The Valuation of American-style Swaptions in a Two-factor Spot-Futures Model[1]

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Abstract

The Valuation of American-style Swaptions in a Two-factor Spot-Futures Model.

We build a no-arbitrage model of the term structure of interest rates using two stochastic factors, the short-term interest rate and the premium of the futures rate over the short-term interest rate. The model provides an extension of the lognormal interest rate model of Black and Karasinski (1991) to two factors, both of which can exhibit mean-reversion. The method is computationally efficient for several reasons. First, the model is based on *Libor* futures prices, enabling us to satisfy the no-arbitrage condition without resorting to iterative methods. Second, we modify and implement the binomial approximation methodology of Nelson and Ramaswamy (1990) and Ho, Stapleton and Subrahmanyam (1995) to compute a multiperiod tree of rates with the no-arbitrage property. The method uses a recombining two-dimensional binomial lattice of interest rates that minimizes the number of states and term structures over time. In addition to these computational advantages, a key feature of the model is that it is consistent with the observed term structure of futures rates as well as the term structure of volatilities implied by the prices of interest rate caps and floors. We use the model to price European-style, Bermudan-style, and American-style swaptions. These prices are shown to be highly sensitive to the existence of the second factor and its volatility characteristics.

1   Introduction

Satisfactory models exist for the pricing of interest-rate dependent derivatives in a single-factor context, where interest rates of various maturities are perfectly correlated. For example, assuming that the short-term interest rate follows a mean-reverting process, Jamshidian (1989) prices options on coupon bonds using an extension of the Vasicek (1977) model. Also, assuming a lognormal process, Black, Derman and Toy (1990) and Black and Karasinski (1991) use a binomial tree of interest rates to price interest-rate derivatives. However, these models, by definition, are not capable of accurately pricing derivatives, such as swaptions and yield-spread options, whose payoffs are sensitive to the shape as well as the level of the term structure. In principle, these options require at least a two-factor model of the interest rate process for pricing and hedging.[1]

One promising approach, used extensively in recent work, has been to build multi-factor forward-rate models of the Heath, Jarrow and Morton (1992) (HJM) type. Since the HJM paper, the required no-arbitrage property of these models has been well known. However, this approach has some drawbacks for the pricing of swaptions and bond options. Most tractable applications require restrictive assumptions on the volatility structure of the forward rates to ensure that the Markov property is satisfied, and for the resulting model to be computable for realistic examples. Hence, while in principle, the forward-rate approach provides a solution, in practice, it is difficult to implement except for certain special cases.

In this paper we present an alternative, no-arbitrage, model based on the London Interbank Offer Rate (*Libor*) futures. By modeling both the *Libor* spot and futures rates, we generate a two-dimensional process for the term structure. We assume that the process for *Libor* is lognormal and that the current term structure of *Libor* futures is given. We derive the no-arbitrage restrictions for such a model, and then approximate the bivariate lognormal diffusion process with a bivariate binomial distribution using a modification of the well known Nelson-Ramaswamy (1990) technique.[2] Since we assume that the *Libor* rate is lognormal and mean reverting, our model can also be seen as an extension of the Black and Karasinski (1991), Brace, Gatarek, and Musiela (1997) (BGM) and Miltersen, Sandmann and Sondermann (1997) (MSS) models. We illustrate the model using realistic examples with a large number of time periods. The computational efficiency is achieved through the

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[1] For a critique of existing methods for the valuation of swaptions see Longstaff, Santa-Clara and Schwartz (1999). Of course, one-factor models are adequate for the valuation of European-style options on the short-term interest rate, such as interest-rate caps and floors.

[2] See Nelson and Ramaswamy (1990) and the multivariate generalisation of Ho, Stapleton and Subrahmanyam (1995)

use of a two-dimensional recombining lattice of interest rates.[3]

We take as given the prices (or equivalently, the implied volatilities) of European-style interest rate caps and floors for all maturities. The problem, as in Black, Derman and Toy (1990) and Black and Karasinski (1991), is to price European-style, Bermudan-style and American-style swaptions, given the prices of the caps and floors.

The computational method introduced to approximate the model builds on previous work by Nelson and Ramaswamy (1990) and Ho, Stapleton and Subrahmanyam (1995) (HSS). Nelson and Ramaswamy approximate a single-variable diffusion with a 'simple' binomial tree, i.e., a binomial tree with the recombining node property. HSS extend this method to multiple, correlated variables in the case of log-normal diffusion processes. In the context of a two-factor interest rate model, preservation of the no-arbitrage condition in a simple bivariate tree requires a modification of this methodology. Although in our model the futures premium is contemporaneously independent of the spot *Libor*, expectations of subsequent spot rates are determined by the futures rate. In a modification of the HSS method, capture this dependence, and hence the no-arbitrage property, in a non-exploding tree structure, by allowing the probabilities of moving up or down to depend upon the outcomes of both the spot and the futures *Libor*.

The outline of the paper is as follows. Section 2 reviews the literature on term-structure models and their relationship to the model developed here. Section 3 presents the spot-futures model, derives its no-arbitrage properties, and discusses its input requirements. Section 4 derives the methodology for approximating the two-dimensional diffusion process for the spot *Libor*. Section 5 establishes the convergence properties of the approximation and presents the results of applying the model to the valuation of Bermudan-style bond options, European-style, Bermudan-style and American-style swaptions. Section 6 concludes with a discussion of the remaining issues of calibration, empirical parameter estimation, and possible extensions of the research.

2 Term-structure Models

In early attempts to value interest rate options, Brennan and Schwartz (1979) and Courtadon (1982) derive equilibrium models of the term structure along the lines of the Vasicek (1977) model. However, since the contribution of Ho and Lee (1986), it has been recognized that interest rate dependent claims can be priced within a no-arbitrage model. Hull and

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[3]Also, since the model is calibrated to the given term structure of futures rates, we avoid the use of iterative methods normally used to calibrate models to the current term structure

White (1994), for example, develop an extended Vasicek model in which interest rates, under the risk-neutral measure, are Gaussian, and exactly match the current term structure. Black, Derman and Toy (1990) and Black and Karasinski (1991) develop lognormal diffusion models for the short rate that have the same no-arbitrage property. Our model follows this no-arbitrage approach; however, in contrast to previous models, we start with the term structure of *futures* rates. We show that the no-arbitrage property is satisfied in a model where *Libor* futures are modelled as a martingale process under the risk-neutral measure. The other difference is that the resulting spot-futures model is based on two factors.

In a no-arbitrage framework, HJM model the evolution of forward rates for various maturities. A similar approach has recently been used in the so-called market model of BGM and MSS. These papers, like this one, model the *Libor*. Since futures rates and forward rates are closely related, our modelling approach can be compared to these papers. In contrast to these reduced-form models where the behaviour of forward rates is exogenous, our model is a structural-type model, where only the behaviour of the short (*Libor*) rate and the premium of the first futures rate over the short rate is exogenous. Although it is possible to develop multifactor forward rate models in the HJM framework, these often require restrictive assumptions to guarantee the Markov property, and the use of Monte-Carlo simulation. Otherwise, the forward rate models would involve complex iterative calculations. The advantage of our methodology is that it is implementable in seconds, for quite general volatility structure assumptions.

A number of authors, in particular Hull and White (1994), Balduzzi, Das and Foresi (1998) and Stapleton and Subrahmanyam (1999), have developed two-factor spot-rate models where the second factor is a shock to the conditional mean of the spot rate. Hull and White define a general class of such models; however, they only implement certain special cases of the class, where the term structure of volatility is restricted. Our incremental contribution is to implement a model where the *Libor* is lognormal, as in Stapleton and Subrahmanyam (1999), and the volatility stucture is general, in contrast to Hull and White.

The lognormal models of Black, Derman and Toy (1990) and Black and Karasinski (1991) are perhaps closest to the model developed in this paper. These papers derive recombining, binomial lattices which match yield volatilities and cap-floor volatilities respectively. In a sense, our model can be viewed as a two-factor extension of the Black and Karasinski model. In their model, the local (conditional) volatilities and the mean reversion of the short rate are given, in addition to the current term structure of zero-coupon bond prices. They build a recombining binomial tree of rates, consistent with this market information, using a technique whereby the length of the time period is changed to accommodate mean reversion and changing local volatilities. Unfortunately, as pointed out by Amin (1991), this 'trick' only works, in general, for a one-factor model. In this paper, we therefore employ

the changing probability technique of Nelson and Ramaswamy (1990), extended to multiple variables by HSS. We are thus able to generalise the Black-Karasinski model to two factors whilst maintaining the recombining property.

One recent paper that deals with the pricing of American-style and Bermudan-style swaptions is by Longstaff, Santa-Clara and Schwartz (1999). Their paper emphasizes the importance of including multiple factors in a pricing model for these claims. Our results support their conclusion. While our analysis only allows for two factors, we are able to price the contingent claims in a much faster, more efficient way, without resorting to the use of Monte-Carlo simulation.[4]

In summary, our model uses many features of previous term-structure models. However, the use and modelling of *Libor* futures in our paper, and the efficient computation that it allows, justifies the introduction of yet another term-structure model into the literature.

3 The Two-factor Model

In this section, we describe our two-factor model and investigate the implications of the no-arbitrage conditions for the model. We first discuss briefly the general approach in the lemmas and propositions that follow. Since our approach involves the calibration of the model using observable futures rates, we first establish the linkage between the spot and futures rates. The key to developing such a link is the observation that in an arbitrage-free economy, futures prices are the expectation, under the risk-neutral measure, of the future spot prices. The other relationship we use is the expression for the mean of the spot interest rate process, based on the assumption of lognormality of the spot interest rate. These restrictions allow us to re-formulate the spot rate process in terms of futures rates. Having specified the spot-rate process, we then derive the process for the one-period futures rate, using similar methods.

The logic of the argument is as follows. First, we show in Lemma 1 that the futures interest rate is the expectation, under the risk-neutral measure, of the future spot interest rate. Since the spot rate is lognormally distributed, the futures rate can be related to the mean and variance of the (log) spot interest rate. Second, in Lemma 2, the spot interest rate process is expressed in terms of observable parameters by taking the expectation and substituting for the futures rate expressed as the mean of the spot interest rate. Third, in Lemma 3, a cross-sectional relation is derived between futures and spot rates. These

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[4]Generally speaking, Monte-Carlo simulation is both inaccurate and slow. Hence, it is used only as a last resort, in most computational problems.

results are combined in Proposition 1 with the requirement that forward bond prices are the expectation, under the risk-neutral measure of the future bond prices. Proposition 1 summarises the no-arbitrage requirements of the model.

