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Factor Risk Premia and Variance Bounds

by

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Abstract

We consider the implications of mean factor risk premia for the variance of admissible (normalized) stochastic discount factors, or pricing kernels. For given mean risk premia, we identify lower bounds on the variance of the pricing kernel which exceed the variance of the projection of the pricing kernel on the (augmented) asset return space: the "Hansen and Jagannathan" variance bound. These lower bounds *increase* with the covariability between the components of the pricing kernel and of the factors which are not explained by asset returns, and *decrease* with the distance between the factors and the (augmented) asset-return space. As an application, we show that the inflation risk premium generated by a consumption-based pricing kernel implies a standard deviation of the kernel which is up to 15% higher than the Hansen and Jagannathan bound.

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1 Introduction

Substantial effort has been produced to identify macroeconomic and financial variables which explain the time-series and cross-sectional variation of asset returns. One common approach assumes a linear multivariate proxy for the stochastic discount factor, and leads to beta-pricing models: expected returns are explained by the sensitivities to given risk factors (betas), and by the risk premia associated with the factors. [See, for example, Ferson and Harvey (1991), and Fama and French (1993)]. A second approach assumes asset returns to be described by a multiple-factor linear model, and leads to pricing implication analogous to those of the multiple-beta models. [See, for example, Chen, Roll, and Ross (1986), and Burmeister and Mc Elroy (1988); and Connor and Korajczyk (1988), and Lehmann and Modest (1988), where risk factors are constructed from asset returns].

While these two approaches have improved our understanding of equilibrium asset pricing, they are subject to some limitations. First, both multiple-beta and multiple-factor models require the identification of all the relevant risk factors, and maintain linearity assumptions which need not be supported by the data.[1] Secondly, both betas and risk premia may be varying over time, and accounting for such time variation is crucial to obtaining meaningful results.[2] Moreover, empirical implementations of multiple-factor models, which rely on factor analysis, assume that the relevant risk factors can be mimicked by the asset returns under consideration.

Recently, several papers have investigated the implications of security-market data for the admissible stochastic discount factors, or pricing kernels, making the only assumption of the law of one price.[3] Hansen and Jagannathan (1991), for example, identify a minimum-variance bound for any admissible pricing kernel, for a given mean of the pricing kernel.

The present paper takes this last approach to make inference on the *cross moments* between an admissible pricing kernel and arbitrary sources of risk. We show that in the context of multiple-beta and multiple-factor models the cross moments of an admissible pricing kernel immediately translate into *risk premia* on the corresponding factors. We then identify a lower bound on the variance of an admissible (normalized) pricing kernel, for *a given mean risk premium* associated with a factor. This lower bound exceeds the variance of the projection of the pricing kernel onto the span of asset returns (augmented of a constant): the "Hansen and Jagannathan" variance bound. The lower bound that we derive *increases* with the covariability between the components of the pricing kernel and of the factor which are not explained by asset returns, and *decreases* with the distance between the factor and the (augmented) asset-return space. In other words, if a pricing kernel correlates with risk factors which are not mimicked by asset returns, then its variance must exceed the variance that is implied by asset returns alone.

[1] Exceptions are Ferson (1990) and Brown and Otsuki (1992), which allow *one* risk variable to be excluded from a multiple-factor model of asset returns. Mei (1994) extends the approach to the case where *more than one* risk factor could be omitted. Also, Bansal and Vishwanathan (1993) extend the multiple-beta model to the case where the stochastic discount factor is a *nonlinear* function of a set of observable asset returns.

[2] Both Ferson and Harvey (1991) and Mei (1994) address the issue of time variation, in the context of multiple-beta and multiple-factor models, respectively.

[3] See Cochrane (1994) for a useful review of many issues in empirical asset pricing from this perspective.

Conversely, for *a given variance* of the pricing kernel, we identify an upper and lower bound for the mean risk premium associated with an arbitrary risk factor. This interval *increases* in size with the variance of the pricing kernel in excess of the Hansen and Jagannathan bound, and with the variance of the component of the risk factor which cannot be mimicked by asset returns. When a risk factor *can* be mimicked by portfolio returns, the bounds on the cross moment reduce to a *single value.*

The risk premium-variance bounds can be compared with the risk premium-variance pairs generated by an explicit asset pricing model, and can be used as diagnostics of the model. A candidate pricing kernel may in fact have mean and variance which are consistent with the Hansen and Jagannathan variance bound, but the risk premia it generates may place it outside of the more restrictive bounds that we derive. Moreover, in situations where we do not observe directly a pricing kernel, but we have *apriori* information on its variability, we can still place bounds on the risk premia it generates.

As stated above, our bounds have special economic meaning when either the pricing kernel or the return-generating process are linear in the risk factors: cross moments of a pricing kernel immediately translate into risk premia. Still, their validity does not depend on the appropriate specification of a pricing model and of a stochastic process for asset returns; and the risk factors that we consider need not lie in the payoff space of traded assets. Also, while we allow for time variation of the cross moments of the pricing kernel, calculating the bounds does not require to explicitly model such time variation. Moreover, conditioning information can be easily introduced along the lines of Hansen and Richard (1987) to effectively *expand* the set of assets under scrutiny, and sharpen the bounds.[4]

As an application, we specialize the analysis to a situation where the candidate normalized pricing kernel is a linear function of consumption growth, and the assets under consideration are those of the "three-factor model" of stock returns of Fama and French (1993). We calculate the risk premium-variance bounds for four variables, which proxy for corporate-default risk, term-structure risk, inflation risk, and business-cycle risk. We find that the risk premium associated with the inflation-risk proxy implies a standard deviation of the normalized kernel which is up to 15% higher than the Hansen and Jagannathan bound. When we introduce returns on large and small stocks, and on high and low book-to-market value stocks, we obtain bounds which are sharper than those obtained using the rate of return on the market and on the riskless asset only. Also, we show how the introduction of conditioning information makes the bounds even sharper.

The approach we take here is related to other recent papers which derive implications for the underlying pricing kernel following the approach of Hansen and Jagannathan (1991). Snow (1991), for example, shows how security-market data contain information on moments of the pricing kernel other than mean and variance, namely its higher-order norms. Our paper is similar in spirit, since we also derive implications for moments of the pricing kernel other than its variance, its *cross moments*; but, differently from his paper, we use information in *addition* to asset returns, and this information is contemporaneous to the realization of asset returns. Also, Cochrane (1992) looks at the implications of observed price-dividend-ratio

[4]See also Gallant, Hansen, and Tauchen (1990), Cochrane and Hansen (1992), and, more recently, Downs and Snow (1994).

behavior for the variance of the unobserved discount factor. Unlike the bounds he derives, though, ours are *at least as sharp* as the Hansen and Jagannathan variance bound. More recently, Hansen and Jagannathan (1994) have shown how the misspecification of a candidate pricing kernel can be measured by looking at the minimum distance between the candidate kernel and an admissible pricing kernel; while Hansen, Heaton, and Luttmer (1995) pursue these ideas further, to allow for market frictions and short-sales constraints.

The paper is organized as follows: Section 2 illustrates how the cross moments of a pricing kernel correspond to factor risk premia when either the pricing kernel or asset returns are linear in the risk factors. Section 3 derives the risk premium-variance bounds, and Section 4 shows how the bounds can be made sharper by including conditioning information. In Section 5 we illustrate possible extensions of the analysis, which consider the implications of the nonnegativity of an admissible pricing kernel, and highlight formal tests of the risk-premia restrictions. Section 6 derives the risk premium-variance implications of a normalized pricing kernel linear in consumption growth. In section 7, we provide an empirical application of our analysis, while section 8 concludes.

2 Cross moments and risk premia

Consider an $N \times 1$ vector of (gross) security returns r. By the law of one price, we have

$$E_t(m_{t+1} r_{t+1}) = \mathbf{1}, \tag{1}$$

for some admissible pricing kernel m, where $\mathbf{1}$ is an $N \times 1$ vector of ones.[5] Note that r can be measured either in units of the consumption basket or in units of the currency; accordingly, m has the interpretation of a *real* or a *nominal* pricing kernel, respectively.

In the following, we shall assume that r includes the possibly time-varying rate of return on a risk-free asset, r_{ft}. We have

$$E_t(m_{t+1} r_{ft}) = 1.$$

For convenience of notation, we define the *normalized* pricing kernel $m_{t+1} r_{ft} \equiv q_{t+1}$, where $E_t(q_{t+1}) = 1$.

