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Abstract

We build a no-arbitrage model of the term structure, using two stochastic factors on each date, the short-term interest rate and the premium of the forward rate over the short-term interest rate. The model can be regarded as an extension to two factors of the lognormal interest rate model of Black-Karasinski. It allows for mean reversion in the short rate and in the forward premium. The method is computationally efficient for several reasons. First, interest rates are defined on a bankers' discount basis, as *linear* functions of zero-coupon bond prices, enabling us to use the no-arbitrage condition to compute bond prices without resorting to iterative methods. Second, the multivariate-binomial methodology of Ho-Stapleton-Subrahmanyam is extended so that a multiperiod tree of rates with the no-arbitrage property can be constructed using analytical methods. The method uses a recombining two-dimensional binomial lattice of interest rates that minimizes the number of states and term structures over time. Third, the problem of computing a large number of term structures is simplified by using a limited number of 'bucket rates' in each term structure scenario. In addition to these computational advantages, a key feature of the model is that it is consistent with the observed term structure of volatilities implied by the prices of interest rate caps and floors. We illustrate the use of the model by pricing American-style and Bermudan-style options on interest rates. Option prices for realistic examples using forty time periods are shown to be computable in seconds.

1 Introduction

Perhaps the most important and difficult problem facing practitioners in the field of interest rate derivatives in recent years has been to build inter-temporal models of the term structure of interest rates, that are both analytically sound and computationally efficient. These models are required, both to help in the pricing, and in the overall risk management of a book of interest rate derivatives. Although many alternative models have been suggested in the literature and implemented in practice, there are serious disadvantages with most of them. For example, Gaussian models of interest rates, which have the advantage of analytical tractability, have the drawbacks of allowing for negative interest rates, as well as failing to take into account the possibility of skewness in the distribution of interest rates. Also, many of the term-structure models used in practice are restricted to one stochastic factor. On the other hand, the Black model which is widely used to value European-style interest rate caps and floors, is not strictly in line with the definition of the contracts and also is not founded on an explicit model of the term structure of interest rates.

Since the work of Ho and Lee (1986), it has been widely recognized that term-structure models must possess the no-arbitrage property. In this context, a no-arbitrage model is one where the forward price of a bond is the expected value of the one-period-ahead spot bond price, under the risk-neutral measure. Building models that possess this property has been a major pre-occupation of both academics and practitioners in recent years. One model that achieves this objective in a one-factor context is the model proposed by Black, Derman and Toy (1990)(BDT), and extended by Black and Karasinski (1991)(BK). In essence, the model which we build in this paper can be thought of as a two-factor extension of this type of model. In our model, interest rates are lognormal and are generated by two stochastic factors. The general approach we take is similar to that of Hull and White (1994)(HW), where the conditional mean of the short rate depends on the short rate and an additional stochastic factor, which can be interpreted as the forward premium. In contrast to HW, and in line with BK, we build a model where the conditional variance of the short rate is a function of time. It follows that the model can be calibrated to the observed term structure of interest rate volatilities implied by interest rate caps/floors. Essentially, the aim here is to build a term structure model which can be applied to value American-style contingent claims on interest rates, which is consistent with the market prices of European-style contingent claims.

An alternative approach to building a no-arbitrage term structure for pricing interest rate derivatives has been pursued by Heath, Jarrow and Morton (1992)(HJM). In this approach, assumptions are made about the volatility of the *forward* interest rates. Since the forward rates are related in a no-arbitrage model to the future spot rates, there is a fairly close relationship between this approach and the one we are taking. In fact, the HJM volatilities can be thought of as the outputs of our model. If the parameters of the HJM model are known, this represents a satisfactory alternative approach. However, the BDT-HW approach has the advantage of requiring as inputs the volatilities of the short rate and of longer bond yields which are more directly observable from market data on the pricing of caps, floors and swaptions.

We would like any model of the stochastic term structure to have a number of desirable properties. Apart from satisfying the no-arbitrage property, we want the output of the model to match the inputs i.e., the conditional volatilities of the variables and the mean-reversion of the short rate and the premium factor . We also require that the short term interest rates be lognormal, so that they are bounded from below by zero and skewed to the right. From a computational perspective, we require the state space to be non-explosive, i.e. re-combining, so that a reasonably large span of time-periods can be covered. The complexities caused by these model requirements are discussed in section 2. We introduce a number of new aspects into our model that allow us to solve these requirements. The most important simplification arises from modeling the bankers' discount interest rate. We then extend and adapt the recombining binomial methodology of Ho, Stapleton and Subrahmanyam (1995)(HSS) to model lognormal rates, rather than prices. The new computational techniques are discussed in detail in section 3. Section 4 presents the basic two-factor model, and discusses some of its principal characteristics. In section 5, we explain how the multiperiod tree of rates is built using a modification of the HSS methodology. In section 6, we present some numerical examples of the output of our model, apply the model to the pricing of American-style and Bermudan-style options and discuss the computational efficiency of our methodology. Section 7 presents our conclusions.

2 Requirements of the model

There are several desirable features of any multi-factor model of the term structure of interest rates. Some of these features are requirements for theoretical consistency and others are necessary for tractability in implementation. Keeping in mind the latter requirements, it is important to recognize that the principal purpose of building a model of the evolution of the term structure is to price interest rate options generally and, in particular, those with path dependent payoffs. The simplest examples of options that need to be valued using such a model are American-style and Bermudan-style options on an interest rate.

First, since we wish to be able to price *any* term-structure dependent claim, it is important that the model output is a probability distribution of the term structure of interest rates at each point in time. A realistic model should be able to project the term structure for ten or twenty years, at least on a quarterly basis. With the order of forty or eighty subperiods, the computational task is substantial and complex. If we do not compute each term structure point at each node of the tree, then we need to be able to interpolate, where necessary, to obtain required interest rates or bond prices. As in the no-arbitrage models of HL, HJM, HW, BDT, and BK, we first build the risk-neutral or martingale distribution of the short-term interest rate, since other maturity rates and bond prices can be computed from the short-term rate.

The second and crucial requirement is that the interest rate process be arbitrage-free. In the context of term structure models, the no-arbitrage requirement, in effect, means that the one-period forward price of a bond of any maturity is the expected value, under the risk-neutral measure, of the one-period-ahead spot price of the bond. Since HL, this requirement has been well-understood, within the context of single factor models, and is a property satisfied by the HW, BDT, and BK models. However, the requirement is more demanding in the two-factor setting as shown by HJM, Duffie and Kan (1993) and by Stapleton and Subrahmanyam (1997). In a two-factor model in which the factors are themselves interest rates, the no-arbitrage condition restricts the behavior of the factors themselves as well as the behavior of bond prices. However, the no-arbitrage property is also an advantage in a computational sense, allowing the computation of bond prices at a node by taking simple expectations of subsequent bond prices under the risk-neutral measure.

The third requirement is that the term structure model should be consistent with the current term structure and with the term structure of volatilities implicit in the price of European-style interest rate caps and floors. Models that are consistent with the current term structure have been common in the literature since the work of Ho and Lee (1986)(HL). For example, the HJM, HW, BDT and BK models are all of this type. The second part of the requirement is rather more difficult to accommodate, since if volatilities are not constant over time, the tree of rates may be non-recombining as in some implementations of the HJM model, leading to an explosion in the number of states. The HW implementation of a two-factor model, in Hull and White (1994), specifically excludes time dependent volatility. BK on the other hand accommodate both time varying volatility and mean reversion of the short rate within a one-factor model by varying the meaning of the time steps in the model. This procedure is difficult to extend to a two-factor case.

