NEW YORK UNIVERSITY
STERN SCHOOL OF BUSINESS
FINANCE DEPARTMENT

Working Paper Series, 1995

The Optimal Dynamic Investment Policy for a Fund Manager Compensated with an Incentive Fee.

Carpenter, Jennifer N.

FIN-95-16

The Optimal Dynamic Investment Policy for a Fund Manager Compensated with an Incentive Fee

Jennifer N. Carpenter*

November 28, 1995

*Department of Finance, Stern School of Business, New York University, 44 West Fourth Street, New York, NY, 10012, (212) 998-0352. This article is based on a chapter of my dissertation. I am grateful for suggestions from Sanford Grossman and Philip Dybvig. I would also like to thank Franklin Allen, Suleyman Basak, Bruce Grundy, Ioannis Karatzas, Richard Kihlstrom, and Anthony Lynch for helpful comments.

The Optimal Dynamic Investment Policy for a Fund Manager Compensated with an Incentive Fee

Abstract

We use martingale methods to solve the investment problem of a risk averse fund manager who charges an incentive fee which he cannot hedge in his personal account. An incentive fee is a share in the positive part of the returns on the client's portfolio net of some benchmark return. The optimal policy is a long-shot; there is always some chance of bankrupting the client, but if the terminal fund value is nonzero, it is in the money by some strictly positive amount. We provide explicit expressions for the optimal trading strategy with either the riskless asset or the market portfolio as benchmark and with either constant relative or absolute risk aversion. Rather than trying to maximize volatility, as earlier literature suggests, the manager dynamically adjusts volatility as the assets move in or out of the money. As the manager accumulates profits, he moderates portfolio risk. For example, if the manager has constant relative risk aversion, volatility converges to the Merton constant as fund value grows large. On the other hand, as bankruptcy approaches, portfolio volatility goes to infinity.

1 Introduction

This paper presents the optimal dynamic trading strategy for a risk averse fund manager who is compensated with an asymmetric incentive fee which he cannot hedge in his personal account. An asymmetric incentive fee is a share, α, for example, 30%, in the positive part of the returns on the client's portfolio net of some benchmark. Such a fee structure is typical for hedge fund and pension fund managers. Grinblatt and Titman (1989) study the fund manager's investment problem under the assumption that the manager can hedge the fee in his personal portfolio, so his objective is to maximize the fee's market value. With this objective, the manager wants to maximize volatility and the problem has no solution. We assume, on the other hand, that the manager cannot hedge the fee in his private account because shorting securities that he purchases on his client's behalf is a breach of fiduciary duty. Now the manager's objective is to maximize his expected utility of the incentive fee. We cast the problem in a standard continuous-time financial market and show that there exists a unique optimal investment policy.[1]

Under the optimal policy, the portfolio has an all-or-nothing payoff, either in the money or zero. The policy is also a long-shot in the sense that the probability of bankruptcy is high, but the payoff, if in the money, is in the money by some strictly positive amount. While this may appear to be extreme, we feel that the essence of the solution, that the manager does not want to end up too near the money, would prevail in a less stylized setting with market frictions or multi-period contracts.

Our martingale approach sheds light on the manager's preference for a long-shot by revealing that the market value of the incentive fee is an increasing function of the probability of bankruptcy under a martingale measure.[2] This relationship implies that

[1]Starks (1987) studies the portfolio manager's problem in a mean-variance framework and concludes that an asymmetric incentive fee will induce the manager to choose a higher beta than he would choose with a symmetric fee.

[2]See Cox and Ross (1976) and Harrison and Kreps (1979).

the contract is inefficient in the sense that there exist lower cost linear contracts that give the manager greater expected utility.

We provide closed-form expressions for the optimal trading strategy for constant relative and absolute risk averse utility functions with either the riskless asset or the market portfolio as benchmarks. Rather than maximizing portfolio risk, the manager dynamically adjusts leverage and volatility in response to changes in the asset value over time. When the manager is near the money, small changes in the value of the mean-variance efficient portfolio lead to large trades as the manager alternates between the desire to gamble and the need to remain solvent. As the manager accumulates profits, so that he is gambling with his own money, he moderates portfolio risk. For example, if the manager has constant relative risk averse utility and the benchmark is riskless, volatility converges to the Merton constant[3] as fund value grows large. On the other hand, as bankruptcy approaches, portfolio volatility and leverage approach infinity.