3.1   No-arbitrage properties of the model

As several authors have noted, one way of introducing a second factor into a spot-rate model of the term structure is to assume that, under the risk-neutral measure, the conditional mean of the spot short-term interest rate is stochastic.[5] In this paper, we take a similar approach. We assume that the logarithm of the short-term interest rate follows a discrete process with a stochastic conditional mean.

We define the short-term, m-year interest rate, on a *Libor* basis as $r_t = [(1/B_{t,t+m}) - 1]/m$, where m is a fixed maturity of the short rate and $B_{t,t+m}$ is the price of a m-year, zero-coupon bond at time t. We then assume that, under the risk-neutral measure, this rate follows the process:

$$\ln(r_t) - \ln(r_{t-1}) = \theta_r(t) - b\ln(r_{t-1}) + \ln(\pi_{t-1}) + \varepsilon_t, \tag{1}$$

where

$$\ln(\pi_t) - \ln(\pi_{t-1}) = \theta_\pi(t) - c\ln(\pi_{t-1}) + \nu_t,$$

and ε_t and ν_t are independently distributed, normal, random variables. π is a shock to the conditional mean of the process, $\theta_r(t)$ and $\theta_\pi(t)$ are time-dependent constants, and b and c are the mean reversion coefficients of r and π respectively. The mean and the unconditional standard deviation of the logarithm of the two factors, r_t and π_t are μ_{r_t}, σ_{r_t} and μ_{π_t}, σ_{π_t} respectively. We assume that the trading interval is one day, and that the *Libor* follows the process in (1) under the daily (rather than the continuous) risk-neutral measure. From here on, we refer to this 'daily' risk-neutral measure as simply *the* risk-neutral measure. We also assume, without loss of generality, that $E(\pi_t) = 1$, where the expectation is again taken under the risk-neutral measure.[6]

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[5]See for example Hull and White (1994), Balduzzi, Das and Foresi (1998), and Jegadeesh and Pennacchi (1996).

[6]Note that the assumed process in equation (1) is the discrete form of the process

$$d\ln(r) = [\theta_r(t) - b\ln(r) + \ln(\pi)]dt + \sigma_r(t)dz_1 \tag{2}$$

where

$$d\ln(\pi) = [\theta_\pi(t) - c\ln(\pi)]dt + \sigma_\pi(t)dz_2$$

The model in equation (1) is attractive because the second factor π is closely related to the futures rate, which is observable. In fact, as we shall show in Appendix A, the futures *Libor* is the expectation of r_t under the risk-neutral measure. Hence, the model lends itself to calibration given market inputs. To see this, we first derive some of the implications of the process assumed in equation (1), in a no-arbitrage economy.

We now state and prove a result that is central to the paper. The result is not new, since a similar result is derived by Sundaresan (1991), and used by MSS (1997) and BGM (1997). However, since it is crucial to the model developed in this paper, we include the proof in Appendix A. The lemma states that, given the definition of the *Libor* futures contract, the futures *Libor* is the expected value of the spot rate, under the risk-neutral measure.

Lemma 1 (Futures Libor) *In a no-arbitrage economy, the time-t futures Libor, for delivery at T, is the expected value, under the risk-neutral measure, of the time-T spot Libor, i.e.*

$$f_{t,T} = E_t(r_T)$$

Also, if r_T is lognormally distributed under the risk-neutral measure, then:

$$\ln(f_{t,T}) = E_t[\ln(r_T)] + \frac{\text{var}_t[\ln(r_T)]}{2},$$

where the operator var *refers to the variance under the risk-neutral measure.*

Proof

See Appendix A.

Lemma 1 allows us to substitute the futures rate directly for the expected value of the *Libor* in the process assumed for the spot rate. In particular, the futures rate has a zero drift, under the risk-neutral measure. We now use this result to solve for the constant parameters in our interest rate process in (1), i.e., to determine the constants $\theta_r(t)$ and $\theta_\pi(t)$. We have,

Lemma 2 (Spot-Libor Process) *Suppose that the short-term interest rate follows the process in equation (1), under the risk-neutral measure, in a no-arbitrage economy. Then,*

In the above equations, $d\ln(r)$ is the change in the logarithm of the short rate, and $\sigma_r(t)$ is the instantaneous volatility of the short rate. The second factor, π, itself follows a diffusion process with mean θ_π, mean reversion c and instantaneous volatility $\sigma_\pi(t)$. dz_1 and dz_2 are standard Brownian motions. If the short rate follows the process in equation (2), it is lognormal over any discrete time period. This is one of the cases considered by Hull and White (1994). Note that the continuous-time process is defined under the continuous-time risk-neutral measure which is different from the 'daily' measure used in this paper.

since $f_{0,t} = E_0(r_t)$, $\forall t$, the short rate process can be specified as

$$\ln(r_t) - \ln(f_{0,t}) = \alpha_{r_t} + [\ln(r_{t-1}) - \ln(f_{0,t-1})](1-b) + \ln(\pi_{t-1}) + \varepsilon_t \tag{3}$$

where

$$\ln(\pi_t) = \alpha_{\pi_t} + \ln(\pi_{t-1})(1-c) + \nu_t,$$

with

$$\alpha_{r_t} = -\sigma_{r_t}^2/2 + (1-b)\sigma_{r_{t-1}}^2 + \sigma_{\pi_{t-1}}^2,$$

and

$$\alpha_{\pi_t} = -\sigma_{\pi_t}^2/2 + (1-c)\sigma_{\pi_{t-1}}^2/2.$$

Proof

See Appendix B.

The result in Lemma 2 is crucial to the implementation of the model developed in this paper, since it defines the parameters of the two-factor interest rate process in terms of potentially observable quantities. The process for the *Libor* depends upon the current futures rates and the volatilities of the *Libor* and of the premium factor. Lemma 2 implies that if the no-arbitrage condition is to be satisfied, the drift of the spot rate process has to reflect the futures *Libor* at time 0 and the volatilities. This is analogous to the no-arbitrage requirement in the HJM model, where the absence of arbitrage implies that the drift of the forward rate depends on the volatility of the forward rates. In our spot rate lognormal model, the volatilities of the spot rate and of the premium factor play a similar role.

However, the condition used in Lemma 2, that $E_0(r_t) = f_{0,t}$, is necessary, but not sufficient, for "no-arbitrage" in our spot-futures model. The "no-arbitrage" requirement is much stronger. From Lemma 1, no-arbitrage requires that the futures *Libor* equals the expected spot rate at *each* date and in *each* state. We then have the following:

Lemma 3 (Futures-Libor Process) *Given that the conditions of Lemma 2 are satisfied and hence that*

$$\ln(r_t) - \ln(f_{0,t}) = \alpha_{r_t} + [\ln(r_{t-1}) - \ln(f_{0,t-1})](1-b) + \ln(\pi_{t-1}) + \varepsilon_t$$

where

$$\ln(\pi_t) = \alpha_{\pi_t} + \ln(\pi_{t-1})(1-c) + \nu_t,$$

and ϵ_t is normally distributed, then the no-arbitrage condition implies

$$\ln(f_{t,t+1}) - \ln(f_{0,t+1}) = \alpha_{f_{t+1}} + [\ln(r_t) - \ln(f_{0,t})](1 - b) + \ln(\pi_t) \tag{4}$$

where

$$\alpha_{f_{t+1}} = \alpha_{r_{t+1}} + \frac{\sigma(t+1)^2}{2} + \ln(f_{0,t+1}).$$

Proof

See Appendix C.

Lemma 3 shows that, in a no-arbitrage economy where the spot rate follows (3), the first futures contract has a rate that follows a two-factor process. The futures rate moves with changes in the spot rate, and in response to the premium factor, π. The futures rate is also affected by the degree of mean reversion in the short rate process. We can interpret the volatility of the premium factor as the part of the volatility of the first futures rate that is not explained by the spot rate.[7]

So far, we have concentrated on the implications of the no-arbitrage condition for the spot-rate process and for futures rates. However, any term-structure model must also satisfy the condition that, under the risk-neutral measure, forward bond prices must equal the expected values of the subsequent period's bond price. This condition is therefore included in the following proposition that summarises the no-arbitrage conditions of our model.

Proposition 1 (No-Arbitrage Properties of the Model) *Suppose that the Libor rate, r_t follows the process:*

$$\ln(r_t) - \ln(r_{t-1}) = \theta_r(t) - b \ln(r_{t-1}) + \ln(\pi_{t-1}) + \varepsilon_t,$$

where

$$\ln(\pi_t) - \ln(\pi_{t-1}) = \theta_\pi(t) - c \ln(\pi_{t-1}) + \nu_t,$$

under the risk-neutral measure, with $E(\pi_t) = 1, \forall t$, and ε_t and ν_t are independently distributed, normal variables. Then, if the model is arbitrage free:

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[7]It is natural to concentrate on the first futures rate, i.e. the futures for delivery at time $t + 1$, since in our spot-rate model, the first futures rate is the expected value of the subsequent spot rate, r_{t+1}. However, it is possible to solve the time-series model for the kth futures rate. Using results from Stapleton and Subrahmanyam (1999), Lemma 1, we have

$$\ln(f_{t,t+k}) - \ln(f_{0,t+k}) = \alpha_{f_{t+k}} + [\ln(r_t) - \ln(f_{0,t})](1 - b)^k + V_t A_{t,k}$$

where V_t is a weighted sum of the innovations in the premium factor, and $A_{t,k}$ is a constant. Hence the kth futures *Libor* also follows a two-factor process similar to that followed by the first futures *Libor*.