Thus, we obtain the familiar orthogonality condition

$$E_t[q_{t+1}(r_{t+1} - \mathbf{1} r_{ft})] = 0. \tag{2}$$

Rearranging, we obtain

$$E_t(q_{t+1})[E_t(r_{t+1}) - \mathbf{1} r_{ft}] = E_t(r_{t+1}) - \mathbf{1} r_{ft} = -\mathrm{Cov}_t(q_{t+1}, r_{t+1}). \tag{3}$$

[5]See Harrison and Kreps (1979). The set of pricing kernels can be interpreted as the set of intertemporal marginal rates of substitution compatible with the distribution of returns.

Consider now a vector of K risk factors

$$y_{k,t+1}, \ k = 1, \dots, K.$$

Without loss of generality, we assume $E_t(y_{k,t+1}) = E_t(y_{k,t+1}y_{j,t+1}) = 0$, and $E_t(y_{k,t+1}^2) = 1$, for any $k, j = 1, \dots, K$.

In the following, we consider situations where the normalized pricing kernel and/or asset returns are linear functions of the K factors.

2.1 Multiple-beta models

The assumption that the pricing kernel is linear in one or more factors features prominently in the asset pricing literature. It is, for example, the assumption of the standard CAPM of Sharpe (1964), Lintner (1965), and Mossin (1968), where the factor is the rate of return on a claim to aggregate wealth, the "market" portfolio.

Let the admissible normalized pricing kernel q be a linear function of the K factors:

$$q_{t+1} = 1 - \lambda_{1t}y_{1,t+1} - \lambda_{2t}y_{2,t+1} - \dots - \lambda_{Kt}y_{K,t+1}, \tag{4}$$

where $\lambda_{kt}, \ k = 1, \dots, K$ are possibly time-varying coefficients.

Using (3) and (4), we have

$$E_t(r_{t+1}) - \mathbf{1}r_{ft} = \lambda_{1t}E_t(y_{1,t+1}r_{t+1}) + \lambda_{2t}E_t(y_{2,t+1}r_{t+1}) + \dots + \lambda_{Kt}E_t(y_{K,t+1}r_{t+1}). \tag{5}$$

Let $\beta_{kt} \equiv E_t(r_{t+1}y_{k,t+1})$. We can rewrite (5) as

$$E_t(r_{t+1}) - \mathbf{1}r_{ft} = \beta_{1t}\lambda_{1t} + \beta_{2t}\lambda_{2t} + \dots + \beta_{Kt}\lambda_{Kt}. \tag{6}$$

The quantity

$$\lambda_{kt} = -E_t(q_{t+1}y_{k,t+1})$$

is the risk premium on the corresponding state variable. Assuming stationarity, we define $\lambda_k \equiv -E(qy_k)$ and, by the law of iterated expectations, we have

$$\lambda_k \equiv -E(q_{t+1}y_{k,t+1}) = E(\lambda_{kt}),$$

which is the *mean* risk premium on y_k. Hence, the *cross moment* between the normalized pricing kernel q and the risk factor y_k equals, with the opposite sign, the mean risk premium on the factor.

Note that the mean risk premia $E(\lambda_{kt})$, for $k = 1, \dots, K$, enter the *unconditional* version of (6). In fact, assuming stationarity, we have

$$E(r_{t+1}) - \mathbf{1}E(r_{ft}) = E(\beta_{1t}\lambda_{1t}) + E(\beta_{2t}\lambda_{2t}) + \dots + E(\beta_{Kt}\lambda_{Kt}),$$

where $E(\beta_{kt}\lambda_{kt}) = E(\beta_{kt})E(\lambda_{kt}) + \text{Cov}(\beta_{kt}, \lambda_{kt})$.

2.2 Multiple-factor models

The linearity of asset returns in a set of risk factors is also a common assumption, see, for example, the APT of Ross (1976) and Connor (1984).

Let r be a linear function of one or more risk factors

$$r_{t+1} = \beta_{0t} + \beta_{1t} y_{1,t+1} + \beta_{2t} y_{2,t+1} + \ldots + \beta_{Kt} y_{K,t+1}, \tag{7}$$

where β_{kt}, $k = 1, \ldots, K$, are possibly time-varying coefficient vectors. Using (3) and (7), we have

$$\begin{aligned} E_t(r_{t+1}) - \mathbf{1} r_{ft} &= -\beta_{1t} E_t(q_{t+1} y_{1,t+1}) - \beta_{2t} E_t(q_{t+1} y_{2,t+1}) - \ldots - \beta_{Kt} E_t(q_{t+1} y_{K,t+1}) \\ &= \beta_{1t}\lambda_{1t} + \beta_{2t}\lambda_{2t} + \ldots + \beta_{Kt}\lambda_{Kt}, \end{aligned}$$

where $\lambda_{kt} \equiv -E_t(q_{t+1} y_{k,t+1})$ is the risk premium on y_k. Under stationarity, $\lambda_k \equiv -E(q y_k)$ is the mean risk premium on y_k. Again, the mean risk premia enter an unconditional version of the pricing relation above.

Note that we can combine the factor-model assumption for the return generating process with the linearity of the normalized pricing kernel. This is the case, for example, of the continuous-time model of Cox, Ingersoll, and Ross (1985), where asset returns are linear in a set of state variables, and marginal utility is *locally* linear in wealth and the state variables. We may assume, for example, $q = 1 - \lambda_{qt} y_{q,t+1}$, where $y_{q,t+1}$ is a factor which drives the pricing kernel and may correlate with the factors driving asset returns. We have

$$\begin{aligned} E_t(r_{t+1}) - \mathbf{1} r_{ft} &= \beta_{1t}\lambda_{qt} E_t(y_{q,t+1} y_{1,t+1}) + \beta_{2t}\lambda_{qt} E_t(y_{q,t+1} y_{2,t+1}) + \ldots + \beta_{Kt}\lambda_{qt} E_t(y_{q,t+1} y_{K,t+1}) \\ &= \beta_{1t}\lambda_{qt}\rho_{q1,t} + \beta_{2t}\lambda_{qt}\rho_{q2,t} + \ldots + \beta_{Kt}\lambda_{qt}\rho_{qK,t}, \end{aligned}$$

where $\rho_{qk,t} = E_t(y_{q,t+1} y_{k,t+1})$ is the conditional correlation coefficient between y_q and y_k. In this case, λ_{qt} is the risk premium on y_q, while $\lambda_{qt}\rho_{qk,t}$ is the risk premium on the factor y_k. Under stationarity, $\lambda_q \equiv -E(q y_q) = E(\lambda_{qt})$ and $\lambda_k \equiv -E(q y_k) = E(\lambda_{qt}\rho_{qk,t}) = E(\lambda_{qt})E(\rho_{qk,t}) + \mathrm{Cov}(\lambda_{qt}, \rho_{qk,t})$ are the corresponding mean risk premia.

3 Factor risk premia and variance bounds

In the following, we implicitly assume linearity of the pricing kernel and/or of the return-generating process. Hence, $\lambda_k \equiv -E(q y_k)$ has the interpretation of a mean risk premium. Still, our analysis is valid even when the linearity assumptions mentioned above do not hold.

Using the definition of q we can rearrange the pricing equation (1) to obtain

$$E_t(q_{t+1} r_{t+1}) = \mathbf{1} r_{ft}. \tag{8}$$

Equation (8) above states that all expected asset returns, after a risk adjustment, should equal the risk-free rate. Assuming stationarity of the risk-free rate r_f, we can multiply both sides of the moment condition $E_t(q_{t+1}) = 1$ by $E(r_f)$ to obtain

$$E_t[q_{t+1}E(r_{ft})] = E(r_{ft}). \tag{9}$$

We define the augmented vectors $r_a \equiv [r', E(r_f)]'$ and $\mathbf{1}_a \equiv [\mathbf{1}, 1]'$. Assuming stationarity, we can apply the law of iterated expectations to equations (8) and (9) to obtain

$$E(qr_a) = \mathbf{1}_a E(r_f). \tag{10}$$

The unconditional moment restriction (10) corresponds to Restriction 1 of Hansen and Jagannathan (1991).

Following Hansen and Jagannathan (1991), we can construct a random variable $q^* \equiv r_a'\alpha$, where α is an $(N+1) \times 1$ coefficient vector, such that

$$E(r_a r_a' \alpha) = \mathbf{1}_a E(r_f).$$

Assuming $E(r_a r_a')$ to be nonsingular, we have $\alpha = E(r_a r_a')^{-1}\mathbf{1}_a E(r_f)$. Note that $E(q^*) = E(q) = 1$. Also, we have $E[r_a(q - q^*)] = 0$, since both q and q^* satisfy (10). Hence, $(q - q^*)$ is orthogonal to r_a, and q^* is the least-squares projection of q onto r_a. We have $\text{Var}(q) = \text{Var}(q^*) + \text{Var}(q - q^*)$, and hence

$$\text{Var}(q) \geq \text{Var}(q^*), \quad E(q) = E(q^*) = 1. \tag{11}$$

The relations above correspond to equation (6) of Hansen and Jagannathan (1991).

