Department of Finance
Working Paper Series 1998

Who Buys and Who Sells Options: The Role and Pricing of Options in an Economy with Background Risk

Marti G Subrahmanyam, Günter Franke, Richard C Stapleton

Who Buys and Who Sells Options: The Role of Options in an Economy with Background Risk

(running title: Who Buys and Who Sells Options)[*]

Günter Franke[†]
Richard C. Stapleton[‡]
Marti G. Subrahmanyam, [§]

First draft: April 1992
This draft: April 7, 1998

[†]Fakultät für Wirtschaftswissenschaften und Statistik, University of Konstanz
  Email: guenter.franke@uni-konstanz.de
[‡]The Management School, Lancaster University
  Email: rcs@staplet.demon.co.uk
[§]Corresponding author, Stern School of Business, New York University
  Phone No. (+1) 212 998- 0348
  Fax No.   (+1) 212 995-4233

1

Abstract

In this paper, we derive an equilibrium in which some investors buy call/put options on the market portfolio while others sell them. Since investors are assumed to have similar risk-averse preferences, the demand for these contracts is not explained by differences in the shape of utility functions. Rather, it is the degree to which agents face other, non-hedgeable, background risks that determines their risk-taking behavior in the model. We show that investors with low or no background risk have a concave sharing rule, i.e., they sell options on the market portfolio, whereas investors with high background risk have a convex sharing rule and buy these options.

1 Introduction

The spectacular growth in the use of derivatives to manage risks has been one of the most significant recent developments in the financial markets. In contrast to the widespread use and importance of options as well as the vast academic and practitioner literature on option pricing, research explaining the motivation for the use of options is quite sparse.[1] In this paper we consider the demand for options on the market portfolio and provide a new explanation for option supply and demand. Our model is based on the assumption that agents in the economy face non-hedgeable background risks. These background risks which could, for example, be associated with labor income or holdings of non-marketable assets, are non- insurable. To this extent, therefore, markets are incomplete in our model. Agents faced with background risks respond by demanding insurance in the form of options on the *marketable* risks.

We assume a pure exchange economy in which agents inherit a portfolio of state-contingent claims on the market portfolio. There is a perfect and complete market for state-contingent claims on this portfolio. All agents in the economy have hyperbolic absolute risk aversion [HARA] utility for wealth at the end of a single time-period.This assumption allows us to compare optimal sharing rules in the presence of background risk with the *linear* sharing rules that exist in an economy with HARA utility and no background risks. The sharing rule tends to be convex for those agents who face high background risk and concave for those with low or no background risk. Thus, the non-linearity in our model is attributable to differential background risks. A convex or concave sharing rule can be obtained by buying or selling options, whereas a linear sharing rule involves only the use of spot or forward contracts.

The analysis in this paper has primary relevance for individual investors' option demand. However, a similar argument applies for corporations, if conditions exist that make hedging relevant for firms. In a pure Modigliani-Miller world, hedging actions by value-maximizing firms are irrelevant since they can be accomplished by shareholders themselves. However, in a more realistic setting, hedging may be relevant for value-maximizing corporations due to non-linear taxes, bankruptcy costs or managerial risk aversion, as discussed by Smith and Stulz [21]. Hedging is also directly relevant for firms owned by entrepreneurs, or for those maximizing managerial utility. In these cases, our model may explain the use of options as opposed to forward contracts. Firms are faced with many risks some of which are hedgeable

3

and some of which are not. In our model, firms facing larger non- hedgeable risks than other firms and agents purchase options from those with smaller background risks.

The organization of the paper is as follows. In section 2, we review the relevant literature on the impact of background risk. In section 3, we assume that a perfect, complete (forward) market exists for state contingent claims on the market portfolio. We define the agent's utility maximization problem in the presence of background risk and illustrate the properties of the precautionary premium given the assumption of HARA preferences. In section 4, we show that, in this economy, the presence of background risk modifies the well-known linear sharing rule.[2] In equilibrium, every agentholds the risk-free asset, the market portfolio and a portfolio of state-contingent claims akin to options on the market portfolio. Agents with high background risk buy these options, whereas those with low background risk sell them. In section 5, we summarize our main conclusions.

2   Previous Work on Background Risk

It has been increasingly recognized in the literature that an agent's behavior towards a marketable risk can be affected by the presence of a second, independent, background risk. Nachman [14], Kihlstrom et. al. [10] and Ross [18] discuss the extent to which the original conclusions of Pratt [16] have to be modified when a background risk is considered. Recent work by Kimball [12] shows that if agents are standard risk averse, i.e., they have positive and declining coefficients of risk aversion and prudence, then the derived risk aversion (of Nachman [14]) of the agent will increase with background risk.[3] Further work by Gollier and Pratt [9], extending results of Pratt and Zeckhauser [17], shows the effect of the introduction of non-positive mean background risks on risk aversion. In this paper, we concentrate on the HARA-class of utility functions, which is a special case of standard risk aversion. This restriction allows us to derive specific results regarding the demand for risky claims by agents in the economy.In deriving the optimal sharing rules in the presence of non-hedgeable risk, we draw also on the work of Kimball [11]. In particular, we use his concept of the precautionary premium. In the special case of the HARA-class of functions considered in this paper, specific statements can be made about the precautionary premium. This allows us, in turn, to specify the optimal sharing rule and identify the role of options.[4]

The above work on background risk has been applied to the analysis of the related question of optimal insurance. Papers by Doherty and Schlesinger [6,7] and Eeckhoudt and Kimball [8] analyzing the optimal deductible and the coinsurance rate show that agents expand the coverage of risks in the presence of background risk. Since insurance contracts can be modeled in terms of options, our results for the demand for options can also be interpreted in terms of the demand for insurance. Finally, there is the related, but distinct work of Leland [13] and Brennan and Solanki [2] in portfolio insurance. These papers investigate differences across the utility functions of agents which cause them to buy or sell options on the market portfolio. They show that agents will demand portfolio insurance if their risk tolerance relative to that of the representative agent increases with the return on the market portfolio. Our analysis is linked to this previous work in the sense that background risk provides a rationale for utility functions to exhibit the properties found to be necessary by Leland. In our economy, differences in the risk-taking behavior of agents arise even though the agents have similar utility functions.

