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Abstract

Models using the *Rational Expectations Hypothesis* (REH) are widely recognized to be inconsistent with the observed behavior of premia in financial markets, as well as other features of asset price dynamics. Moreover, many reasons have been advanced as to why the REH cannot generally represent, even approximately, the expectations behavior of *individually rational* agents.

In this paper, we develop a new model of the equilibrium premium in the foreign exchange market that replaces the REH with the *Imperfect Knowledge Forecasting* (IKF) framework. Because we maintain that agents must cope with imperfect knowledge *and* that they are *not* grossly irrational, our IKF approach imposes *only qualitative* conditions on the formation of individual forecasting models and their updating.

We also develop a dynamic extension of the original formulation of Kahneman and Tversky's prospect theory. We find that under IKF and *dynamic prospect theory*, the equilibrium premium on foreign exchange is positively related to the gap between the aggregate forecast of the exchange rate and its historical benchmark level. We test this implication, using survey data on the German mark-U.S. dollar exchange rate, and find that the behavior of the *ex ante* premium on foreign exchange is consistent with our model of the premium.

Of course, compared with the precise predictions we have learnt to expect in the physical sciences, this sort of mere pattern predictions is a second best...I am anxious to repeat, we will still achieve predictions which can be falsified and which therefore are of empirical significance... Yet the danger of which I want to warn is precisely the belief that in order to be accepted as scientific it is necessary to achieve more. This way lies charlatanism and more. To act on the belief that we possess the knowledge,...which in fact we do not possess, is likely to make us do much harm...I confess that I prefer true but imperfect knowledge, even if it leaves much indetermined and unpredictable, to a pretence of exact knowledge that is likely to be false. (Excerpts from the Nobel lecture of Friedrich Hayek, 1978, p.33).

1 Introduction

The almost universal use of the *Rational Expectations Hypothesis* (REH) testifies to the widespread belief among economists that the REH provides *the* solution to the enduring problem of modeling the expectations of rational agents. Yet it is precisely in settings in which agents' expectations matter most, such as asset markets, that REH-based models encounter their greatest difficulties.

In the foreign exchange market, Dornbusch and Frankel surmised that

> The chief problem with the overshooting theory, and indeed with the more general rational expectations approach, is that it does not explain well the shorter-term [long-swings] dynamics (Dornbusch and Frankel, 1988, p. 16).[1]

But until recently, such observations have been largely ignored – despite research of the past two decades that indicates that the implications of the REH approach are inconsistent with much of the empirical evidence on exchange rates and other asset prices.[2,3]

[1] For early studies documenting the difficulties of the REH approach in explaining the observed movements of the term structure of interest rates and stock prices, see Shiller (1979, 1981).

[2] For examples of recent surveys of anomalous behavior (by REH standards) of exchange rates see Frankel and Rose (1995) and Engel (1996) and references therein.

[3] It should also be noted that the REH *does not*, in general, represent, even *approximately*, the expectations of *individually rational* agents. For an early discussion of the

The widespread belief that the REH is *the* model for the expectations of rational agents has recently given rise to a view that – to explain the observed asset-price dynamics – departures from "rationality" need to be introduced into models of asset markets.[4] We believe, however, that a different factor may be key to understanding the apparently anomalous behavior of asset prices.

In this paper, we develop a new model of the equilibrium premium in the foreign exchange market. The key assumption underlying our approach (dubbed the *Imperfect Knowledge Forecasting,* IKF, framework) is that economic agents – in formulating and revising their forecasting models – are limited by imperfect knowledge.[5] While we maintain that agents must cope with imperfect knowledge, we do not differ in the presumption that, *on the whole*, economic agents *are not* grossly *irrational*, in the sense that they *do not* pass up – *endlessly* – profit opportunities. Consequently, the IKF framework – in contrast to the extant approaches to the modeling of expectations, including the REH – imposes only *qualitative* conditions on the formulation of individual forecasting models and their revisions.[6]

The second distinctive feature of our approach is that we develop a dynamic extension of the original formulation of Kahneman and Tversky's prospect theory (Kahneman and Tversky, 1979 and Tversky and Kahneman, 1992). This *dynamic prospect theory* assumes that agents are not only loss averse, but that they are more sensitive to *changes* in potential losses than to *changes* in potential gains of the same size. We find that under dynamic prospect theory, both bulls and bears hold open positions in foreign

assumptions under which each of the *individually rational* agents would "choose" to use only one *common* model and other related epistemological problems besetting the REH approach, see Frydman (1982), Frydman and Phelps (1983), Frydman (1983) and Phelps (1983).

[4]See Akerlof (2002), and references therein, for a recent discussion on the important role of "irrationality" for understanding asset price dynamics.

[5]This IKF framework has its roots in Keynes (1936), whose ideas on the behavior of asset markets influenced much of the analysis in this paper. Hayek's (1948) penetrating critique of socialist planning and his insights on the use of knowledge in society have also guided us in the development of our approach.

[6]This qualitative approach is put forth in Frydman and Goldberg (2003a). In that paper we argue that if an economist were to characterize *fully* individual forecasting models and their revisions (i.e. in terms of *fixed* rules or generalizations of those rules that depend on observable variables or factors), then the resulting formulations would be, in general, *inconsistent* with the postulate of individual rationality. See the concluding remarks below for further discussion of this point.

exchange only if they expect a positive excess return on their open positions, to compensate them for their extra sensitivity to the potential losses. This result leads to a new, *momentary*, equilibrium condition for the foreign exchange market, which we call *uncertainty adjusted uncovered interest rate parity* (UAUIP). The UAUIP condition implies that equilibrium in the foreign exchange market is associated with an aggregate premium on foreign exchange – dubbed an equilibrium *uncertainty premium* – that depends on the relative assessments of bulls and bears concerning the potential losses from foreign exchange speculation.

In our model, equilibrium is defined, at each point in time, for a *given* set of individual forecasting models. Revisions of these models, then, cause the equilibrium uncertainty premium to move over time. To model this movement, we follow the IKF framework and impose only qualitative assumptions on the process by which individual forecasting models are revised. To this end, we build on an idea put forth by Keynes (1936) and assume that the gap between agents' conditional forecasts of the exchange rate and its perceived historical benchmark plays a key role in how agents' revise their forecasting models and their assessments of the potential losses. We find that these individual gap effects lead to a positive relationship between the equilibrium uncertainty premium and an aggregate measure of the gap.

We test this implication, using one-month forecasts of the German mark-U.S. dollar exchange rate from Money Market Services International (MMSI), and show that the observed time path of the *ex ante* premium on foreign exchange is indeed consistent with our gap effect. We conjecture that this new empirical finding – that the gap from the historical benchmark plays an important role in understanding the dynamics of the premium on foreign exchange– may also be important in understanding the behavior of equilibrium premia in other asset markets. We also show that our IKF-based model provides a simple explanation of the sign reversals that have been observed in foreign exchange premia. These positive findings stand in sharp contrast to the widely known difficulties of standard REH models of the risk premium in explaining the time path, volatility and sign reversals of foreign exchange premia over the modern period of floating.[7]

Although prospect theory has been incorporated recently into models of asset prices, all of the extant applications make use of the representative-

[7] For review articles, see Lewis (1995) and Engel (1996).

agent assumption and the REH.[8] The REH-based models tie expectations rigidly to the model formulated by an economist, thereby restricting the updating of expectations to the arrival of new information on macroeconomic fundamentals. Because these fundamentals are much less volatile than asset returns, the REH models often rely on modifications of preferences to generate a greater variance of asset returns.[9] This consideration leads Barberis, Huang and Santos (2001) to adopt a particular interpretation of Kahneman and Tversky's (1979) notion of reference dependence, called a "house-money" effect: "how loss averse the investor is depends on his prior investment performance (Barberis, Huang and Santos, 2001, p. 2)."

To compare the implications of a house-money effect with those of our gap effect, we formalize a house-money effect in the context of our model of foreign exchange speculation. In sharp contrast to the implication of our gap effect, we show that a house money effect implies a *negative* relationship between the premia on foreign exchange speculation and the expected gap. This negative relationship is rejected by empirical evidence.[10]

The literature in finance and economics has also produced many models with expectations that depart from the REH. In moving away from the REH – which attributes to agents a forecasting rule based on one specific model – economists must confront this key question: which non-REH forecasting models should be attributed to agents, given that the universe of such models is, in principle, unbounded?

To limit the universe of potential forecasting models, extant departures from the REH have relied on important insights from behavioral economics concerning the behavior of agents in real-world markets.[11] Although these approaches are superior to the REH models from an empirical point of view, they share one crucial feature with the REH approach: they attribute to agents specific quantitative forecasting models and fixed mechanisms for how

[8]See Benartzi and Thaler (1995), Barberis, Huang and Santos (2001), Barberis and Huang (2001) and references therein.

[9]Barberis, Huang and Santos (2001) recognize that, even with the assumption of loss aversion, the standard, REH-based setup is unable to explain the main "perplexing features of the aggregate data."

[10]This finding, however, does not necessarily imply that a house-money effect is irrelevant; it may simply mean that the gap effect is stronger than a house-money effect.