Consider the least-squares projection of y_k onto r_a. Assuming stationarity, we have

$$y_k^* \equiv r_a'[E(r_a r_a')]^{-1}E(y_k r_a).^6$$

Also, let $\lambda_k^* \equiv -E(qy_k^*) = -E(q^* y_k^*)$ denote the mean risk premium on y_k^*. Using the definitions of q^* and y_k^*, we have

$$\begin{aligned} \lambda_k^* &= E(E(r_f)\mathbf{1}_a'[E(r_a r_a')]^{-1} r_a r_a' [E(r_a r_a')]^{-1} E(y_k r_a)) \\ &= E(r_f)\mathbf{1}_a'[E(r_a r_a')]^{-1}E(y_k r_a). \end{aligned}$$

We turn now to the mean risk premium λ_k. We have

$$\begin{aligned} \lambda_k &\equiv -E(qy_k) \\ &= -E([q^* + (q - q^*)][y_k^* + (y_k - y_k^*)]) \\ &= -E(q^* y_k^*) - E[q^*(y_k - y_k^*)] - E[(q - q^*)y_k^*] - E[(q - q^*)(y_k - y_k^*)]. \end{aligned}$$

Since both $y_k - y_k^*$ and $q - q^*$ are orthogonal to r_a, $E[q^*(y_k - y_k^*)] = E[(q - q^*)y_k^*] = 0$, and we obtain $\lambda_k = \lambda_k^* - E[(q - q^*)(y_k - y_k^*)]$. Rearranging, we have

$$E[(q - q^*)(y_k - y_k^*)] = \lambda_k^* - \lambda_k. \tag{12}$$

[6] Since $E[r_a(y_k - y_k^*)] = 0$, we also have $E[E(r_f)(y_k - y_k^*)] = 0$ and $E(y_k) - E(y_k^*) = 0$. Hence, $E(y_k^*) = E(y_k) = 0$.

By the Cauchy-Schwarz inequality, we have

$$|E[(q-q^*)(y_k-y_k^*)]| \leq \sqrt{E[(q-q^*)^2]E[(y_k-y_k^*)^2]}. \tag{13}$$

Since $E(q) = E(q^*) = 1$, we have $E(q-q^*)^2 = E(q^2) - E[(q^*)^2] = \mathrm{Var}(q) - \mathrm{Var}(q^*)$. Taking the square of both sides of (13), using (12), and rearranging, we have

$$\mathrm{Var}(q) \quad \geq \quad \mathrm{Var}(q^*) + \frac{[E(q-q^*)(y_k-y_k^*)]^2}{E[(y_k-y_k^*)^2]} = \mathrm{Var}(q^*) + \frac{(\lambda_k^*-\lambda_k)^2}{E[(y_k-y_k^*)^2]}. \tag{14}$$

The right-hand side of (14) describes a *parabola* in the risk premium-variance space, with a minimum of $\mathrm{Var}(q^*)$ at $\lambda_k = \lambda_k^*$.

Hence, to the extent that the components of q and y_k, which *are not* explained by r_a, exhibit *covariability*, the variance of q must exceed the variance of the projection q^*. This "excess" variance, $\mathrm{Var}(q) - \mathrm{Var}(q^*)$, increases with $(\lambda_k^* - \lambda_k)^2$, and decreases with the distance between a risk factor and the span of asset returns as measured by $E[(y_k - y_k^*)^2]$, where $0 \leq E[(y_k-y_k^*)^2] \leq E(y_k^2) = 1$.

The bounds (14) may find an application as diagnostics of an explicit pricing model. When "diagnosing" a pricing kernel, we should not limit ourselves to the first and second moments. Snow (1991) extends the analysis of Hansen and Jagannathan (1991) to the higher-order norms of a pricing kernel. This paper considers the *cross moments* of a pricing kernel which are of special economic interest: under linearity of the kernel and/or of the return-generating process, they correspond to the mean risk premia on the corresponding risk factors.

Note that when q is *spanned* by r_a, the bound reduces to $\mathrm{Var}(q) = \mathrm{Var}(q^*)$. For example, if the candidate pricing kernel q were a linear function of some asset returns and $q = q^*$, we would have $\lambda_k = \lambda_k^*$, for $k = 1, \ldots, K$. Also, when r_a does not capture *any* variability of y_k, we have $y_k^* = \lambda_k^* = 0$ and the bound (14) simplifies to $\mathrm{Var}(q) \geq \mathrm{Var}(q^*) + \lambda_k^2$.

Also, note that

$$-\lambda_{kt} \equiv E_t(m_{t+1} r_{ft} y_{k,t+1}) = p_{kt} r_{ft},$$

where p_k is the price of an asset with payoff y_k: $p_{kt} = E_t(m_{t+1} y_{k,t+1})$. Assuming stationarity and using the law of iterated expectations, we have

$$-\lambda_k = E(p_k r_f). \tag{15}$$

Equation (15) highlights the close relation between the bound (14) and the regions derived in Hansen and Jagannathan (1991). In deriving the mean-standard deviation frontiers, Hansen and Jagannathan (1991) assume the risk-free rate to be unobservable. Hence, they assign average prices to a unit payoff and obtain the corresponding minimum variance of an admissible pricing kernel. In our derivation, it is the *mean factor risk premium* which is unobservable. Hence, we *assign* mean factor risk premia, and derive the corresponding minimum variance of an admissible (normalized) pricing kernel.

Alternatively, the risk premium-variance bounds can be used to place restrictions on the a risk premium for *a given* variance of a normalized pricing kernel. In general, we would expect the bounds around a risk premium to increase as the variance of a pricing kernel increases, and thus the kernel is "allowed" to exhibit a stronger covariability with the risk factor. This intuition is made more precise when we reconsider the inequality (13). From (12) we have $E[(q-q^*)(y_k-y_k^*)] = \lambda_k - \lambda_k^*$. Moreover, we have $E(q-q^*)^2 = \text{Var}(q) - \text{Var}(q^*)$. Substituting in (13), we obtain

$$\lambda_k^* - \sqrt{[\text{Var}(q) - \text{Var}(q^*)]E[(y_k - y_k^*)^2]} \leq \lambda_k \leq \lambda_k^* + \sqrt{[\text{Var}(q) - \text{Var}(q^*)]E[(y_k - y_k^*)^2]}. \quad (16)$$

The size of the interval for λ_k increases with the variance of the normalized pricing kernel in excess of $\text{Var}(q^*)$, and with the distance between a risk factor and r_a. In situations where we do not observe directly a pricing kernel, but we have *apriori* information as to its variability, we can still place bounds on the risk premia it generates. Such bounds are tighter, the closer are the risk factors to r_a, and thus the more "traded" are the factors. Note that, again, if $y_k^* = 0$ the bound (16) simplifies to $|\lambda_k| \leq \sqrt{\text{Var}(q) - \text{Var}(q^*)}$.[7]

In some situations a risk factor may be identified with the return on a portfolio of assets: a "mimicking" portfolio. When a factor *lies* in r_a, the bound (16) reduces to $\lambda_k = \lambda_k^*$. Hence, rather than an admissible interval we obtain a *point estimate* of the mean factor risk premium.

4 Conditioning information

While the moment restriction (10) is *unconditional*, there is a simple way to incorporate conditional information. Consider again the conditions (8) and (9), and assume we multiply r and q_{t+1} by a vector of instrument z_t observed at time t. We have

$$E_t[q_{t+1}(r_{t+1} \otimes z_t)] = (\mathbf{1} \otimes z_t) r_{ft} \quad (17)$$

$$E_t[q_{t+1} E(r_{ft}) z_t] = z_t E(r_{ft}). \quad (18)$$

Let $r^z_{a,t+1} \equiv r_{a,t+1} \otimes z_t$ and $\iota_t \equiv [(\mathbf{1} \otimes z_t)' r'_{ft}, z'_t E(r_{ft})]'$. Assuming stationarity, we can apply the law of iterated expectations to (17) and (18) to obtain

$$E(q r^z_a) = E(\iota).$$

The risk premium-variance bounds (14) and (16) still hold, but the projections q^* and y_k^* are defined as

$$q^* \equiv (r^z_a)'(E[r^z_a(r^z_a)'])^{-1} E(\iota)$$

$$y_k^* \equiv (r^z_a)'(E[r^z_a(r^z_a)'])^{-1} E(y_k r^z_a).$$

[7]Remember that $E(y_k^2) = 1$ by assumption, while $\lambda_k^* \equiv -E(q^* y_k^*) = 0$.