3 The Demand for Risky Assets and the Precautionary Premium

We assume a two-date, pure-exchange economy, where the dates are indexed 0 and 1. There are I agents, $i = 1, 2, \ldots, I$, in the economy. X is the time 1 measurable payoff on the market portfolio and is assumed to be continuous on R^+. Agents have homogeneous expectations with regard to X. We assume a perfect and complete market for claims on X, so that each agent can buy state-contingent claims on the market portfolio.[5] This means that an agent can buy a claim paying one unit of cash if $X > K$, and zero if $X \leq K$.[6] Hence, as in Leland [13], the agent is able to choose a payoff function, which we denote as $g_i(X)$. The function relates the agent's payoff from holding state-contingent claims on the market portfolio to the aggregate payoff, X. Given the complete market for claims on X, a unique pricing kernel denoted $\phi = \phi(X)$ exists, with $E(\phi) = 1$. Initially, ϕ is given exogenously. In addition to the investment in the marketable state-contingent claims, the agent also faces a non-insurable background risk. This risk has a non-positive mean and is independent of the market portfolio payoff, X. This background risk is also a time 1 measurable random variable, denoted $e_i = \sigma_i \varepsilon_i$, where ε_i is a random variable with non-positive mean and

5

unit variance. σ_i is a constant measuring the size of the background risk. We assume that e_i is bounded from below: $e_i \geq \underline{e_i}$. The agent's total income at time 1 is

$$y_i = g_i(X) + e_i. \tag{1}$$

The background risk is unavoidable and cannot be traded. The agent can only take this risk into account in designing an optimal portfolio of claims on X. Hence, we investigate the effect of the background risk, e_i, on the optimal payoff function, $g_i(X)$. We assume that the utility function $\nu_i(\cdot)$ is of the hyperbolic absolute risk aversion (HARA) form

$$\nu_i(y_i) = \frac{1 - \gamma_i}{\gamma_i} \left[\frac{A_i + y_i}{1 - \gamma_i} \right]^{\gamma_i} \tag{2}$$

where γ_i and A_i are constants.[7] We restrict our analysis to cases where $-\infty \leq \gamma_i < 1$, i.e. those exhibiting constant or decreasing absolute risk aversion. In the case of $0 \leq \gamma_i < 1$, we also assume that any attainable payoff function yields finite expected utility for the agent, and that $\exists\ \theta_i > 0$ such that $E[\phi^{(1-R_i)/R_i}] < \infty$, where $R_i = 1 - \gamma_i - \theta_i$ and $R_i > 0$. We choose the HARA-class since it is the only class of utility functions that implies linear sharing rules for all agents, in the absence of background risk. Also, we assume that it is feasible, given the agent's endowment, background risk, and the pricing kernel, to choose $g_i(X)$ so that $A_i + y_i > 0$, for all possible e_i. Defining $x_i = g_i(X)$, and dropping the subscript i, the agent solves the following maximization problem:

$$\max_x E\left[E_e[\nu(x + e)] \right] \tag{3}$$

$$\text{s.t.} \quad E\left[(x - x^0)\phi \right] = 0$$

where $\nu(\cdot)$ is the utility function of the agent. In equation (3), $E_e(\cdot)$ is the expectation over e, and $E(\cdot)$ is the expectation over X. In the budget constraint, $x^0 = x^0(X)$ is the agent's endowment of claims on the market portfolio payoff X. Given the HARA assumption, the agent's optimization problem can be written as

$$\max_x E\left[E_e \left[\frac{1 - \gamma}{\gamma} \left[\frac{A + x + e}{1 - \gamma} \right]^\gamma \right] \right] \tag{4}$$

$$\text{s.t.} \quad E\left[(x - x^0)\phi \right] = 0$$

6

We first establish that the assumptions we have made are sufficient to guarantee the differentiability of the expected utility function in the presence of background risk. We have:

Lemma 1: *Assume that*

$$\nu(x+e) = \frac{1-\gamma}{\gamma} \left[\frac{A+x+e}{1-\gamma} \right]^{\gamma}, -\infty \leq \gamma < 1 \tag{5}$$

where $e \geq \underline{e}$. Also, assume that it is possible to choose x so that $A+x+e > 0$, for all e. Then, $E_e[\nu(x+e)]$ is a three- times differentiable and strictly concave function of x.

Proof: See Appendix A.

We now establish

Theorem 1: *Assume that*

$$\nu(x+e) = \frac{1-\gamma}{\gamma} \left[\frac{A+x+e}{1-\gamma} \right]^{\gamma}, -\infty \leq \gamma < 1 \tag{6}$$

where $e \geq \underline{e}$. Also, assume that it is possible to choose x so that $A+x+e > 0$, for all e. Then, the first order condition for a solution of (3) is

$$E_e[\nu'(x+e)] = \lambda\phi, \tag{7}$$

where λ is the Lagrangian multiplier of the budget constraint. The solution is optimal and unique.

Proof: From Lemma 1, the first derivative of $E_e[\nu(x+e)]$ exists and is given by $E_e[\nu'(x+e)]$. First, take the case where $-\infty \leq \gamma < 0$. In this case, the relative risk aversion of $E_e[\nu(x+e)]$ is > 1, as $x \to \infty$. It follows then, by Back and Dybvig [1, Theorem 1], that a solution of (7) exists and is optimal. In the case where $0 \leq \gamma < 1$, the assumption that there exists $\theta > 0$ such that $E[\phi^{(1-R)/R}] < \infty$ with $R = 1 - \gamma - \theta$, $R > 0$, and the assumption that expected utility is finite, also imply that a solution of (7) exists and is optimal. This follows from Back and Dybvig [1, Theorem 2].[8] Finally, uniqueness follows from the strict concavity of $E_e[\nu(x+e)]$ established in Lemma 1. Also, the first order condition holds as an equality since

$$E_e \left[\frac{A+x+e}{1-\gamma} \right]^{\gamma-1} \to 0, \text{as } x \to \infty \tag{8}$$