[11]Early approaches to modeling departures from the REH in financial markets include the seminal studies by Frankel and Froot (1987) and Delong, Shleifer, Summers and Waldman (1990a,b). For more recent studies, see Gourinchas and Tornell (2001) and Hong and Stein (2002, 2003).

agents update these models. Each one of these formulation is tantamount to an assumption of one over-arching model with a structure that is *unchanging over time*. Such an unchanging structure may involve a finite number of *pre-specified* forecasting models along with a *fixed* rule governing how agents switch between models in updating their forecasts.[12]

For example, a large class of standard learning models assumes that all agents learn and forecast on the basis of a *common* model and a fixed updating mechanism throughout the period of learning.(e.g., Evans and Honkapohja, 2001). In the foreign exchange market, Frankel and Froot (1987) assume an unchanging structure for expectations and their updating: a representative agent updates his expectations by switching – according to a "fixed" (Bayesian) rule – between a chartist model and a fundamental model.[13]

These behaviorally-motivated departures from the REH are an advance over the REH approach; they do not ignore the fact that agents in real-world markets may change their forecasting models – rather than merely respond to the arrival of new information – when updating their forecasts. However, these departures from the REH neglect the fact that in a world of imperfect knowledge, agents, in general, not only face a choice among extant models – which an economist tries to capture by a *pre-specified set of* models – but they also *invent* new models. Moreover, agents also *invent* new ways of revising their models that differ from the fixed updating rules attributed to them by an economist.

Because extant departures from the REH largely ignore the creative aspect of the process of the acquisition of knowledge in a market economy, they also suffer from another difficulty. Though various forms of irrationalities introduced into these models are well established, and can be plausibly justified on behavioral grounds, the assumption of an unchanging structure for expectations *and their updating* is actually equivalent to a much stronger assumption about the degree of agents' irrationality: it assumes agents per-

[12] This interpretation of "one model" as involving a number (or a class) of *pre-specified* models with a fixed rule governing switches between models has also been adopted by the REH literature. For examples, see Engel and Hamilton (1990) and Hansen and Sargent (2001a,b). For a critical discussion of this less restrictive, but still REH-based approach to the modeling of expectations, see Frydman and Goldberg (2002, 2003a) and footnote 61 below.

[13] For a related approach that also assumes an unchanging structure – consisting of speculators, who form expectations according to the REH and feedback traders, who trade on the basis of simple rules – see Delong, Shleifer, Summers and Waldman (1990b). This model features fully informed .

sist, in perpetuity, in the particular forms of irrationality attributed to them.

In contrast, our IKF approach is predicated on the presumption that under imperfect knowledge economic agents do not adhere to one unchanging structure in forming and updating their forecasting models. To do so would imply *gross* irrationality, in the sense of *endlessly* passing up apparent profit opportunities. Thus, although the extant departures from the REH shed light on particular episodes of the empirical record, they do not offer a general approach that can replace the REH as a model of the forecasting process. This lack of generality may explain why many economists are reluctant to abandon the REH.

The IKF approach aims to provide a general framework for modeling the forecasting process, that can replace the REH. The most important distinction between the non-IKF departures from the REH and our IKF approach is that we do not attribute specific forecasting mechanisms to agents and we do not treat the revisions of forecasting models as a *mechanical process*. Instead, we recognize that knowledge is imperfect in general and, thus, economic agents must *choose* from among the myriad of existing forecasting models and decide on whether and how to develop new models.

Because our position is that economic agents are *not* grossly irrational, we characterize this *creative process* in a *qualitative* as opposed to a quantitative manner. This qualitative approach frees the economist from having to specify precisely agents' forecasting models and how these models are revised. Thus, the IKF approach is consistent with the creative process of model formulation and discovery; it is not only compatible with a wide class of existing models – including those from behavioral economics – but it also allows for forecasting models yet to be invented.

Contrary to the implicit presumption of the REH approach – that quantitative restrictions are needed to impose "discipline" on the analysis – we show in this paper that qualitative restrictions on the revisions of the individual forecasting models are able to generate implications that can be rejected by the data. Moreover, as we demonstrate in this paper, the qualitative IKF approach is able to explain asset market dynamics that the REH models deem anomalous.

2 An Overview of the Paper

Modeling the dynamics of the equilibrium premium on foreign exchange involves two steps: a definition of momentary equilibrium at a point in time and a specification of the updating of forecasting models, and other factors, that move this momentary equilibrium over time. In section 3 of this paper, we use dynamic prospect theory to derive our new momentary equilibrium at a point in time, UAUIP, while in section 4, we use our IKF framework to model the movement of this momentary equilibrium over time.

Section 3.1 provides a formal characterization of the individual forecasting models and conditional forecasts of the one-period ahead excess return, $R_{t+1} = S_{t+1} - S_t - FP_t$, where S_t and FP_t denote the log levels of the exchange rate and the forward premium, respectively. We represent individual i's forecasting model of R_{t+1} in terms of a conditional probability distribution of the one-period ahead exchange rate, denoted by $\mathcal{P}_t^i(S_{t+1}|\mathcal{X}_t^i, \theta_t^i)$, where $\mathcal{X}_t^i$ and θ_t^i are the time-t information set and set of parameters of this conditional distribution..[14]

An individual exchange rate forecast at time t, $\tilde{s}_{t+1}^i$, is the mean of this conditional distribution, evaluated at x_t^i, and θ_t^i. The conditional forecast of the return on a long (short) position of *one unit* of foreign exchange held from time t to $t+1$ is then defined as $\tilde{r}_{t+1}^i = \tilde{s}_{t+1}^i - s_t - fp_t$ $(-\tilde{r}_{t+1}^i = fp_t + s_t - \tilde{s}_{t+1}^i)$.[15]

At a point in time, our assumptions of individual rationality and imperfect knowledge imply that, in general, individual agents use different forecasting models and therefore form heterogeneous forecasts. In this paper we assume that the heterogeneity of forecasts entails the presence in the market of both bulls and bears – speculators who hold long and short positions, respectively. To clarify the important elements in this overview, we replace a diversity of forecasting models *within* the groups of bulls and bears with two forecasting models, one for a representative bull and the other for a representative bear, denoted by $i =$L,S, respectively.[16]

Because we assume the presence of both bulls and bears in the market,

[14] As is customary, we will use upper and lower case letters to distinguish between random variables and their realizations, respectively.

[15] A long (short) position in foreign exchange is one in which a rise (fall) in the value of foreign exchange leads to a positive return on the open position.

[16] In sections 3 and 4 of this paper, we examine aggregation under the heterogeneity of preferences and forecasting models within each group of bulls and bears and show that under mild aggregation conditions, our conclusions remain valid.

a well-defined momentary equilibrium requires that the amount of capital each speculator chooses to place at risk (i.e., his position size) is limited. We model this decision problem on position size in section 3.2, by augmenting the original formulation of prospect theory with an assumption we call *dynamic loss aversion*. Dynamic loss aversion assumes that as speculators contemplate larger open positions – and, thus, contemplate larger potential gains and losses – they are more sensitive to the *increase* in the potential losses than to the concomitant *increase* in the potential gains.

We next use the standard decomposition of next period's rate of return, $R_{t+1} = R^{+}_{t+1} + R^{-}_{t+1}$[17], to represent potential losses of bulls and bears. Because bulls (bears) view negative (positive) realizations of R_{t+1} as potential losses, we represent these losses by the variable $R^{-}_{t+1}(-R^{+}_{t+1})$. Furthermore, we represent an individual assessment of the potential losses, which we refer to an *expected loss* and denote by l^{i}_{t}, as the mean of the "loss-part" – R^{-}_{t+1} for bulls and $-R^{+}_{t+1}$ for bears – of the individual probability distributions.[18]

In section 3.2, we show that dynamic loss aversion implies that all agents limit the size of their gambles when an expected profit opportunity arises.[19] We find that the optimal position size for bulls and bears – f^{L}_{t} and f^{S}_{t}, respectively – implies that speculators hold open positions (either long or short) in foreign exchange, only if they expect a positive return *in excess* of some minimum value, which we call an individual uncertainty premium and denote by up^{i}_{t}. Thus, the optimal position size for bulls and bears, $f^{\text{L}}_{t} = f^{\text{L}}_{t}(\tilde{r}^{\text{L}}_{t+1} - up^{\text{L}}_{t})$ and $f^{\text{S}}_{t} = f^{\text{S}}_{t}(\tilde{r}^{\text{S}}_{t+1} - up^{\text{S}}_{t})$, is increasing in the conditional forecast of the return and decreasing in the uncertainty premium.

Our solution for the optimal position size also shows that the individual uncertainty premia can be interpreted as compensation for potential losses: individual uncertainty premia depend positively on the degree of loss aversion and negatively on agents' expected losses. This result implies that the optimal positions of bulls and bears, $f^{\text{L}}_{t} = f^{\text{L}}_{t}(\tilde{s}^{\text{L}}_{t+1}, s_t, fp_t, l^{\text{L}}_{t})$ and $f^{\text{S}}_{t} =$

[17] $R^{+}_{t+1} = R_{t+1}I(R_{t+1} > 0)$ and $R^{-}_{t+1} = R_{t+1}I(R_{t+1} < 0)$, where $I(.)$ is an indicator function.