At a computational level, it is necessary that the state space of the model does not explode, producing so many states that the computations become infeasible. Even in a single-factor model, this fourth requirement means that the tree of interest rates or bond prices must recombine. This is a property of the binomial models of HL, BDT, and BK, and also of the trinomial model of HW. A number of computational methods have been suggested to guarantee this property, including the use of different time steps and state-dependent probabilities. In the context of a two-factor model, the requirement is even more important. In our bivariate-binomial model we require that the number of states is no more than $(n+1)^2$, after n time steps. This is the bivariate analog of the "simple" recombining one-variable binomial tree of Cox, Ross and Rubinstein (1979).

Based on the empirical evidence as well as on theoretical considerations, the fifth requirement for our two-factor model is that the interest rates are lognormally distributed. It is well known that the class of Gaussian models, where the interest rate is normally distributed, are analytically tractable, allowing closed-form solutions for bond prices. However, apart from admitting the significant probability of negative interest rates, they are not consistent with the skewness that is considered important, at least for some currencies. Furthermore, such a model would be inconsistent with the widely-used Black model to value interest rate options, which assumes that the short-term rate is lognormally distributed. Of the models in the literature, the BDT and BK models explicitly assume that the short rate is lognormal. The HJM and HW multi-factor models are general enough to allow for lognormal rates, but at the expense of computational complexity.

In addition to these five requirements, there is an overall necessity that the model be computable for realistic scenarios, efficiently and with reasonable speed. In this context, we aim to compute option prices in a matter of seconds rather than minutes. To achieve this we need a number of modelling innovations, compared to the techniques used in prior models. These methodological innovations are discussed in the next section.

3 Particular features of the methodology

The dynamics of the term structure of interest rates can be modeled in terms of one of three alternative variables: zero-coupon bond prices, interest rates, or forward interest rates. If the objective of the exercise is to price contingent claims on interest rates, it is sufficient to model forward rates, as demonstrated by HJM. However, there are some problems with adopting this approach in a multi-factor setting. First, from a computational perspective, for general forward rate processes, the tree may be non-recombining, which implies that a large number of time-steps becomes practically infeasible. Second, it is difficult to estimate the volatility inputs for the model directly from market data. Usually, the implied volatility data are obtained from the market prices of options on Euro- currency futures, caps, floors and swaptions, which cannot be easily transformed into the volatility inputs required to build the forward rates in the HJM model.

We choose to model interest rates rather than prices because existing methodologies, introduced by HSS, can be employed to approximate a process with a log-binomial process. One major problem arises in modelling rates rather than prices, however, and that concerns the no- arbitrage property. Under the risk-neutral measure, forward bond prices are related to one-period-ahead spot prices, but the relationship for interest rates is more complex, as shown by HJM. This non-linearity makes the implementation of the binomial lattice much more cumbersome. We overcome this problem by modelling interest rates defined on a bankers' discount basis, as suggested in Stapleton and Subrahmanyam (1993) and Stapleton and Subrahmanyam (1997). In this case, the short term interest rate is a linear function of the price of a zero-coupon bond of the same maturity. Further, we assume that the three-month rate, defined on a bankers' discount basis, is lognormally distributed.[1]

We choose here to model interest rates, and then derive bond prices, forward prices and forward rates, as required, from the spot rate process. We prefer to directly model the short rate, which we interpret here as the three-month interest rate, since it is used to determine the payoffs on many contracts such as interest rate caps, floors and swaptions. One advantage of doing so is that implied volatilities from caplet/floorlet floor prices may be used to determine the volatility of the short-rate process, in a fairly straightforward manner.

As in HW, and Stapleton and Subrahmanyam (1997), we model the short rate, under the risk-neutral measure, as a two-dimensional AR process. In particular, we assume that the logarithm of the short rate, follows such a process where the second factor is an independent shock to the forward premium. The short rate itself and the premium factor each mean revert, at different rates, allowing for quite general shifts and tilts in the term structure. Also, in contrast to HW, we assume time dependent volatility functions for both the short rate and the premium factor. Stapleton and Subrahmanyam (1997) explore the properties of this model in detail. Taking conditional expectations of the process they show that the term structure of futures rates is given by a log-linear model in any two futures rates. In this model, the bond prices and forward rates for all maturities can be computed by backward induction, using the no-arbitrage property. Since the interest rate process is the risk-neutral distribution forward bond prices and interest rates (defined as above as linear functions of zero-bond prices) are expectations of one- period-ahead bond prices under this measure. This property permits the rapid computation of bond prices of all maturities by backward induction, at each point in time.

[1] Note that we only assume that the short-term interest rate is lognormal. If the short-term interest rate is represented by the three-month rate, we would expect a price in the region of 0.95-0.99. It follows that the probability of a negative price is negligible.

The principal computational problem is to build a tree of interest rates, which has the property that the conditional expectation of the rate at any point depends on the rate itself and the premium factor. A methodology available in the literature, which allows the building of a multivariate tree, approximating a lognormal process with non-stationary variances and covariances, is described in Ho, Stapleton and Subrahmanyam (1995)(HSS). In HSS, the expectation of a variable depends on its current value, but not on the value of a second stochastic variable. However, as we show in section 5, the methodology is easily extended to this more general case. The HSS methodology is itself a generalization to two or more variables of the method advocated by Nelson and Ramaswamy (1990), who devised a method of building a 'simple' or re-combining binomial tree for a single variable. Essentially, the HSS method relies on fixing the conditional probabilities on the tree to accommodate the mean reversion of the interest rates, the changing volatilities of the variables and the covariances of the variables. In the case of interest rates, it is crucial to model changes in the short rate so as to reflect the second, premium factor. This is the key, in a two-factor model, to maintaining the no-arbitrage property, while avoiding an explosion in the number of states. Using our extension of the HSS methodology allows us to model the bivariate distribution of short rate and the premium factor, with $n + 1$ states for each variable after n time steps, and a total of $(n + 1)^2$ term structures after n time steps. This is achieved by allowing the probabilities to vary in such a manner as to guarantee that the no- arbitrage property is satisfied and the tree is consistent with the given volatilities and mean reversion of the process.

One problem with extending the typical interest rate tree building methods of HW, BDT, and BK to two or more factors arises from the forward induction methodology normally employed in these models. The tree is built around the current term structure and the calculation proceeds by moving forward period-by-period. This is expensive in computing time, and could become prohibitively so, in the case of multiple factors. To avoid this problem, we devise a new dynamic method of implementation of the HSS tree, which allows us to compute the multivariate tree in a matter of seconds for up to eighty periods. This method uses the feature of HSS which allows a variation in the density of the tree over any given time step. A forward, dynamic procedure is used whereby a two-period tree with changing density is converted into the required multi-period tree. For example, when the eightieth time step is computed, the program computes a two-period tree with a density over the first period of seventy-nine and a density over the second period of one. This allows us to compute the tree nodes and the conditional probabilities analytically, and without recourse to iterative methods.