1. *the spot-Libor process can be written as:*

$$\ln(r_t) - \ln(f_{0,t}) = \alpha_{r_t} + [\ln(r_{t-1}) - \ln(f_{0,t-1})](1-b) + \ln(\pi_{t-1}) + \varepsilon_t,$$

2. *the process for the 1-period futures-Libor can be written as:*

$$\ln(f_{t,t+1}) - \ln(f_{0,t+1}) = \alpha_{f_{t+1}} + [\ln(r_t) - \ln(f_{0,t})](1-b) + \ln(\pi_t),$$

3. *zero-coupon bond prices are given by the relation:*

$$B_{s,t} = B_{s,s+1} E_s(B_{s+1,t}),\ 0 \le s < t \le T.$$

Proof

Parts 1 and 2 of the proposition follow from Lemmas 2, 3. As shown by Pliska (1997), Part 3 is a requirement of any no-arbitrage model. □

Proposition 1 summarises the conditions that have to be met for the spot-futures model to be arbitrage-free. Also, as noted above, the further implication of Lemma 1, is that the futures rate is a martingale, under the risk-neutral measure. Hence, we can easily calibrate the model to the given term structure of futures rates, and thereby guarantee that the no-arbitrage property holds.

Finally, for completeness, we should note that the process followed by the spot and futures rates in this model can be written in difference form:-

Corollary 1 (The Bi-Variate Spot-Futures Process) *The bi-variate process for the spot-Libor and the one-period futures-Libor can be written as*

$$\Delta\ln(r_t) = \alpha'_{r_t} - b\,\ln(r_{t-1}) + \ln(\pi_{t-1}) + \varepsilon_t$$

$$\Delta\ln(f_{t,t+1}) = \alpha'_{f_t} + [\ln(r_t) - \ln(r_{t-1})](1-b) + \ln(\pi_t) - \ln(\pi_{t-1}),$$

for some constants α'_{r_t} and α'_{f_t}.

<u>*Proof*</u>

Write equation (3) for r_{t+1} and for r_t and subtract the second equation from the first. Then the first part of the corollary follows with

$$\alpha'_{r_t} = \alpha_{r_{t+1}} - \alpha_{r_t} - (1-b)\ln(f_{0,t}) + (1-b)\ln(f_{0,t-1}).$$

Similarly, write equation (4) for $f_{t+1,t+2}$ and for $f_{t,t+1}$ and subtract the second equation from the first. Then the second part of the corollary follows with

$$\alpha'_{f_t} = \alpha_{f_{t+1}} - \alpha_{f_t} - (1-b)\ln(f_{0,t}) + (1-b)\ln(f_{0,t-1}).$$

$\square$

The first part of the corollary shows that the spot rate follows a one dimensional mean-reverting process. The second part shows that the 1-period futures rate follows a two-dimensional process, depending partly on the change in the spot rate and partly on the change in the premium factor.

3.2   Regression Properties of the Model

The two-factor model of the term structure described above has the characteristic that the conditional mean of the short rate is stochastic, as does the Hull and White (1994) model. Since the futures rate directly depends on the conditional mean, there is an imperfect correlation between the short rate and the futures rate. In this section, we establish the regression properties of the model, using the covariances of the short rate and premium process. These properties are required as inputs for the construction of a binomial approximation model of the term structure. In the following proposition, we denote the covariance of the logarithm of the short rate and the premium factor as σ_{r_t,π_t}. The process assumed in Lemma 2 has the following properties:

Proposition 2 (Multiple Regression Properties) *Assume that*

$$\ln(r_t) - \ln(f_{0,t}) = \alpha_{r_t} + [\ln(r_{t-1}) - \ln(f_{0,t-1})](1-b) + \ln(\pi_{t-1}) + \varepsilon_t$$

where

$$\ln(\pi_t) = \alpha_{\pi_t} + \ln(\pi_{t-1})(1-c) + \nu_t,$$

with $E_0(\pi_t) = 1$, $\forall t$.

Then,

1. *the multiple regression*

$$\ln\left[\frac{r_t}{f_{0,t}}\right] = \alpha_{r_t} + \beta_{r_t}\ln\left[\frac{r_{t-1}}{f_{0,t-1}}\right] + \gamma_{r_t}\ln(\pi_{t-1}) + \varepsilon_t \tag{5}$$

has coefficients

$$\alpha_{r_t} = (-\sigma_{r_t}^2 + \beta_{r_t}\sigma_{r_{t-1}}^2 + \gamma_{r_t}\sigma_{\pi_{t-1}}^2)/2$$

$$\beta_{r_t} = (1 - b)$$

$$\gamma_{r_t} = 1$$

2. *the regression*

$$\ln(\pi_t) = \alpha_{\pi_t} + \beta_{\pi_t}\ln(\pi_{t-1}) + \nu_t \tag{6}$$

has coefficients

$$\alpha_{\pi_t} = [-\sigma_{\pi_t}^2 + \sigma_{\pi_{t-1}}^2(1 - c)]/2$$

$$\beta_{\pi_t} = (1 - c)$$

3. *the conditional variance of* $ln(r_t)$ *is given by*

$$\mathrm{var}_{t-1}(\varepsilon_t) = \sigma_{r_t}^2 - (1-b)^2\sigma_{r_{t-1}}^2 - \sigma_{\pi_{t-1}}^2 - 2(1-b)\sigma_{r_{t-1},\pi_{t-1}},$$

where $\sigma_{r,\pi}$ *denotes the annualized covariance of the logarithms of the short rate and the premium factor.*

Proof

See Appendix D.

Note that we require the multiple regression coefficients $\alpha_{r_t}, \beta_{r_t}$, and γ_{r_t} in order to build the binomial approximation of the multi-variate process, using our modification of the method of Ho, Stapleton and Subrahmanyam (1995). From Part 1 of the proposition, the β_r coefficients simply reflect the mean-reversion of the short rate. The γ_r coefficients are all unity, reflecting the one-to-one relationship between π, the futures premium factor and the expected spot rate. The α_r coefficients reflect the drift of the lognormal distribution, which depends on the variances of the variables. Part 2 of the proposition shows that the regression relation for π_t is a simple regression, where the β_π coefficients reflect the constant mean reversion of the premium factor. Lastly, Part 3 of the proposition gives an expression for the conditional variance of the logarithm of the short rate.

3.3   Determining the Volatility Inputs of the Model

In order to build the model outlined above, we need the parameters of the premium process, as well as those for the short rate process itself. The result in Proposition 2, part 3 gives the relationship of the conditional volatility of the short rate to the unconditional volatilities of the short rate, the volatility of the premium factor, and the mean reversion of the short rate. We assume that the unconditional volatilities of the short rate are given, for example, observable from caplet/floorlet volatilities, and that the mean reversion is also given. The premium process, π_t, on the other hand, determines the extent to which the first futures rate differs from the spot rate in the model. Note that it is the first futures rate that is relevant, since it is this futures rate that determines the expectation of the subsequent spot rate, in the model. Since the premium factor is not directly observable, we need to be able to estimate the mean and volatility of the premium factor from the behavior of futures rates. In order to discuss this, we first establish the following general result:

Lemma 4 *Assume that*

$$\ln(r_t) - \mu_{r_t} = [\ln(r_{t-1}) - \mu_{r_{t-1}}](1 - b) + \ln(\pi_{t-1}) + \varepsilon_t$$

where

$$\ln(\pi_t) - \mu_{\pi_t} = \ln(\pi_{t-1} - \mu_{\pi_{t-1}})(1 - c) + \nu_t,$$

with $E_0(\pi_t) = 1$, $\forall t$, then the conditional volatility of π_t is given by

$$\sigma_\pi(t) = [\sigma_x^2(t) - (1 - b)^2 \sigma_r^2(t)]^{\frac{1}{2}}$$

where $x = E_t(r_{t+1})$ and $\mathrm{var}_{t-1}[\ln(x_t)] = \sigma_x^2(t)$.

Proof

See Appendix E.

Lemma 4 relates the volatility of the premium factor to the volatility of the conditional expectation of the short rate. To apply this in the current context, we first assume that the short rate follows the process assumed in the lemma under the risk-neutral process. We then use the fact that the unconditional expectation of the $t + 1$ th rate is $f_{t,1} = E_t(r_{t+1})$, i.e., the first futures (or forward) rate is the expected value of the next period spot rate. This implication of no-arbitrage leads to

$$\ln f_{t,1} = \mu_{r_{t+1}} + [\ln r_t - \mu_{r_t}](1 - b) + \ln \pi_{t-1}(1 - c) + \nu_t + \sigma_r^2(t)/2. \qquad (7)$$

It follows that the conditional logarithmic variance of the first futures rate is given by the relationship

$$\sigma_f^2(t) = (1 - b)^2 \sigma_r^2(t) + \sigma_\pi^2(t). \qquad (8)$$

Hence, the volatility of the premium factor is potentially observable from the volatility of the first futures rate. This, in turn, could be estimated empirically or implied from the prices of options on the *Libor* futures rate.

4   The Multivariate-Binomial Approximation of the Process

In order to implement the model with a binomial approximation, we need to construct a recombining lattice for the spot rate, r_t, and the futures rate, $f_{t,t+1}$. A number of methods have been suggested in the literature. For example, Hull and White (1994) use a trinomial tree, but they assume a special case of non-time-dependent volatility, which is not realistic, in general. Amin (1991) and Black and Karasinski (1991) redefine the time interval between points on the grid to cope with changing local volatility. However, as noted by Amin (1991), this technique only works in the univariate case, or when the volatility functions and mean reversions are the same for each variable. In his multivariate implementation, Amin (1991) assumes time-independent volatilities. Nelson and Ramaswamy (1990) use a transformation of the process and state-dependent probabilities, to approximate a univariate diffusion. In an extension to multivariate diffusions, and in the special case, relevant here, of lognormal diffusions, Ho, Stapleton and Subrahmanyam (1995) use the regression properties of the multivariate diffusion to compute the appropriate probabilities of up-moves on the multivariate binomial tree. This allows them to capture both the time series and cross-sectional properties of the process. In this section we use a modification of their methodology.

4.1   The HSS approximation

The general approach we take to building a bivariate-binomial lattice, representing a discrete approximation of the process in equation (2), is to construct two separate recombining

binomial trees for the short-term interest rate and the futures-premium factors. The no-arbitrage property and the covariance characteristics of the model are then captured by choosing the conditional probabilities at each node of the tree. The recombining nature of the bi-variate tree is illustrated in Figure 1 for a two-period example and in Figure 2 for a three-period example. As shown in the figures, there are two possible outcomes emanating from each node. However, since the tree is required to recombine, it does not result in an explosive state space.