[18] Note that the expected losses of bulls and bears – l^{L}_{t} and l^{S}_{t} – are *both* negative magnitudes. We will refer to decreases in l^{L}_{t} and l^{S}_{t} as increases in the magnitudes of the potential losses.

[19] In contemplating models with heterogeneous expectations, the behavioral economics literature has relied solely on the assumption of risk aversion to obtain what is called *limits to arbitrage* (e.g., see Barberis and Thaler, 2002). Our analysis with dynamic prospect theory shows that limits to speculation can also be achieved solely on the basis of dynamic loss aversion.

$f_t^{\mathrm{S}}(\tilde{s}_{t+1}^{\mathrm{S}}, s_t, fp_t, l_t^{\mathrm{S}})$, are decreasing in the magnitudes of the potential losses.

In section 3.3, we set the total of long positions equal to the total of short positions to derive UAUIP. We find that this equilibrium relationship is characterized by an equality between the market's expected return (or average opinion) on foreign exchange, $\tilde{r}_{t+1} = \tilde{r}_{t+1}^{\mathrm{L}} + \tilde{r}_{t+1}^{\mathrm{S}}$, and the aggregate uncertainty premium, $up_t = up_t^{\mathrm{L}} - up_t^{\mathrm{S}}$. The equilibrium uncertainty premium, then, is the uncertainty premium of bulls in excess of the uncertainty premium of bears. The upward-sloping 45^o line in figure 1 is a locus of such equilibrium points in the model.

One of the implications of UAUIP is that the sign of the equilibrium premium on foreign exchange is determined by the relative magnitudes of $\tilde{r}_{t+1}^{\mathrm{L}}$ and $\tilde{r}_{t+1}^{\mathrm{S}}$, i.e., in equilibrium, if bulls (bears) dominate in determining the sign of the average opinion – $\tilde{r}_{t+1}^{\mathrm{L}} > -\tilde{r}_{t+1}^{\mathrm{S}}(\tilde{r}_{t+1}^{\mathrm{L}} < -\tilde{r}_{t+1}^{\mathrm{S}})$ – then a balance between long and short positions in the market obtains when bulls (bears) require a higher minimum return for holding open positions – $up_t^{\mathrm{L}} > up_t^{\mathrm{S}}$ $(up_t^{\mathrm{L}} < up_t^{\mathrm{S}})$. Figure 1 assumes bears dominate in terms of the average opinion at the initial equilibrium point (point A), i.e., $\tilde{r}_{t+1} = up_t < 0$.

The f_t^{L} and f_t^{S} curves in figure 2 plot at time t the total of long positions held by bulls and the total of short positions held by bears as negative and positive functions of the exchange rate, s_t, respectively. The intersection of these two curves at point A, then, determines the equilibrium exchange rate. The equilibrium at point A in figures 1 and 2 is defined for *a given set* of forecasting models, i.e., for given $\mathcal{P}_t^i(S_{t+1}|\mathcal{X}_t^i, \theta_t^i)$'s.

The logic behind the slopes and shifts of the f_t^{L} and f_t^{S} curves is central to our analysis and its implications for the behavior of the uncertainty premium over time. Both shifts in and movements along these curves are associated with revisions of forecasting models. In our IKF framework, the process of revising forecasts does *not* just entail a *mechanical* updating of the means of agents's conditional distributions, but it is also compatible with a simultaneous revision of the higher moments of these distributions;.Under IKF revisions of forecasts, in general, involve revisions of the forecasting models.[20]

[20] The standard REH approach to the updating of forecasts is to model this process as a *passive recomputation* of conditional forecasts triggered by new realizations of the variables in agents' information sets. The IKF framework not only allows for such conventional, information-driven, updating, but it is also compatible with revisions of forecasts that involve changes in the composition of the set of variables included in the model and/or revisions of the values of the model parameters.

Figure 1

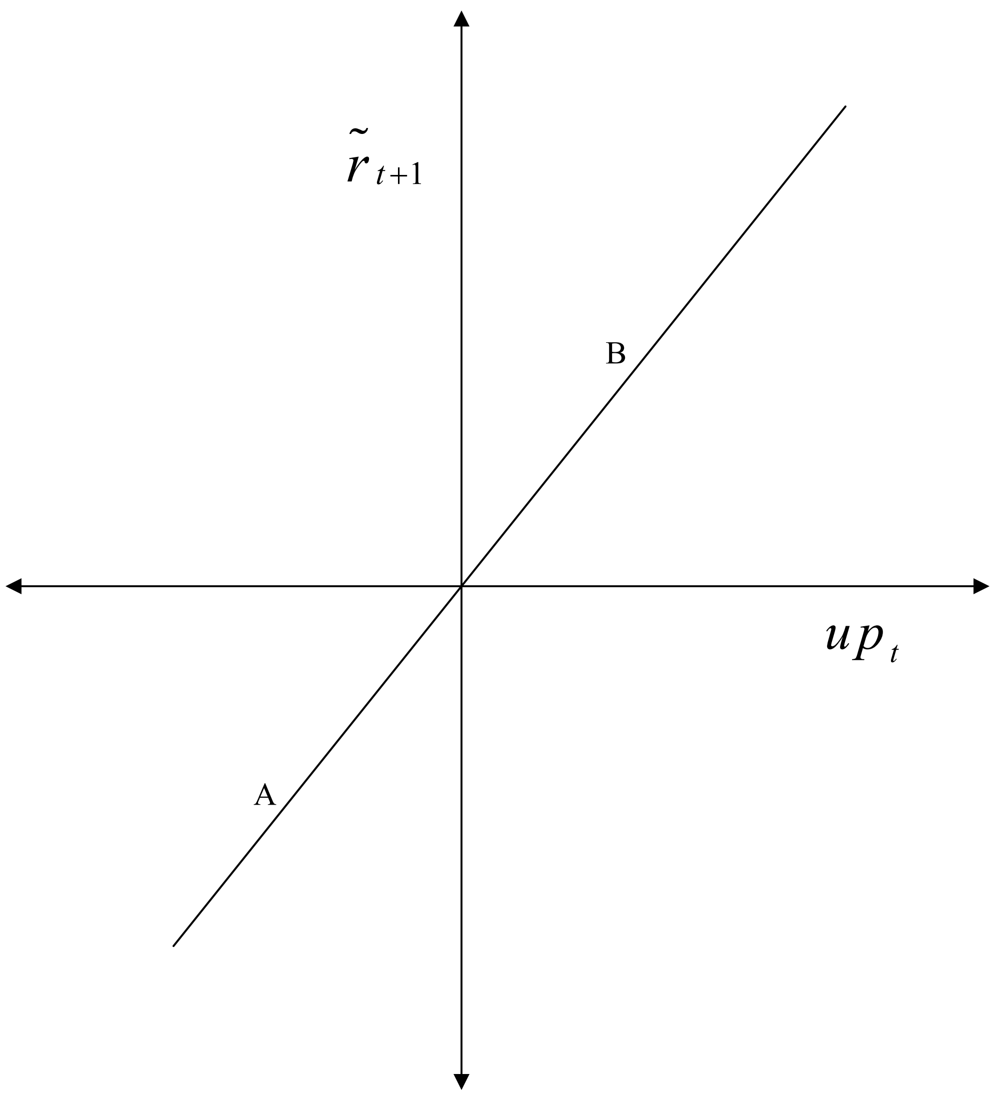
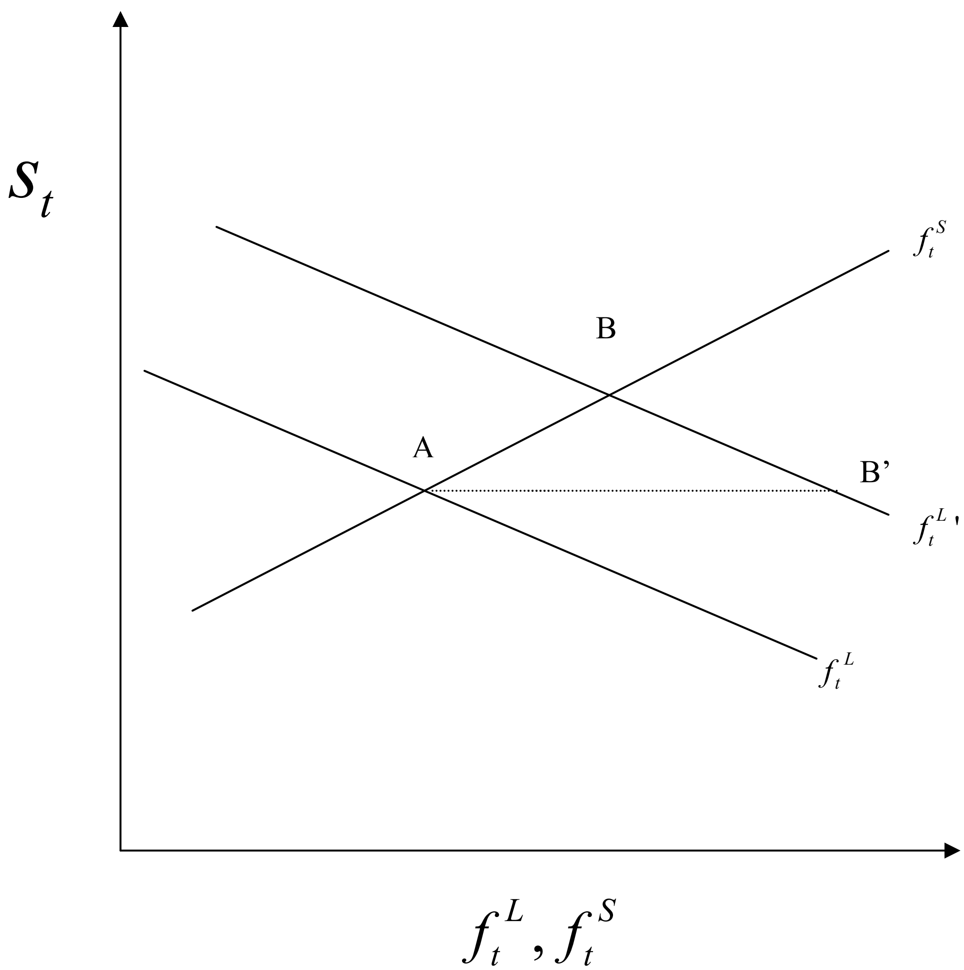