In practice much of the skill in building realistic models rests in deciding exactly what to compute. Potentially, in a tree covering eighty time steps, we could compute bond prices for between one and eighty maturities at each node of the tree. Not only is this a vast number of bond prices, but also, most of the bond prices will not be required for the solution of any given option valuation problem. We assume here that it will be sufficient to compute bond prices for maturities one year apart from each other. Intermediate maturity prices, if required, can always be computed by interpolation. Hence, in our eighty-time step examples, where each time step is a quarter of a year, we compute at most twenty bond prices. This saving reduces the number of calculations by almost seventy-five percent. For large numbers of time steps, it can turn an almost infeasible computational task into one that can be accomplished within a reasonable time frame. For example, with three hundred time steps, the number of bond price calculations can be reduced from approximately twenty-seven million to approximately one million.

In spite of the computational savings that are made by having a recombining tree methodology and reducing the number of bond prices that need to be calculated, it may still be the case that the computation time is excessive for a given problem. For example, for a Bermudan-style bond option that is exercisable every year for the first six years of the underlying bond's twenty year life, we only require bond prices at the end of each of the first six years. One computational advantage of the HSS methodology, is that the binomial density can be altered so that this problem is reduced to a seven-period problem with differential density (numbers of time steps). The binomial density ensures sufficient accuracy in the computations, while the number of bond and option price calculations is minimized.

4 The Two-factor Model

As in Stapleton and Subrahmanyam (1997), we assume that, under the risk- neutral measure, the logarithm of the short-term interest rate, for loans of maturity m, follows the process

$$d\ \mathrm{ln}r = [\theta_r(t) - a\mathrm{ln}r + \mathrm{ln}\pi]dt + \sigma_r(t)dz_1 \tag{1}$$

where

$$d\ \mathrm{ln}\pi = [\theta_\pi(t) - b\mathrm{ln}\pi]dt + \sigma_\pi(t)dz_2$$

In the above equations d ln$r$ is the change in the logarithm of the short rate, $\theta_r(t)$ is a time-dependent constant term that determines the mean, a is the speed at which the short rate mean reverts, π is the forward premium factor and $\sigma_r(t)$ is the instantaneous volatility of the short rate. The forward premium factor itself follows a diffusion process with mean θ_{pi}, mean reversion b and instantaneous volatility $\sigma_\pi(t)$. Although this structure is broadly similar to HW, note that we do not restrict the volatilities: σ_r and σ_π to be constant. Also, we assume that $r_t = (1 - B_t)/m$, where m is a fixed maturity of the short rate and B_t is the price of a m-year, zero-coupon bond at time t. dz_1 and dz_2 are standard Brownian motions.

In discrete form, equation (1) can be written[2]

$$\ln r_{t+1} - \mu_{t+1} = (\ln r_t - \mu_t)(1 - a) + \ln \pi_t - \mu_{\pi,t} + \varepsilon_{t+1} \tag{3}$$

where

$$\ln \pi_t - \mu_{\pi,t} = (\ln \pi_{t-1} - \mu_{\pi,t-1})(1 - b) + \nu_t,$$

and $\mu_t = E(\ln r_t)$ is the unconditional expectation of $\ln r_t$, and $\mu_{\pi,t} = E(\ln \pi_t)$ under the risk-neutral measure. In equation (3), ε_t and ν_t are mean-zero, independent, normally distributed shocks. Since the short-term interest rate is defined on a bankers' discount basis, the interest rate at time t is

$$r_t = (1 - B_{t,t+1})/m, \tag{4}$$

where $B_{t,t+1}$ is the value of a one-period zero-coupon bond and m is the length of one period measured in years.

4.1 Mean and Volatility Inputs for the Model

One important requirement of any financial model is that the parameters should be observable, or at least capable of being estimated, from market data. In the case of the model in equation (2) the parameters are the expected values of the short-term interest rate, μ_t

[2] In discrete form equation (1) is

$$\ln r_{t+1} - \ln r_t = \theta_r(t) - a \ln r_t + \ln \pi_t + \varepsilon_{t+1}, \tag{2}$$

where

$$\ln \pi_t - \ln \pi_{t-1} = \theta_\pi(t) - b \ln \pi_{t-1} + \nu_t,$$

Taking expectations and subtracting from equation (2) immediately yields equation (3).

and the conditional volatilities of the interest rate and the forward premium, $\sigma_r(t)$ and $\sigma_\pi(t)$. It also requires estimates of the degree of mean reversion of the short rate and of the premium, a and b respectively. The use of the banker's discount definition of the rate means that the expected value of the short-term interest rate, under the risk neutral measure, is directly observable from market prices of futures contracts. In the analysis that follows, we assume that the futures prices of zero-coupon bonds for the relevant maturity dates are either directly observable from the market for traded futures contracts, or estimable from the prices of bonds.[3]

The following argument can be used to back out estimates of the mean parameter, μ_t, from market prices of futures contracts. First, we know, from market data, the Libor futures rate, for delivery of an m-maturity loan at date t. We denote this rate as $l_{0,t}$. The corresponding futures price of an m-period zero-coupon bond is denoted $F_{0,t}$ and is given by

$$F_{0,t} = \frac{1}{1 + l_{0,t}m'}$$

where m' is the loan maturity, adjusted for the Libor day-count convention. We now define the futures rate, on a banker's discount basis, as

$$f_{0,t} = (1 - F_{0,t})/m. \tag{5}$$

From now on, for simplicity, we refer to this rate as the futures rate.

As argued by Cox, Ingersoll and Ross (1981)(CIR), the futures price of an asset is related directly to the expectation of the spot price on the maturity date, under the risk-neutral measure. Specifically, from CIR, proposition 2, it follows that the t-period futures price of a zero- coupon bond is the expected value, under the risk-neutral measure, of the time t spot price of the bond, i.e.

$$F_{0,t} = E_0(B_{t,t+1}). \tag{6}$$

Combining equations (4), (5) and (6) it follows immediately then that the corresponding futures rate is,

[3]For the major currencies, Eurocurrency (based on Libor rates) futures prices are observable from the market. For US \$, futures prices of the Eurodollar contract can be observed for up to ten years, with substantial liquidity. However, for most currencies, some estimation will be required, using market forward prices of bonds.

$$f_{0,t} = E_0(r_t)$$

Since the futures rate, defined on a bankers' discount basis, is equal to the expected spot rate under the risk-neutral measure, for all t, it follows that we can implement the model in (3) by constructing a tree of short-term rates where the expectation of the time-t rate is equal to the time-0 futures rate for delivery at time t. It also follows, using the above result and the lognormality of the interest rate, that the process in (3) can be re-written in the form

$$\ln r_{t+1} - [\ln f_{0,t+1} - \mathrm{var}(r_{t+1})] = (\ln r_t - [\ln f_{0,t} - \mathrm{var}(r_t)])(1-a) + \ln\pi_t + \mathrm{var}(\pi_t) + \varepsilon_{t+1} \quad (7)$$

where

$$\ln\pi_t - \mathrm{var}(\pi_t) = [\ln\pi_{t-1} - \mathrm{var}(\pi_{t-1})](1-b) + \nu_t,$$

and where $\mathrm{var}(r_t)$ and $\mathrm{var}(\pi_t)$ are the unconditional variances of the interest rate and the premium factor over the period $(0, t)$. Equation (7) shows that the only required inputs for the model are futures rates and the volatilities of the spot rate and of the premium factor.