Consistent with these dynamics, Brown, Harlow, and Starks (1994) find evidence in the mutual fund industry that managers with relatively poor performance in the first half of their performance evaluation period increase fund volatility in the second half of the period more than managers who have done well. While, in a given year, mutual fund managers typically earn a fixed proportion of initial asset value, Sirri and Tufano (1992) show that new money tends to flow into winning funds faster than old money flows out of losers. Mutual fund managers' long run compensation may therefore still be convex in fund performance even though there is no explicit incentive fee. Indeed, Chevalier and Ellison (1995) estimate a nonlinear relationship between one year's performance and the next year's flow of new money for a large set of mutual funds and find that, for young funds, the function is relatively flat for moderately poor performance and then increasing for better performance. They conclude that this provides incentives for funds with moderately poor performance to gamble to recover losses. Then they study the relationship between performance from January to September and changes in

[3]See Merton (1969) and (1971).

portfolio riskiness from September to December and find that funds that are somewhat behind do tend to increase risk.

Our paper proceeds as follows. §2 describes the manager's preferences and opportunity set. §3 uses martingale methods to transform the manager's dynamic trading problem strategy into a static problem of choosing an optimal random terminal portfolio value. §4 solves the transformed problem. §5 gives examples of the optimal trading strategy. §6 explores implications of our results for contract theory.

2 Assumptions

At time zero, the client hires the manager for a fixed length of time T, and agrees to pay him an incentive fee. The manager's total terminal wealth, Y, is equal to his incentive fee plus a constant, K, that includes any fixed fees and personal wealth. Letting X_t represent fund value and B_t represent the value of a benchmark portfolio at time t,

$$Y = \alpha(X_T - B_T)^+ + K \ , \tag{1}$$

where $0 < \alpha < 1$.

The manager chooses an investment policy to maximize his expected utility of terminal wealth. His utility function U is strictly increasing, strictly concave, at least twice continuously differentiable, and defined on a domain containing $(0, \infty)$. U'' is nondecreasing and $U'(W)$ approaches zero as W approaches infinity. Consequently, the function $I = U'^{-1}$ is a well-defined, strictly decreasing, convex, continuously differentiable function from $(0, \infty)$ onto a range containing $(0, \infty)$. For example, both the constant absolute and relative risk averse classes of utility functions satisfy these hypotheses.

The financial market consists of a riskless asset with interest rate r, and n risky assets. The risky asset prices, $P_i, i = 1, \ldots, n$ are diffusion processes governed by the equations

$$\frac{dP_{i,t}}{P_{i,t}} = (r + \mu_i)\, dt + \sigma_i'\, dWt \ ,$$

where $\mu_i \in \mathcal{R}$ and $\sigma_i \in \mathcal{R}^n$ are constants and W is standard n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathcal{R}^n$, let σ be the matrix whose ith row is σ_i', and assume that σ is nondegenerate. Let $\{\mathcal{F}_t\}$ denote the $\mathcal{P}$-augmentation of the filtration generated by the Brownian motion; $\mathcal{F}_t$ represents the manager's information at time t.

A trading strategy for the manager is an n-dimensional process $\{\pi_t : 0 \leq t \leq T\}$ whose ith component, $\pi_{i,t}$, is the value of the holdings of risky asset i in the portfolio at time t. An admissible trading strategy, π, must be progressively measurable with respect to $\{\mathcal{F}_t\}$, must prevent fund value from falling below zero, and must satisfy $\int_0^T ||\pi_t||^2 \, dt < \infty$, a. s. Under an admissible trading strategy π, portfolio value evolves according to

$$dX_t = (rX_t + \pi_t'\mu)\, dt + \pi_t'\sigma \, dW_t \ . \tag{2}$$

The benchmark portfolio value, B_t, is a geometric Brownian motion that can be replicated with a self-financing trading strategy involving the market securities:

$$\frac{dB_t}{B_t} = (r + \pi_B'\mu)\, dt + \pi_B'\sigma \, dWt \ ,$$

where π_B is a constant.

3 The Manager's Investment Problem

The manager's dynamic problem is to choose an admissible trading strategy for the fund to maximize his expected utility of terminal wealth:

$$\begin{aligned} \max_{\pi} \quad & EU(\alpha(X_T - B_T)^+ + K) \\ \text{subject to} \quad & dX_t = (rX_t + \pi_t'\mu)\, dt + \pi_t'\sigma \, dW_t \\ \text{and} \quad & X_t \geq 0 \ \forall t \in [0, T] \ . \end{aligned} \tag{3}$$

Using martingale methods, we recast (3) as a static problem of choosing an optimal terminal fund value:[4]

$$\begin{aligned} \max_{X_T} \quad & EU(\alpha(X_T - B_T)^+ + K) \\ \text{subject to} \quad & E\zeta_T X_T \le X_0 \\ \text{and} \quad & X_T \ge 0 \ . \end{aligned} \tag{4}$$

where $\zeta_t \equiv e^{-rt-\theta' W_t - \|\theta\|^2 t/2}$ and $\theta \equiv \sigma^{-1}\mu$.