Figure 2



In section 4, we use the IKF framework to model revisions of the individual conditional probability distributions and examine the implications of this process for the movements of the equilibrium uncertainty premium over time. To illustrate our analysis, consider an autonomous (*for a given* s_t) upward revision of the mean of the conditional distribution for a representative bull, $\tilde{s}^{\text{L}}_{t+1}$, assuming that fp_t is given.[21] The impact of this revision is shown as a rightward shift of the f^{L}_t curve in figure 2. The movement of the equilibrium from point A to point B can be decomposed into two steps; the shift of f^{L}_t from A to B', *for a given* s_t, and the resulting movement of s_t, along the f^{L}_t and f^{S}_t curves to the new equilibrium at point B.

In section 4.1, we build on a profound – yet neglected – idea put forth by Keynes (1936) to model the first-step shift in $f^{\text{L}}_t(\tilde{s}^{\text{L}}_{t+1}, s_t, l^{\text{L}}_t)$. We assume that an autonomous increase in the bulls' conditional forecast $\tilde{s}^{\text{L}}_{t+1}$ not only causes $\tilde{r}^{\text{L}}_{t+1}$ to rise, but it also causes l^{L}_t to fall (i.e. its magnitude rises) because of a *gap effect*: a bull's (bear's) expected loss depends negatively (positively) on the divergence, or gap, between his conditional forecast and the perceived historical benchmark level, which we denote by $\tilde{s}^{\text{HB}i}_t$.

To understand the logic behind the gap effect, assume for concreteness that $s_t > \tilde{s}^{\text{HB}i}_t$ for all agents, i.e., all agents believe that the foreign currency is overvalued relative to the perceived historical benchmark. As bulls increase $\tilde{s}^{\text{L}}_{t+1}$, they therefore expect a greater overvaluation of the foreign currency. This upward revision can occur despite the assumed belief, on the part of all agents, that the asset price will eventually adjust back to its perceived benchmark level. The forecasting problem all agents face, therefore, is that the asset price may move persistently away from its perceived benchmark over an extended time period, prior to "reverting" back. The gap effect implies that as bulls raise their expectation of an overvaluation, they become more fearful of a movement back to the historical benchmark. This, then, leads bulls to increase their assessment of the magnitude of the potential losses.

As bulls raise their assessments of the potential losses, they simultaneously raise their uncertainty premia, up^{L}_t. Thus, in general, $\tilde{r}^{\text{L}}_{t+1} - up^{\text{L}}_t$ may rise or fall. However, we assume that although agents' loss aversion limits their willingness to risk capital, the degree of loss aversion is not so high

[21] We explore the movement of the equlibrium premium following a change in fp_t in Frydman and Goldberg (2002), where we find that our IKF model of the uncertainty premium sheds new light on the forward-premium anomaly.

as to prevent agents from changing the size of their positions in the same direction as the change in their conditional forecasts, i.e., $\tilde{r}^{\mathrm{L}}_{t+1} - up^{\mathrm{L}}_t$ rises above $\tilde{r}^{\mathrm{S}}_{t+1} - up^{\mathrm{S}}_t$. This increase in the expected excess return leads bulls to increase their long positions, as depicted by the shift of f^{L}_t from point A to B' in figure 2. At the initial exchange rate, then, the total of long positions is greater than the total of short positions, causing s_t to rise.

In section 4.2, we model the second step movement of s_t from its initial equilibrium level to its new equilibrium level at point B. An increase in s_t leads to revisions of the distribution of next period's return, R_{t+1}, for both bulls and bears. In general, these revisions involve not only changes in the means, $\tilde{r}^{\mathrm{L}}_{t+1}$ and $\tilde{r}^{\mathrm{S}}_{t+1}$, but also changes in l^{L}_t and l^{S}_t, and, thus, changes in up^{L}_t and up^{S}_t. The slopes of the f^{L}_t and f^{S}_t curves in figure 2 follow from two qualitative assumptions on the updating of $\tilde{s}_{t+1}$ as a consequence of changes in s_t.[22] Under these assumptions, the rise in s_t causes $\tilde{r}^{\mathrm{L}}_{t+1} - up^{\mathrm{L}}_t$ to fall and $\tilde{r}^{\mathrm{S}}_{t+1} - up^{\mathrm{S}}_t$ to rise, thereby working to reestablish equilibrium and a balance between the total of long and short positions.

The equilibrium movements of $\tilde{r}_{t+1}$ and up_t are shown in figure 1 as a movement from point A to point B. The figure assumes that the initial first-step increase in $\tilde{s}^{\mathrm{L}}_{t+1}$ (and therefore in $\tilde{r}^{\mathrm{L}}_{t+1}$) was large enough so that, after the equilibrium movement in s_t, the average opinion not only increases, but its algebraic sign changes from negative to positive. Equilibrium, then, requires that the aggregate uncertainty premium not only increase, but that its algebraic sign also change from negative to positive, as the dominant weight in the average opinion shifts from bears to bulls. Thus, our IKF model of the uncertainty premium provides a simple explanation of the sign reversals in foreign exchange premia that have been observed in the literature.[23]

The shifts in figures 1 and 2 also illustrate another implication of our IKF model of the equilibrium uncertainty premium, which we refer to as an aggregate gap effect: *up_t increases with the expected gap from benchmark levels,* $\widetilde{gap}_t = \tilde{s}_{t+1} - \tilde{s}^{\mathrm{HB}}_t$.

[22]The first assumption is that $\frac{d\tilde{s}^i_{t+1}}{ds_t} < 1$, which we need for stability of the momentary equilibrium. The second qualitative assumption is $\frac{d\tilde{s}^i_{t+1}}{ds_t} > 0$.We refer to this assumption as an *average opinion effect.* We motivate this assumption by drawing on the extensive evidence in the behavioral economics literature that this assumption captures the behavior of agents in the real world markets.

[23]See Lewis (1995) and Mark and Woo (1998) for the inability of standard REH models to explain the observed sign reversals in foreign exchange premia.

In section 4.3, we formulate a model of foreign exchange premia with a house-money effect and dynamic loss aversion, and show that this model implies a *negative* relationship between the equilibrium uncertainty premium and the expected gap from benchmark levels.

Section 5 provides empirical evidence in support of our IKF model with an aggregate gap effect. Figure 3 illustrates this new empirical finding well. The figure plots six-month averages of up_t, using the MMSI data, and the expected gap, in a sample that spans the period from January 1983 through December 1996.[24] Although not a statistical test, the time plots are rather suggestive that the relationship between up_t and $\widetilde{gap}_t$ is indeed positive. Formal tests reveal that such a positive relationship is supported at very high significance levels. Thus, we are able to reject the IKF model with a house money effect in favor of the IKF model with a gap effect;.but see footnote 10 above.

Section 6, which concludes the paper, discusses the relative merits of the qualitative IKF and the quantitative REH approaches to the modeling of the forecasting process in a world of imperfect knowledge.

3 Modeling the Decisions of Speculators at a Point in Time

In this section we model the decisions of speculators in the foreign exchange market under the assumptions of a diversity of individual forecasting models and a heterogeneity of expectations. The introduction of heterogeneous expectations raises a problem that has been noted in behavioral-finance models such as Delong, Shleifer, Summers and Waldman (1990a,b) and Shleifer and Summers (1990): heterogeneity of forecasts and unlimited short selling require limits to speculation in order to obtain a well-defined equilibrium.[25]

[24] To obtain a measure of $\widetilde{gap}_t$, we used the Big Mac PPP exchange rate reported in the April 1993 edition of the *Economist* (which was 2.02), and then used inflation differentials (based on CPI series from the IFS data bank) to estimate the PPP exchange rate both forwards and backwards.