We now outline our method for approximating the two-factor process interest rate process, described above. We use three types of inputs: first, the unconditional means of the short-term rate, $E_0(r_t)$, $t = 1, ..., T$, second, the volatilities of ε_t, i.e., the conditional volatility of the short rate, given the previous short rate and the previous futures rate, denoted by $\sigma_r(t)$, and the conditional volatilities of the premium, denoted by $\sigma_\pi(t)$, and third, estimates of the mean reversion of the short rate, b, and the mean reversion, of the premium factor, c. The process in (3) is then approximated using an adaptation of the methodology described in Ho, Stapleton and Subrahmanyam (1995) (HSS). HSS show how to construct a multiperiod multivariate-binomial approximation to a joint-lognormal distribution of M variables with a recombining binomial lattice. However, in the present case, we need to modify the procedure, allowing the expected value of the interest rate variable to depend upon the premium factor. That is, we need to model the two variables r_t and π_t, where r_t depends upon π_{t-1}. Furthermore, in the present context, we need to implement a multiperiod process for the evolution of the interest rate, whereas HSS only implement a two-period example of their method. In this section, these modifications and the resulting multiperiod algorithm are presented in detail.

We divide the total time period into T periods of equal length of m years, where m is the maturity period, in years, of the short-term interest rate. Over each of the periods from t to $t + 1$, we denote the number of binomial time steps, termed the *binomial density*, by n_t. Note that, in the HSS method, n_t can vary with t allowing the binomial tree to have a finer density, if required for accurate pricing, over a specified period. This might be required, for example, if the option exercise price changes between two dates, increasing the likelihood of the option being exercised, or for pricing barrier options.

We use the following result, adapted from HSS:

Proposition 3 (Approximation of a Two-factor Lagged Diffusion Process) *Suppose that X_t, Y_t follows a joint lognormal process, where $E_0(X_t) = 1, E_0(Y_t) = 1 \ \forall t$, and where*

$$E_{t-1}(x_t) = a_x + bx_{t-1} + y_{t-1}$$

$$E_{t-1}(y_t) = a_y + cy_{t-1}.$$

Let the conditional logarithmic standard deviation of Z_t be $\sigma_z(t)$ for $Z = (X, Y)$. If Z_t is approximated by a log-binomial distribution with binomial density $N_t = N_{t-1} + n_t$ and if the proportionate up and down movements, u_{z_t} and d_{z_t} are given by

$$d_{z_t} = \frac{2}{1 + \exp(2\sigma_z(t)\sqrt{1/n_t})}$$

$$u_{z_t} = 2 - d_{z_t}$$

and the conditional probability of an up-move at node r of the lattice is given by

$$q_{z_t} = \frac{E_{t-1}(z_t) + (N_{t-1} - r)\ln(u_{z_t}) - (n_t + r)\ln(d_{z_t})}{n_t[\ln(u_{z_t}) - \ln(d_{z_t})]}$$

then the unconditional mean and conditional volatility of the approximated process approach their true values, i.e., $\hat{E}_0(Z_t) \to 1$ and $\hat{\sigma}_{z_t} \to \sigma_{z_t}$ as $n \to \infty$.

Proof

If $E_0(Z_t) = 1$, $\forall t$, then we obtain the result as a special case of HSS(1995), Theorem 1.□

4.2   Computing the nodal values

In this section, we first describe how the vectors of the short-term rates and the premium factor are computed. We approximate the process for the short-term interest rate, r_t, with a binomial process, i.e., moves up or down from its expected value, by the multiplicative factors d_{r_t} and u_{r_t}. Following HSS, equation (7), these are given by

$$d_{r_t} = \frac{2}{1 + \exp(2\sigma_r(t)\sqrt{1/n_t})}$$

$$u_{r_t} = 2 - d_{r_t}.$$

We then build a *separate* tree of the futures premium factor π. The up-factors and down-factors in this case are given by

$$d_{\pi_t} = \frac{2}{1 + \exp(2\sigma_\pi(t)\sqrt{1/n_t})}$$

$$u_{\pi_t} = 2 - d_{\pi_t}.$$

At node j at time t, the interest rates r_t and premium factors π_t are calculated from the equations

$$
\begin{aligned}
r_{t,j} &= u_{r_t}^{(N_t-j)} d_{r_t}^j E_0(r_t), \\
\pi_{t,j} &= u_{\pi_t}^{(N_t-j)} d_{\pi_t}^j, \\
j &= 0, 1, ..., N_t,
\end{aligned}
\tag{9}
$$

where $N_t = \sum_t n_t$. In general, there are $N_t + 1$ nodes, i.e., states of r_t and π_t, since both binomial trees are recombining. Hence, there are $(N_t + 1)^2$ states after t time steps.

4.3   Computing the conditional probabilities

In general, as in Hull and White (1994), the covariance of the two approximated diffusions may be captured by varying the conditional probabilities in the binomial process. Since the trees of the rates and the futures premium are both recombining, the time-series properties of each variable must also be captured by adjusting the conditional probabilities of moving up or down the tree, as in HSS and in Nelson and Ramaswamy (1990). Since, increments in the premium variable are independent of r_t, this is the simplest variable to deal with. Using the results of Proposition 2, we compute the conditional probability using HSS, equation (10). In this case the probability of a up-move, given that π_{t-1} is at node j, is

$$
q_{\pi_t} = \frac{\alpha_{\pi_t} + \beta_{\pi_t} \ln \pi_{t-1,j} - (N_{t-1} - j)\ln u_{\pi_t} - (j + n_t)\ln d_{\pi_t}}{n_t \left(\ln u_{\pi_t} - \ln d_{\pi_t}\right)}
\tag{10}
$$

where

$$
\beta_{\pi_t} = (1 - c)
$$

$$
\alpha_{\pi_t} = (-\sigma_{\pi_t}^2 + \beta_{\pi_t}\sigma_{\pi_{t-1}}^2)/2
$$

and where b is the coefficient of mean reversion of π, and $\sigma_{\pi_t}^2$ is the unconditional logarithmic variance of π over the period $(0 - t)$.

The key step in the computation is to fix the conditional probability of an up-movement in the rate r_t, given the outcome of r_{t-1}, the mean reversion of r, and the value of the premium factor π_{t-1}. In discussing the multiperiod, multi-factor case, HSS present the formula for the conditional probability when a variable x_2 depends upon x_1 and a contemporaneous

variable, y_2. Again using the regression properties derived in Proposition 2, and adjusting HSS, equation (13) to the present case, we compute the probability

$$q_{r_t} = \frac{\alpha_{r_t} + \beta_{r_t} \ln(r_{t-1,j}/E(r_{t-1})) + \gamma_{r_t} \ln \pi_{t-1,j} - (N_{t-1} - j) \ln u_{r_t} - (j + n_t) \ln d_{r_t}}{n_t (\ln u_{r_t} - \ln d_{r_t})} \qquad (11)$$

where

$$\beta_{r_t} = (1 - b)$$
$$\gamma_{r_t} = 1$$
$$\alpha_{r_t} = [-\sigma_{r_t}^2 + \beta_{r_t} \sigma_{r_{t-1}}^2 + \gamma_{r_t} \sigma_{\pi_t}^2]/2.$$

Then, by Proposition 3, the process converges to a process with the given mean and variance inputs.

4.4 The multiperiod algorithm

HSS(1995) provide the equations for the computation of the nodal values of the variables, and the associated conditional probabilities, in the case of two periods t and $t+1$. Efficient implementation requires the following procedure for the building of the T period tree. The method is based on forward induction. First, compute the tree for the case where $t=1$. This gives the nodal values of the variables and the conditional probabilities, for the first two periods. Then, treat the first two periods as one new period, but with a binomial density equal to the sum of the first two binomial densities. The computations are carried out for period three nodal values and conditional probabilities. Note that the equations for the up-movements and down-movements of the variables always require the *conditional* volatilities of the variables in order to compute the vectors of nodal values. The following steps are implemented:

1. Using equation (9), compute the $[n_1 \text{ x } 1]$ dimensional vectors of the nodal outcomes of r_1, π_1 with inputs $\sigma_r(1)$, $E(r_1)$, $\sigma_\pi(1)$, $E(\pi_1)$ and binomial density n_1. Also, compute the $[(n_1 + n_2) \text{ x } 1]$ dimensional vectors r_2, π_2 using inputs $\sigma_r(2)$, $E(r_2)$, $\sigma_\pi(2)$, $E(\pi_2)$ and binomial density n_2. Assume the probability of an up-move in r_1 is 0.5 and then compute the conditional probabilities q_{π_1} using equation (10) with $t=1$. Then, compute the conditional probabilities q_{r_2}, q_{π_2}, using equations (10) and (11), with $t=2$.

2. Using equation (9), compute the $[(N_2 + n_3) \text{ x } 1]$ dimensional vectors r_3, π_3 using inputs $\sigma_r(3)$, $E(r_3)$, $\sigma_r(3)$, $E(r_3)$ and binomial density, n_3. Then, compute the conditional probabilities q_{r_3}, q_{π_3} using equations (10) and (11) with $t=3$.

3. Continue the procedure until the final period T.

In implementing the above procedure, we first complete step 1, using $t = 1$ and $t = 2$, and with the given binomial densities n_1 and n_2. To effect step 2, we then redefine the period from $t = 0$ to $t = 2$ as period 1 and the period 3 as period 2 and re-run the procedure with a binomial densities $n_1^* = n_1 + n_2$ and $n_2^* = n_3$. This algorithm allows the multiperiod lattice to be built by repeated application of equations (9), (10) and (11).

4.5 A summary of the approximation method

We will summarize the methodology by using a two-period and a three-period example. Figure 1 shows the recombining nodes for the two-factor process in the two-period case. The interest rate goes up to $r_{1,0}$ or down to $r_{1,1}$ at $t = 1$. The futures premium factor goes up to $\pi_{1,0}$ or down to $\pi_{1,1}$ at $t = 1$, with probability q_{π_1}. In the second period, there are just three nodes of the interest rate tree, together with three possible premium factor values. There are nine possible states, and the probability of an r_2 value materialising is q_{r_2}. Note that this probability depends on the level of the premium factor and of the interest rate at time $t = 1$. The recombining property of the lattice, which is crucial for its computability, is emphasised in Figure 2, where we show the process for the interest rate over periods $t = 2$ and $t = 3$. After two periods, there are three interest rate states and nine states representing all the possible combinations of the interest rate and premium factor. The interest rate then goes to four possible states at time $t = 3$ and there are sixteen states representing all the possible combinations of rates and premium factor. Note that the probability of reaching an interest rate at $t = 3$ depends on both the interest rate and the premium factor at $t = 2$. These are the probabilities that allow the no-arbitrage property of the model to be fulfilled. In the model, the term structure at time t is determined by the two factors, one representing the short rate and the premium factor. Thus, with a binomial density of $n = 1$, there are $(t + 1)^2$ term structures generated by the binomial approximation, at time t.