One final transformation illuminates the key difference between the manager's problem and the standard terminal wealth problem. Observe that under an optimal policy, $X_T \in \{0\} \cup (B_T, \infty)$, a.s.; whenever X_T takes on values in $(0,\ B_T]$, it uses resources without adding to utility, so an optimal choice cannot do so with positive probability. Consequently, the manager's terminal wealth Y is invertible for the terminal fund value B_T from (1), so we may treat Y as the choice variable.

In addition, when the assets have the all-or-nothing terminal distribution described above, the market value of the incentive fee is $\alpha(X_0 - B_0 + B_0\tilde{\mathcal{P}}\{X_T = 0\})$, where $\tilde{\mathcal{P}}$ is the measure defined by $\frac{d\tilde{\mathcal{P}}}{d\mathcal{P}} = e^{rT}\zeta_T \frac{B_T}{B_0}$. For instance, if the benchmark is riskless, then $\tilde{\mathcal{P}}$ is the usual risk-neutral martingale measure. If the benchmark is the reciprocal of the pricing kernal ζ_T, then $\tilde{\mathcal{P}} = \mathcal{P}$, the true probability measure. Now the manager's problem is

$$\begin{aligned} \max_{Y} \quad & EU(Y) \\ \text{subject to} \quad & E\zeta_T Y \le \alpha(X_0 - B_0 + B_0\tilde{\mathcal{P}}\{Y = K\}) + K \\ \text{and} \quad & Y \ge K \ . \end{aligned} \tag{5}$$

So the manager's problem is like the standard terminal wealth problem except that his budget, $\alpha(X_0 - B_0 + B_0\tilde{\mathcal{P}}\{Y = K\}) + K$ is a function of his strategy–a "longer-shot"

[4]See, for example, Harrison and Kreps (1979), Harrison and Pliska (1981), Pliska (1986), Karatzas, Lehoczky, and Shreve (1987), and Cox and Huang (1989) for the development of these methods and their application to optimal portfolio choice. See also the review article Karatzas (1989) for these and additional applications.

has more value. We know from first principles that the convexity of the incentive fee makes risky strategies relatively more attractive to the manager, and budget constraint in problem (5) quantifies exactly how.

4 The Optimal Terminal Portfolio Value

We solve the problem by concavifying the objective function.[5] Define $u : \mathcal{R} \times (0, \infty) \to \mathcal{R}$ by $u(x, b) = U(\alpha(x - b)^+ + K)$, for $x \geq 0$, and $u(x, b) = -\infty$, otherwise. In terms of u, the manager's problem is

$$\max_{X_T} Eu(X_T, B_T) \text{ subject to } E\zeta_T X_T \leq X_0 \ . \tag{6}$$

Let $u'(x, b) = \frac{\partial u(x,b)}{\partial x}$, for $x > b$, and let $f(x, b) = u(x, b) - u(0, b) - xu'(x, b)$, for all $b > 0$ and $x > b$.

Lemma *For every b, there exists a unique $x > b$ such that $f(x, b) = 0$.*

Proof *Fix b and let $x > b$. $f(x, b) = U(\alpha(x - b) + K) - U(K) - \alpha x U'(\alpha(x - b) + K)$ is strictly increasing in x, for its derivative with respect to x is $-\alpha^2 x U''(\alpha(x - b) + K) > 0$. As $x \to b$, $f(x, b) \to -\alpha x U'(K) < 0$. As $x \to \infty$, $f(x, b)$ approaches a strictly positive limit, possibly infinity. To see this, rewrite f as*

$$f(x, b) = [U(\alpha(x - b) + K) - U(K) - \alpha(x - b)U'(\alpha(x - b) + K)] - \alpha b U'(\alpha(x - b) + K) \ .$$

The term in brackets above is strictly positive and increasing for all $x > b$, while the remaining term above approaches zero as x approaches ∞. Therefore, $f(\cdot, b)$ has a unique zero on (b, ∞).

Let $\hat{x}(b)$ be the unique $x > b$ such that $f(x, b) = 0$. Then $\tilde{u} : \mathcal{R} \times (0, \infty) \to \mathcal{R}$ defined by

$$\tilde{u}(x, b) = \quad -\infty \qquad \text{for } x < 0$$

[5]See Aumann and Perles (1965).

$$u(0) + u'(\hat{x}(b), b)x \quad \text{for } 0 \le x \le \hat{x}(b)$$
$$u(x, b) \quad \text{for } x > \hat{x}(b)$$

is concave in x. Furthermore, $\tilde{u}(x, b) \ge u(x, b)$ for all $(x, b) \in \mathcal{R} \times (0, \infty)$ and $\tilde{u}(x, b) = u(x, b)$ for $x = 0$ and for all $x \ge \hat{x}(b)$.