The scaled returns $r^z_{a,t+1}$ have the interpretation of cash flows generated by *managed* portfolios: investors who observe z_t can invest in an asset according to the value of z_t.[8] The projections q^* and y_k^* above are likely to capture more of the variability of q and y_k, and to make the risk premium-variance bounds more stringent.[9]

5 Extensions

This Section extends the analysis of Section 3. We consider the implications of the requirement of nonnegativity of an admissible pricing kernel, and we highlight how the restrictions implied by mean risk premia can be formally tested.

5.1 Nonnegativity

While the projection q^* satisfies condition (10), in general it will not satisfy the no-arbitrage condition $q^* > 0$, Restriction 2 of Hansen and Jagannathan (1991). Here, we limit ourselves to the weaker requirement of *nonnegativity* of an admissible pricing kernel: $q \geq 0$.

Let $\tilde{\alpha}$ denote an $(N+1)\times 1$ coefficient vector. Following Hansen and Jagannathan (1991), we define $\tilde{q} \equiv (r_a'\tilde{\alpha})^+ \equiv \max\{r_a'\tilde{\alpha}, 0\}$. Assume

$$E(\tilde{q}r_a) = \mathbf{1}_a E(r_f). \tag{19}$$

The random variable $\tilde{q}$, if it exists, has the smallest variance among all nonnegative random variables q satisfying restriction (19). In fact, we have $E(qr_a) = E(\tilde{q}r_a) = \mathbf{1}_a E(r_f)$. Since q is nonnegative, we have

$$E(\tilde{q}q) \geq \tilde{\alpha}'E(r_a q) = \tilde{\alpha}'E(r_a\tilde{q}) = E(\tilde{q}^2). \tag{20}$$

By the Cauchy-Schwarz inequality we have $E(q^2) \geq [E(\tilde{q}q)]^2/E(\tilde{q}^2)$. This, combined with the inequality (20), leads to

$$\mathrm{Var}(q) \geq \mathrm{Var}(\tilde{q}), \quad E(q) = E(\tilde{q}) = 1. \tag{21}$$

The relations above correspond to equation (22) of Hansen and Jagannathan (1991).

When we turn to the mean risk premium λ_k, we can write

$$\begin{aligned} \lambda_k &\equiv -E(qy_k) \\ &= -E([\tilde{q} + (q - \tilde{q})][y_k^* + (y_k - y_k^*)]) \\ &= -E(\tilde{q}y_k^*) - E[\tilde{q}(y_k - y_k^*)] - E[(q - \tilde{q})y_k^*] - E[(q - q^*)(y_k - y_k^*)]. \end{aligned} \tag{22}$$

[8]See Hansen and Richard (1987).

[9]The extent to which the projection q^* is altered by introducing conditioning information can be explicitly tested, as shown in Downs and Snow (1994).

Since $E(qr_a) = E(\tilde{q}r_a)$, we have $E[(q-\tilde{q})r_a] = 0$ and $q - \tilde{q}$ is orthogonal to r_a. Hence, $E[(q-\tilde{q})y_k^*] = 0$, but, in general, $E[\tilde{q}(y_k - y_k^*)] \neq 0$. Let $\tilde{\lambda}_k \equiv -E(\tilde{q}y_k^*) - E[\tilde{q}(y_k - y_k^*)] = \lambda_k^* - E[\tilde{q}(y_k - y_k^*)]$. Hence, we can write (22) as $\lambda_k = \tilde{\lambda}_k - E[(q-q^*)(y_k - y_k^*)]$. Rearranging, we have

$$E[(q-q^*)(y_k - y_k^*)] = \tilde{\lambda}_k - \lambda_k.$$

Following the analysis of Section (3), we obtain bounds analogous to (14) and (16). We have

$$\mathrm{Var}(q) \;\geq\; \mathrm{Var}(\tilde{q}) + \frac{[E(q-\tilde{q})(y_k - y_k^*)]^2}{E[(y_k - y_k^*)^2]} = \mathrm{Var}(q^*) + \frac{(\tilde{\lambda}_k - \lambda_k)^2}{E[(y_k - y_k^*)^2]}, \tag{23}$$

and

$$\tilde{\lambda}_k - \sqrt{[\mathrm{Var}(q) - \mathrm{Var}(\tilde{q})]E[(y_k - y_k^*)^2]} \leq \lambda_k \leq \tilde{\lambda}_k + \sqrt{[\mathrm{Var}(q) - \mathrm{Var}(\tilde{q})]E[(y_k - y_k^*)^2]}. \tag{24}$$

Following Hansen and Jagannathan (1991), it is possible to show that $\mathrm{Var}(\tilde{q}) \geq \mathrm{Var}(q^*)$. In general, whether (23) is more stringent than (14) depends on the sign of $E[\tilde{q}(y_k - y_k^*)]$ and on the magnitude of λ_k.[10] On the other hand, $\mathrm{Var}(q) - \mathrm{Var}(\tilde{q}) \leq \mathrm{Var}(q) - \mathrm{Var}(q^*)$, and (24) is at least as stringent as its counterpart (16), for *any* value of $E[\tilde{q}(y_k - y_k^*)]$.

5.2 Tests

So far, we limited ourselves to the analysis of the risk premium-variance regions. In the following, we highlight how formal tests of the restrictions implied by factor risk premia could be performed.

Assume the mean risk premium on the factor y_k to be known *apriori*, and to equal $\bar{\lambda}_k$. An admissible pricing kernel q must satisfy the set of restrictions

$$\begin{aligned} E(qr_a) &= \mathbf{1}_a E(r_f) \\ E(qy_k) &= -\bar{\lambda}_k. \end{aligned}$$

We can test whether the minimum-variance kernel q^* that satisfies the moment condition (10), also prices correctly (on average) the risk factor y_k. This amounts to a test of $\lambda_k^* = \lambda_k$. Such test can be performed using the generalized method of moments, GMM [see Hansen (1982)], along the same lines of Snow (1991) and Downs and Snow (1994).

The approach above can be easily generalized to the case where we consider *several* risk premia at the same time. In this case, we have

$$E(qr_a) = \mathbf{1}_a E(r_f)$$

[10]We have $(\tilde{\lambda}_k^2 - \lambda_k^2) \geq (\tilde{\lambda}_k^2 - \lambda_k^2)$, and hence (23) is more stringent than (14), i) for $\lambda_k \geq (\tilde{\lambda}_k + \lambda_k)/2$ when $E[\tilde{q}(y_k - y_k^*)] \geq 0$, and ii) for $\lambda_k \leq (\tilde{\lambda}_k + \lambda_k)/2$ when $E[\tilde{q}(y_k - y_k^*)] \leq 0$.

$$E(qy) = -\bar{\lambda},$$

where $\bar{\lambda}$ is a $K \times 1$ vector. In this case, we want to test whether the minimum-variance kernel q^* that satisfies the moment condition (10), also prices correctly (on average) *all* the risk factor y_k, for $k = 1, \ldots, K$. This amounts to a test of $\lambda_k^* = \lambda_k$, for $k = 1, \ldots, K$. Equally straightforward (at least from a conceptual standpoint) is imposing nonnegativity [see, again Snow (1991) and Downs and Snow (1994)].

Alternatively, we may consider an explicit asset pricing model and a random variable x which is a *candidate* normalized kernel. Hence, x does not necessarily price correctly neither asset returns nor risk factors. Following Hansen and Jagannathan (1994), we may want to assess the misspecification of x by measuring the minimum distance between any admissible kernel q, and x:

$$\delta \equiv \min_q E[(x-q)^2]. \tag{25}$$

Let $f \equiv [r_a', y']'$. Also, let $s \equiv [\mathbf{1}_a' E(r_f), -\bar{\lambda}']'$. Hansen and Jagannathan (1994) show that the distance δ is equal to the distance between the least-squares projections of q and x onto f. If we also impose positivity of the admissible pricing kernel q, we have

$$\tilde{\delta} \equiv \inf_{q>0} E[(x-q)^2] = \min_{q \geq 0} E[(x-q)^2]. \tag{26}$$

Hansen and Jagannathan (1994) show the problem (26) to be equivalent to the conjugate maximization problem

$$(\tilde{\delta})^2 = \max_{\eta} E(x^2) - E([(x - f'\eta)^+]^2) - 2\eta' E(s),$$

where η is an $(N+1+K)\times 1$ coefficient vector. The asymptotic distribution of the empirical counterparts of δ and $\tilde{\delta}$ is derived in Hansen, Heaton, and Luttmer (1995). Hansen, Heaton, and Luttmer (1995) also generalize the analysis to the case where $E(qr) \geq \mathbf{1}E(r_f)$ because of market frictions, and where some of the elements of η are restricted to be nonnegative because of short-sale constraints. It is worth noting that the requirement that some cross moment $E(qy_k)$ *exceeds* the value $-\bar{\lambda}_k$ could be handled in the same way as market frictions, and we would have $E(qy) \geq -\bar{\lambda}$.