[25] A detailed examination of this problem is outside the scope of this paper. In Frydman and Goldberg (2003b) we examine this problem within the context of the monetary models of Dornbusch (1976) and Frankel (1979). We find that under heterogeneous expectations, equilibrium in this class of models is no longer characterized by uncovered interest rate parity.

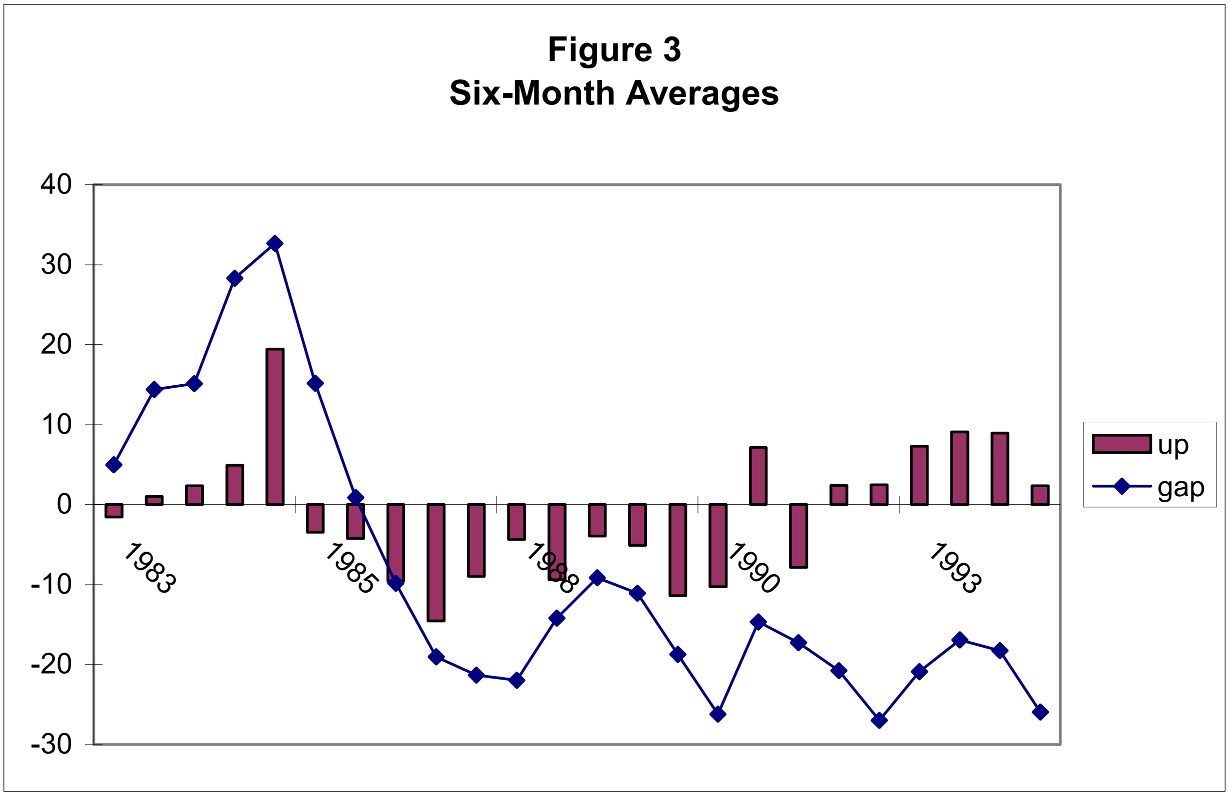


Figure 3
Six-Month Averages
40
30
20
10
0
-10
-20
-30
1983
1985
1988
1990
1993
up
gap

To solve this problem, the literature assumes that all speculators are risk averse.[26] We show in this section that limits to speculation can also be obtained solely on the basis of prospect theory, which assumes that utility functions are specified over *the change* in wealth, provided that the original formulation of prospect theory is extended to the dynamic setting of financial market speculation.

We begin with a description of the speculative decision in the foreign exchange market under the standard assumption of perfect capital mobility. At every point in time, both domestic and foreign speculators operating in the foreign exchange market face a decision on whether to take a long or a short position in foreign exchange. The assumption of perfect capital mobility implies that only pure speculation matters, i.e., the activity of borrowing capital so as to take simultaneously a short position in the borrowed currency and a long position of equal size in the other currency.[27]

The *ex post value* of the return on a long position of *one unit* of foreign exchange from time t to $t+1$, which we denote by r_{t+1}, can be approximated as follows:

$$r_{t+1} = (s_{t+1} - s_t) - fp_t \tag{1}$$

Since $-r_{t+1}$ is the *ex post* return from a short position of *one unit* of foreign exchange, the decision of speculator i on whether to take a long or short position in foreign exchange depends on *his individual forecast at time t* of r_{t+1} in equation (1), $\tilde{r}^i_{t+1}$.

If agent i were risk neutral, then he would take a long (short) position in foreign exchange if $\tilde{r}^i_{t+1} > 0$ ($\tilde{r}^i_{t+1} < 0$). In developing our model, we maintain that at every point in time bulls and bears *coexist* in the market. Since the forecasts of bulls and bears are necessarily of the opposite sign, the coexistence of bulls and bears in the market necessarily implies a *heterogeneity* of expectations.

[26] This result, that risk aversion leads to *limits to arbitrage*, is viewed as one of the main pillars of behavioral finance (see Shleifer and Summers, 1990 and Barberis and Thaler, 2002). Since in a world of imperfect knowledge no speculator knows the "true fundamental value," we will use the term *limits to speculation*.

[27] Perfect capital mobility implies that pure speculative flows are infinitely elastic with respect to the expected return on foreign exchange, eliminating the need to consider capital flows originating from all other sources. It is based on the assumption of no barriers to the international flow of capital, including no constraints on short selling. See Mundell (1963) and Dornbusch (1976).

3.1 Diversity of Forecasting Models

To develop our model, we need a formal characterization of the *creative* process by which individual agents form their forecasts of r_{t+1}. In general, this process can be conceptualized as involving three basic steps:

- **A description of the information set on which individual agents form forecasts**

  The information set used by agent i at time t, denoted by $\mathcal{X}_t^i$, can be divided into two basic subsets: a subset, $\mathcal{X}_t^{i,f}$, consisting of variables capturing information contained in fundamental factors (e.g. money supply or GDP growth) and a subset, $\mathcal{X}_t^{i,nf}$, consisting of non-fundamental factors (e.g., those based on technical trading or market psychology) considered relevant by agent i for forecasting the return to foreign currency speculation. We note that the fundamental and/or non-fundamental subsets of the information set are indexed by t and thus the composition of either of these subsets is *not* restricted to remain unchanged over time. This reflects the recognition that, in general, individual agents may use different sets of variables in forming their forecasts at different points in time.[28] To streamline the exposition, we treat all of the variables in $\mathcal{X}_t^i$ as random variables, even though some of these (particularly those in $\mathcal{X}_t^{i,nf}$) may be qualitative in nature or constants, and thus, may have degenerate marginal probability distributions. In what follows, we assume that the information set for each agent includes S_t and FP_t.[29]

- **A formal representation of the individual forecasting model**

  To form forecasts in a world of imperfect knowledge, individual agents need to form assessments of the likelihood of potential realizations of next period's return, R_{t+1}. They arrive at such judgements on the

[28] For empirical evidence that agents use different variables in their forecast functions at different points in time see Goldberg and Frydman (1996b,2001). See also Lyons (2001) and the survey studies of Cheung and Chinn (2001) and Cheung *et al* (1999).

[29] This plausible assumption simplifies our notation considerably, in that it allows us to express, in a particularly simple way, the conditional distribution of R_{t+1} , $\mathcal{P}_t^{\mathcal{R},i}$, in terms of the conditional distribution of S_{t+1}: $\mathcal{P}_t^{\mathcal{R},i}(R_{t+1}|\mathcal{X}_t^i, \theta_t^i) = \mathcal{P}_t^i(S_{t+1}|\mathcal{X}_t^i, \theta_t^i) - S_t - FP_t$. Note that if either S_t and/or FP_t were excluded from $\mathcal{X}_t^i$, we would need separate notation for the conditional distributions for R_{t+1} and S_{t+1}. Because keeping the two distributions distinct would not affect any of our conclusions, we ignore this possibility.

basis of the time-t realizations of the variables *they choose* to include in their information set, and by some *formal or informal* procedure *they choose* to map these realizations into their assessments of the likelihood of potential values of R_{t+1}. One way to model this forecasting process, which we follow in this paper, is to represent an *individual forecasting model* of R_{t+1} in terms of a conditional probability distribution over S_{t+1}.[30] We denote this latter distribution by $\mathcal{P}_t^i(S_{t+1}|\mathcal{X}_t^i, \theta_t^i)$, where θ_t^i is a set of parameters of this conditional distribution.