Now define the set-valued function $\tilde{u}'$ on $[0, \infty) \times (0, \infty)$ by

$$\begin{aligned} \tilde{u}'(x, b) = \ & (\infty, u'(\hat{x}(b), b)] && \text{for } x = 0 \\ & \{u'(\hat{x}(b), b)\} && \text{for } 0 < x \le \hat{x}(b) \\ & \{u'(x, b)\} && \text{for } x > \hat{x}(b) \ . \end{aligned}$$

Then, for every $x' \in \mathcal{R}$ and every $m \in \tilde{u}'(x, b)$, $\tilde{u}(x', b) - \tilde{u}(x, b) \le m(x' - x)$. Strict inequality holds whenever $x > \hat{x}(b)$ and $x' \ne x$. For each b, $\tilde{u}'(\cdot, b)$ is the subdifferential of $\tilde{u}(\cdot, b)$.[6]

Next, define $i : (0, \infty) \times (0, \infty) \to [0, \infty)$ by

$$i(y, b) = [(I(y/\alpha) - K)/\alpha + b] 1_{\{y < u'(\hat{x}(b), b)\}}$$

where 1_A is the indicator function of the set A. Then $y \in \tilde{u}'(i(y, b), b)$ for all $b > 0$.

Finally, let $\mathcal{X}(\lambda) = E\zeta_T i(\lambda\zeta_T, B_T)$ for $\lambda > 0$. Assume that

$$\mathcal{X}(\lambda) < \infty \text{ for all } \lambda \ . \tag{7}$$

This will hold for constant absolute and relative risk aversion with the choices of benchmarks we make below. Then $\mathcal{X}(\lambda)$ is continuous and strictly decreasing, $\mathcal{X}(\lambda) \to \infty$ as $\lambda \to 0$, and $\mathcal{X}(\lambda) \to 0$ as $\lambda \to \infty$. Therefore, there exists a unique $\lambda^* > 0$ such that $\mathcal{X}(\lambda^m) = X_0$.

Proposition 1 *Under assumption (7), $X_T^* \equiv i(\lambda^*\zeta_T, B_T)$ is the unique optimal solution to problem (6).*

---

[6]See Rockafellar (1970, p.214-215).

Proof *If X' is any other feasible strategy that is not almost surely equal to X_T^*, then*

$$\begin{aligned} E\{u(X', B_T) - u(X_T^*, B_T)\} &= E\{u(X', B_T) - \tilde{u}(X_T^*, B_T)\} \\ &\leq E\{\tilde{u}(X', B_T) - \tilde{u}(X_T^*, B_T)\} \\ &< E\{\lambda^* \zeta_T (X' - X_T^*)\} \\ &\leq \lambda^*(E\zeta_T X' - X_0) \leq 0 \ . \end{aligned}$$

Notice that under the optimal policy, the incentive fee is either out of the money, or else it is in the money by at least $\alpha(\hat{x}(B_T) - B_T) > 0$. It does not pay for the manager to be just marginally in the money, since he must expend substantial resources to bring fund value into the money at all.

One typical benchmark is a constant, or, in other words, a riskless portfolio. Another is a market index such as the S&P 500. A representation for a market index in this model is the portfolio $M_t \equiv 1/\zeta_t$, since risk averse investors solving standard investment/consumption problems in this framework will always divide their investments between M and the riskless asset. In both cases, the manager's optimal terminal fund value will be a simple function of the pricing kernal.

If the manager is measured against a riskless benchmark, $B_T = B_0 e^{rT}$, the manager's optimal terminal fund value is

$$X_T^1 = [(I(\lambda_1 \zeta_T/\alpha) - K)/\alpha + B_0 e^{rT}] 1_{\{\zeta_T < z_1\}} \ ,$$

where λ_1 solves $E\zeta_T i(\lambda\zeta_T, B_0 e^{rT}) = X_0$ and $z_1 = \alpha U'(\alpha(\hat{x}(B_0 e^{rT}) - B_0 e^{rT})/\lambda_1$. A sketch of the optimized incentive fee appears in Figure 1. Notice that the manager's payoff is a discontinuous function of the pricing kernal, ζ_T.

With B_0/ζ_T as the benchmark, the form of the optimal policy is similar. Let λ_2 solve $E\zeta_T i(\lambda\zeta_T, B_0/\zeta_T) = X_0$ and let $g(\zeta_T) = u'(\hat{x}(B_0/\zeta_T), B_0/\zeta_T) - \lambda_2\zeta_T$.

Proposition 2 *Under assumption (7), the optimal policy for problem (6) with benchmark $B_T \equiv B_0/\zeta_T$ is*

$$X_T^2 = [(I(\lambda_2 \zeta_T/\alpha) - K)/\alpha + B_0/\zeta_T] 1_{\{\zeta_T < z_2\}} \ ,$$

where z_2 is the unique zero of g.