6 A consumption-based model

Here we apply the analysis of Section 3 to a specific pricing kernel.

Let c denote gross real per-capita consumption growth and let $y_{c,t+1} \equiv \frac{c_{t+1} - E_t(c_{t+1})}{\sqrt{\text{Var}_t(c_{t+1})}}$. We consider the following candidate pricing kernel

$$x_{t+1} = 1 - \lambda_c y_{c,t+1}, \tag{27}$$

which can be seen as a linearization of $c_{t+1}^{-\gamma}/E_t(c_{t+1}^{-\gamma})$.[11] Note that $\lambda_c > 0$ because of the diminishing marginal utility of consumption.

According to the candidate kernel x, expected excess returns are given by

$$E_t(r_{t+1}) - \mathbf{1}r_{ft} = \beta_{ct}\lambda_c,$$

where $\beta_{ct} \equiv E_t(r_{t+1}y_{c,t+1})$, and λ_c is the risk premium on consumption risk.

The assumed specification for x implies

$$\text{Var}(x) = \lambda_c^2, \tag{28}$$

and λ_c coincides with the standard deviation of x. Hence, the value of λ_c which would make the candidate consumption-based pricing kernel consistent with the Hansen and Jagannathan variance bound is

$$\lambda_c^{(HJ)} \equiv \sqrt{\text{Var}(q^*)}.$$

Consider now a risk factor y_k, where $E_t(y_{k,t+1}) = 0$ and $\text{Var}_t(y_{k,t+1}) = 1$. Also, assuming stationarity, let $\rho_{ck} \equiv E(y_{ct}y_{kt}) = E(\rho_{ck,t})$, where $\rho_{ck,t} \equiv E_t(y_{c,t+1}y_{k,t+1})$. The candidate kernel x implies the mean risk premium

$$\lambda_k \equiv -E(xy_k) = \lambda_c E(y_c y_k) = \lambda_c \rho_{ck}. \tag{29}$$

We can combine equations (28) and (29) above to obtain the following relation between the variance of x and mean risk premium on y_k:

$$\text{Var}(x) = \frac{\lambda_k^2}{\rho_{ck}^2}. \tag{30}$$

Note that the risk premium λ_k must have the *opposite* sign of ρ_{ck} for it to be consistent with the restriction $\lambda_c > 0$. Equation (30) describes a *parabola* in the risk premium-variance space, with a minimum of zero at $\lambda_k = 0$.

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[11] The coefficient λ_c can be translated into a measure of relative-risk-aversion measure as follows. Assume c to be distributed *iid*. We have

$$\frac{c^{-\gamma}}{Ec^{-\gamma}} \approx \frac{(Ec)^{-\gamma}}{Ec^{-\gamma}} - \gamma\frac{(Ec)^{-\gamma-1}}{Ec^{-\gamma}}\sigma_c y_c.$$

If we ignore Jensen's inequality, and we replace $(Ec)^{-\gamma}$ with $Ec^{-\gamma}$, we obtain

$$\frac{c^{-\gamma}}{Ec^{-\gamma}} \approx 1 - \gamma\frac{\sigma_c}{Ec}y_c.$$

Hence, $\lambda \approx \gamma\sigma_c/Ec$.

For the consumption-based pricing kernel to satisfy the bounds (14) we need

$$\frac{\lambda_k^2}{\rho_{ck}^2} \geq \mathrm{Var}(q^*) + \frac{(\lambda_k^* - \lambda_k)^2}{E(y_k - y_k^*)^2}.$$

Let $A \equiv 1/E(y_k - y_k^*)^2 - 1/(\rho_{ck})^2$, $B \equiv -2\lambda_k^*/E(y_k - y_k^*)^2$, and $C \equiv \mathrm{Var}(q^*) + (\lambda_k^*)^2/E(y_k - y_k^*)^2$. The minimum value, if any, of λ_k which satisfies the inequality above is given by

$$\frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

where the only relevant root is the one with the opposite sign of ρ_{ck}. Correspondingly, the minimum value of λ_c satisfying the risk premium-variance bound is

$$\lambda_c^{(BK)} \equiv -\frac{-B \pm \sqrt{B^2 - 4AC}}{2A\rho_{ck}} \geq \lambda_c^{(HJ)}.$$

While $\lambda_c^{(HJ)}$ is always well defined, we may have $B^2 - 4AC < 0$, and the consumption-based pricing kernel may not be able to satisfy the risk premium-variance bound for any value of λ_c.

7 An Application

In this application, we consider the candidate pricing kernel introduced in the previous Section, and we compare the risk premium-variance pairs that it generates with the implications of actual asset returns, along the lines of Section 3.

We look at monthly-returns data on five stock portfolios and one bond portfolio, which are borrowed from the study of Fama and French (1993).[12] The five stock portfolios are: a market portfolio proxy, a portfolio of small stocks, a portfolio of big stocks, a portfolio of high book-to-market-value stocks, and a portfolio of low book-to-market-value stocks. The bond portfolio is the one-month Treasury bill.

In the following, we briefly describe the portfolio-returns data with reference to the series of Fama and French (1993). (Further details on the construction of the stock portfolios can be found in Fama and French (1993), pp.8-10.) The *market* portfolio returns, r_M, correspond to the RM series of Fama and French (1993). The *small-firm* portfolio returns, r_S, are the simple average of the returns on the three small-firm portfolios S/L, S/M, and S/H. Similarly, the *big-firm* portfolio returns, r_B, are the simple average of the returns on the three big-firm portfolios B/L, B/M, and B/H. The *high book-to-market-value* portfolio returns, r_H, are the average of the returns on the two high book-to-market-value portfolios S/H and B/H. The *low book-to-market-value* portfolio returns, r_L, are the average of the returns on the two low book-to-market-value portfolios S/L and B/L. The *bond-portfolio* return, r_f, is the one-month Treasury bill rate, series TB, which we use as a proxy for the risk-free rate.

[12]We thank Eugene Fama for kindly providing us these data.

All returns are deflated using a CPI inflation series, which we constructed from the series PZUNEW from CITIBASE.

Table 1 reports mean real rates of return of the stock and bond portfolios and their standard deviations, for the period 1963:7–1991:12.

Table 1 about here

This particular choice of assets allows us to investigate the information contained in the returns on small and large stocks, and in the returns on high and low book-to-market-value stocks, on the variance and risk premia of an admissible pricing kernel.

We also consider the four risk factors

DEF: the change in the yield on Baa corporate bonds minus the change in the yield on long-term Treasury bonds (annualized yields, percentage points);

TS: the change in the difference between the mean monthly yield of a ten-year Treasury note and a three-month Treasury bill (annualized yields, percentage points);

INF: the monthly rate of inflation (percentage points per month);

IP: the monthly growth of the industrial production index (percentage points per month).

The yields on Baa corporate bonds, long-term Treasury bonds, ten-year Treasury notes, and three-month Treasury bills are from the CITIBASE tape, series FYBAAC, FYGL, FYGT10, and FYGM3.

These variables have been chosen on the basis of earlier studies which found them to command non-zero risk premia in the context of empirical investigations of multiple-beta and of multiple-factor models (similar variable have been used, for example, in Chen, Roll, and Ross (1986), McElroy and Burmeister (1988), and Ferson and Harvey (1991)). Of course, we do not claim that these variables are a unique and exhaustive representation of the relevant economic risks driving asset returns.

We also construct a series of the monthly growth rate of per-capita real consumption of nondurables and services, CG (percentage points per month). The series used to construct consumption data are from the CITIBASE tape. Monthly real consumption of nondurable goods and services are the GMCN and GMCS series deflated by the corresponding deflator series GMDN and GMDS. Per-capita quantities are obtained using data on resident population, series POPRES.

Our analysis has assumed the risk factors driving the pricing kernel and/or asset returns to have (conditionally) mean zero and unit variance, and to be (conditionally) orthogonal to each other. In order to generate a set variables which satisfy this requirement, we regress the four risk factors and consumption growth on one lag of the risk factors, where the covariance matrix of the residuals is assumed to be constant over time. The residuals from the first four equations (DEF, TS, INF, and IP) are made orthogonal by means of a

Choleski decomposition of the residual covariance matrix, and normalized by their standard deviations.[13] The transformed residuals are denoted y_{DEF}, y_{TS}, y_{INF}, and y_{IP}. The residual from the consumption equation is also normalized by its standard deviation, and proxies for the variable y_c in the consumption-based pricing kernel (28).