5 Model Validation and Examples of Inputs and Outputs

This section documents the results from several numerical examples based on the two-factor term structure model described in previous sections. First, we show an example of how well the binomial approximation converges to the mean and unconditional volatility inputs, illustrating the accuracy of our methodology. Second, we show that a two-factor term structure model can be implemented in a speedy and efficient manner. Third, we

discuss the input and output for an eight-period example, showing the illustrative output of zero-coupon bond prices, and conditional volatilities. Finally, we present the output from running a forty-eight quarter model, including the pricing of European-style, Bermudan-style and American-style swaptions.

In the numerical examples that follow, we choose a period length of three months. This is convenient for two reasons. First, we can model three-month Libor and then compute the corresponding maturity bond prices up to a given horizon without the added complexity of overlapping periods. Also, it enables the computational time to be reduced compared to a daily time interval model. However, changing the time interval does introduce one approximation. Theoretically, we need to use futures prices from contracts that are marked-to-market at the same periodicity as the time interval in the model; otherwise, lemma 1 does not strictly apply. However, only daily marked-to-market prices are widely available. In calibrating the three-month period model to market data, a convexity adjustment may be required to adjust futures prices from a daily to a quarterly marked-to-market basis. In practice, this adjustment is likely to be very small, especially compared with the problems of obtaining long-maturity futures prices.[8]

5.1 Convergence of Model Statistics to Exogenous Data Inputs

The first test of the two-factor model is how quickly the mean and variance of the short rates generated converge to the exogenous input data. Table 1 shows an example of a twenty-period model, where the input mean of the spot rate is 5% p.a., with a 10% conditional volatility. There is no mean reversion and the premium has a volatility of 1%. Note first that for a binomial density of 1, the accuracy of the binomial approximation deteriorates for later periods. This is due to the premium factor increasing with maturity and the difficulty of coping with the increased premium by adjusting the conditional probabilities.

One way to increase the accuracy of the approximation is to increase the binomial density. In the last three columns of the table we show the effect of increasing the binomial density to 2, 3, and 4 respectively. By comparing different binomial densities in a given row of the table we observe the convergence of the binomial approximation to the exogenous inputs as the density increases. Even for the 20-period case, high accuracy is achieved by increasing the binomial density to 4.

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[8] The difference between daily and three-monthly marked-to-market futures Libor is probably less than one basis point. For long maturities, lack of liquid futures contracts means that we have to estimate forward rates and apply a convexity adjustment. In this case the convexity adjustment is far more significant. See Gupta and Subrahmanyam (2000), for empirical estimates.

Table 1 here

5.2   Computing Time

Apart from the accuracy of the model, the most important feature of the methodology for implementing a two-factor model proposed in this paper is the computation time. It goes without saying that with two stochastic factors rather than one, the computation time can easily increase dramatically. In Table 2, we illustrate the efficiency of our model by showing the time taken to compute the zero-coupon bond prices and option prices. With a binomial density of one, the 48-period model takes 4.8 seconds and the 72-period model takes 17.2 seconds. Doubling the number of periods increases the computer time by a factor of six. There is clearly a trade-off between the number of periods, the binomial density of each period, and the computation time for the model. This is illustrated by the second line in the table, showing the effect of using a binomial density of two. Again the computation time increases more than proportionately as the density increases. The time taken for the 24-period model, when the binomial density is two, is roughly the same as that for the 48-period model with a density of one.

Table 2 here

5.3   Numerical Example: An Eight-Period Zero-Coupon Bond

This subsection shows a numerical example of the input and output of the two-factor term structure model, in a simplified eight-quarter example. It illustrates the large amount of data produced by the model, even in this small scale case, with just eight periods and a binomial density of one. The input is shown in Table 3. We assume a rising curve of futures rates, starting at 5%p.a. and increasing to 6%p.a.. These values are used to fix the means of the short rate for the various periods. The second row shows the conditional volatilities assumed for the short rate. These start at 14% and fall through time to 12%. We then assume a constant mean reversion of the short rate, of 10%, and constant conditional volatilities and mean reversion of the premium factor, of 2% and 40% respectively. While this example shows the flexibility of the model in coping with varying inputs, in more realistic examples the number of periods would be greater, the binomial density could change and the parameters might vary even more over different time periods.

Table 3 here

Tables 4 and 5 show a selection of the basic output of the model. For a binomial density of one, there are four states at time 1, nine states at time 2, sixteen states at time 3, and so on. In each state the model computes the whole term structure of zero-bond prices, using the no-arbitrage bond condition in Proposition 1, part 3. In Table 4, we show just the longest bond price, paying one unit at period eight. These are shown for the four states at time 1, in the first block of the table. The subsequent blocks show the nine prices at time 2, the sixteen prices at time 3, and so on.

Table 4 here

One of the most important features of the methodology is the way that the no-arbitrage property is preserved, by adjusting the conditional probabilities at each node in the tree of rates. In Table 5, we show the probability of an up-move in the interest rate given a state, where the state is defined by the short rate and the premium factor. In the first block of the table is the set of probabilities conditional on being in one of four possible states at time 1. The second block shows the conditional probabilities at time 2, in the nine possible states, and so on.

Table 5 here

5.4   An Example of a Payer Swaption

An important application of the model is to price and hedge contingent claims such as options with American and path-dependent features. A good example is a pay-fixed, receive-floating swaption, referred to as a payer swaption, since its value depends upon the possible movement of several interest rates over time.[9] We illustrate our methodology by pricing European, Bermudan and American swaptions, and compare these prices with those produced by models with fewer parameters, such as a one-factor model, and a two-factor model

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[9] The swap rate is computed using the standard definition

$$s_{t,n,m} = \frac{1 - B_{t,t+n}}{[B_{t,t+1} + B_{t,t+2} + ... + B_{t,t+n}]m},$$

where $s_{t,n,m}$ is the swap rate for a n year, m-month, swap at time t. The swaption payoffs are computed from $max[s_{t,n,m} - k, 0]$, where k is the strike rate.

with no mean reversion for one of the factors. The Bermudan-style option has the feature that it is exercisable at the end of each year up to the option maturity in year five. The American-style option is exercisable at the end of any quarter over the same period. The European-style swaptions are one-year options on one-year to five-year swaps. Note that the model uses twenty-four quarterly time periods, to cover the six-year life of the bond. Table 6 shows the values of European, Bermudan and American swaptions at differing depths-in-the-money, for four different models. The two-factor model is the one where the current *Libor* is 5%, all the futures rates are 5%, and with constant cap volatility of 15%, the coefficient of mean reversion of the short rate is 20%, and volatility of the premium is 3% with a 30% coefficient of mean reversion. All rates are on an annualised basis. The one-factor model is the same model without the premium factor. The third model is the two-factor model without mean reversion of the short rate, and the final model is the two-factor model without mean reversion of the premium factor.

Table 6 here

Table 6 shows that the Bermudan and American options are worth considerably more than the European one-year option on a five-year swap. The table also shows that using restricted models to price these options can produce incorrect prices for all options across different depths-in-the-money. However, the errors for in-the-money and out-of-the-money options are much smaller than those for at-the-money options options. For example, the one-factor model prices the option on the one-year swap, exercisable once in one years' time at 27 basis points, when the strike rate is at-the-money (5%), compared to the two-factor model's price of 31 basis points. In-the-money and out-of-the-money prices vary little from those produced by the two-factor model for this swaption. However, as the term of the swap increases, the errors grow larger. In-the-money swaption prices produced by the model with no mean reversion of the short rate produce similar results to the complete two-factor model. The at-the-money and out-of-the-money prices reveal larger errors. Out-of-the-money prices produced by the last model, which omits mean reversion of the premium, are similar to those from the complete model, although the errors appear to increase for increasing depths-in-the-money.

6   Conclusions

In this paper we have presented a model of the term structure of interest rates which can be regarded as a two-factor extension of the Black-Karasinski lognormal-rate model. However,

in this model, we assume that the short-term *Libor* follows a lognormal process. We have shown that, by calibrating to the current term structure of futures rates, the model is arbitrage-free in the sense of Ho and Lee (1986) and Pliska (1997).

The model has been implemented by using a multivariate-binomial tree approach. By extending previous work of Nelson and Ramaswamy (1990) and Ho, Stapleton and Subrahmanyam (1995), we have developed a recombining bivariate-binomial tree, which has a non-exploding number of nodes.

We have applied the model to the valuation of European-style, American-style and Bermudanstyle payer swaptions. We have shown that prices are computable in seconds, for examples with a realistic number of periods. Also, prices in the two-factor model exceed those in the one-factor model, calibrated to the same data.

A number of related research issues remain to be resolved in future work. First, the relationship between the Hull-White, Black-Karasinski type model developed here and the Heath-Jarrow-Morton multifactor model needs to be explored further. Specifically, we can generate forward-rate volatilities as outputs of our model in an extension of the analysis in the paper. Second, it is not clear to what extent a two-factor model, as opposed to a one-factor model, is necessary for the accurate valuation of specific interest-rate related contingent claims. The properties of two-factor models need to be further examined by applying them to a range of American-style, Bermudan-style and exotic options on bonds and interest rates. Third, the application of models of this type to derive risk management measures for interest-rate dependent claims should be studied further.

One important further extension of the model would apply to the pricing of credit derivatives. The pricing of options on defaultable bonds, for example, would ideally require the modelling of a two-factor risk-free rate process and a credit spread. Given the efficiency of the two-factor model presented here, it should be possible to approximate such a threefactor model, at least for a limited number of time periods. This is a subject for further research.

References

1. Balduzzi, P., S.R.Das, and S. Foresi (1998), "The Central Tendency: A Second Factor in Bond Yields," *The Review of Economics and Statistics*, **80**, 62–72.