Proof *Given proposition (1), it remains only to show that $g(\zeta_T) > 0 \iff \zeta_T < z_2$, for some constant z_2. g cannot be nonpositive everywhere, by construction of λ_2, and $g(\zeta_T) \to -\infty$ as $\zeta_T \to \infty$, so, by continuity, g must have a zero.*

$$g'(\zeta_T) = g(\zeta_T)\frac{B_0/\zeta_t^2}{\hat{x}(B_0/\zeta_T)} - \lambda_2(1 - \frac{B_0/\zeta_T}{\hat{x}(B_0/\zeta_T)}) ,$$

so, whenever $g \leq 0$, g is decreasing. Therefore, g has a unique zero, z_2, and $g(\zeta_T) > 0 \iff \zeta_T < z_2$.

5 Examples of Optimal Trading Strategies

We now take the benchmark to be either the riskless asset or the market portfolio and derive closed-form expressions for the manager's optimal trading strategy in the cases of constant absolute and relative risk aversion. At issue is whether or not we may use Ito's lemma to obtain a stochastic differential equation for the optimal portfolio value process, X_t^*, despite the fact that X_T^* is a discontinuous function of ζ_T. With both benchmarks, final portfolio value $X_T^* = \psi(\zeta_T)$ for some function $\psi : (0,\infty) \to \mathcal{R}$, so intermediate portfolio value, $X_t^* = E((\zeta_T/\zeta_t)X_T^*|\mathcal{F}_t)$, is equal to $x^*(t,\zeta_t)$ for some function $x^* : [0,T) \times (0,\infty) \to \mathcal{R}$, because ζ is a Markov Process. Set $x^*(T,\zeta) \equiv \psi(\zeta)$. If the function $x^* : [0,T] \times (0,\infty) \to \mathcal{R}$ were continuous on $[0,T] \times (0,\infty)$ and $C^{1,2}$ on $[0,T) \times (0,\infty)$, then we could apply Ito's lemma to get an expression for dx^* and equate the resulting diffusion coefficient with the quantity $\pi_t'\sigma$ from (2) to arrive at the equation

$$\pi_t^* = \rho(t,\zeta_t) = -\zeta_t x_\zeta^*(t,\zeta_t)\Sigma^{-1}\mu \ , \tag{8}$$

where $\rho : [0,T) \times (0,\infty) \to \mathcal{R}^n$, x_ζ^* is the partial derivative of x^* with respect to its second argument, and the matrix $\Sigma = \sigma\sigma'$ is the covariance matrix of instantaneous stock returns.[7]

In the case of the manager's optimal policy, $x^*(T,\cdot)$ is not continuous. Nevertheless, in the cases of constant absolute and relative risk aversion, x^* is $C^{1,2}$ on $[0,T)\times(0,\infty)$, so (2) holds, with $X \equiv X^*$ and π^* defined by (8), for all $t < T$. In addition, for all values of $\zeta \neq z_i$, $i = 1$ for the riskless benchmark and $i = 2$ for the market benchmark, $x^*(\cdot,\zeta)$ is continuous on $[0,T]$. Furthermore, $\rho(\cdot,\zeta)$ defined by (8) has a continuous extension to $[0,T]$. Letting π^* be given by this extension, we have $X_T^* = X_0 + \int_0^T (rX_s^* + \pi_s^{*\prime}\mu)\, ds + \int_0^T \pi_s^{*\prime}\sigma\, dW_s$ a.s. because the equality holds for all $t < T$ and both the wealth process and the integrals are almost surely path-continuous.

5.1 Constant Relative Risk Aversion

Let $U(X) = \frac{X^{1-A}}{1-A}$ where $A > 0$ and $A \neq 1$.

5.1.1 The Riskless Portfolio as Benchmark

For this section, set $B_T \equiv B_0 e^{rT}$. Then portfolio value is the process

$$X_t^1 = e^{-r(T-t)}(B_T - \frac{K}{\alpha})N(d_{1,t}) + (\frac{e^{-r(T-t)}}{\alpha})^{1-1/A} e^{||\theta||^2(T-t)(1-A)/2A^2}(\lambda_1\zeta_t)^{-1/A}N(d_{2,t})$$

and the manager's optimal trading strategy is

$$\begin{aligned}\pi_t^1 = \quad & \{\frac{1}{A}[X_t^1 - e^{-r(T-t)}(B_T - \frac{K}{\alpha})N(d_{1,t})] \\ & + e^{-r(T-t)}(\frac{(\lambda_1 z_1/\alpha)^{-1/A} - K}{\alpha} + B_T)\frac{N'(d_{1,t})}{||\theta||\sqrt{T-t}}\}\Sigma^{-1}\mu \ ,\end{aligned}$$

where N is the standard cumulative normal distribution, $d_{1,t} = \frac{\ln(z_1/\zeta_t)+(r-||\theta||^2/2)(T-t)}{||\theta||\sqrt{T-t}}$, and $d_{2,t} = d_{1,t} + ||\theta||\sqrt{T-t}/A$.