Table 2 reports means and standard deviations of the four state variables, and of consumption growth. It also reports the correlation coefficients between y_c and the risk factors. The period is 1963:7–1991:12.

Table 2 about here

As we would expect, the two interest-rate factors, y_{DEF} and y_{TS}, correlate with y_c (in absolute value) less than the two macroeconomic factors, y_{INF} and y_{IP}.

Using the approach outlined in Section 3, we now turn to calculate the projections of q and y_k onto the (augmented) space of returns. Based on these projections, we then calculate the risk premium-variance bounds. Figures 1-4 illustrate the risk premium-variance bounds (the "B-K bounds"), which are contrasted with the Hansen and Jagannathan variance bound (the "H-J bound"), and the risk premium-variance pairs generated by the consumption-based pricing kernel (the "C-CAPM").

Figures 1-4 about here

In calculating projections, bounds, and the risk premium-variance pairs, theoretical moments are replaced by their sample counterparts, using data for the 1963:7–1991:12 period.

Figure 1 is obtained using the returns on the market and the bond portfolio only; Figure 2 includes the returns on big and small stocks; while Figure 3 uses returns on all six portfolios. In Figure 4, the six asset returns are scaled by the realizations of conditioning variables, as illustrated in Section 4. The conditioning vector z_t is given by $[1,\ DEF_t,\ TS_t,\ INF_t,\ IP_t,\ CG_t]$. Again, while these variables are commonly used [Downs and Snow (1994), for example, use a similar conditioning set], they are not meant to represent all the relevant conditioning information.

Table 3 reports the minimum values of λ_c consistent with the Hansen and Jagannathan bounds, $\lambda_c^{(HJ)}$, the distances of the four factors from the (augmented) span of asset returns, $E[(y_k - y_k^*)^2]$, the risk premia λ_k^*, and the minimum values of λ_c consistent with the risk premium-variance bounds, $\lambda_c^{(BK)}$.

Table 3 about here

The four figures and Table 3 clearly show that the inflation risk premium implies the highest variance of the kernel. Specifically, for the inflation-risk factor, the ratio $[\lambda_c^{(BK)} - \lambda_c^{(HJ)}]/\lambda_c^{(HJ)}$ goes from a minimum of 5.01% (asset returns: r_M, r_f, r_S, and r_B) to a maximum of 15.73% (asset returns: all portfolios, scaled returns).

[13]The order of the decomposition is DEF, TS, INF, and IP.

This result has the following "geometrical" interpretation. The right-hand side of the inequality (14) is a parabola with a minimum of $\text{Var}(q^*)$ for $\lambda_k = \lambda_k^*$. Hence, if the cross moment-variance parabola [the right-hand-side of (14)] is "flat," while the relevant branch of the consumption-based-kernel locus (30) is "steep," the intersection between the two curves takes place for a value of λ_k close to λ_k^*. This, in turn, implies a value of $\text{Var}(q)$ close to the Hansen and Jagannathan variance bound, $\text{Var}(q^*)$. By contrast, if the cross moment-variance parabola is steep and the branch of the consumption-based-kernel locus is flat, the intersection between the two curves takes place for a value λ_k different from λ_k^*; which implies a value of $\text{Var}(q)$ larger than $\text{Var}(q^*)$.

In fact, the absolute value of the slope of the cross moment-variance parabola is given by

$$\frac{2}{E[(y_k - y_k^*)^2]} \left|\lambda_k - \lambda_k^*\right|,$$

which is high when $E[(y_k - y_k^*)^2]$ (which measures the distance of the risk factor from r_a) is low (remember that $0 \leq E[(y_k - y_k^*)^2] \leq 1$). Table 3 shows how the three factors y_{DEF}, y_{TS}, and y_{IP} are more distant from r_a than the factor y_{INF}.

The absolute value of the slope of the consumption-based-kernel locus is given by

$$2\left|\frac{\lambda_k}{\rho_{ck}}\right|,$$

which is high when $|\rho_{ck}|$ is low. Hence, the risk factors y_{DEF}, y_{TS}, and y_{IP}, which exhibit less correlation (in absolute value) with the consumption-based kernel than y_{INF}, generate steeper consumption-based-kernel loci, and imply a smaller increase of the variance of the kernel.

One additional result from our analysis is that the introduction of returns on small and large stocks (Figure 2), and of returns on high and low book-to-market-value stocks (Figure 3) shifts all bounds upward, relative to the case where only the market and the risk-free asset proxies are considered. The figures of Table 3 show that the inclusion of returns on small and large stocks increases $\lambda_c^{(HJ)}$ by roughly 98%; and the inclusion of returns on high and low book-to-market-value stocks further increases $\lambda_c^{(HJ)}$ by about 22%. In other words, returns on small and large stocks, and returns on high and low book-to-market-value stocks do carry information on admissible pricing kernels beyond that of the returns on the market and the risk-free asset.

When the set of asset returns is expanded by introducing conditional information, the bounds are made even sharper (Figure 4). Table 3 shows that the increase in $\lambda_c^{(HJ)}$ when going from "all portfolios" to "all portfolios, scaled returns" is about 61%.

8 Conclusions

This paper shows how information in *addition* to asset returns, and *contemporaneous* to the realization of asset returns, can be used to learn about the properties of an admissible pricing kernel. Namely, we derived restrictions on the *risk premium-variance* pairs of a candidate pricing kernel. For given risk premia, these restrictions translate into a lower bound on the variance of a normalized pricing kernel which is *at least as high* as the Hansen and Jagannathan variance bound. Specifically, we have shown that the inflation risk premium generated by a consumption-based pricing kernel implies a standard deviation of the normalized kernel which is up to 15% higher than the Hansen and Jagannathan bound.

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Table 1: asset returns, summary statistics

We report summary statistics for the real rates of return on the stock and bond portfolios, for the 1963:7–1991:12 period (percentage points per month).

Return	Mean	Standard Deviation
r_M	0.5055	4.521
r_S	0.8478	5.959
r_B	0.5760	4.385
r_H	0.9368	4.961
r_L	0.5069	5.654
r_f	0.0984	0.312

Table 2: risk factors, summary statistics

We report summary statistics for the four risk factors and consumption growth (percentage points per month). We also report estimates of the correlation coefficients between y_c and the factors y_k. The period is 1963:7–1991:12.

Variable	Mean	Standard Deviation
DEF	0.0032	0.1733
TS	0.0010	0.4504
INF	0.4426	0.3457
IP	0.2815	0.8562
CG	0.1714	0.4019

Factor	ρ_{ck}
y_{DEF}	0.05
y_{TS}	-0.08
y_{INF}	-0.25
y_{IP}	0.14

Table 3: risk premium-variance bounds

We report estimates of $\lambda_c^{(HJ)}$, $E(y_k - y_k^*)^2$, λ_k^* and $\lambda_c^{(BK)}$, for the 1963:7–1991:12 period.

Asset returns: r_M, and r_f

$\lambda_c^{(HJ)}$
0.0984

Factor	$E(y_k - y_k^*)^2$	λ_k^*	$\lambda_c^{(BK)}$
y_{DEF}	0.98	-0.0087	0.0985
y_{TS}	1.00	-0.0020	0.0989
y_{INF}	0.44	0.0044	0.1036
y_{IP}	0.99	-0.0037	0.0989

Asset returns: r_M, r_f, r_S, and r_B

$\lambda_c^{(HJ)}$
0.1957

Factor	$E(y_k - y_k^*)^2$	λ_k^*	$\lambda_c^{(BK)}$
y_{DEF}	0.95	-0.0141	0.1958
y_{TS}	0.98	-0.0225	0.1996
y_{INF}	0.44	0.0099	0.2056
y_{IP}	0.99	-0.0059	0.1969

