,1) with Gaussian/Student-T shocks on daily(D) and weekly(W) SP500 returns for.[01/01/1990-12/31/2004].

	D	W	D	W	D	W	D	W
α	0.056	0.044	0.007	0.007	0.047	0.045	0.006	0.004
α_D	-	-	0.100	0.112	-	-	0.095	0.094
β	0.941	0.953	0.933	0.918	0.951	0.948	0.941	0.927
ν	-	-	-	-	7.25	7.77	8.21	8.31
After trimming 0.1% of extreme positive and negative returns								
α	0.062	0.094	0.023	0.032	0.062	0.088	0.022	0.030
α_D	-	-	0.067	0.112	-	-	0.073	0.104
β	0.935	0.896	0.941	0.895	0.936	0.90	0.938	0.90
ν	-	-	-	-	8.82	10.64	9.63	11

Table 3
Estimated parameters of GARCH(1,1)/TARCH(1,1) with Gaussian/Student-T shocks on daily(D) and weekly(W) SP500 returns for.[01/01/1962-12/31/2004].

	D	W	D	W	D	W	D	W
α	0.076	0.107	0.029	0.037	0.064	0.09	0.027	0.032
α_D	-	-	0.081	0.136	-	-	0.072	0.106
β	0.923	0.886	0.928	0.877	0.934	0.897	0.934	0.894
ν	-	-	-	-	7.86	9.26	8.44	10.19
After trimming 0.1% of extreme positive and negative returns								
α	0.051	0.044	0.004	0.007	0.046	0.045	0.004	0.004
α_D	-	-	0.093	0.112	-	-	0.096	0.094
β	0.945	0.953	0.940	0.918	0.952	0.947	0.941	0.926
ν	-	-	-	-	7.78	7.77	9.011	8.31

C Monte Carlo Simulation of Portfolio Loss under LHP Assumption

Because of the one factor structure of the model we can use LHP setup described in proposition 1 to calibrate the loss of a large homogeneous portfolio using the distribution of the aggregated market return generated by TARCH(1,1) model. The latent variables are assumed to have symmetric one factor structure with the factor following TARCH(1,1) model. We calibrate the loss of the portfolio using the one factor GARCH model described in and the formula 35 for LHP loss

$$L_T = \left(1 - \bar{R}\right) \Phi\left(\frac{d_T - bR_{m,T}}{\sqrt{1-b^2}}\right)$$

where

- $R_{m,T} = \frac{1}{\sqrt{T}} \sum_{u=1}^{T} r_{m,u}$ is return over horizon T generated using aggregation of simulated TARCH(1,1) returns with unconditional volatility equal to 1
- d_T is calibrated so that the probability of $R_{i,T} = bR_{m,T} + \sqrt{1-b^2}E_T$ hitting d_T is equal to single name default probability p_T

$$P\left(bR_{m,T} + \sqrt{1-b^2}E_T \leq d_T\right) = p_T$$

- b is the factor loading that is chosen to match a given unconditional linear correlation $\rho = b^2$

To calculate the expected tranche losses generated by the model and to calibrate d_T we use $I = 100,000$ independent Monte Carlo simulations of the factor and then use corresponding sample moments:

$$d_T \text{ solves } \frac{1}{I}\sum_{i=1}^{I} \Phi\left(\frac{d_T - bR^{(i)}_{m,T}}{\sqrt{1-b^2}}\right) = p_T$$

$$EL_{(0,K]} = \frac{1}{I}\sum_{i=1}^{I} f_{(0,K]}\left(\left(1 - \bar{R}\right) \Phi\left(\frac{d_T - bR^{(i)}_{m,T}}{\sqrt{1-b^2}}\right)\right)$$