7

and $E_e[\nu'(x+e)] \to \infty$ as $A + x + e \to 0$. $\square$

In order to analyze the impact of background risk on the agent's optimal demand for claims on the market payoff, it is useful to introduce Kimball's concept of the precautionary premium. Kimball [11] defines a precautionary premium, ψ, analogous to the Arrow-Pratt risk premium, except that it applies to the *marginal* utility function rather than the utility function itself. In the present context, we define

$$E_e[\nu'(x+e)] \equiv \nu'(x-\psi) \tag{9}$$

where $\psi = \psi(x, \sigma)$. The precautionary premium is a function of the market payoff of the agent and the scale of the background risk.[9] It is the amount of the deduction from x, which makes the marginal utility equal to the conditional expected marginal utility of the agent in the presence of the background risk.[10]

From equations (7) and (2) it follows that

$$\nu'(x - \psi(x, \sigma)) = \left[\frac{A + x - \psi(x, \sigma)}{1 - \gamma} \right]^{\gamma - 1} = \lambda\phi \tag{10}$$

Equation (10) reveals that, given the market pricing kernel, $\phi = \phi(X)$, the payoff function $x = g(X)$ depends directly on the precautionary premium ψ. We begin,therefore, by analyzing the effect of the x and σ on the precautionary premium.For fairly general utility functions, a number of properties of the precautionary premium, ψ, have been established in the literature. Most of these follow from the analogy between the risk premium, π, defined on the utility function, and the precautionary premium, ψ, defined on the marginal utility function. From the analysis of Pratt-Arrow, π is positive and decreasing in x, if the coefficient of absolute risk aversion, $a(y) = -\nu''(y)/\nu'(y)$ is positive and decreasing in y. Similarly, ψ is positive and decreases in x, if the coefficient of the absolute prudence, defined as $\eta(y) = -\nu'''(y)/\nu''(y)$ is positive and decreases in y (see Kimball [11]). The correspondence can be taken further. For *small* risks with a zero-mean, the risk premium [precautionary premium] is equal to one-half the product of the coefficient of absolute risk aversion [absolute prudence] and the variance of the payoff on the small risk. For larger risks, higher absolute risk aversion [prudence] implies a higher risk premium [precautionary premium]. Since, for the HARA-class of utility functions, the coefficient of absolute prudence is strictly proportional to the coefficient of absolute risk aversion, $\gamma < 1$

8

implies also positive decreasing absolute prudence and hence, standard risk aversion as defined in Kimball [12]. We now establish the following results regarding the shape of the $\psi(x, \sigma)$ function:

Lemma 2: *In the presence of background risk, if $\nu(y)$ is of the HARA family with $-\infty < \gamma < 1$, ψ is twice differentiable and*

$$\psi > 0,$$
$$\frac{\partial \psi}{\partial x} < 0,$$
$$\frac{\partial^2 \psi}{\partial x^2} > 0,$$

For $\gamma = -\infty$ (exponential utility), $\psi > 0$ and $\partial \psi / \partial x = 0$.

<u>Proof</u>: See Appendix B.

The significance of Lemma 2 is that it implies that, given a level of background risk, its effect, measured by the precautionary premium, declines at a decreasing rate in the income from the marketable assets. In other words, the precautionary premium is a positive, decreasing, convex function of the marketable income. The first two statements are implied by positive, decreasing absolute prudence.[11] The exception is the case of the exponential utility function, for which the precautionary premium is independent of the marketable income. We are interested also in the effect of the scale of the non-hedgeable background risk, which is indexed by σ. Hence, we now establish

Lemma 3: *In the presence of background risk, if $\nu(y)$ is of the HARA family with $\infty < \gamma < 1$,*

$$\frac{\partial \psi}{\partial \sigma} > 0,$$
$$\frac{\partial^2 \psi}{\partial \sigma \partial x} < 0,$$
$$\frac{\partial^3 \psi}{\partial \sigma \partial x^2} > 0.$$

For $\gamma = -\infty$ (exponential utility), $\partial \psi / \partial \sigma > 0$, but independent of x.

<u>Proof</u>: See Appendix B.

In other words, the increase in the precautionary premium due to an increase in background risk is smaller, the higher the income x; moreover, the convexity of the premium increases as the background risk increases. The first statement in Lemma 3 is implied by positive prudence. The significance of Lemma 3 is that it allows us to compare the effect of background risk on the optimal sharing rules of different agents. Other things being equal, an agent with a higher background risk (larger σ_i) will have a more convex precautionary premium function than one with a lower background risk (σ_i small). These statements are correct as long as the agent has non-exponential HARA utility.

4 Optimal Demand for Marketable Risks: An Equilibrium Analysis

We now analyse the optimal demand of agents $i = 1, 2, ..., I$ with different levels of background risk and derive equilibrium prices of state- contingent claims in this economy. As above, we assume a complete market for state-contingent claims on the market portfolio payoff, X. Individual agents choose $x_i = g_i(X)$ claims on X. Agents have HARA utility functions with declining absolute risk aversion $-\infty < \gamma_i < 1$ and homogeneous expectations regarding the market portfolio payoff.[12] In equilibrium, we require that individual demands, x_i, sum to X, the market portfolio payoff. Agents face different levels of background risks. The differing levels of background risk affect the agents' demands for shares of the market portfolio payoff.

Solving equation (10) for x_i, aggregating over all agents in the economy and imposing the equilibrium market clearing condition $\sum_i x_i = X$, we have

$$X = \sum_i \left[\psi_i(x_i, \sigma_i) + [\lambda_i \phi]^{\frac{1}{\gamma_i - 1}} (1 - \gamma_i) - A_i \right], \quad \forall X \qquad (11)$$

In principle, (11) can be solved to endogenously determine the market pricing kernel, $\phi = \phi(X)$, and then, by substituting back in the individual demand condition, equation (10), to determine the equilibrium optimal demand function, $x_i = g_i(X)$, for agent i. However, in general, the resulting expressions for ϕ and x_i are complex functions of the parameters γ_i, A_i, and the variables, λ_i, ψ_i, for all the agents in the economy. Further insight into the portfolio behavior of agents can be gained by assuming that all the agents have the same risk aversion coefficient, γ, but face different levels of background risk, σ_i. This allows us to isolate the effect of the background

risk in the portfolio behavior of the agent.[13] If all the agents have the same γ, we can derive a simpler equation for x_i. In this case, we have:

<u>Theorem 2</u>: *Suppose that agents in the economy have homogeneous expectations and have HARA utility functions with $-\infty < \gamma_i < 1$ and with the same γ. Then the optimal sharing rule of agent i is*

$$x_i = A_i^* + \alpha_i X + \alpha_i [\psi_i^*(x_i) - \psi(X)] \tag{12}$$

wherea) $A_i^* = \alpha_i A - A_i$ *is the agent's risk free income at time 1, where* $A \equiv \sum_i^I A_i$ *and*

$$\alpha_i = \frac{\lambda_i^{\frac{1}{\gamma-1}}}{\sum_{h=1}^I \lambda_h^{\frac{1}{\gamma-1}}}, \qquad \sum_{i=1}^I \alpha_i = 1,$$

b) $\alpha_i X$ *is the agent's linear share of the market portfolio payoff,c)* $\alpha_i [\psi_i^*(x_i) - \psi(X)]$ *is the agent's payoff from contingent claims, where*$\psi_i^* = \frac{\psi_i}{\alpha_i}$ *and*

$$\psi(X) = \sum_{i=1}^I \psi_i(x_i).$$

<u>Proof</u>: Solving (11) for $\phi(X)$ in the special case where $\gamma_i = \gamma, \forall i$, and substituting in (10) yields (12). □

Theorem 2 does not provide an explicit solution for x_i, since $\psi_i^* = \psi_i/\alpha_i$ and the ψ_i themselves depend on the x_i. However, it permits us to separate the demand of the agent for claims on X into three elements. The first two provide a linear share of the market portfolio payoff. If there were no background risk for all agents in the economy, the third element would be zero and the individual agent would have a linear sharing rule (as in Rubinstein [20]). Note that the linear share represented by the first two elements can be achieved by arranging forward contracts on the market portfolio, or, equivalently, by aggregate borrowing/lending and investment in shares of the market portfolio. The non-linear element is provided by the third term in equation (12). This is non-linear because we know that the precautionary premium ψ_i is a convex function of x_i (Lemma 2). However, in equilibrium, it is the *relative* convexity of $\psi_i^* = \psi_i/\alpha_i$ compared to the aggregate ψ of all agents in the market that determines the convexity (or

concavity) of the sharing rule. Since the third element in the sharing rule is non-linear, it must be achieved by the agent buying or selling option-like contingent claims on the market portfolio. However, whether an individual agent buys or sells such claims depends upon $\psi_i^* = \psi_i/\alpha_i$ compared to the aggregate $\psi(X)$.[14]

In order to evaluate the sharing rule for a particular agent and to ask whether that agent is, for example, a buyer or seller of options, we need to investigate the convexity of the pricing function $\phi = \phi(X)$. For that purpose, we investigate the shape of $\phi^{\frac{1}{\gamma-1}}$, as a function of X.[15] First we have to establish the differentiability of the sharing rule $x_i = g_i(X)$ and of the pricing function $\phi = \phi(X)$. We have:

Lemma 4: *In equilibrium, the sharing rule defined by the function $x_i = g_i(X)$, and the pricing function $\phi = \phi(X)$ are twice differentiable.*

Proof: see Appendix C.

Differentiating equation (10) with respect to X yields

$$\frac{\partial x_i}{\partial X} = (1 - \gamma) \left[\frac{\partial \phi^{\frac{1}{\gamma-1}}}{\partial X} \right] \lambda_i^{\frac{1}{\gamma-1}} \left[1 - \frac{\partial \psi_i}{\partial x_i} \right]^{-1} \tag{13}$$

Aggregating over all the agents in the market, we find[16]

$$\left[\frac{\partial \phi^{\frac{1}{\gamma-1}}}{\partial X} \right]^{-1} = (1 - \gamma) \sum_i \lambda_i^{\frac{1}{\gamma-1}} \left[1 - \frac{\partial \psi_i}{\partial x_i} \right]^{-1} \tag{14}$$

Since $\gamma < 1$ and $\partial \psi_i/\partial x_i < 0$, $\forall i$, it follows that $\frac{\partial \phi^{\frac{1}{\gamma-1}}}{\partial X} > 0$, and hence that $\partial \phi/\partial X < 0$. This result confirms our intuition that contingent claims on states where X is low are relatively expensive. This is also true in the presence of the non-hedgeable risks. Differentiating (14) with respect to X and taking the sign, we have

$$sgn \left[\frac{\partial^2 \phi^{\frac{1}{\gamma-1}}}{\partial X^2} \right] = sgn \left[- \sum_i \lambda_i^{\frac{1}{\gamma-1}} \left[1 - \frac{\partial \psi_i}{\partial x_i} \right]^{-2} \frac{\partial^2 \psi_i}{\partial x_i^2} \frac{\partial x_i}{\partial X} \right] \tag{15}$$

Since $\partial^2 \psi_i/\partial x_i^2 > 0$ from Lemma 2, and $\partial x_i/\partial X > 0$, it follows that the sign in equation (15) is negative. Therefore $\phi^{\frac{1}{\gamma-1}}$ is an increasing, strictly concave

function. Now, from the aggregate equation (11), it follows immediately that $\psi(X) = \sum_i \psi_i(X)$ is strictly convex.

Background risk changes $\phi^{[1/(\gamma-1)]}$ from a linear function of X to a concave function. An agent without background risk reacts to this concavity by selling claims in states where X is low or X is high and by buying claims in the other states. This implies a concave sharing rule:

<u>Theorem 3</u>: *Suppose that there is an agent who has no background risk in an economy where other agents face background risk. The sharing rule of this agent is strictly concave.*

<u>Proof</u>: Since the agent has no background risk, this follows by placing $\psi_i^* = 0$ in equation (12). Since $\psi(X)$ is convex, as has been shown above, $-\alpha_i\psi(X)$ is concave and the optimal sharing rule for this agent is concave. $\square$

In order to obtain a concave sharing rule, the agent has to sell call and put options at different strike prices. Strictly speaking, options with infinitely many strike prices would be required to exactly construct the desired sharing rule. The essential point is that although the agent may take positions in linear claims such as forward contracts, options are also required to produce the desired sharing rule. This is also true of agents with positive background risk who have a non-linear demand for claims on the market portfolio. This non-linear element is the difference between two functions, $\psi_i^*(X)$ and $\psi(X)$. It is difficult to be precise, therefore, about an agent's sharing rule except to say that it will tend to be convex if the agent's precautionary premium (caused by relatively high σ_i) is more convex than that of the average agent in the market. Those agents with relatively high σ_i tend to buy claims with convex payoffs and those with relatively low σ_i tend to sell those claims. This is parallel to Leland's conclusion that agents whose risk tolerance increases rapidly with income buy convex claims from agents whose risk tolerance increases less rapidly. These agents achieve this by purchasing put and/or call options. It follows that background risk could explain why some agents buy and others sell portfolio insurance.