2. Black, F., E. Derman, and W. Toy (1990), "A One-Factor Model of Interest Rates and its Application to Treasury Bond Options", *Financial Analysts' Journal*, **46**, 33–39.

3. Black, F. and P. Karasinski (1991), "Bond and Option Pricing when Short Rates are Lognormal", *Financial Analysts Journal*, **47**, 52–59

4. Brace, A., D. Gatarek, and M. Musiela, (1997), "The Market Model of Interest Rate Dynamics," *Mathematical Finance*, **7**, 127–155.

5. Brennan, M.J. and E.S. Schwartz (1979), "A Continuous-Time Approach to the Pricing of Bonds," *Journal of Banking and Finance*, **3**, 135–155.

6. Brennan,M.J. and E.S. Schwartz (1982), "An Equilibrium Model of Bond Pricing and a Test of Market efficiency", *Journal of Financial and Quantitative Analysis*, **17**, 301-329.

7. Courtadon, G. (1982), "The Pricing of Options on Default-Free Bonds", *Journal of Financial and Quantitative Analysis*, **17**, 75-100.

8. Cox, J.C., J.E. Ingersoll, and S.A. Ross (1981), "The Relationship between Forward Prices and Futures Prices", *Journal of Financial Economics*, **9**, 1981, 321-46.

9. Gupta, A. and M.G. Subrahmanyam (2000), "An Empirical Examination of the Convexity Bias in the Pricing of Interest Rate Swaps," *Journal of Financial Economics*, forthcoming

10. Heath, D., R.A. Jarrow, and A. Morton (1992), "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, **60**, 1, January, 77–105.

11. Ho, T.S.Y. and S.B. Lee (1986), "Term Structure Movements and Pricing of Interest Rate Claims," *Journal of Finance*, **41**, December, 1011–1029.

12. Ho. T.S., R.C. Stapleton, and M.G. Subrahmanyam (1995), "Multivariate Binomial Approximations for Asset Prices with Non- Stationary Variance and Covariance Characteristics," *Review of Financial Studies*, **8**, 1125-1152.

13. Hull, J. and A. White (1994), "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models," *Journal of Derivatives*, **2**, 37–48.

14. Jamshidian, F. (1989), "An Exact Bond Option Formula," *Journal of Finance*, **44**, 205–209.

15. F.Longstaff, P. Santa-Clara, and E.S. Schwartz (1999), "Throwing Away a Billion Dollars: The Cost of Suboptimal Exercise Strategies in the Swaptions Market", UCLA Working paper.

16. Miltersen, K.R., K. Sandmann and D. Sondermann (1997), "Closed Form Solutions for Term Structure Derivatives with Log-Normal Interest Rates," *Journal of Finance*, **1**, 409–430.

17. Nelson, D.B. and K. Ramaswamy (1990), "Simple Binomial Processes as Diffusion Approximations in Financial Models," *Review of Financial Studies*, **3**, 393-430.

18. Peterson, S.J. (1999), "The Application of Binomial Trees to Calculate Complex Option Prices, Two-factor Stochastic Interest Rate Option Prices and Value-at-Risk", *Ph.D Thesis*, Lancaster University.

19. Pliska, *Introduction to Mathematical Finance: Discrete Time Models*, Oxford: Blackwell, 1997.

20. Stapleton, R.C. and M.G. Subrahmanyam (1999) "The Term Structure of Interest-Rate Futures Prices", working paper, New York University.

21. Sundaresan, S. (1991), "Futures Prices on Yields, Forward Prices, and Implied Forward Prices from Term Structure", *Journal of Financial and Quantitative Analysis*, **26**, 409-424.

22. Vasicek, O., (1977), "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics*, **5**, 177–188.

Figure 1: **A Recombining Two-factor Process for the Short-term Interest Rate (Two-period case)**

t = 1 : 4 states   t = 2 : 9 states
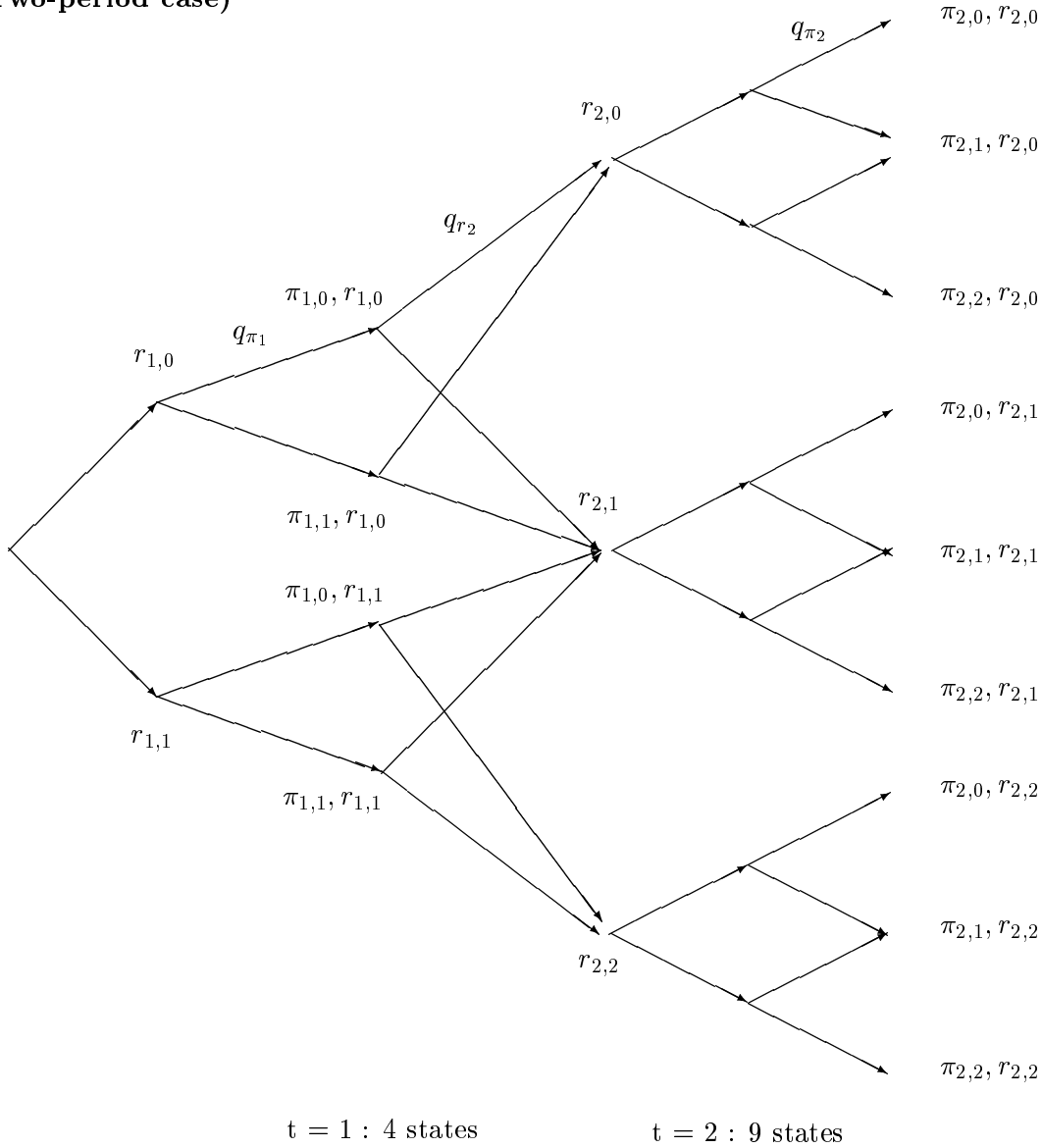
[1] The probability of moving, for example, to $r_{2,0}$ given $(\pi_{1,0}, r_{1,0})$ is q_{r_2}, defined in Equation (11).
[2] The probability of moving, for example, to $(\pi_{1,0}, r_{1,0})$ given $r_{1,0}$ and $\pi_{1,0}$ is q_{π_1}, defined in Equation (10).

Figure 2: **A Recombining Two-factor Process for the Short-term Interest Rate (Three-period case)**

t = 2 : 9 states   t = 3 : 16 states
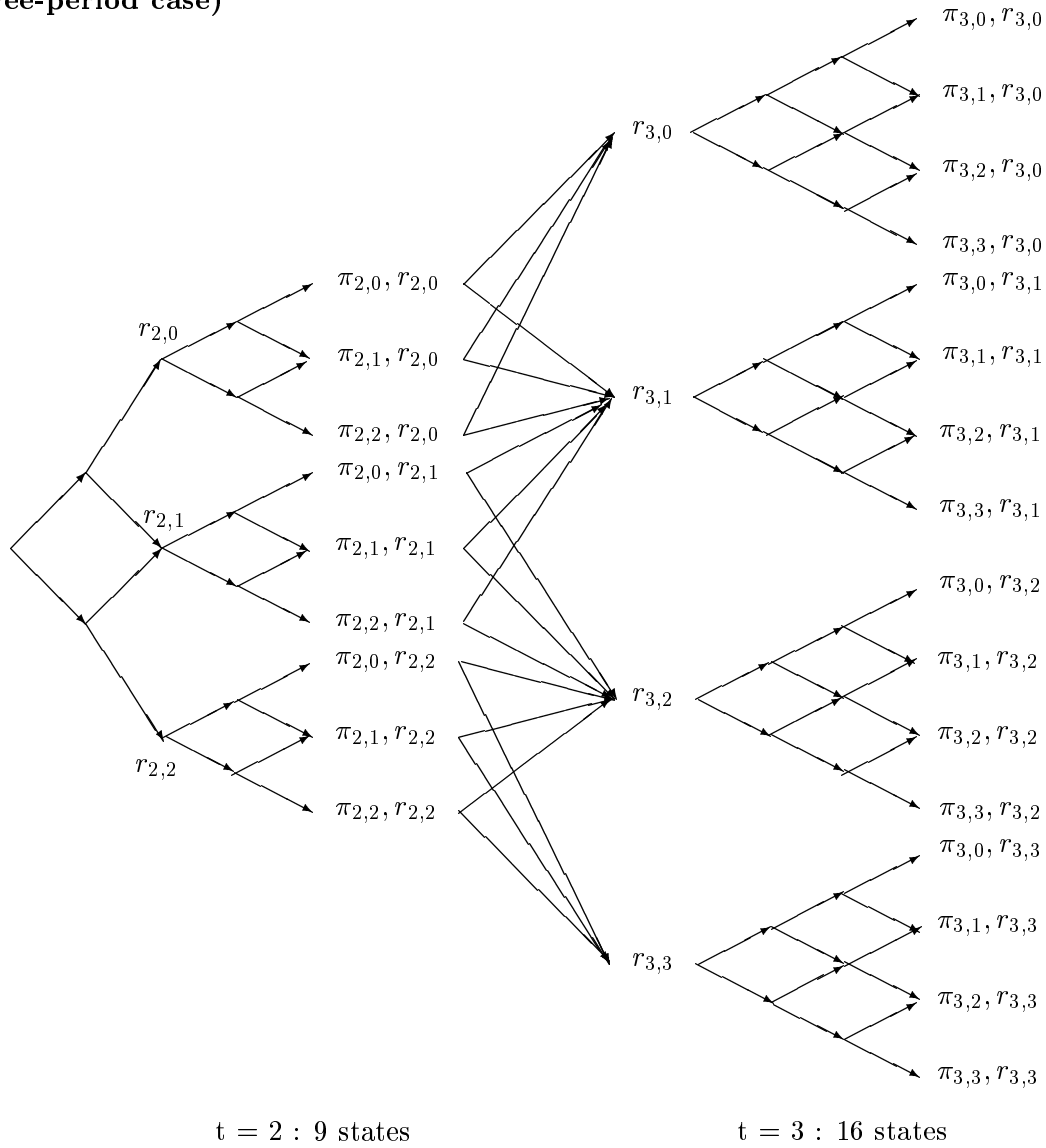
[1] The probability of moving, for example, to $r_{3,0}$ given $(\pi_{2,0}, r_{2,0})$ is q_{r_2}, defined in Equation (11).
[2] The probability of moving, for example, to $(\pi_{3,0}, r_{3,1})$ given $r_{3,1}$ and $\pi_{2,0}$ is q_{π_3}, defined in Equation (10).