We now consider the volatility inputs for the model. First, the conditional volatility of the short rate, $\sigma_r(t)$, is assumed to be given exogenously. The volatility of the short rate is potentially observable directly from market data on the pricing of European interest rate options, in particular, interest rate caps and floors, or from options on Libor futures contracts. The other required volatility parameter is the conditional volatility of the premium, $\sigma_\pi(t)$. As we now show, this volatility is closely related to the volatility of the first futures rate, and hence, also potentially observable. First, taking the conditional expectation of $\ln r_{t+1}$ in (3) and substituting for π we find

$$E_t(\ln r_{t+1}) = \mu_{t+1} + [\ln r_t - \mu_t](1-a) + \ln\pi_{t-1}(1-b) + \nu_t. \quad (8)$$

Next, using the no-arbitrage condition for the futures rate at time t, $f_{t,1} = E_t(r_{t+1})$, and the expression for the expected value of a lognormal variable yields, after taking logarithms,

$$\ln f_{t,1} = \mu_{t+1} + [\ln r_t - \mu_t](1-a) + \ln\pi_{t-1}(1-b) + \nu_t + \frac{m}{2}\sigma_r^2(t). \quad (9)$$

It follows that the conditional logarithmic variance of the first futures rate is given by the relation

$$\sigma_f^2(t) = (1-a)^2\sigma_r^2(t) + \sigma_\pi^2(t). \tag{10}$$

Hence, the volatility of the premium factor is potentially observable from the volatility of the first futures rate. In summary then, the time-dependent conditional volatilities of the two factors can be estimated from market data on the pricing of European- style interest rate options. Also, the time-dependent means of the short rate under the risk-neutral measure can be estimated using futures prices. Finally, the mean reversion of the short rate and the forward premium, each of which is assumed to be constant, can be estimated directly from market data. All the inputs of the model are therefore potentially observable.

4.2 Covariance Characteristics of the Model

In order to build the model of the term structure, using the HSS methodology, we need to know the unconditional (logarithmic) covariances between the variables in the model. For simplicity, from now on we will refer to the logarithmic covariance as the covariance, and denote the covariance of the logarithm of the short rate and the premium, for example, as $\text{cov}(r_t, \pi_t)$. The annualized covariance is denoted conventionally, as σ_{r_t,π_t}. In fact, we need two types of covariances: the contemporaneous covariances between the interest rate and the forward premium factor and the serial covariances of the short rate and of the premium factor, denoted on an annualized basis by σ_{r_{t+1},r_t} and σ_{π_{t+1},π_t}. Although we assume that the errors ε_t and ν_t are independent, this does not mean that the short rate and the forward premium factor, π are also independent. In fact, the solution of the time-series, difference equation (3), shows that the covariance of the r_t and π_t is, in general, a function of t. This, in turn, implies that the covariance of the spot rate with both forward and futures rates is time-dependent.

We first investigate the covariance of the spot rate at time $t+1$ with its previous realization, at time t. It follows immediately from (2) that the relevant covariances are given by the recursive relation

$$\text{cov}(r_{t+1}, r_t) = (1-a)\text{var}(r_t) + \text{cov}(r_t, \pi_t) \tag{11}$$

where $\text{var}(r_t)$ is the variance of the logarithm of r_t, and where the covariance of the short rate, with the premium factor, is given by

$$\text{cov}(r_t, \pi_t) = (1-a)(1-b)\text{cov}(r_{t-1}, \pi_{t-1}) + (1-b)\text{var}(\pi_{t-1}), \tag{12}$$

Secondly, from equation (3), the autocovariance of the premium factor is given by

$$\text{cov}(\pi_{t+1}, \pi_t) = (1-b)\text{var}(\pi_t). \tag{13}$$

Finally, we need

$$\text{cov}(r_t, \pi_{t-1}) = (1-a)\text{cov}(r_{t-1}, \pi_{t-1}) + \text{var}(\pi_{t-1}). \tag{14}$$

4.3 Futures rates, forward rates, and bond prices in the model

The no-arbitrage property requires that the forward price of a zero-coupon bond price must equal the expected one-period-ahead spot price of the bond, where the expectations are taken with respect to the risk- neutral measure. This property must hold in each state and at each date. In our discrete time implementation of the model, this requires that bond prices satisfy the following set of recursive equations.

The full set of recursive equations is

$$B_{t,t+k+1} = E_t(B_{t+1,t+k+1})B_{t,t+1} \tag{15}$$

and

$$B_{t,t+1} = 1 - r_t m$$

where

$$t = 0, 1, ..., T-1,$$
$$k = 0, 1, ..., T-1-t.$$

In equation (15), the bond prices for all maturities are given by the no-arbitrage relation, where the bond forward price is the expectation of the one-period-ahead spot price. The one-period bond price $B_{t,t+1}$ is related linearly to the short term interest rate r_t, defined on a bankers' discount basis. Note, however that only the short rate is defined on this basis. The longer maturity bond prices have to be determined by the no-arbitrage condition via equation (15). We now define the forward rate, again on a bankers' discount basis, by

$$g_{t,k} = 1 - \frac{B_{t,t+k+1}}{B_{t,t+1}}/m, \tag{16}$$

$$g_{t,0} = r_t$$

where

$$t = 0, 1, ..., T-1,$$
$$k = 0, 1, ..., T-1-t.$$

In equation (16), the forward price of the zero-coupon bond is first determined by forward parity as the ratio of the two relevant maturity spot bond prices. Then, the ratio is annualized using the bankers' discount convention. The forward rate for period 0 delivery, $g_{t,0}$ is the spot rate at time t.

Unfortunately, there is no simple closed-form relationship between forward rates of varying maturities. In contrast to the Gaussian model, the lognormal form of the term-structure model is not analytically tractable.[4] However, a simple closed-form solution does exist for futures prices and for futures rates, when interest rates are defined on a bankers' discount basis. This has been shown by Stapleton and Subrahmanyam (1997). We define futures rates in the model by

$$f_{t,k} = E_t(f_{t+1,k-1}), \tag{17}$$
$$f_{t,0} = r_t$$

where

$$t = 0, 1, ..., T-1,$$
$$k = 0, 1, ..., T-1-t.$$

Here, we denote the futures rate at time t, for k periods delivery as $f_{t,k}$. Since the futures rate is again defined on a bankers' discount basis, it is a linear function of the futures price. Since the futures price itself is a martingale under the risk-neutral measure, it follows immediately that the futures rate is a martingale. In the appendix A, we employ some results from Stapleton and Subrahmanyam (1997) to show that a log-linear relationship exists between futures rates of various maturities. This linear property, which holds exactly for futures rates, holds as an approximation for forward rates in the model.

[4]Various examples of Gaussian models can be analyzed following along the lines of Vasicek (1977).

5 A Methodology for approximating the interest-rate process

We now outline the method used to approximate the interest rate process, whose characteristics have been discussed in the previous section. We use three types of inputs. These are first, the unconditional means $E(r_t)$, $t = 1, ..., T$, of the short-term rate. Second, we require the volatilities of ε_t (the conditional volatility of the short rate, given the short rate and the one-period forward premium, denoted $\sigma_r(t)$) and the conditional volatilities of ν_t (the volatility of the premium, denoted $\sigma_\pi(t)$). Third, we need estimates of the mean reversion, a, of the short rate, and the mean reversion, b, of the premium factor. The process in (1) is then approximated using an adaptation of the methodology described in Ho, Stapleton and Subrahmanyam (1995) (HSS). HSS show how to construct a multivariate-binomial approximation to a joint lognormal distribution of M variables with a re-combining binomial lattice. However, in the present case we need to modify the procedure, allowing the expected value of the interest rate variable to depend upon the premium factor. That is, we need to model the two variables r_t and π_t, where r_t depends upon π_{t-1}. Furthermore, in the present context, we need to implement a multi-period process for the evolution of the interest rate, whereas HSS only implement a two-period example of their method. In this section, these modifications and the multi-period algorithm are presented in detail.