Next, we can relate our result in Theorem 2 directly to the literature on sharing rules where a two-fund separation is established. Two-fund separation refers to the agent buying a portfolio of riskless securities and a share of a portfolio of risky assets.[17] Theorem 2 indicates that the existence of background risk destroys the two-fund separation property. It is not possible to generalize the result to three-fund separation since the third "fund" varies

across agents. To see this, note that agents' holdings in the third "fund" net out to zero and hence have the nature of "side-bets". These side bets are similar, however, for those agents with "similar" precautionary premia ψ_i.

5 Concluding Comments

We derive an equilibrium in which some agents supply and others demand convex claims such as options, even though their risk preferences are similar. Background risk, which is assumed to be non-insurable, has the effect of changing the risk aversion of the agent. Given HARA preferences, agents with relatively high background risk become more averse to marketable risk, i.e., the risk involved in holding marketable claims. However, they not only become more risk averse, but also become relatively more risk averse in the states with low marketable income. Across investors, the effect is that investors with high background risk have a convex demand for contingent claims and purchase options from investors with low background risk.

Appendix A

Proof of Lemma 1

The assumptions that $e \geq \underline{e}$ and $A + x + e > 0$ imply that

$$\nu'(x + e) = \left[\frac{A + x + e}{1 - \gamma} \right]^{\gamma - 1} \leq \nu'(x + \underline{e}) < \infty, \forall e. \tag{16}$$

Hence, $\nu'(x + e)$ is uniformly integrable and it follows that

$$\frac{\partial}{\partial x} E_e[\nu(x + e)] = E_e[\nu'(x + e)] \tag{17}$$

and therefore, $E_e[\nu(x + e)]$ is differentiable. By a similar argument, the second and the third derivative of $E_e[\nu(x + e)]$ exist and equal $E_e[\nu''(x + e)]$ and $E_e[\nu'''(x + e)]$ respectively. Also, since $-\infty < \nu''(x + e) < 0$, $E_e[\nu''(x + e)] < 0$. Hence, $E_e[\nu(x + e)]$ is strictly concave in x. $\square$

16

Appendix B

Properties of the Precautionary Premium for the HARA Class of Preferences with $\gamma < 1$

Note that in this appendix we write $E(.)$ for $E_e(.)$, since all expectations are taken with respect to the background risk $e = \sigma \varepsilon$. First, the differentiability of ψ follows from the differentiability of the HARA utility. For the HARA class of preferences, with $\gamma < 1$,

$$\nu(y) = \frac{1-\gamma}{\gamma} \left[\frac{A+y}{1-\gamma}\right]^{\gamma} \tag{18}$$

It follows that

$$\nu'(y) = \left[\frac{A+y}{1-\gamma}\right]^{\gamma-1} > 0 \tag{19}$$

$$\nu''(y) = -\left[\frac{A+y}{1-\gamma}\right]^{\gamma-2} < 0 \tag{20}$$

$$\nu'''(y) = \frac{\gamma-2}{\gamma-1}\left[\frac{A+y}{1-\gamma}\right]^{\gamma-3} > 0 \tag{21}$$

$$a(y) = \left[\frac{A+y}{1-\gamma}\right]^{-1} > 0 \tag{22}$$

$$\eta(y) = \frac{\gamma-2}{\gamma-1}\left[\frac{A+y}{1-\gamma}\right]^{-1} > 0 \tag{23}$$

We can now prove the various statements of Lemmas 2 and 3.

1) Proof that $\psi > 0$.

For the HARA utility function, the marginal utility function ν' is a strictly convex function since $\nu''' > 0$. As a result, we have from Jensen's inequality

$$
\begin{aligned}
\nu'[x - \psi(x,\sigma)] &\equiv E[\nu'(x + e)] \\
&> \nu'[E(x + e)] = \nu'(x + \sigma\, E(\varepsilon))
\end{aligned}
\tag{24}
$$

Hence,

$$\psi > -\sigma \, E(\varepsilon) \geq 0 \tag{25}$$

since the risk ε has a non-positive mean and ν' is a strictly decreasing function of x. $\square$

2) Proof that $\partial\psi/\partial x < [=] 0$. We have for a HARA utility function

$$\eta(x) \equiv -\frac{\nu'''}{\nu''} = \frac{\gamma - 2}{\gamma - 1} a(x) \tag{26}$$

where $a(x)$ is the Arrow-Pratt measure of risk aversion. Hence,

$$\text{sgn } \eta(x) = \text{sgn } a(x), \quad \text{sgn } \frac{\partial\eta(x)}{\partial x} = \text{sgn } \frac{\partial a(x)}{\partial x} \tag{27}$$

It follows from arguments of Pratt (1964) about $a(x)$ that

$$\frac{\partial\psi}{\partial x} < [=] 0 \tag{28}$$

where the inequality holds for decreasing absolute risk aversion and the equality holds for exponential utility ($\gamma = -\infty$) for which $a(x)$ is constant. $\square$

3) Proof that $\partial\psi/\partial\sigma > 0$.

By analogy with the arguments of Pratt [16] and Rothschild and Stiglitz [19] , since

$$\nu' > 0, \ \nu'' < 0 \implies \partial\pi/\partial\sigma > 0$$

we can write that

$$\nu'' < 0, \ \nu''' > 0 \implies \partial\psi/\partial\sigma > 0 \qquad \square$$

4) Proof that $\partial^2\psi/\partial x\partial\sigma < 0$.