- **An individual (conditional) forecast**

  An individual forecast of R_{t+1} is based on the mean of the conditional distribution $\mathcal{P}_t^i(S_{t+1}|\mathcal{X}_t^i, \theta_t^i)$, evaluated at the time-$t$ realizations of the variables in $\mathcal{X}_t^i$, denoted by x_t^i, and the time-t values of the parameters θ_t^i:

$$\tilde{r}_{t+1}^i = E_{\mathcal{P}_t^i}(S_{t+1}|x_t^i, \theta_t^i) - s_t - fp_t = \tilde{s}_{t+1}^i - s_t - fp_t \tag{2}$$

Beyond providing a formal representation of the individual forecasting models, the foregoing formulation highlights the two potential sources of the heterogeneity of expectations: 1) a diversity among the individual conditional distributions, $\mathcal{P}_t^i(.|.)$; and/or 2) differences in the measurement of the variables in some common information set, $\mathcal{X}_t$, on which forecasts are based, for example, as in the Lucas (1973) island model.

We note that our formulation of individual forecasting models subsumes the REH-based expectation function as a special case: under the REH, all agents form expectations based on *the common*, sometimes referred to as the "objective", probability distribution, $\mathcal{P}_t$. Thus, to rationalize the heterogeneity of expectations, conventional REH models *must* appeal to differences in the measurement of the variables appearing in the common forecasting model.[31]

In contrast, heterogeneity in the IKF framework can be rationalized by appealing to the implications of the postulate of economic rationality in a

[30] The formal representation of the individual forecasting models as conditional expectations is analytically convenient. The analysis in this paper can be extended to incorporate one of the elements of prospect theory: that instead of attaching probabilities to prospects, agents attach *decision weights* to them. See Kahneman and Tversky (1979) and Tversky and Kahneman (1991,1992). Also see footnote 33 below for a remark on this point.

[31] See Lyons (2001) for such an approach in the foreign exchange market. Although an extensive discussion of the difficulties involved with such rationalizations is outside the scope of this paper, we return to some of these problems in section 5.

world of *imperfect knowledge*. For example, Frydman (1982) formally shows that *rational* agents, who *do not* pass up opportunities for gain, *will not*, in general, use a *common* model and/or rely on the same information when forming forecasts in a world of imperfect knowledge. Thus, a heterogeneity of forecasting models as well as differences in the information used by agents can be linked to the *rationality* of agents coping with the fact that they must forecast on the basis of *imperfect knowledge*.

3.2 Position Size at a Point in Time: Dynamic Prospect Theory for a Given Set of Individual Forecasting Models

In this subsection we build on the seminal formulation of prospect theory due to Kahneman and Tversky (1979) and Tversky and Kahneman (1992). Kahneman and Tversky developed prospect theory to explain the often gross inconsistencies between the actual choices of individuals who are faced with risky or uncertain outcomes and the predictions implied by the conventional expected utility hypothesis. The typical choice problems considered by Kahneman and Tversky were static, in that they involved a rich variety of monetary gambles involving choices between *exogenously fixed* outcomes with respect to the potential gains and losses. The large literature in psychology and economics has presented striking evidence that in such situations, prospect theory is able to explain the observed choices of individual agents much more successfully than the conventional expected utility hypothesis.

Although gambles with exogenously fixed outcomes characterize a large class of decision problems under uncertainty, such gambles do not characterize speculation in financial markets. Outcomes from financial speculation are endogenous, in the sense that, given speculators' forecasting models of next period's return, they must decide on the amount of capital to place at risk. This decision will then determine the overall potential gains and losses.

In applying prospect theory to speculative decisions, however, there is one *special* case – the case of homogeneous expectations – in which the decision of how much to gamble when a profit opportunity is perceived need not be modeled. With homogeneous expectations, there is a unique price at which agents are willing to hold a given supply and this equilibrium level is independent of the size of speculative positions outside of equilibrium. Since recent applications of prospect theory to the modeling of asset markets all make use of the standard representative-agent assumption (e.g., Benartzi and Thaler, 1995, Barberis, Huang and Santos, 2001 and Barberis and Huang, 2001), they are able to sidestep the problem of modeling the size of speculative positions.

In this paper, we assume a heterogeneity of forecasting models and expectations. In this setting, we find that the original, static formulation of prospect theory does not provide a basis for modeling the decision of how much to gamble, given the individual forecasting models. This consideration leads us to our assumption of *dynamic loss aversion*, which imposes a condition on the curvature of the utility function over gains relative to its curvature over losses. In this section, we make use of a particularly simple version of dynamic loss aversion, which connects the degree of loss aversion to position size, thereby making preferences dynamic.[32] We show that with dynamic loss aversion, loss-averse agents limit the size of their gambles when an expected profit opportunity arises. This result leads to our new equilibrium condition for the foreign exchange market,

3.2.1 Dynamic Prospect Theory

According to Tversky and Kahneman (1991, pp. 1039-1040), prospect theory implies that risky prospects are evaluated by a value function that has the following three basic characteristics:[33]

- **Reference Dependence**

  "The carriers of value are gains and losses defined relative to a reference point and the effect of the *level of wealth* on utility is assumed to be

[32] In applying prospect theory to financial markets, Barberis, Huang and Santos (2001) and Barberis and Huang (2001) also make preferences dynamic by combining the assumption of loss aversion with the house-money effect. As we show in this subsection, such a specification of dynamic preferences would not *by itself* limit the size of speculative positions. However, Barberis, Huang and Santos (2001) and Barberis and Huang (2001) embed their version of dynamic preferences under loss aversion into a general utility function that is also defined over the level of consumption. Since this part of the utility function is assumed to be concave, these studies would in principle be able to obtain limits to speculation based on the standard assumption of risk aversion if the assumption of homogenous expectations were dropped.

[33] Although Kahneman and Tversky "restrict" their original discussion to prospects with so-called "objective or standard probabilities" they note that "the theory can also be... [applied] to the *typical* situations of choice, where the probabilities of outcomes are not explicitly given (Kahneman and Tversky, 1979, p. 263)."

relatively negligible."

- **Loss Aversion**

  The disutility from losses exceeds the utility from gains of the same magnitude. As they put it, "losses loom larger than corresponding gains."

- **Diminishing Sensitivity**

  "The marginal value of both gains and losses decreases with their size"

Benartzi and Thaler (1995) extend the notion of loss aversion in an insightful and important way by observing that "when decision makers are loss-averse they will be more willing to take risks if they evaluate their performance...infrequently (Benartzi and Thaler, 1995, p. 75)." This leads Benartzi and Thaler to a notion they call "myopic loss aversion," which combines loss aversion with *frequent* evaluation. We follow Benartzi and Thaler by adopting the assumption of myopic loss aversion as one of the key assumptions of our approach. In what follows we assume that speculators on foreign exchange evaluate their performance every period.

The distinction between gains and losses plays a central role in prospect theory and so we need expressions for agents' forecasts of the potential gains and losses from speculation. Using the standard decomposition of R_{t+1}, we have

$$\tilde{r}^{i}_{t+1} = E_{\mathcal{P}^i_t}(R_{t+1}|x^i_t,\theta^i_t) = E_{\mathcal{P}^i_t}(R^{+}_{t+1}|x^i_t,\theta^i_t) + E_{\mathcal{P}^i_t}(R^{-}_{t+1}|x^i_t,\theta^i_t) = \tilde{r}^{+,i}_{t+1} + \tilde{r}^{-,i}_{t+1} \tag{3}$$

Recalling, that for a bull (bear), positive realizations of R_{t+1} imply gains (losses) and negative (positive) realizations imply losses (gains) on an open position equal to *one unit* of foreign exchange, we can express the total expected gain and loss of a bull and a bear as follows:

Remark 1 *Let $g^{i_L}_t$ ($g^{i_S}_t$) and $l^{i_L}_t$ ($l^{i_S}_t$) denote the expected gain and loss at time t of a bull (bear) who has a one-unit long (short) position in foreign exchange. Also, let f^i denote the number of units of foreign exchange that speculator i has decided to gamble, to be referred to as the size of speculator i's position. Then the* ***total*** *expected gain and the* ***total*** *expected loss on a position of size $f^i_t \geq 0$ are*[34]

[34] Note that although we use upper case G and L to denote the total expected gains and losses, it is clear from (4)-(7) that G and L *are not* random variables but rather the *values* of the total expected gains and losses evaluated at time t.