We can show that as $\zeta_t \to 0$, $X_t^1 \to +\infty$, $||\pi_t^1|| \to +\infty$, and $\frac{\pi_t^1}{X_t^1} \to \frac{\Sigma^{-1}\mu}{A}$. On the other hand, as $\zeta_t \to +\infty$, $X_t^1 \to 0$, $\pi_t^1 \to 0$, but $||\frac{\pi_t^1}{X_t^1}|| \to \infty$.

---

[7]Similar arguments appear in Harrison and Pliska (1981, Subsection 5.3), Pliska (1986, Section 5), Karatzas, Lehoczky, and Shreve (1987, Section 7), and Karatzas (1989, Example 5.6).

5.1.2 The Market Portfolio as Benchmark

Now set $B_T \equiv B_0/\zeta_T$. Then portfolio value is the process

$$X_t^2 = (B_0/\zeta_t)N(d_{5,t}) - e^{-r(T-t)}\frac{K}{\alpha}N(d_{3,t}) + (\frac{e^{-r(T-t)}}{\alpha})^{1-1/A} e^{||\theta||^2(T-t)(1-A)/2A^2}(\lambda_2\zeta_t)^{-1/A}N(d_{4,t})$$

and the manager's optimal trading strategy is

$$\pi_t^2 = \{\frac{1}{A}[X_t^2 - (B_0/\zeta_t)N(d_{5,t}) + e^{-r(T-t)}\frac{K}{\alpha}N(d_{3,t})] + e^{-r(T-t)}(\frac{(\lambda_2 z_2/\alpha)^{-1/A} - K}{\alpha} + B_0/z_2)\frac{N'(d_{3,t})}{||\theta||\sqrt{T-t}} + (B_0/\zeta_t)N(d_{5,t})\}\Sigma^{-1}\mu \ ,$$

where $d_{3,t} = \frac{\ln(z_2/\zeta_t)+(r-||\theta||^2/2)(T-t)}{||\theta||\sqrt{T-t}}$, $d_{4,t} = d_{3,t} + ||\theta||\sqrt{T-t}/A$, and $d_{5,t} = d_{3,t} + ||\theta||\sqrt{T-t}$.

As $\zeta_t \to 0$, $X_t^2 \to +\infty$, $||\pi_t^2|| \to +\infty$, and $\frac{\pi_t^2}{X_t^2}$ goes either to $\frac{\Sigma^{-1}\mu}{A}$ if $A < 1$ or $\Sigma^{-1}\mu$ if $A > 1$. Yet as $\zeta_t \to +\infty$, $X_t^2 \to 0$, $\pi_t^2 \to 0$, but $||\frac{\pi_t^2}{X_t^2}|| \to \infty$.

5.2 Constant Absolute Risk Aversion

Now let $U(W) = -e^{-AW}$ where $A > 0$.

5.2.1 The Riskless Portfolio as Benchmark

For this section, set $B_T \equiv B_0 e^{rT}$. Then portfolio value is the process

$$X_t^{1'} = e^{-r(T-t)}\{[\frac{1}{\alpha A}(\ln\frac{\alpha A}{\lambda_1'\zeta_t} + (r-||\theta||^2/2)(T-t) + B_T - K/\alpha]N(d_{1,t}') + \frac{||\theta||\sqrt{T-t}}{\alpha A}N'(d_{1,t}')\} \ ,$$

and the manager's optimal trading strategy is

$$\pi_t^{1'} = e^{-r(T-t)}\{\frac{N(d_{1,t}')}{\alpha A} + \frac{N'(d_{1,t}')}{||\theta||\sqrt{(T-t)}}[\frac{1}{\alpha A}\ln\frac{\alpha A}{\lambda_1' z_1'} + B_T - K/\alpha]\}\Sigma^{-1}\mu \ ,$$

where $d_{1,t}' = \frac{\ln(z_1'/\zeta_t)+(r-||\theta||^2/2)(T-t)}{||\theta||\sqrt{T-t}}$.

In this case, as $\zeta_t \to 0$, $X_t^{1'} \to +\infty$, $\pi_t^{1'} \to \frac{\Sigma^{-1}\mu}{\alpha A}$, and $\frac{\pi_t^{1'}}{X_t^{1'}} \to 0$, while as $\zeta_t \to +\infty$, $X_t^{1'} \to 0$, $\pi_t^{1'} \to 0$, but $||\frac{\pi_t^{1'}}{X_t^{1'}}|| \to \infty$.