Table 1: Convergence of Term Structure Model

	Binomial Density	1	2	3	4
Period					
[0, 1]	mean	5.0	5.0	5.0	5.0
	volatility	10.00	10.00	10.00	10.00
[0, 2]	mean	4.9999	4.99997	4.99998	4.99998
	volatility	9.97	9.99	9.99	9.99
[0, 3]	mean	4.9998	4.99992	4.99994	4.99996
	volatility $_3$	9.95	9.97	9.99	9.99
[0, 4]	mean $_4$	4.9997	4.99985	4.9999	4.99992
	volatility	9.92	9.97	9.98	9.98
[0, 5]	mean	4.9996	4.9997	4.9998	4.9998
	volatility	9.93	9.96	9.97	9.97
[0, 10]	mean	4.998	4.9992	4.9995	4.9996
	volatility	9.88	9.94	9.96	9.97
[0, 20]	mean	4.996	4.998	4.998	4.999
	volatility	9.85	9.92	9.94	9.96

The numbers in the table are the computed means and volatilities, in percent, for the short rate over periods 1,2,3,4,5,10, and 20, using the output of the two-factor model. The means are calculated using the possible outcomes and the nodal probabilities. The volatilities are the annualized standard deviations of the logarithm of the short rate. The binomial density refers to the grid size of the binomial tree of the short rate and the premium factor, over each sub interval. The input parameters in this case are a constant mean of 5% for each period, and conditional volatility of 10% with no mean reversion of the short rate. The premium factor has a volatility of 1%, a mean of 1, and no mean reversion.

Table 2: Computing Time for Bond and Option Pricing (seconds)

Number of Periods	8	12	24	48	72
Binomial Density 1	0.1	0.2	0.9	4.8	17.2
Binomial Density 2	0.2	0.6	5.0	28.0	102.9
Binomial Density 3	0.6	1.7	14.8	87.0	-

The table shows the time taken to compute all the zero-bond prices, swaption prices, given the tree of rates for different levels of binomial density, for different numbers of periods. The computer speed is 550 MHZ, and the processor is Pentium III.

Table 3: 8-period Example Input

Period	1	2	3	4	5	6	7	8
Futures rate	5.0	5.2	5.4	5.6	5.7	5.8	5.9	6.0
Conditional volatility (r)	14.0	14.0	13.5	13.0	13.0	12.5	12.5	12.0
Mean reversion (r)	10.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0
Conditional volatility (π)	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
Mean reversion (π)	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0

All numbers are in percent on an annualized basis. The table shows the exogenous data input for an 8-period example, with a binomial density of 1. The short rate is the quarterly rate, so the period length is quarter of one year. Input data relating to the short rate appears in the first three rows; data relating to the premium appear in the last two rows.

Table 4: Illustrative Output of Zero-Coupon Bond Prices

0.9016606 0.9053589
0.9121947 0.9155353

0.9074258 0.9105445 0.9135762
0.9175077 0.9203237 0.9230596
0.9265372 0.9290763 0.9315416

0.9162768 0.9187132 0.9210944 0.9234185
0.9251191 0.9273254 0.9294793 0.9315810
0.9330661 0.9350606 0.9370067 0.9389042
0.9401996 0.9420003 0.9437568 0.9454675

0.9282870 0.9299746 0.9316477 0.9332908 0.9349043
0.9355540 0.9370946 0.9386109 0.9400998 0.9415616
0.9421182 0.9435164 0.9448895 0.9462374 0.9475607
0.9480383 0.9493036 0.9505457 0.9517651 0.9529554
0.9533705 0.9545143 0.9556373 0.9567394 0.9578075

0.9429089 0.9437140 0.9446889 0.9456537 0.9466083 0.9475538
0.9485605 0.9494144 0.9502989 0.9511741 0.9520400 0.9528975
0.9537454 0.9545554 0.9553573 0.9561506 0.9569353 0.9577125
0.9584548 0.9591885 0.9599150 0.9606336 0.9613444 0.9620482
0.9626968 0.9633611 0.9640187 0.9646692 0.9653126 0.9658693
0.9665154 0.9671165 0.9677114 0.9682999 0.9688820 0.9693053

In this example the binomial density is 1. All prices are for a zero-coupon bond paying one unit of currency at the end of period 8. The first set of 4 numbers are the time 1 bond prices $B_{1,8}$, the second set of 9 numbers are the time 2 bond prices $B_{2,8}$, through to the time 5 set of 36 prices $B_{5,8}$. The 49 prices, $B_{6,8}$, and 64 prices, $B_{7,8}$, are not presented for reasons of space. Bond prices are computed using Proposition 1, part 3.

Table 5: Illustrative Output of Conditional Probabilities

0.5530794674 0.4087419001
0.5932346449 0.4488970776

0.6622172425 0.5086479906 0.3550787388
0.6514562049 0.4978869530 0.3443177012
0.6406951673 0.4871259154 0.3335566636

0.7780765989 0.6126741059 0.4472716129 0.2818691198
0.7548745812 0.5894720882 0.4240695952 0.2586671021
0.7316725635 0.5662700705 0.4008675775 0.2354650844
0.7084705458 0.5430680528 0.3776655597 0.2122630667

0.8091566018 0.6377849460 0.4664132901 0.2950416343 0.1236699785
0.8248723262 0.6535006704 0.4821290145 0.3107573587 0.1393857029
0.8405880506 0.6692163947 0.4978447389 0.3264730831 0.1551014272
0.8563037749 0.6849321191 0.5135604633 0.3421888074 0.1708171516
0.8720194993 0.7006478435 0.5292761877 0.3579045318 0.1865328760

1.0000000000 0.8614994841 0.6744916140 0.4874837439 0.3004758739 0.1134680038
1.0000000000 0.8248200186 0.6378121486 0.4508042785 0.2637964084 0.0767885384
0.9751484233 0.7881405532 0.6011326831 0.4141248131 0.2271169430 0.0401090730
0.9384689578 0.7514610878 0.5644532177 0.3774453477 0.1904374776 0.0034296075
0.9017894924 0.7147816224 0.5277737523 0.3407658822 0.1537580122 0.0000000000
0.8651100270 0.6781021569 0.4910942869 0.3040864168 0.1170785467 0.0000000000

In this example the binomial density is 1. All the probabilities are conditional probabilities of an up-move in the interest rate, given the short rate and the premium factor. The first set of 4 numbers are the probabilities at time 1, the second set of 9 numbers are the conditional probabilities at time 2, through to the time 5 set of 36 conditional probabilities. The 49 probabilities at time 6, and the 64 probabilities at time 7, are not presented for reasons of space. In each case the columns show the probabilities for different (increasing to the right) values of the short rate. The rows show the values (increasing downwards) for different values of the premium factor.

Table 6: Swaption Prices (basis points)

Model	Strike rate	One	year	option	on	(years)	Bermudan (24 qtrs)	American (24 qtrs)
		1	2	3	4	5		
2 factor	5%	31	61	141	223	288	367	376
	6%	5	10	53	102	137	246	251
	4%	99	193	315	440	547	578	615
1 factor	5%	27	49	65	77	86	150	171
$(\sigma_\pi = 0)$	6%	4	5	4	3	3	49	57
	4%	99	191	278	358	434	441	511
2 factor	5%	26	86	153	226	302	355	364
$b = 0$	6%	2	26	60	101	146	218	222
	4%	98	204	317	436	557	577	620
2 factor	5%	31	83	143	219	305	376	386
$c = 0$	6%	5	25	54	93	140	242	250
	4%	99	204	317	441	575	604	645

The above table shows the values of European, Bermurdan and American swaptions at differing levels of moneyness, for 4 different models. The European swaptions are 1 year options on 1 to 5 year swaps. The Bermudan swaption is exercisable yearly for 5 years on a 6 year underlying bond, and the American swaption is exercisable quarterly for 5 years on the same bond. The 2-factor model is the model where the short rate of the bond is 5%, futures rate of 5% for each maturity, a cap volatility of 15%, a coefficient of mean reversion of the short rate of 20%, and volatility of the premium at 3% with 30% coefficient of mean reversion. The one factor model is the same model without the premium factor. The third model is the 2-factor model without mean reversion of the short rate, and the final model is the 2-factor model without mean reversion of the premium factor. The swap rate is computed using

$$s_{t,n,m} = \frac{1 - B_{t,t+n}}{[B_{t,t+1} + B_{t,t+2} + ... + B_{t,t+n}]m}$$

where $s_{t,n,m}$ is the swap rate for a n year, m-month, swap at time t. The swaption payoffs are computed from $max[s_{t,n,m} - k, 0]$, where k is the strike rate.