We divide the total time period into T periods of equal length of m years, where m is the maturity period, in years, of the short-term interest rate. Over each of the periods t to $t+1$ we assume that there are an exogenously given number of binomial time steps, termed the *binomial density*, and denoted by n_t. Note that, in the HSS method, n_t can vary with t allowing the binomial tree to have a finer density, if required for accurate pricing, over a specified period. This might be required, for example if the option exercise price changes between two dates, increasing the likelihood of the option being exercised, or for pricing barrier options.

5.1 Computing the nodal values

In this section, we first describe how the vectors of the short-term rates and the premium factor are computed. We approximate the process for the short-term interest rate, r_t, with a binomial process, which moves up or down from its expected value, by the multiplicative factors d_{r_t} and u_{r_t}. Following HSS, equation (7), these are given by

$$\begin{aligned} d_{r_t} &= \frac{2}{1+\exp(2\sigma_r(t)\sqrt{m/n_t})} \\ u_{r_t} &= 2 - d_{r_t} \end{aligned}$$

We then build a *separate* tree of the forward premium factor π. The up and down factors in this case are given by

$$\begin{aligned} d_{\pi_t} &= \frac{2}{1+\exp(2\sigma_\pi(t)\sqrt{m/n_t})} \\ u_{\pi_t} &= 2 - d_{\pi_t} \end{aligned}$$

The vectors of rates r_t and π_t are calculated from the equations

$$\begin{aligned} r_{t,j} &= u_{r_i}^{(N_t-j)} d_{r_t}^{j} E(r_t), \\ \pi_{t,j} &= u_{\pi_t}^{(N_t-j)} d_{\pi_t}^{j}, \\ j &= 0, 1, ..., N_t. \end{aligned} \tag{18}$$

where $N_t = \sum_t n_t$. In general, there are $N_t + 1$ nodes, i.e., states of r_t and π_t, since both binomial trees are re-combining. Hence, there are $(N_t + 1)^2$ states after t time steps.

5.2 Computing the conditional probabilities

As in HW, in general, the covariance of the two variables may be captured by varying the conditional probabilities in the binomial process. Since the trees of the rates and the forward premium are both re-combining, the time-series properties of each variable must also be captured by adjusting the conditional probabilities of moving up or down the tree, as in HSS and in Nelson and Ramaswamy (1991). Since, increments in the premium variable are independent of r_t, this is the simplest variable to deal with. We compute the conditional probability using HSS, equation (10). In this case the probability of a up-move, given that π_{t-1} is at node j, is

$$q_{\pi_t} = \frac{\alpha_{\pi_t} + \beta_{\pi_t} \ln \pi_{t-1,j} - (N_{t-1} - j)\text{ln}u_{\pi_t} - j\text{ln}d_{\pi_t} - n_t\text{ln}d_{\pi_t}}{n_t\,(\text{ln}u_{\pi_t} - \text{ln}d_{\pi_t})} \tag{19}$$

where

$$\beta_{\pi_t} = (1 - b)$$

$$\alpha_{\pi_t} = -\frac{t\sigma^2_{\pi_t}}{2} + \beta_{\pi_t}\frac{(t-1)\sigma^2_{\pi_{t-1}}}{2}$$

and where b is the mean reversion of π, and $\sigma^2_{\pi_t}$ is the unconditional volatility of π over the period $(0 - t)$.

The key step in the computation is to fix the conditional probability of an up-movement in the rate r_t, given the outcome of r_{t-1}, the mean reversion of r, and the value of the premium factor π_{t-1}. In discussing the multi-period, multi-factor case, HSS present the formula for the conditional probability when a variable x_2 depends upon x_1 and a contemporaneous variable, y_2. Adjusting HSS, equation (13) to the present case, we compute the probability

$$q_{r_t} = \frac{\alpha_{r_t} + \beta_{r_t} \ln(r_{t-1,j}/E(r_{t-1})) + \gamma_{r_t} \ln \pi_{t-1,j} - (N_{t-1} - j)\text{ln}u_{r_t} - j\text{ln}d_{r_t} - n_t\text{ln}d_{r_t}}{n_t(\text{ln}u_{r_t} - \text{ln}d_{r_t})} \tag{20}$$

where

$$\beta_{r_t} = (1 - a)$$

$$\gamma_{r_t} = 1$$

$$\alpha_{r_t} = -\frac{t\sigma^2_{r_t}}{2} + \beta_{r_t}\frac{(t-1)\sigma^2_{r_{t-1}}}{2} + \gamma_{r_t}\frac{(t-1)\sigma^2_{\pi_t}}{2}$$

where σ_{r_t} and σ_{π_t} refer to the unconditional volatilities of r_t and π_t for the period 0 to t. We show in the appendix B that the coefficient γ_{r_t} is equal to one given the covariances of the model derived in section 4.

5.3 The multiperiod algorithm

HSS provide the equations for the computation of the nodal values of the variables, and the associated conditional probabilities, in the case of two periods t and $t + 1$. Efficient implementation requires the following procedure for the building of the T period tree. The method is based on the following forward induction. We first compute the tree for the case where t=1. This gives us the nodal values of the variables and the conditional probabilities, for the first two periods. We then treat the first two periods as one new period, but with a binomial density equal to the sum of the first two binomial densities. The computations are then made for period three nodal values and conditional probabilities. In order to be consistent, we always use the conditional volatilities to build the vectors of nodal values. The following steps are followed:-

1. Using equation (18), compute the $[n_1 \text{ x } 1]$ vectors r_1, π_1 using inputs $\sigma_r(1)$, $E(r_1)$ and binomial density n_1. Also compute the $[(n_1 + n_2) \text{ x } 1]$ vectors r_2, π_2 using inputs $\sigma_r(2)$, $E(r_2)$ and binomial density n_2. Assume the probability of an up-move in r_1 is 0.5 and then compute the conditional probabilities q_{π_1} using equation (19) with t=1. Then compute the conditional probabilities q_{r_2}, q_{π_2}, using equations (19) and (20), with t=2.

2. Using equation (18), compute the $[(N_2 + n_3) \text{ x } 1]$ vectors r_3, π_3 using inputs $\sigma_r(3)$, $E(r_3)$ and binomial density, n_3. Then compute the conditional probabilities q_{r3}, $q_{\pi3}$ using equations (19) and (20) with t=3.

3. Continue the procedure until the final period T.

In implementing the above procedure, we first complete step 1, using $t = 1$ and $t = 2$, and with the given binomial densities n_1 and n_2. To effect step 2, we then redefine the period from $t = 0$ to $t = 2$ as period 1 and the period 3 as period 2 and re-run the procedure with a binomial densities $n_1^* = n_1 + n_2$ and $n_2^* = n_3$. This algorithm allows the multiperiod lattice to be built by repeated application of equations (18), (19) and (20).