Asset returns: all portfolios

$\lambda_c^{(HJ)}$
0.2394

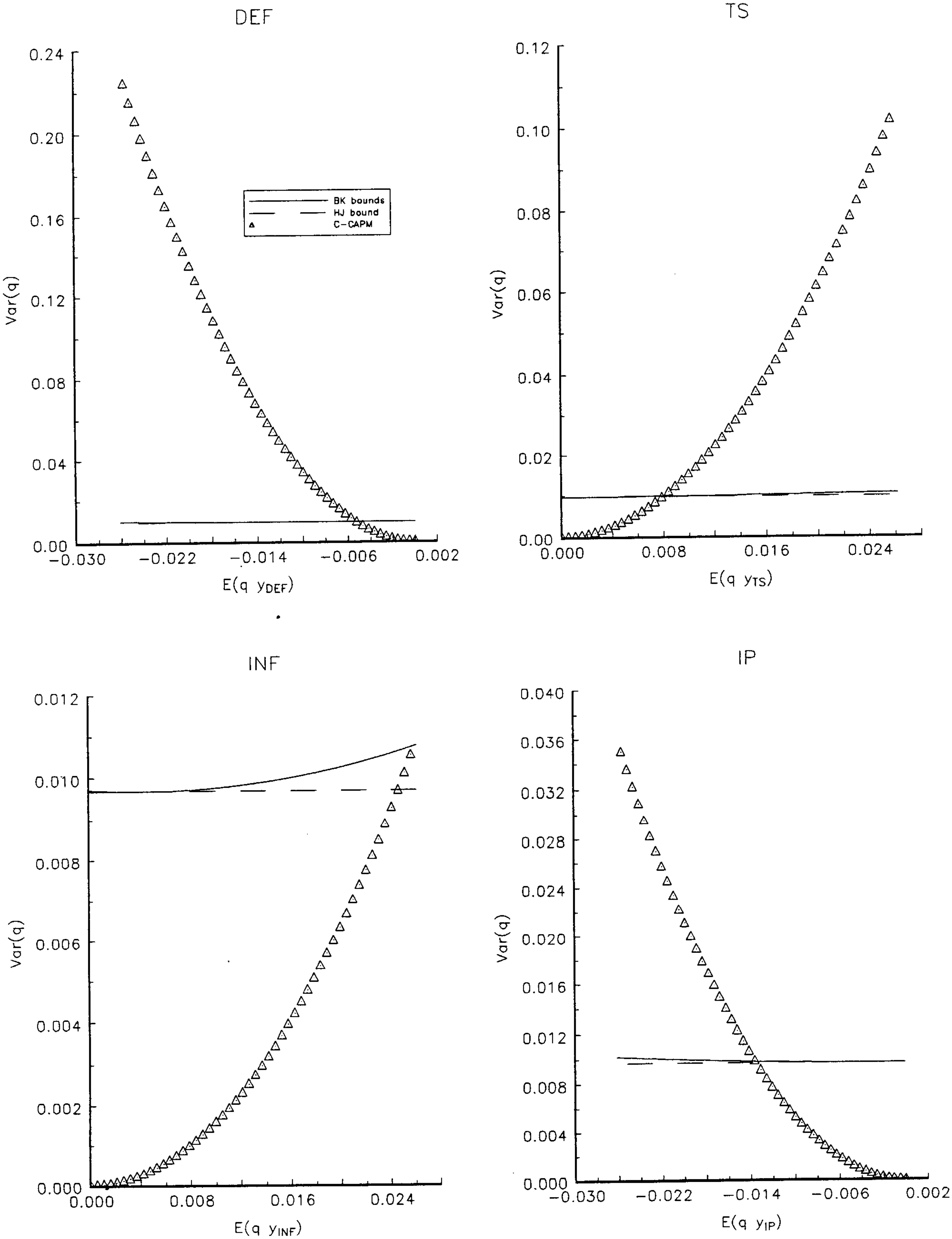
Factor	$E(y_k - y_k^*)^2$	λ_k^*	$\lambda_c^{(BK)}$
y_{DEF}	0.95	-0.0071	0.2395
y_{TS}	0.96	-0.0413	0.2474
y_{INF}	0.43	-0.0011	0.2597
y_{IP}	0.98	-0.0076	0.2408

Asset returns: all portfolios, scaled returns

$\lambda_c^{(HJ)}$
0.3872

Factor	$E(y_k - y_k^*)^2$	λ_k^*	$\lambda_c^{(BK)}$
y_{DEF}	0.81	-0.0457	0.3881
y_{TS}	0.73	-0.0380	0.3957
y_{INF}	0.34	-0.0191	0.4481
y_{IP}	0.88	0.0029	0.3919

Figure 1



DEF
TS
INF
IP
BK bounds
HJ bound
C-CAPM
Var(q)
E(q y_{DEF})
E(q y_{TS})
E(q y_{INF})
E(q y_{IP})
0.24
0.20
0.16
0.12
0.08
0.04
0.00
−0.030
−0.022
−0.014
−0.006
0.002
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0.000
0.008
0.016
0.024
0.012
0.010
0.008
0.006
0.004
0.002
0.000
0.000
0.008
0.016
0.024
0.040
0.036
0.032
0.028
0.024
0.020
0.016
0.012
0.008
0.004
0.000
−0.030
−0.022
−0.014
−0.006
0.002

Figure 2

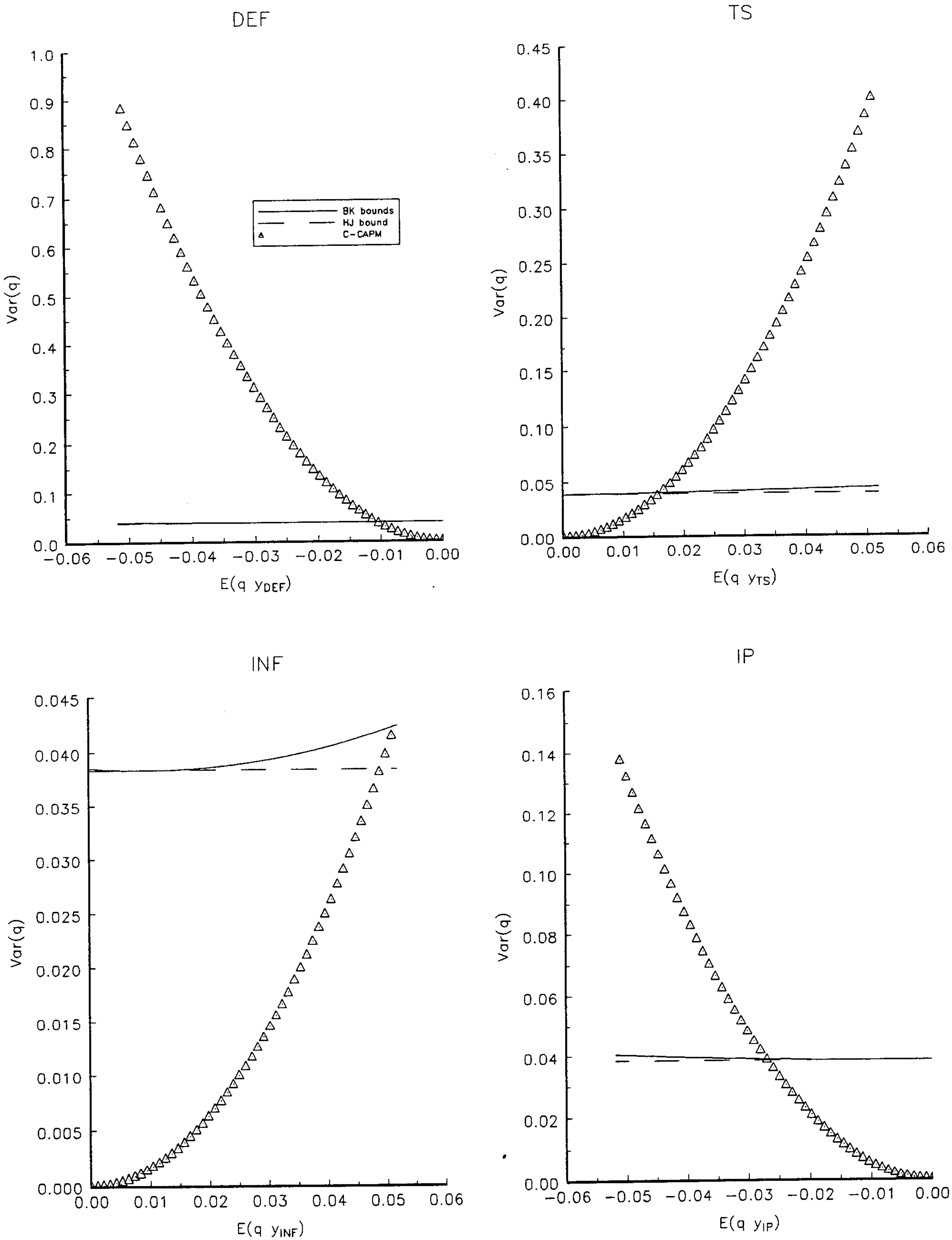

DEF
TS
INF
IP
BK bounds
HJ bound
C-CAPM
Var(q)
E(q y_DEF)
E(q y_TS)
E(q y_INF)
E(q y_IP)