for a bull:

$$G_t^{i_{\mathrm{L}}} = g_t^{i_{\mathrm{L}}} f_t^{i_{\mathrm{L}}} = \tilde{r}_{t+1}^{+,i} f_t^{i_{\mathrm{L}}} = E_{\mathcal{P}_t^i}(R_{t+1}^{+}|x_t^i, \theta_t^i) f_t^{i_{\mathrm{L}}} \geq 0 \tag{4}$$

$$L_t^{i_{\mathrm{L}}} = l_t^{i_{\mathrm{L}}} f_t^{i_{\mathrm{L}}} = \tilde{r}_{t+1}^{-,i} f_t^{i_{\mathrm{L}}} = E_{\mathcal{P}_t^i}(R_{t+1}^{-}|x_t^i, \theta_t^i) f_t^{i_{\mathrm{L}}} \leq 0 \tag{5}$$

and for a bear:

$$G_t^{i_{\mathrm{S}}} = g_t^{i_{\mathrm{S}}} f_t^{i_{\mathrm{S}}} = -\tilde{r}_{t+1}^{-,i} f_t^{i_{\mathrm{S}}} = -E_{\mathcal{P}_t^i}(R_{t+1}^{-}|x_t^i, \theta_t^i) f_t^{i_{\mathrm{S}}} \geq 0 \tag{6}$$

$$L_t^{i_{\mathrm{S}}} = l_t^{i_{\mathrm{S}}} f_t^{i_{\mathrm{S}}} = -\tilde{r}_{t+1}^{+,i} f_t^{i_{\mathrm{S}}} = -E_{\mathcal{P}_t^i}(R_{t+1}^{+}|x_t^i, \theta_t^i) f_t^{i_{\mathrm{S}}} \ \leq 0 \tag{7}$$

We can now specify the preferences of a loss-averse speculator as a function of his expected gain and loss. Tversky and Kahneman (1992) proposed the following specific functional form for the utility function of loss-averse agents,

$$V^i = \begin{cases} G_t^{i\beta} \\ -\lambda_i \left(-L_t^i\right)^{\alpha} \end{cases} \tag{8}$$

where the parameter λ^i captures the *degree of loss aversion*, and β and α determine the curvature of the utility function over total gains and losses, respectively. A value of λ^i greater than 1 implies loss aversion on the part of agents, namely that losses loom larger than corresponding gains. Based on experimental evidence involving one-shot, fixed gambles, Kahneman and Tversky (1992) estimated the value of λ^i to be in excess of 2.[35] Since the curvature of the utility function over potential gains *relative* to its curvature over potential losses plays no role in static decision problems, Kahneman and Tversky (1992) set $\beta = \alpha$. Under this assumption, Kahneman and Tversky found $\beta = \alpha = .88$, implying that V^i is concave over gains and convex over losses.

In applying prospect theory to financial markets, Barberis, Huang and Santos (2001) and Barberis and Huang (2001) use a simpler, linear version of the utility function in (8):

$$V^i(G_t^i, L_t^i) = g_t^i f_t^i. + \lambda^i l_t^i f_t^i = v_{\mathrm{G}t}^i f_t^i + v_{\mathrm{L}t}^i f_t^i = V_{\mathrm{G}t}^i + V_{\mathrm{L}t}^i \tag{9}$$

where we express the utility function for a loss averse speculator in terms of the expected gain and loss from a one-unit position in foreign exchange, g_t^i

[35] This result has been found in a wide range of experimental contexts. See Barberis and Thaler (2002) and references therein.

and l_t^i, multiplied by the size of the position, f_t^i. Thus, $\left(v_{\mathrm{G}_t}^i + v_{\mathrm{L}_t}^i\right)$ denotes the expected utility from an open position of *one unit* and $\left(V_{\mathrm{G}_t}^i + V_{\mathrm{L}_t}^i\right)$ is the expected utility on a position of size f_t^i.

The utility function in (9) captures loss aversion, but ignores the assumption of diminishing sensitivity. Barberis and Thaler argue that such a simplification is justified "because it is difficult to incorporate all these features into a fully dynamic framework; but also, it is based on Benartzi and Thaler's observation that it is mainly loss aversion that drives their results (Barberis and Thaler, 2002, p. 26)."

A utility function that is linear in total gains and losses, however, does not give rise to limits to speculation. Note that once a speculator formulates his forecasting model at time t and observes the realizations of the variables in his information set, the values of the conditional forecast, $\tilde{r}_{t+1}^i$, and the expected gain and loss, g_t^i and l_t^i, are pre-determined from the point of view of determining the optimal size of his position. Let $f_t^{\mathrm{O}_i}$ denote the position size of speculator i that maximizes his utility. Thus, if speculator i believes that $g_t^i + \lambda^i l_t^i < 0$, he will be unwilling to take an open position in foreign exchange of any size (i.e., $f_t^{\mathrm{O}_i} = 0$), whereas if $g_t^i + \lambda^i l_t^i > 0$, he will want to take an open position of unlimited size (i.e., $f_t^{\mathrm{O}_i} = \infty$).

Thus, to justify limits to speculation, solely on the basis of prospect theory, the utility function of loss-averse agents must be concave in f_t^i, i.e., the disutility of losses must grow faster than the utility of gains as the position size increases. Given the utility function in (8), this requires that $\beta < \alpha$, i.e., the degree of concavity of V^i over gains must be larger than its degree of convexity over losses. This leads to the following extension of prospect theory to the dynamic setting of financial-market speculation:

- **Dynamic Loss Aversion**

  The disutility of losses grows faster than the utility of gains as the magnitudes of losses and gains increase proportionately.

Remark 2 *We note that the assumption of dynamic loss aversion is in the spirit of the original formulation of prospect theory: as the magnitudes of losses and gains increase proportionately, the total* ***change*** *in losses is assumed to loom larger in utility than the total* ***change*** *in gains.*

To incorporate the assumption of dynamic loss aversion in a convenient way, we retain the additive form of the utility function in (9) with respect to the total expected gain and loss. We also ignore the third postulate of prospect theory, but unlike Barberis, Huang and Santos (2001) and Barberis and Huang (2001), we develop a specification that is linear in the expected gain and concave in the expected loss.[36] Such a specification follows from the assumption that there exists a positive relationship between the disutility of losses on a unit position, $v^i_{\mathrm{L}t}$, which Kahneman and Tversky (1979) assumed to be a constant, and position size. The idea behind this assumption is a natural one; agents' fear of potential losses grows as position size grows.

To keep the analysis simple, without affecting any of our conclusions, we assume that

$$v^i_{\mathrm{L}t} = \lambda^i l^i_t = \lambda^i_1 l^i_t - \lambda^i_2 f^i_t, \quad 1 < \lambda^i < \lambda^{\max} \tag{10}$$

where now the degree of loss aversion is a function of position size, i.e., $\lambda^i = \lambda^i_1 + \lambda^i_2 \frac{f^i_t}{(-l^i_t)}$, and $\lambda^{\max}$ is some constant.[37] Plugging (10) into (9) yields the following expression for the expected *total* utility on an open position of size f^i_t held by a loss-averse agent:

$$V^i_t = g^i_t f^i_t + \lambda^i_1 l^i_t f^i_t - \lambda^i_2 (f^i_t)^2 \tag{11}$$

The assumption of dynamic loss aversion, which implies concavity over f^i_t, ensures that there is a finite position size that maximizes speculator i's utility. Differentiating V^i_t with respect to position size, and setting the result equal to zero, yields the following expression for $f^{\mathrm{O}i}_t$:

$$f^{\mathrm{O}i}_t = \frac{1}{2\lambda^i_2}(g^i_t + \lambda^i_1 l^i_t) \tag{12}$$

Note that a negative value for $f^{\mathrm{O}i}_t$ occurs for speculator i only when $g^i_t + \lambda^i_1 l^i_t < 0$. In this case, speculator i expects that an open position in foreign exchange will not increase his utility and therefore decides to stay out of the market, i.e., $f^{\mathrm{O}i}_t = 0$. Given that $\lambda^i_1 > 1$, the speculator will stay out of the market even when $g^i_t + l^i_t > 0$, i.e., although speculator i may

[36] All we need is a concave function in f^i_t. Thus, our main results, although more complicated, would be unchanged if we were to maintain diminishing sensitivity over both gains and losses, as in Kahneman and Tversky (1979), and assumed that the curvature of V was greater over gains.

[37] Having made the degree of loss aversion a function of position size, we need to restrict the parameters λ^i_1 and λ^i_2 in (10) so that all values of λ^i lie within a range between 1 and $\lambda^{\max}$. This can be done easily under mild assumptions. See Frydman and Goldberg (2003b) for a derivation of such conditions.

expect to earn a positive return from an open position in foreign exchange, the size of this return may not be large enough to compensate him for his greater sensitivity to losses. The solution in (12) implies, therefore, that a positive $f_t^{o_i}$ is necessarily associated with $g_t^i + l_t^i > 0$. This reasoning leads to one of the major conclusions of our analysis:

Conclusion 1 *All dynamically loss-averse speculators require an expected return in excess of some individually determined positive value in order to take open positions in foreign exchange.*

In the next section, we connect this minimum expected return to the uncertainty faced by agents concerning the timing of countermovements in the exchange rate back to its historical benchmark level. We refer to this minimum expected return, therefore, as an individual uncertainty premium, up_t^i. To obtain a simple expression for up_t^i, we rewrite (12) as follows:

$$f_t^{o_i} = \frac{(g_t^i + l_t^i) + (\lambda_1 - 1)\, l_t^i}{2\lambda_2} = \frac{1}{2\lambda_2}\left[I(type^i)\tilde{r}_{t+1}^i - up_t^i\right] \tag{13}$$

where $I(type^i)$ is an indicator function such that if $type^i = bull$, $I = 1$, and if $type_i = bear$, $I = -1$. From (13), the minimum expected return agent i needs to take an open position is:

$$up_t^i = \left(1 - \lambda_1^i\right) l_t^i > 0 \tag{14}$$

The expression in (14) shows that the individual uncertainty premium can be written as a simple function of agent i's expected loss and degree of loss aversion as captured by the parameter $\lambda_1^i > 1$. Agent i will be willing to take an open position in foreign exchange only when his conditional forecast of next period's return is large enough to compensate him for his greater sensitivity to losses (i.e., $\tilde{r}_{t+1}^i > up_t^i$).