6 Model Validation and Examples of Inputs and Outputs

This section shows the results from several numerical examples and examines the two factor term structure model described in previous sections in detail. Firstly, we show an example of how well the binomial approximation converges to the mean and unconditional volatility inputs. Secondly, we show that a two-factor term structure model can be run in a speedy and efficient manner. Thirdly, we discuss the input and output for an eight- period example, showing illustrative output of zero-coupon bond prices, and conditional volatilities. Finally we show the output from running a forty-eight quarter model, including the pricing of European and Bermudan style options on coupon bonds.

6.1 Convergence of Model Statistics to Exogenous Data Inputs

The first test of the two-factor model is how quickly the mean and variance of the short rates generated converge to the exogenous data. Table 1 shows an example of a twenty-period model, where the input mean of the spot rate is 5%, with a 10% conditional volatility. There is no mean reversion and the premium has a volatility of 1%. Note first that for a binomial density of 1, the accuracy of the binomial approximation deteriorates for later periods. This is due to the premium factor increasing with maturity and the difficulty of coping with the increased premium, by adjusting the conditional probabilities.

One way to increase the accuracy of the approximation is to increase the binomial density. In the last three columns of the table we show the effect of increasing the binomial density to 2, 3, and 4 respectively. By comparing different binomial densities in a given row of the table we observe the convergence of the binomial approximation to the exogenous inputs as the density increases. Even for the 20 period case, high accuracy is achieved by increasing the binomial density to 4.

Table 1 here

6.2 Computing Time

The most important feature of the two-factor model proposed in this paper is the computation time. With two stochastic factors rather than one, the computation time can easily increase dramatically. In table 2, we illustrate the efficiency of our model by showing the time taken to compute the zero-coupon bond prices and option prices. With a binomial density of 1, the 48-period model takes 12.3 seconds. Doubling the number of periods increases the computer time by a factor of six. There is a trade-off between the number of periods, the binomial density of each period, and the computation time for the model. This is illustrated by the second line in the table, showing the effect of using a binomial density of 2. Again the computation time increases more than proportionately as the density increases. The time taken for the 28-period model, when the binomial density is 2, is roughly the same as that for the 48-period model with a density of 1.

Table 2 here

6.3 Numerical Example: An Eight-Period Bond

This subsection shows a numerical example of the input and output of the two-factor term structure model, in a simplified eight quarter example . It illustrates the large amount of data produced by the model, even in this small scale case. The input is shown first in Table 3. We assume a rising curve of futures rates, starting at 5% and rising to 6%. These are used to fix the means of the short rate for the various periods. The second row shows the conditional volatilities assumed for the short rate. These start at 14% and fall through time to 12%. We then assume a constant mean reversion of the short rate, of 10%, and constant conditional volatilities and mean reversion of the premium factor, of 2% and 40% respectively.

Table 3 here

Tables 4 and 5 show a selection of the basic output of the model. For a binomial density of one, there are 4 states at time 1, 9 states at time 2, 16 states at time 3, and so on. In each state the model computes the whole term structure of zero-bond prices. In Table 4, we show just the longest bond price, paying one unit at period 8. These are shown for the 4 states at time 1, in the first block of the table. The subsequent blocks show the 9 prices at time 2, the 16 prices at time 3, and so on.

Table 4 here

One of the most important features of the methodology is the way that the no-arbitrage property is preserved, by adjusting the conditional probabilities at each node in the tree of rates. In Table 5, we show the probability of an up move in interest rate given a state, where the state is defined by the short rate and the premium factor. In the first block of the table is the set of probabilities conditional on being in one of four possible states at time 1. The second block shows the conditional probabilities at time 2, in the 9 possible states, and so on.

Table 5 here

6.4 An Example of an Option on a Coupon-Bond

The main purpose of the model is to price path-dependent payoff options. A good example is a Bermudan option on a coupon bond. We illustrate our methodology by pricing a six-year option on a twelve-year coupon bond. The option has the Bermudan feature that it is exercisable, at par, at the end of each year up to the option maturity in year six. We first build the tree of rates, assuming the data detailed in the notes to Table 6. Note that the model uses 48 quarterly time periods, to cover the twelve-year life of the coupon bond.

Table 6 here

Table 6 shows that the Bermudan option on the coupon bond is worth considerably more than the European option. Also there is a small positive effect of valuing the options with the binomial tree with a density of two. Illustrative output of the price of the underlying coupon bond is shown in Table 7.

Table 7 here

7 Conclusions

In this paper we have to shown that an arbitrage-free two-factor model of the term structure of interest rates can be implemented quickly and effectively. We have illustrated that the model's output as a probability distribution of the term structure of interest rates at each point in time, and hence we are able to price *any* term-structure dependent claim.. In doing so, we have shown that the output of the model approximates the inputs: the conditional volatilities of the variables and the mean-reversion of the short rate and the premium factor. We have shown how to extend and adapt the recombining binomial methodology of HSS (1995) to model lognormal rates, rather than prices. We have used the model to price a relatively complex claim: a Bermudan-style call option on a coupon bond.

In a subsequent paper we will show how to calibrate the model to the observed term structure of interest rate volatilities implied by interest rate caps/floors.

Appendix A: Properties of futures rates in the two-factor model

In a lognormal model of the type that we assume, there are no simple closed-form solutions for bond prices or forward rates. However, the martingale property of futures rates does mean that relatively simple relationships exist between futures rates of different maturities. In fact, Stapleton and Subrahmanyam (1997) (SS) derive the closed-form, cross-sectional relationship between the futures rates. They show that a simple log-linear relationship holds at each future date. The factor rates in *this* relationship can be *any* two futures rates. We begin by discussing the stochastic properties of futures rates in the model. In particular, we look at the volatility of the futures rate, and then derive the correlation of the spot and futures rates.

From SS(1997), Lemma 3 we have

Lemma 1 *In a no-arbitrage economy, if the spot interest rate, defined on a 'banker's discount' basis, is lognormally distributed under the martingale measure, the k-period futures rate at time t for an m− year loan is*

$$\ln f_{t,k} = \mu_{t,t+k} + \frac{1}{2}\mathrm{var}_{t,t+k} \tag{21}$$

The lemma states that lognormality of the futures rate follows from lognormality of the spot rate. This is because the conditional logarithmic mean of the spot rate, $\mu_{t,t+k}$, is normally distributed and the conditional variance, of the spot rate is a constant. Lemma 1 also restricts the correlation of the spot and the futures rates. SS also show that the kth futures rate can be found by solving the time series model, by successive substitution. They find

$$\begin{aligned} \ln f_{t,k} &= \mu_{t+k} + [\ln r_t - \mu_t](1-a)^k \\ &+ \sum_{\tau=0}^{t-1} \nu_{t-\tau}(1-b)^\tau \sum_{\tau=1}^{k}(1-a)^{k-\tau}(1-b)^{\tau-1} \\ &+ \frac{1}{2}\mathrm{var}_{t,t+k} \end{aligned} \tag{22}$$

and the conditional variance of the futures rate is therefore

$$\text{var}_t[f_{t,k}] = (1-a)^{2k}\text{var}_{t-1}[r_t] + \left[\sum_{\tau=1}^{k}(1-a)^{k-\tau}(1-b)^{\tau-1}\right]^2 \text{var}_{t-1}[\nu_t] \qquad (23)$$

Also, it follows that the conditional covariance of the spot and the k th futures rate is

$$\text{cov}_{t-1}[r_t, f_{t,k}] = (1-a)^k \text{var}_{t-1}[r_t]$$

and the correlation of the spot and the kth futures rates is therefore

$$\rho(r, f_k) = \frac{(1-a)^k \text{var}_{t-1}(r_t)}{\text{var}_{t-1}(f_{t,k})}$$

This expression for the correlation of the short rate and the kth futures rate illustrates an important implication of the no-arbitrage model. Given the volatilities of the spot and futures rates, we cannot independently choose both the correlation and the degree of mean reversion. Thus, the no-arbitrage model restricts the correlation between the two factors to be a function of the degree of mean reversion of the short rate.