Figure 3

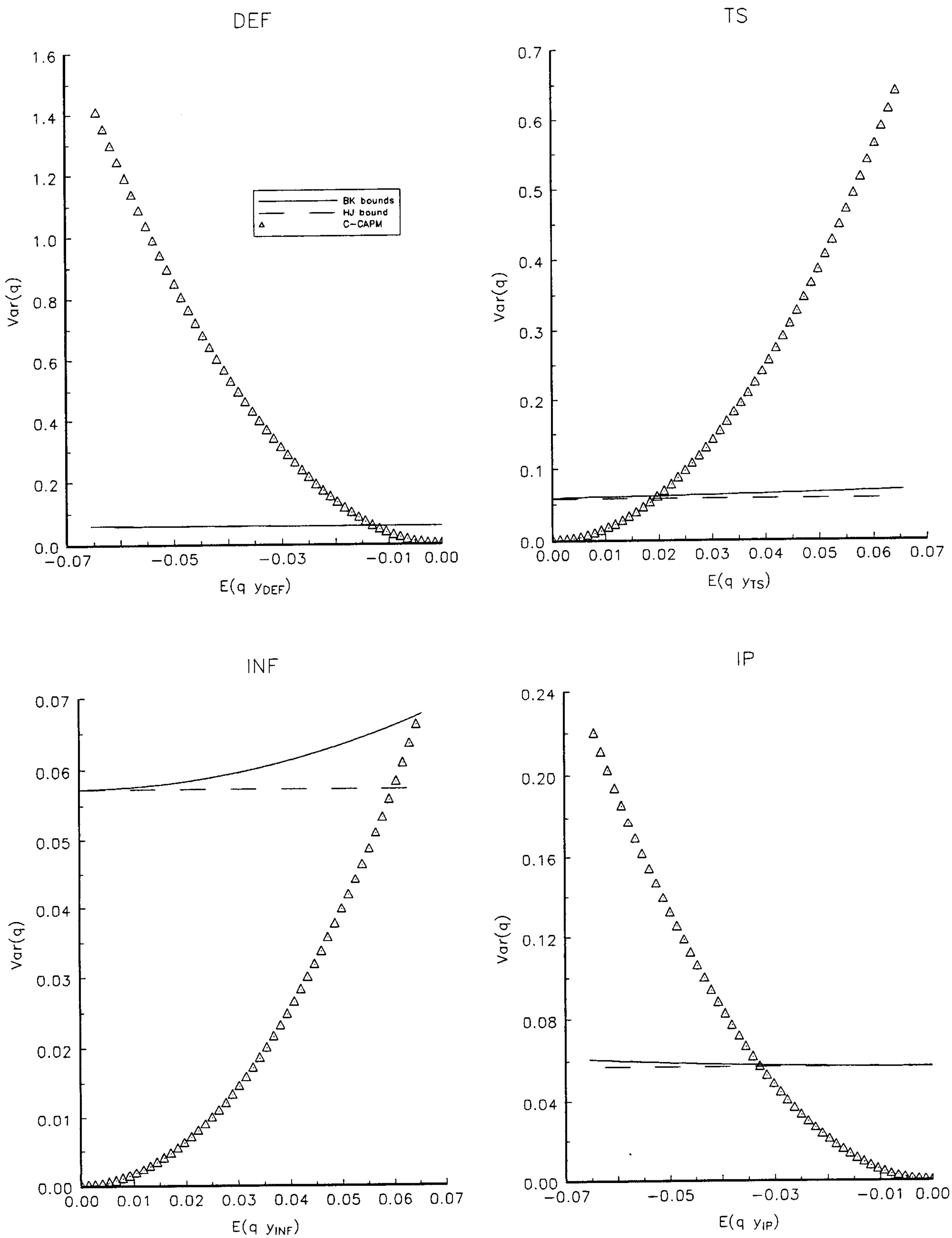
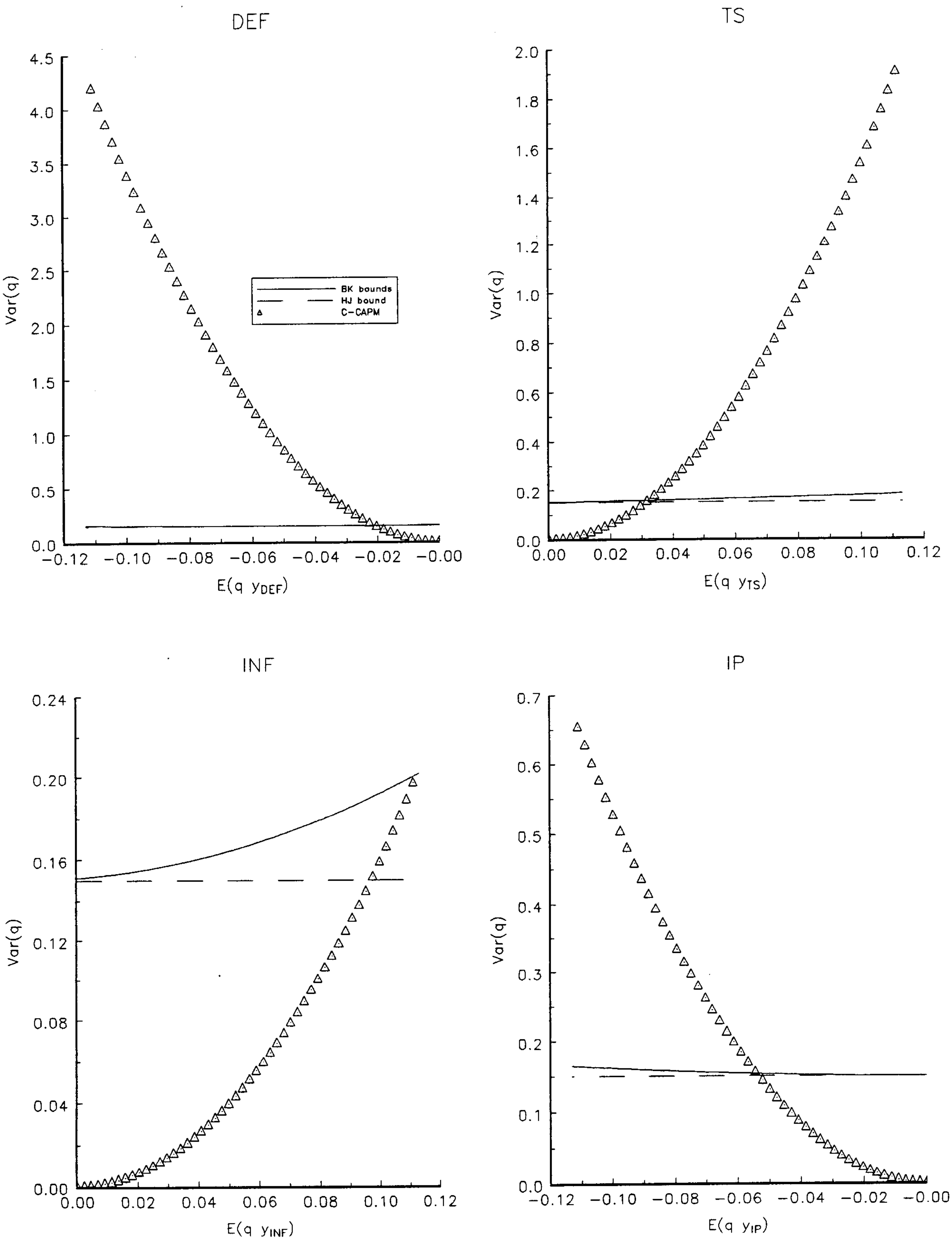

DEF
TS
INF
IP
BK bounds
HJ bound
C-CAPM
Var(q)
E(q y_DEF)
E(q y_TS)
E(q y_INF)
E(q y_IP)
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.07
−0.05
−0.03
−0.01
0.00
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
0.24
0.20
0.16
0.12
0.08
0.04
0.00

Figure 4


DEF
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−0.12 −0.10 −0.08 −0.06 −0.04 −0.02 −0.00
Var(q)
E(q y_{DEF})
BK bounds
HJ bound
C-CAPM
TS
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.00 0.02 0.04 0.06 0.08 0.10 0.12
Var(q)
E(q y_{TS})
INF
0.24
0.20
0.16
0.12
0.08
0.04
0.00
0.00 0.02 0.04 0.06 0.08 0.10 0.12
Var(q)
E(q y_{INF})
IP
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
−0.12 −0.10 −0.08 −0.06 −0.04 −0.02 −0.00
Var(q)
E(q y_{IP})