Differentiate the definitional equation

$$\nu'[x - \psi] \equiv E[\nu'(x + e)] \tag{29}$$

18

with respect to σ and obtain

$$\frac{\partial \psi}{\partial \sigma} = \frac{E[\nu''(x + e)\varepsilon]}{-\nu''[x - \psi]} \tag{30}$$

$$= \frac{E[\nu''(x + e)\varepsilon]}{E[-\nu''(x + e)]} \cdot \frac{E[-\nu''(x + e)]}{-\nu''[x - \psi]} \tag{31}$$

The second term on the right hand side of equation (31) is positive, given the assumption of risk aversion. Since the left hand side is positive, both fractions on the right hand side of (31) are positive. We now show that both fractions decline in x.

Differentiate the first fraction with respect to x. The differential is negative if and only if

$$E[\nu''(x + e)]E[\nu'''(x + e)\varepsilon] > E[\nu''(x + e)\varepsilon]E[\nu'''(x + e)] \tag{32}$$

which is the same as

$$\frac{E[\nu'''(x + e)\varepsilon]}{E[\nu'''(x + e)]} < \frac{E[-\nu''(x + e)\varepsilon]}{E[-\nu''(x + e)]} \tag{33}$$

sinc e $E[\nu''(x + e)] < 0$ and $E[\nu'''(x + e)] > 0$.Consider an agent facing the choice between a riskless and a risky asset, where the excess return on the risky asset is equal to $\check{\mu} + \varepsilon$, and $\check{\mu} + E(\varepsilon)$ is the expected excess return of the risky asset over the riskless rate. Let σ denote the optimal dollar investment in the risky asset, given another utility function with marginal utility being equal to $-\nu''(\cdot)$. Then, the optimality condition is that the right hand side of inequality (33) equals $-\check{\mu}$, with x being the riskfree income plus $\sigma\check{\mu}$. For a utility function with higher absolute risk aversion the same fraction would be smaller than $-\check{\mu}$, since the optimal investment in the risky asset would be smaller. As for the HARA class with $\gamma < 1$, $-\nu''''(\cdot)/\nu'''(\cdot) > -\nu'''(\cdot)/\nu''(\cdot) > 0$, inequality (33) holds. This proves that the first fraction on the right hand side of (31) declines in x.

In order to show the same for the second fraction, define

$$\nu''[x - \varphi] \equiv E[\nu''(x + e)] \tag{34}$$

where $\varphi = \varphi(x, \sigma)$ is the premium defined by the second derivative of the utility function. [π is the premium defined by the utility function (risk premium) and ψ is the premium defined by the first derivative (precautionary

19

premium)]. Then, the second fraction in (31) can be rewritten as

$$\frac{E[-\nu''(x+e)]}{-\nu''[x-\psi]} = \frac{\nu''[x-\varphi]}{\nu''[x-\psi]} \tag{35}$$

For the HARA class of preferences, the right hand side of (35) can be written as

$$\frac{\nu''[x-\varphi]}{\nu''[x-\psi]} = \left(\frac{A+x-\varphi}{A+x-\psi}\right)^{\gamma-2} \tag{36}$$

Differentiate the right hand side of (36) with respect to x. The differential is negative (since $\gamma < 1$), if

$$(A+x-\varphi)^{-1}\left(1-\frac{\partial\varphi}{\partial x}\right) > (A+x-\psi)^{-1}\left(1-\frac{\partial\psi}{\partial x}\right) \tag{37}$$

We substitute for $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial\psi}{\partial x}$ by differentiating (29) and (34) to obtain

$$\left[1-\frac{\partial\psi}{\partial x}\right] = \frac{E[\nu''(x+e)]}{\nu''[x-\psi]} \tag{38}$$

$$\left[1-\frac{\partial\varphi}{\partial x}\right] = \frac{E[\nu'''(x+e)]}{\nu'''[x-\varphi]} \tag{39}$$

We substitute (38) and (39) in (37) to yield

$$\frac{E\left[(A+x+e)^{\gamma-3}\right]}{[A+x-\varphi]^{\gamma-2}} > \frac{E\left[(A+x+e)^{\gamma-2}\right]}{[A+x-\psi]^{\gamma-1}} \tag{40}$$

Substitute for the denominators in the two sides of the inequality from (29) and (34) and obtain

$$E\left[(A+x+e)^{\gamma-3}\right]E\left[(A+x+e)^{\gamma-1}\right] > \left[E\left\{(A+x+e)^{\gamma-2}\right\}\right]^2 \tag{41}$$

Since

$$(A+x+e)^{\gamma-3}(A+x+e)^{\gamma-1} = \left[(A+x+e)^{\gamma-2}\right]^2 \tag{42}$$

it follows from Cauchy's inequality that (41) holds. Hence $\partial^2\psi/\partial x\partial\sigma < 0$.
$\square$

5) Proof that $\partial^2\psi/\partial x^2 > 0$.

20

From equation (38), it follows that

$$\frac{\partial^2 \psi}{\partial x^2} > 0 \tag{43}$$

if and only if the right-hand side in equation (38) decreases as x increases. We have already shown this to be true in equations (35) through (42). $\square$

6) Proof that $\partial^3 \psi / \partial \sigma \partial x^2 > 0$.

First, note that convexity of ψ approaches 0 as $\sigma \to 0$. Since ψ is convex for any positive value of σ, it follows that convexity increases with σ for small changes from $\sigma = 0$. We now use a monotonicity result to show that convexity increases with σ for any value of σ. We rewrite equation (29) for the HARA class and multiply throughout by $(1/\sigma)^{\gamma-1}$ to obtain

$$\left[\frac{[A+x]}{\sigma} - \frac{\psi}{\sigma}\right]^{\gamma-1} = E\left[\left(\frac{[A+x]}{\sigma} + \varepsilon\right)^{\gamma-1}\right] \tag{44}$$

Multiply and divide equation (44) throughout by q, where $q > 0$, to yield

$$\left[\frac{q[A+x]}{q\sigma} - \frac{q\psi}{q\sigma}\right]^{\gamma-1} = E\left[\left(\frac{q[A+x]}{q\sigma} + \varepsilon\right)^{\gamma-1}\right] \tag{45}$$