5.2.2 The Market Portfolio as Benchmark

Now set $B_T \equiv B_0/\zeta_T$. Portfolio value is the process

$$X_t^{2'} = \quad e^{-r(T-t)}[\frac{1}{\alpha A}(\ln \frac{\alpha A}{\lambda_2' \zeta_t} + (r - ||\theta||^2/2)(T-t) + -K/\alpha]N(d'_{3,t})$$
$$+(B_0/\zeta_t)N(d'_{5,t}) + \frac{e^{-r(T-t)}||\theta||\sqrt{T-t}}{\alpha A}N'(d'_{3,t}),$$

and the manager's optimal trading strategy is

$$\pi_t^{2'} = \{\frac{e^{-r(T-t)}N(d'_{1,t})}{\alpha A} + (B_0/\zeta_t)N(d'_{5,t}) + \frac{e^{-r(T-t)}N'(d'_{3,t})}{||\theta||\sqrt{(T-t)}}[\frac{1}{\alpha A}\ln\frac{\alpha A}{\lambda_2' z_2'} - K/\alpha + \zeta/z_2']\}\Sigma^{-1}\mu\,,$$

where $d'_{3,t} = \frac{\ln(z_2'/\zeta_t)+(r-||\theta||^2/2)(T-t)}{||\theta||\sqrt{T-t}}$ and $d'_{5,t} = d'_{3,t} + ||\theta||\sqrt{T-t}$..

Here, as $\zeta_t \to 0$, $X_t^{2'} \to +\infty$, $||\pi_t^{2'}|| \to \infty$, and $\frac{\pi_t^{2'}}{X_t^{2'}} \to 1$, while as $\zeta_t \to +\infty$, $X_t^{2'} \to 0$, $\pi_t^{2'} \to 0$, but $||\frac{\pi_t^{2'}}{X_t^{2'}}|| \to \infty$.

In all cases, when the portfolio value is very high so that the manager is deep in the money, his portfolio choice looks like the choice he would make if the performance fee were linear, that is, if he were maximizing $EU[\alpha(X_T - B_T) + K]$. For instance, with the riskless asset as benchmark and constant relative risk aversion, the proportional portfolio holdings approach the Merton constant, $\frac{\Sigma^{-1}\mu}{A}$.

The effect of the convexity of the incentive fee becomes dramatic as wealth level falls to zero. As the manager gets farther out of the money, he takes on as much risk as possible, subject to the constraint that wealth must be nonnegative (in all cases, $||\frac{\pi_t^*}{x_t^*)}|| \to \infty$, even though $\pi_t^* \to 0$). To illustrate, Figure 2 plots asset volatility as a function of asset value for the case of constant relative risk aversion and the riskless benchmark.

The manager's trading position becomes very unstable if he is near the money as the evaluation date draws near. As ζ_t vibrates around the critical point z_i, the manager's portfolio π^m oscillates between zero and a strictly positive value. Thus, small changes in the value of the market portfolio precipitate large trades as the manager alternates between the desire to gamble and the need to remain solvent.

6 Inefficiency of the Incentive Fee Contract

Though we focus primarily on the manager's investment problem, our results have some implications for contract theory. We show below that the incentive fee is an expensive way to provide the manager with a given level of expected utility because it forces the manager to bear substantial risk in order to extract value from the contract. If the client is a profit-maximizer, a simple linear fee would be more efficient. This raises the question of why corporate shareholders so commonly compensate executives with options instead of restricted shares of stock. Although the principal-agent literature does not say that convex sharing rules are necessarily best,[8] the wide-spread use of option-like compensation suggests that options have an advantage over stock that outweighs this inefficiency. Perhaps they better motivate the manager to exert effort on the client's behalf.

Corollary *There exists a linear performance fee that has lower cost to the client than the incentive fee and gives the manager greater expected utility.*

Proof *Let $p = \tilde{\mathcal{P}}\{X_T^* > 0\}$ and let $\alpha' = \alpha - pB_0/X_0$. Under any investment policy, the linear fee $\alpha' X_T$, has the same market value as the optimized incentive fee. Under the optimal policy, this fee gives the manager strictly greater expected utility. Indeed, letting $y = \alpha' X_T + K$, the manager's problem is*

$$\begin{array}{rl} \max\limits_{y} & EU(y) \\ \textit{subject to} & E\zeta_T y \leq \alpha(X_0 - pB_0) + K \\ \textit{and} & y \geq K \ . \end{array} \tag{9}$$

This is just like the incentive fee problem, (5), without the constraint on the probability of bankruptcy. Relaxing that constraint allows the manager to achieve a better policy. Reducing α' just slightly will still leave the manager better off than he is under the incentive fee and will cost the client less.

[8]The optimal shape of the sharing rule can be arbitrary. See Holmstrom and Hart (1987).

References

Aumann, R. and M. Perles, 1965, A Variational Problem Arising in Economics, *Journal of Mathematical Analysis and Applications*, 11, 488-503.

Brown, K., W. Harlow, and L. Starks, 1994, Of Tournaments and Temptations: An Analysis of Managerial Incentives in the Mutual Fund Industry, Working Paper, University of Texas at Austin.