The AR(1) process assumed for the conditional-mean factor π_t:

$$\ln\pi_t = \ln\pi_{t-1}(1-b) + \nu_t,$$

is the same as that assumed in Proposition 3 of Stapleton and Subrahmanyam (1997). It follows that futures rates at time t for delivery at time $t+k$ are given by a two-factor cross-sectional model.

Proposition 1 *In a no-arbitrage economy in which the short rate of interest follows a lognormal process of the form*

$$\ln r_{t+1} - \mu_{t+1} = (\ln r_t - \mu_t)(1-a) + \ln\pi_t + \varepsilon_{t+1} \qquad (24)$$

where

$$\ln\pi_t = \ln\pi_{t-1}(1-b) + \nu_t,$$

the term structure of futures rates at time t is generated by a two- factor model. The kth futures rate is given by

$$\begin{aligned} \ln f_{t,k} &= \mu_{0,t,k} + a_k[r_t - \mu_t] \\ &+ b_k[f_{t,1} - \mu_{0,t,1}] \end{aligned} \quad (25)$$

where

$$b_k = [(1-a)^{k-1} + \ldots + (1-b)^{k-1}]$$

and

$$a_k = (1-a)^k - (1-a)b_k.$$

Also, a short rate process in the form of (24) is necessary for the two-factor model in equation (25).

Proposition 1 relates the kth futures rate to the spot rate r_t and the first futures rate, $f_{t,1}$. If $m = 91/365$, for example, this means that the kth three–month futures rate is related to the spot three–month rate and the one period futures, three–month rate. In a recent contribution, Duffie and Kan (1993) have pointed out that if the model is linear in two such rates, it can always be expressed in terms of *any* two forward rates. In our context, it may be more practical to express the kth futures rate as a function of the spot rate and the nth futures rate. Hence, we derive the following implication of Proposition 1:

Corollary 1 *Suppose we choose any two futures rates as factors, where N_1 and N_2 are the maturities of the factors then the following linear model holds:*

$$\begin{aligned} f_{t,k} &= \mu_{0,t,k} + A_k(N_1, N_2)[f_{t,N_1} - \mu_{0,t,N_1}] \\ &+ B_k(N_1, N_2)[f_{t,N_2} - \mu_{0,t,N_2}] \end{aligned} \quad (26)$$

where

$$B_k(N_1, N_2) = (a_k b_{N_1} - b_k a_{N_1})/(a_{N_2} b_{N_1} - b_{N_2} a_{N_1}),$$

$$A_k(N_1, N_2) = (-a_k b_{N_1} + b_k a_{N_1})/(a_{N_2} b_{N_1} - b_{N_2} a_{N_1}),$$

and

$$b_k = [(1-a)^{k-1} + \ldots + (1-b)^{k-1}],$$

and

$$a_k = (1-a)^k - (1-a)b_k.$$

Corollary 1 follows by solving equation (25) for $k = N_1$, and $k = N_2$ and then substituting back into equation (25). Stapleton and Subrahmanyam (1997) also show that if the spot rate follows a two-dimensional process, so does the futures rate.

Appendix B: Proof of Multiple Regression Coefficients

In this appendix we establish that the regression coefficients $\beta_{r_t} = (1-a)$ and $\gamma_{r_t} = 1$.

From the multiple regression

$$\ln r_t = \alpha_{r_t} + \beta_{r_t} \ln(r_{t-1,j}/E(r_{t-1})) + \gamma_{r_t} \ln \pi_{t-1,j} + \varepsilon \tag{27}$$

the regression coefficients are

$$\beta_{r_t} = \frac{\text{cov}(r_t, r_{t-1})\text{var}(\pi_{t-1}) - \text{cov}(r_t, \pi_{t-1})\text{cov}(r_{t-1}, \pi_{t-1})}{\text{var}(r_{t-1})\text{var}(\pi_{t-1}) - [\text{cov}(r_{t-1}, \pi_{t-1})]^2}$$

$$\gamma_{r_t} = \frac{\text{cov}(r_t, \pi_{t-1})\text{var}(r_{t-1}) - \text{cov}(r_t, r_{t-1})\text{cov}(r_{t-1}, \pi_{t-1})}{\text{var}(r_{t-1})\text{var}(\pi_{t-1}) - [\text{cov}(r_{t-1}, \pi_{t-1})]^2}$$

Substituting from equations (11) to (14),

$$\text{cov}(r_{t+1}, r_t) = (1-a)\text{var}(r_t) + \text{cov}(r_t, \pi_t),$$

$$\text{cov}(r_t, \pi_t) = (1-a)(1-b)\text{cov}(r_{t-1}, \pi_{t-1}) + (1-b)\text{var}(\pi_{t-1}),$$

$$\text{cov}(\pi_{t+1}, \pi_t) = (1-b)\text{var}(\pi_t),$$

and

$$\text{cov}(r_t, \pi_{t-1}) = (1-a)\text{cov}(r_{t-1}, \pi_{t-1}) + \text{var}(\pi_{t-1}).$$

and simplifying yields

$$\beta_{r_t} = (1-a)$$

and

$$\gamma_{r_t} = 1$$

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Table 1: Convergence of Term Structure Model

Period	Binomial Density	**1**	**2**	**3**	**4**
[0, 1]	mean	5.0	5.0	5.0	5.0
	volatility	10.00	10.00	10.00	10.00
[0, 2]	mean	4.9999	4.99997	4.99998	4.99998
	volatility	9.97	9.99	9.99	9.99
[0, 3]	mean	4.9998	4.99992	4.99994	4.99996
	volatility 3	9.95	9.97	9.99	9.99
[0, 4]	mean 4	4.9997	4.99985	4.9999	4.99992
	volatility	9.92	9.97	9.98	9.98
[0, 5]	mean	4.9996	4.9997	4.9998	4.9998
	volatility	9.93	9.96	9.97	9.97
[0, 10]	mean	4.998	4.9992	4.9995	4.9996
	volatility	9.88	9.94	9.96	9.97
[0, 20]	mean	4.996	4.998	4.998	4.999
	volatility	9.85	9.92	9.94	9.96

The numbers in the table are the computed means and volatilities, in percent, for the short rate over periods 1,2,3,4,5,10, and 20, using the output of the two-factor model. The means are calculated using the possible outcomes and the nodal probabilities. The volatilities are the annualized standard deviations of the logarithm of the short rate. The binomial density refers to the denseness of the binomial tree of the short rate and the premium factor, over each sub interval. The input parameters in this case are a constant mean of 5%, and conditional volatility of 10% with no mean reversion of the short rate. The premium factor has a volatility of 1%, a mean of 1, and no mean reversion.