Define x_1 such that

$$q[A + x_0] \equiv A + x_1.$$

Then, using subscript 0 for x in equation (45) yields

$$\left[\frac{[A+x_1]}{q\sigma} - \frac{q\psi(x_0,\sigma)}{q\sigma}\right]^{\gamma-1} = E\left[\left(\frac{q[A+x_0]}{q\sigma} + \varepsilon\right)^{\gamma-1}\right] \tag{46}$$

In words, if σ changes from σ to $q\sigma$ and x changes from x_0 to x_1, then the new precautionary premium $\psi = \psi(x_1, q\sigma) = q\psi(x_0, \sigma)$. In order to show that the convexity of ψ increases with σ, suppose that σ is raised from a level arbitrarily close to 0. Then, the convexity of $\psi = \psi(x_0, \sigma)$ increases. Hence, the convexity of $\psi = \psi(x_1, q\sigma)$ increases by the factor q. As q can be arbitrarily large, the convexity of $\psi = \psi(x_1, q\sigma)$ increases monotonically with $q\sigma$. $\square$

Appendix C

Proof of Lemma 4

The first step is to prove that the function $x = f(\phi)$ is twice differentiable. From Theorem 1 a unique, optimal solution of the first order condition (7) exists and can be written as

$$F(x, \phi) = E_e[\nu'(x + e)] - \lambda\phi = 0. \tag{47}$$

From Lemma 1, the partial derivative F_x exists and is continuous. Also, $F_x \neq 0$ for $x < \infty$. Since F_ϕ also exists, it follows from the implicit function theorem that $x = f(\phi)$ is defined and is differentiable with

$$f'(\phi) = \frac{-F_\phi}{F_x}. \tag{48}$$

Since $F_\phi = \lambda$ is differentiable and, given the three- times differentiability of $E_e[\nu(x + e)]$ from lemma 1, F_x is differentiable, then $f''(\phi)$ exists.

The second step is to establish the differentiability of the function $\phi = \phi(X)$ and of the sharing rule $x_i = g_i(X)$. For investor i,

$$\frac{\partial x_i}{\partial \phi} = \frac{\lambda_i}{E_e[\nu_i''(x_i + e_i)]}, \tag{49}$$

and since $\lambda_i > 0$, and $\nu_i'' < 0$, $\frac{\partial x_i}{\partial \phi} < 0, \forall i$. In equilibrium $X = \Sigma_i x_i$ and

$$\frac{\partial X}{\partial \phi} = \sum_i \frac{\partial x_i}{\partial \phi} < 0. \tag{50}$$

Since $X = h(\phi)$ is strictly decreasing, h is one-to-one and $\phi = h^{-1}(X)$ is a function. Also, since $\frac{\partial X}{\partial \phi}$ exists and is non zero, then $\frac{\partial \phi}{\partial X}$ exists. Hence, by the chain rule

$$\frac{\partial x_i}{\partial X} = \frac{\partial \phi}{\partial X} \frac{\partial x_i}{\partial \phi} \tag{51}$$

exists.

Since $\frac{\partial^2 x_i}{\partial \phi^2}$ exists ,

$$\frac{\partial^2 X}{\partial \phi^2} = \sum_i \frac{\partial^2 x_i}{\partial \phi^2} \tag{52}$$

exists. Also, $\frac{\partial^2 x_i}{\partial \phi^2} > 0$ and hence $\frac{\partial^2 X}{\partial \phi^2} > 0$. Since $\frac{\partial X}{\partial \phi}$ is strictly decreasing, it is a one-to-one function. Hence, since $\frac{\partial^2 X}{\partial \phi^2} \neq 0$, then $\frac{\partial^2 \phi}{\partial X^2}$ exists. Finally, it follows from differentiating (51), that $\frac{\partial^2 x_i}{\partial X^2}$ also exists. $\square$

References

1. K. Back and P.H. Dybvig, On Existence of Optimal Portfolios in Complete Markets, Working Paper, Washington University in St. Louis, 1993.

2. M.J. Brennan and R. Solanki , Optimal Portfolio Insurance, *J. Finan. Quant. Anal.*, **16** (1981), 279-300.

3. E. Briys and H. Schlesinger , Risk Aversion and the Propensities forSelf-Insurance and Self-Protection, *Southern Econ. J.*, **57** (1990), 458-467.

4. E. Briys, M. Crouhy and H. Schlesinger, Optimal Hedging in a Futures Market With Background Noise and Basis Risk, *Europ. Econ. Rev.*, **37** (1993), 949-960.

5. D. Cass and J. Stiglitz , The Structure of Investor Preference and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds, *J.Econ. Theory*, **2** (1970), 122-160.

6. N. Doherty and H. Schlesinger, Optimal Insurance In Incomplete Markets, *J. Polit. Economy*, **91** (1983), 1045-1054.

7. N. Doherty and H. Schlesinger, The Optimal Deductible For An Insurance Policy When Initial Wealth Is Random, *J. Bus.*, **56** (1983), 555-565.

8. L. Eeckhoudt and M. Kimball. , Background Risk, Prudence, and the Demand for Insurance, in: "Contributions to Insurance Economics", ed. by G. Dionne, Kluwer Academic Publishers, Dordrecht, (1992), 239-254.

9. C. Gollier and C. J.W. Pratt, Risk Vulnerability and the Tempering Effect of Background Risk, *Econometrica*, **64** (1996), 1109- 1123.

10. R. E. Kihlström, D. Romer and S. Williams , Risk Aversion withRandom Initial Wealth, *Econometrica*, **49** (1981), 911-920.

11. M. S. Kimball, Precautionary Saving in the Small and in the Large, *Econometrica*, **58** (1990), 53-73.

12. M.S. Kimball, Standard Risk Aversion, *Econometrica*, **61** (1993), 589-64.

13. H.E. Leland, Who Should Buy Portfolio Insurance ?, *J. Finance*, **35** (1980), 581-594.

14. D.C. Nachman, Preservation of 'More Risk Averse' under Expectations, *J. Econ. Theory*, **28** (1982), 361-368.

15. D.C. Nachman, Spanning and Completeness with Options, *Rev. Finan. Stud.* , **1** (1988), 311-328.

16. J.W. Pratt, Risk Aversion in the Small and in the Large, *Econometrica*, **32** (1964), 122-136.

17. J.W. Pratt and R.J. Zeckhauser, Proper Risk Aversion, *Econometrica*, **55** (1987), 143-154.

18. S. Ross, Some Stronger Measures of Risk Aversion in the Small and Large with Applications, *Econometrica*, **49** (1981), 621-638.