Chevalier, J. and G. Ellison, 1995, Risk Taking by Mutual Funds as a Response to Incentives, Working Paper, University of Chicago.

Cox, J., and C. Huang, 1989, Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process, *Journal of Economic Theory*, 49, 33-83.

Cox, J., and S. Ross, 1976, The Valuation of Options for Alternative Stochastic Processes, *Journal of Financial Economics*, 3, 145-166.

Grinblatt, M. and S. Titman, 1989, Adverse Risk Incentives and the Design of Performance-Based Contracts, *Management Science*, 35, 807-822.

Harrison, J., and D. Kreps, 1979, Martingales and Arbitrage in Multiperiod Securities Markets, *Journal of Economic Theory*, 20, 381-408.

Harrison, J., and S. Pliska, 1981, Martingales and Stochastic Integrals in the Theory of Continuous Trading, *Stochastic Processes and Their Applications*, 11, 215-260.

Holmstrom, B., and O. Hart, 1987, Theory of Contracts. In *Advances in Economic Theory*, ed. Truman F. Bewley, 71-155. New York and Melbourne: Cambridge University Press.

Karatzas, I., 1989, Optimization Problems in the Theory of Continuous Trading, *SIAM J. Control and Optimization*, 27, 1221-1259.

Karatzas, I., J. Lehoczky, and S. Shreve, 1987, Explicit Solution of a General Consumption/Investment Problem, *SIAM J. Control and Optimization*, 25, 1557-1586.

Merton, R., 1969, Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case, *The Review of Economics and Statistics*, 51, 247-257.

Merton, R., 1971, Optimum Consumption and Portfolio Rules in a Continuous-Time Model, *Journal of Economic Theory*, 51, 373-413.

Pliska, S., 1986, A Stochastic Calculus Model of Continuous Trading: Optimal Portfolios, *Mathematics of Operations Research*, 11, 371-382.

Rockafellar, R. T. (1970), *Convex Analysis*. Princeton, New Jersey: Princeton University Press.

Sirri, E., and P. Tufano, 1992, The Demand for Mutual Fund Services by Individual Investors, Working Paper, Harvard University.

Starks, L., 1987, Performance Incentive Fees: An Agency Theoretic Approach, *Journal of Financial and Quantitative Analysis*, 22, 17-32.

Figure 1

The Optimal Policy for the Manager's Problem

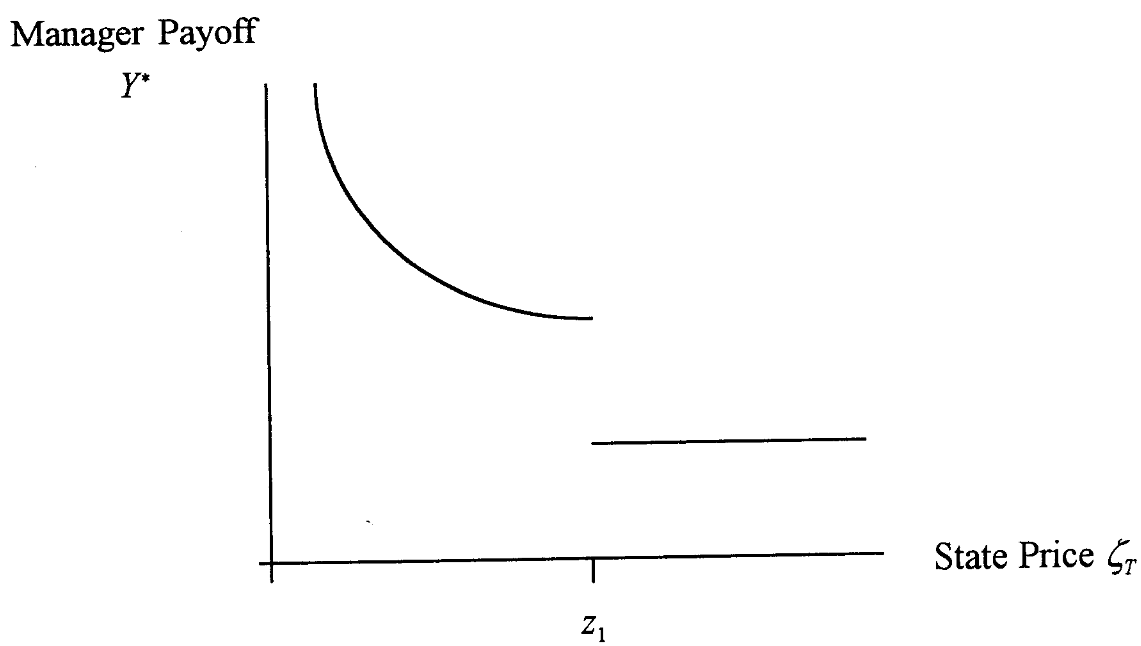

Figure 2

Asset Volatility vs. Asset Value