Table 2: Computing Time for Bond and Option Pricing (seconds)

Number of Periods	**8**	**12**	**28**	**48**
Binomial Density 1	0	0.3	2.5	12.3
Binomial Density 2	0.3	1	12.7	-

The table shows the time taken to compute the all zero-bond prices, coupon bond prices and option prices, given the tree of rates.The computer speed is 266 MHz, and the processor is Pentium.

Table 3: 8-period Example Input

Period	1	2	3	4	5	6	7	8
Futures rate	5.0	5.2	5.4	5.6	5.7	5.8	5.9	6.0
Conditional volatility (r)	14.0	14.0	13.5	13.0	13.0	12.5	12.5	12.0
Mean reversion (r)	10.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0
Conditional volatility (π)	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
Mean reversion (π)	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0

All numbers are in percent. The table shows the exogenous data input for an 8-period example. The short rate is the quarterly rate, so the period length is quarter of one year. Input data relating to the short rate appears in the first three rows; data relating to the premium in the last two rows.

Table 4: Illustrative Output of Zero-Coupon Bond Prices

0.9002411	0.9040475
0.9110627	0.9144909

0.9059610	0.9091793	0.9123043
0.9163451	0.9192414	0.9220527
0.9256151	0.9282188	0.9307455

0.9148397	0.9173619	0.9198222	0.9222217
0.9239718	0.9262463	0.9284646	0.9306274
0.9321501	0.9341996	0.9361978	0.9381458
0.9394686	0.9413137	0.9431126	0.9448656

0.9269570	0.9287221	0.9304547	0.9321554	0.9338241
0.9344906	0.9360850	0.9376500	0.9391859	0.9406928
0.9412640	0.9427034	0.9441161	0.9455023	0.9468622
0.9473504	0.9486492	0.9499236	0.9511740	0.9524006
0.9528166	0.9539880	0.9551371	0.9562645	0.9573702

0.9416127	0.9426358	0.9436478	0.94464880.9456387	0.9466177	
0.9476198	0.9485447	0.9494596	0.95036440.9512591	0.9521438	
0.9530188	0.9538544	0.9546810	0.95549850.9563069	0.9571062	
0.9578691	0.9586239	0.9593704	0.96010880.9608389	0.9615607	
0.9622248	0.9629064	0.9635804	0.9642470	0.9649062	0.9655578
0.9661353	0.9667504	0.9673588	0.96796050.9685553	0.9691435	

All prices are for a zero-coupon bond paying one unit of currency at the end of period 8. The first set of 4 numbers are the time 1 bond prices $B_{1,8}$, the second set of 9 numbers are the time 2 bond prices $B_{2,8}$, through to the time 5 set of 36 prices $B_{5,8}$. The 49 prices, $B_{6,8}$, and 64 prices, $B_{7,8}$, are not shown for reasons of space.

Table 5: Illustrative Output of Conditional Probabilities

0.5530794674	0.4087419001
0.5932346449	0.4488970776

0.6622172425	0.5086479906	0.3550787388
0.6514562049	0.4978869530	0.3443177012
0.6406951673	0.4871259154	0.3335566636

0.7780765989	0.6126741059	0.4472716129	0.2818691198
0.7548745812	0.5894720882	0.4240695952	0.2586671021
0.7316725635	0.5662700705	0.4008675775	0.2354650844
0.7084705458	0.5430680528	0.3776655597	0.2122630667

0.8091566018	0.6377849460	0.4664132901	0.2950416343	0.1236699785
0.8248723262	0.6535006704	0.4821290145	0.3107573587	0.1393857029
0.8405880506	0.6692163947	0.4978447389	0.3264730831	0.1551014272
0.8563037749	0.6849321191	0.5135604633	0.3421888074	0.1708171516
0.8720194993	0.7006478435	0.5292761877	0.3579045318	0.1865328760

1.0000000000	0.8614994841	0.6744916140	0.4874837439	0.3004758739	0.1134680038
1.0000000000	0.8248200186	0.6378121486	0.4508042785	0.2637964084	0.0767885384
0.9751484233	0.7881405532	0.6011326831	0.4141248131	0.2271169430	0.0401090730
0.9384689578	0.7514610878	0.5644532177	0.3774453477	0.1904374776	0.0034296075
0.9017894924	0.7147816224	0.5277737523	0.3407658822	0.1537580122	0.0000000000
0.8651100270	0.6781021569	0.4910942869	0.3040864168	0.1170785467	0.0000000000

All the probabilities are conditional probabilities of an up move in the interest rate, given the short rate and the premium factor. The first set of 4 numbers are the probabilities at time 1, the second set of 9 numbers are the conditional probabilities at time 2, through to the time 5 set of 36 conditional probabilities. The 49 probabilities at time 6, and the 64 probabilities at time 7, are not shown for reasons of space. In each case the columns show the probabilities for different (increasing to the right) values of the short rate. The rows show the values (increasing downwards) for different values of the premium factor

Table 6: 48 period Option Price ($)

Binomial Density	1	2
Bermudan Call	0.867	0.874
European Call	0.505	0.509

The above table shows the value of a 6-year call option on a coupon bond which pays $100 in 12 years time. The coupons are due annually at a rate of 5%. The interest rate tree is built assuming a futures rate of 5% for each maturity, a volatility of the short rate of 10%, mean reversion of the short rate of 10%, and volatility of the premium at 1% with 50% mean reversion.

Table 7: Illustrative Output of Coupon Bond Prices

0.9341104	0.9418602	0.9495420	0.9571580	0.9647215
0.9544404	0.9619713	0.9694343	0.9768317	0.9841763
0.9740350	0.9813479	0.9885932	0.9957732	1.0029000
0.9929119	1.0000080	1.0070380	1.0140030	1.0209150
1.0110900	1.0179730	1.0247890	1.0315410	1.0382420

0.8710285	0.8859522	0.8969290	0.9049382	0.9128766	0.9207478	0.9285752	0.9391028	0.9502901
0.8924639	0.9070311	0.9177220	0.9255130	0.9332327	0.9408847	0.9484923	0.9587294	0.9710779
0.9131774	0.9273819	0.9377851	0.9453570	0.9528575	0.9602904	0.9676783	0.9776250	0.9904720
0.9331792	0.9470176	0.9571320	0.9644850	0.9717672	0.9789817	0.9861511	0.9958085	1.0085100
0.9524815	0.9659519	0.9757781	0.9829136	0.9899787	0.9969769	1.0039290	1.0133000	1.0256380
0.9710987	0.9842008	0.9937407	1.0006610	1.0075110	1.0142950	1.0210330	1.0301200	1.0420980
0.9894373	1.0017820	1.0110380	1.0177450	1.0243830	1.0309560	1.0374830	1.0462910	1.0579130
1.0074770	1.0187120	1.0276880	1.0341850	1.0406150	1.0469800	1.0533000	1.0618330	1.0731050
1.0255210	1.0351460	1.0437110	1.0500020	1.0562270	1.0623880	1.0685060	1.0767690	1.087696

The coupon-bond prices are computed at each of 25 nodes at the end of year 1, shown in the first block in the table. 91 prices are computed at the end of year 2, shown in the second block of the table. The coupon bond has an annual coupon of 5% and a par value of 1 unit. Further price are computed at the end of year 3, year 4, year 5, and year 6, but are not shown here.