19. M. Rothschild and J.E. Stiglitz, Increasing Risk I: A Definition, *J. Econ. Theory*, **2** (1970), 225-243.

20. M. Rubinstein , An Aggregation Theorem for Securities Markets, *J. Finan. Econ.*, **1** (1974), 225-244.

21. C.W. Smith and R. Stulz , The Determinants of Firms' Hedging Policies, *J. Finan. Quant. Anal.*, **20** (1985), 391-405.

Footnotes

1. Exceptions are the papers by Leland [13] and Brennan and Solanki [2], who also examine the demand for options on the market portfolio assuming that agents have utility functions with different rates of declining relative risk aversion.

2. See Cass and Stiglitz [5] and Rubinstein [20].

3. The coefficient of risk aversion is defined as the negative of the ratio of the second to the first derivative of the utility function. The coefficient of prudence is defined as the negative of the ratio of the third to the second derivative of the utility function.

4. It should be noted that Briys, Crouhy and Schlesinger [4] and Briys and Schlesinger [3] have also previously employed the precautionary premium in the context of hedging.

5. We are concerned, in this paper, with the effect of non-marketable background risk on the agents' portfolio behavior. Standard results from portfolio theory would apply to the choice between various marketable assets, and hence, this simplification does not affect the results here.

6. See Nachman [15].

7. Most commonly-used utility functions such as the quadratic, constant absolute risk aversion and constant proportional risk aversion cases can be obtained as special cases of the HARA family, by choosing particular values of γ_i and A_i. In the case of constant absolute risk aversion, $\gamma_i = -\infty$ and $\nu_i(y_i) = -\exp(A_i y_i)$. With $\gamma_i = 0$, we obtain the generalized logarithmic utility function, $\nu_i(y_i) = \ln(A_i + y_i)$.

8. For HARA utility functions, relative risk aversion of $E_e[\nu(x + e)]$ exceeds R for high levels of x, as required by Back and Dybvig [1, Theorem 2].

9. Note that ψ is differentiable since, from lemma 1, the expected marginal utility function is differentiable.

10. The precautionary premium is not constrained to the case of zero-mean background risk as noted by Kimball [11, p 56].

11. The statements of Lemma 2 regarding $\frac{\partial \psi}{\partial x}$ and $\frac{\partial^2 \psi}{\partial x^2}$ hold also if the background risk has a positive mean, since the mean has the same effect as adding a constant to x. This is also true for the results in section 4.

12. Hence we exclude the trivial case of constant absolute risk aversion, $\gamma = -\infty$, where the precautionary premium is not a function of x.

13. Leland [13] focuses on the other case, where there is no background risk, but agents differ in terms of their risk aversion coefficients. He shows that the sharing rule of agent i is convex if and only if γ_i is less than the γ of the representative agent, assuming HARA preferences.

14. One might conjecture that under appropriate conditions there exists an agent with a linear sharing rule. This is very doubtful, however. The following example shows a situation in which such an agent cannot exist. Suppose there exist three agents. Agent 1 has no background risk. The other two agents have *small* background risks so that

$$\psi_i(x_i, \sigma_i) = 1/2\eta_i(x_i)\sigma_i^2; \qquad i = 2, 3.$$

Now suppose that agent 2 has a linear sharing rule. Then

$$\psi_2(x_2, \sigma_2) = \alpha_2(\psi_2(x_2, \sigma_2) + \psi_3(x_3, \sigma_3))$$

follows from his sharing rule, or

$$\psi_2(x_2, \sigma_2)(1 - \alpha_2) = \psi_3(x_3, \sigma_3)).$$

For small risks it follows

$$\eta_2(x_2)\sigma_2^2(1 - \alpha_2) = \eta_3(x_3)\sigma_3^2.$$

In the HARA-case, this yields

$$\frac{\sigma_2^2(1 - \alpha_2)}{A_2 + x_2} = \frac{\sigma_3^2}{A_3 + x_3}$$

so that x_3 is linear in x_2. Hence linearity of $x_2 = g_2(X)$ implies linearity of $x_3 = g_3(X)$. But then agent 1 must also have a linear sharing rule in equilibrium which contradicts Theorem 3. Therefore, in this example, a representative agent, i.e. an agent with a linear sharing rule, cannot exist.
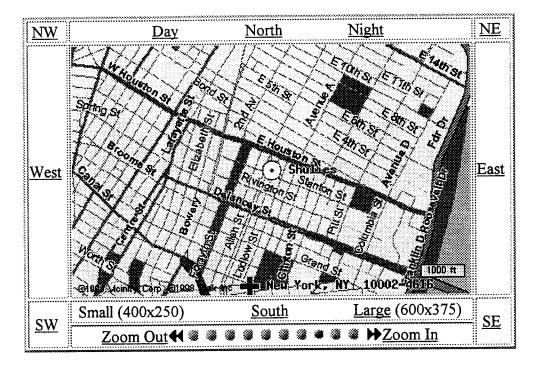
26

15. $\phi^{\frac{1}{\gamma-1}}$ is a linear function of $X - \psi(X)$, the aggregate wealth reduced by the aggregate precautionary premium. As $\psi(X)$ is non-linear in X, $\phi^{\frac{1}{\gamma-1}}$ is not linear in X, given the background risk.

16. Note that $\sum_i \partial x_i / \partial X = 1$.

17. See, for example, Cass and Stiglitz [5] and Rubinstein [20].

YAHOO!
YELLOW PAGES

Click here to get Yahoo! Online
Fast and reliable Internet access for only **$14.95 a month.**

YAHOO! Online

Shulies, 175 Orchard St, New York, NY 10002-1417(212-473-2480)

NW	Day	North	Night	NE
West				East
SW	Small (400x250)	South	Large (600x375)	SE
	Zoom Out		Zoom In	

[BACK to Main Categories] | Print Preview | [SEARCH from another location]
[FAQs | Need Help?]