Appendices

A Proof of Lemma 1

The price of the futures Libor contract is by definition

$$F_{t,T} = 1 - f_{t,T} \tag{12}$$

and its price at maturity is

$$F_{T,T} = 1 - f_{T,T} = 1 - r_T. \tag{13}$$

From Cox, Ingersoll and Ross (1981), the futures price $F_{t,T}$ is the value, at time t, of an asset that pays

$$V_T = \frac{1 - r_T}{B_{t,t+1} B_{t+1,t+2} ... B_{T-1,T}} \tag{14}$$

at time T, where the time period from t to $t+1$ is one day. In a no-arbitrage economy, there exists a risk-neutral measure, under which the time-t value of the payoff is

$$F_{t,T} = E_t(V_T B_{t,t+1} B_{t+1,t+2} ... B_{T-1,T}). \tag{15}$$

Substituting (14) in (15), and simplifying then yields

$$F_{t,T} = E_t(1 - r_T) = 1 - E_t(r_T). \tag{16}$$

Combining (16) with (12) yields the first statement in the lemma. The second statement in the lemma follows from the assumption of the lognormal process for r_T and the moment generating function of the normal distribution. □

B Proof of Lemma 2

Taking the unconditional expectation of equation (1),

$$\mu_{r_t} - \mu_{r_{t-1}} = \theta_r(t) - b\mu_{r_{t-1}} + \mu_{\pi_{t-1}},$$

$$\mu_{\pi_t} - \mu_{\pi_{t-1}} = \theta_\pi(t) - c\mu_{\pi_{t-1}}.$$

Then, substituting for $\theta_r(t)$ and $\theta_\pi(t)$ in (1) yields

$$\ln(r_t) - \mu_{r_t} = [\ln(r_{t-1}) - \mu_{r_{t-1}}](1 - b) + \ln(\pi_{t-1}) - \mu_{\pi_{t-1}} + \varepsilon_t$$

and
$$\ln(\pi_t) = \mu_{\pi_t} - \mu_{\pi_{t-1}}(1-c) + (1-c)\ln(\pi_{t-1}) + \nu_t.$$

Since r_t and π_t are lognormally distributed, it follows from the moment generating function of the normal distribution that

$$E_0(r_t) = \exp(\mu_{r_t} + \sigma_{r_t}^2/2)$$

$$E_0(\pi_t) = \exp(\mu_{\pi_t} + \sigma_{\pi_t}^2/2)$$

Lemma 1 implies:

$$\ln[f_{o,t}] = \ln[E_0(r_t)] = \mu_{r_t} + \sigma_{r_t}^2/2,$$

and using $E(\pi_t) = 1$,
$$\ln[E_0(\pi_t)] = 0 = \mu_{\pi_t} + \sigma_{\pi_t}^2/2.$$

Substitution for μr_t, $\mu_{r_{t-1}}$, and μ_{π_t}, $\mu_{\pi_{t-1}}$ then yields the statement in the lemma. $\square$

C Proof of Lemma 3

From lemma 1, the no-arbitrage condition implies

$$f_{t,t+1} = E_t(r_{t+1})$$

in all states and for all t. From the lognormality of r_{t+1},

$$E_t(r_{t+1}) = \exp\{E[\ln(r_{t+1})] + \frac{\sigma(t+1)^2}{2}\}.$$

Hence, the no-arbitrage condition requires

$$\ln(f_{t,t+1}) = E[\ln(r_{t+1})] + \frac{\sigma(t+1)^2}{2}. \tag{17}$$

But, taking the expectation of equation (3), for r_{t+1} yields:

$$E[\ln(r_{t+1})] = \ln(f_{0,t+1}) + \alpha_{r_{t+1}} + (1-b)[\ln(r_t) - \ln(f_{0,t})] + \ln\pi_t. \tag{18}$$

Hence, substituting (18) into (17) yields:

$$\ln(f_{t,t+1}) = \ln(f_{0,t+1}) + \alpha_{r_{t+1}} + (1-b)[\ln(r_t) - \ln(f_{0,t})] + \ln\pi_t + \frac{\sigma(t+1)^2}{2}.$$

The lemma follows with

$$\alpha_{f_{t+1}} = \alpha_{r_{t+1}} + \frac{\sigma(t+1)^2}{2} + \ln(f_{0,t+1}).$$

□

D   Proof of Proposition 2

First, we derive the following covariances from equation (3)

$$\sigma_{r_{t+1},r_t} = (1-b)\sigma_{r_t}^2 + \sigma_{r_t,\pi_t},$$

$$\sigma_{r_t,\pi_t} = (1-c)\sigma_{\pi_{t-1}}^2 + (1-b)(1-c)\sigma_{r_{t-1},\pi_{t-1}}$$

and

$$\sigma_{\pi_{t+1},\pi_t} = (1-c)\sigma_{\pi_t}^2,$$

$$\sigma_{r_t,\pi_{t-1}} = (1-b)\sigma_{r_{t-1},\pi_{t-1}} + \sigma_{\pi_{t-1}}^2.$$

Now, from the multiple regression

$$\ln\left[\frac{r_t}{f_{0,t}}\right] = \alpha_{r_t} + \beta_{r_t}\ln\left[\frac{r_{t-1}}{f_{0,t-1}}\right] + \gamma_{r_t}\ln(\pi_{t-1}) + \varepsilon_t \tag{19}$$

the regression coefficients are

$$\beta_{r_t} = \frac{\sigma_{r_t,r_{t-1}}\sigma_{\pi_{t-1}}^2 - \sigma_{r_t,\pi_{t-1}}\sigma_{r_{t-1},\pi_{t-1}}}{\sigma_{r_{t-1}}^2\sigma_{\pi_{t-1}}^2 - (\sigma_{r_{t-1},\pi_{t-1}})^2},$$

$$\gamma_{r_t} = \frac{\sigma_{r_t,r_{t-1}}\sigma_{r_{t-1}}^2 - \sigma_{r_t,r_{t-1}}\sigma_{r_{t-1},\pi_{t-1}}}{\sigma_{r_{t-1}}^2\sigma_{\pi_{t-1}}^2 - (\sigma_{r_{t-1},\pi_{t-1}})^2}.$$

Substituting the covariances and simplifying yields

$$\beta_{r_t} = (1 - b) \tag{20}$$

and

$$\gamma_{r_t} = 1. \tag{21}$$

From the lognormality of r_t and π_t we can write equation (3) as

$$
\begin{aligned}
\ln(r_t) - \ln[f_{0,t}] + \sigma_{r_t}^2/2 &= \{\ln(r_{t-1}) - \ln[f_{0,t-1}] + \sigma_{r_{t-1}}^2/2\}(1 - c) \\
&+ \ln(\pi_{t-1}) - \left\{\sigma_{\pi_{t-1}}^2/2\right\} + \varepsilon_t.
\end{aligned}
$$

Re-arranging terms yields

$$
\begin{aligned}
\ln\left[\frac{r_t}{f_{0,t}}\right] &= [-\sigma_{r_t}^2 + \sigma_{r_t}^2(1 - c) + \sigma_{\pi_t}^2]/2 \\
&+ \ln\left[\frac{r_{t-1}}{f_{0,t-1}}\right](1 - c) + \ln(\pi_{t-1}) + \varepsilon_t.
\end{aligned}
$$

Given (19), (20), and (21), we have α_{r_t} as stated in the lemma. Similarly, with $E_0(\pi_t) = 1$, we have

$$\ln(\pi_t) = \alpha_{\pi_t} + \ln(\pi_{t-1})(1 - c) + \nu_t,$$

and

$$
\begin{aligned}
\alpha_{\pi_t} &= E_0[\ln(\pi_t)] - (1 - c)[E_0[\ln(\pi_t)] \\
\alpha_{\pi_t} &= [-\sigma_{\pi_t}^2 + (1 - c)\sigma_{\pi_{t-1}}^2]/2.
\end{aligned}
$$

Finally, the variance of ε_t, given t, is

$$
\begin{aligned}
\mathrm{var}_{t-1}(\varepsilon_t) &= \mathrm{var}_0\left\{\ln\left[\frac{r_t}{f_{0,t}}\right]\right\} - \beta_{r,t}^2 \mathrm{var}_0\left\{\ln\left[\frac{r_{t-1}}{f_{0,t-1}}\right]\right\} \\
&- \gamma_{r_t}^2 \mathrm{var}_0[\ln(\pi_{t-1})] - \beta_{r_t}\gamma_{r_t} \mathrm{cov}[\ln(r_{t-1}), \ln(\pi_{t-1})]
\end{aligned}
$$

or,

$$\mathrm{var}_{t-1}(\varepsilon_t) = \sigma_{r_t}^2 - (1 - b)^2\sigma_{r_{t-1}}^2 - \sigma_{\pi_{t-1}}^2 - 2(1 - b)\sigma_{r_{t-1}, \pi_{t-1}}.$$

$\square$

E   Proof of Lemma 4

Taking the conditional expectation of equation (3) at t

$$E_t[\ln(r_{t+1})] - \mu_{r_{t+1}} = [\ln(r_t) - \mu_{r_t}](1-b) + \ln(\pi_t) - \mu_{\pi_t}.$$

Given $x_t = E_t(r_{t+1})$ and using the lognormal property of r_{t+1},

$$
\begin{aligned}
\ln(x_t) &= E_t[\ln(r_{t+1})] + \sigma_r^2(t+1)/2 \\
&= \sigma_r^2(t+1)/2 + \mu_{r_{t+1}} + [\ln(r_t) - \mu_{r_t}](1-b) + [\ln(\pi_{t-1}) - \mu_{\pi_{t-1}}](1-c) + \nu_t.
\end{aligned}
$$

Hence

$$\sigma_x^2(t) = \text{var}_{t-1}[\ln(x_t)] = (1-b)^2 \text{var}_{t-1}[\ln(r_t)] + \text{var}_{t-1}[\nu_t]$$

or

$$\sigma_x^2(t) = \sigma_\pi^2(t) + (1-b)^2 \sigma_r^2(t)$$

and the statement in the lemma follows. □