It is useful to compare the speculative decision under dynamic loss aversion with the decision under risk neutrality. With risk neutrality, a $\tilde{r}_{t+1}^i > 0$ ($\tilde{r}_{t+1}^i < 0$) leads agent i to take a long (short) position in foreign exchange of unlimited size. But with dynamic loss aversion, agent i limits his willingness to commit capital to the speculative game because his fear of losses grows as position size grows. The assumption of dynamic loss aversion, therefore, leads to limits to speculation. Moreover, as we noted earlier, the fear of losses may be large enough so that even though a speculator may expect a positive excess return, he may decide, nonetheless, to stay out of the market. This is never the case under risk neutrality.

We are now ready to use our solution of the speculative decision under dynamic prospect theory, and its implications for uncertainty premia and limits to speculation, to derive the momentary equilibrium condition in the foreign exchange market.

3.2.2 Momentary Equilibrium under Dynamic Prospect Theory and Diversity of Forecasting Models: Uncertainty Adjusted Uncovered Interest Parity

In deriving the momentary equilibrium condition for the foreign exchange market with heterogeneous expectations, we adopt an insight due to Keynes. Keynes argued in his analysis of speculation in the market for long-term debt that

> The market price will be fixed at the point at which the sales of the "bears" and the purchases of the "bulls" are balanced (Keynes, 1936, p. 170).

In terms of our model of speculation, the "sales of bears" translates into the cumulative total of short positions in foreign exchange and the "purchases of the bulls" translates into the cumulative total of long positions in foreign exchange. Aggregating (13) for bulls yields

$$\sum_{i}^{n^{\mathrm{L}}} f_t^{\mathrm{O}_{i_{\mathrm{L}}}} = w \sum_{i}^{n^{\mathrm{L}}} w^{i_{\mathrm{L}}} \tilde{r}_{t+1}^{i_{\mathrm{L}}} - w \sum_{i}^{n^{\mathrm{L}}} w^{i_{\mathrm{L}}} [1 - \lambda_1^{i_{\mathrm{L}}}] l_t^{i_{\mathrm{L}}} = w \left(\tilde{r}_{t+1}^{\mathrm{L}} - up_t^{\mathrm{L}} \right) \tag{15}$$

where

$$w = \frac{1}{\sum_i^{n^{\mathrm{L}}+n^{\mathrm{S}}} \frac{1}{2\lambda_2^i}} \quad \text{and} \quad 0 < w^{i_{\mathrm{L}}} = \frac{1}{2\lambda_2^{i_{\mathrm{L}}}} \frac{1}{w} < 1 \text{ and } \sum_{i}^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i = 1 \tag{16}$$

Analogously for bears

$$\sum_{i}^{n^{\mathrm{S}}} f_t^{\mathrm{O}_{i_{\mathrm{S}}}} = -w \sum_{i}^{n^{\mathrm{S}}} w^{i_{\mathrm{S}}} \tilde{r}_{t+1}^{i_{\mathrm{S}}} - w \sum_{i}^{n^{\mathrm{S}}} w^{i_{\mathrm{S}}} [1 - \lambda_1^{i_{\mathrm{S}}}] l_t^{i_{\mathrm{L}}} = w \left(-\tilde{r}_{t+1}^{\mathrm{S}} - up_t^{\mathrm{S}} \right) \tag{17}$$

where w^{i_S} is defined in a way similar to (16). The weights w^{i_L} and w^{i_S} have straightforward interpretations. Note that these weights are negative functions of the degree of loss aversion as measured by the parameter λ_2^i. From (13), however, higher values of λ_2^i, *ceteris paribus*, imply smaller open positions in foreign exchange. Thus, the aggregation weights used in expressions (15) and (17) reflect the fact that as λ_2^i rises, agent i's forecast, $\tilde{r}_{t+1}^i$, and uncertainty premium, up_t^i, will have less of an influence on the exchange rate, and therefore less of an influence on the average expected return and uncertainty premium.

Let

$$\tilde{r}_{t+1} = \tilde{r}_{t+1}^{L} + \tilde{r}_{t+1}^{S} = \sum_{i}^{n^L+n^S} w^i \tilde{r}_{t+1}^i \text{ and } up_t = up_t^L - up_t^S \tag{18}$$

then the equality between the cumulative total of long positions in (15) and short positions in (17), and thus the condition for momentary equilibrium, can be expressed as follows:

$$\tilde{r}_{t+1}^{L} + \tilde{r}_{t+1}^{S} = up_t^L - up_t^S \Longleftrightarrow \tilde{r}_{t+1} = up_t \tag{19}$$

The terms up_t^L and up_t^S are weighted sums that move one-for-one with the weighted averages of the uncertainty premia of the group of bulls and bears, respectively. These terms represent the minimum expected return that will induce the group of bulls and the group of bears to take open positions in foreign exchange. Thus, equilibrium in the foreign exchange market obtains when on average, the expected return of the bulls, *in excess* of the minimum expected return needed to take long positions, $\tilde{r}_{t+1}^L - up_t^L$, equals the expected return of bears, *in excess* of the minimum expected return needed to take short positions, $\tilde{r}_{t+1}^S - up_t^S$.

As with UIP, then, equation (19) says that equilibrium in the foreign exchange market occurs when *in the aggregate*, the expected return on long positions equals the expected return on short positions, but only after these expected returns have been adjusted for agents' greater sensitivity to potential losses. We refer to this equilibrium condition, therefore, as uncertainty adjusted uncovered interest rate parity (UAUIP).

The UAUIP condition shows that the algebraic sign of up_t is determined by the algebraic sign of the average opinion or "market" expectation of R_{t+1}, i.e., $\tilde{r}_{t+1}$. If, for example, the average opinion is positive, so that the expectation of bulls dominates that of bears ($\tilde{r}_{t+1}^L > -\tilde{r}_{t+1}^S$), then equilibrium requires that bulls are either more "fearful" of losses or contemplate larger potential losses relative to bears, i.e., $up_t^{\text{L}} > up_t^{\text{S}}$. It is this higher expected loss that allows the market to reach a balance between the total of long and short positions. This leads to one of the major conclusions of our analysis:

Conclusion 2 *The algebraic sign of the equilibrium uncertainty premium, up_t, will change whenever the dominant weight behind the average opinion shifts between bulls and bears.*

4 Revisions of Individual Forecasting Models and the Movement of the Uncertainty Premium Over Time: A Qualitative Approach

We have shown that dynamic prospect theory provides a basis for modeling part of the decision problem faced by financial-market speculators, namely how much capital to gamble at each point in time. The analysis led to an equilibrium condition for the foreign exchange market in which the market's expected return on open positions in foreign exchange is, in general, nonzero, i.e., $up_t \neq 0$. However, the equilibrium defined in (19) is *only momentary*, because it is defined for a *given* set of forecasting models, and, therefore, for given values of the aggregate forecast and the expected losses.

As individual agents revise their forecasts and their expected losses, they will, according to (13), adjust the size of their long and short positions. These adjustments of the speculative positions disturb the balance between the weight of bulls and bears and move the momentary equilibrium exchange rate and the equilibrium uncertainty premium.

The main objective of this section is to model the evolution of the equilibrium uncertainty premium, up_t, over time.[38] Because up_t in (19) is a weighted average of agents' expected losses, we need to model the evolution of these expected losses as individual forecasting models are revised. To this end, we follow the IKF approach and develop *qualitative* restrictions on the

[38] For the use of the IKF approach to model the movement of the momentary equlibrium exchange rate, see Frydman and Goldberg (2003a). In that paper, we examined the widely noted tendency of floating exchange rates to exhibit long swings (e.g., see Dornbusch and Frankel, 1988). We found that under imperfect knowledge and behaviorally plausible *qualitative* conditions on the revisons of forecasting models, a monetary model implied a tendency for the exchange rate to move persistently away from PPP over an extended time period.

formulation of the individual forecasting models and their revisions. In the next subsection, we build on an idea, put forth by Keynes (1936), and formulate qualitative conditions that link revisions of forecasts with revisions of expected losses.

4.1 The Gap Effect

In discussing the question of why an agent might hold cash rather than interest-bearing bonds, Keynes argued that

> [the demand for cash] will not have a definitive quantitative relation to a given rate of interest of r; what matters is not the absolute level of r but the degree of its divergence from what is considered a fairly safe level of r, having regard to those calculations of probability which are being relied on (Keynes [1936, p.201].

Furthermore, in what is apparently the first explicit reference to loss aversion in the economics literature, Keynes connected agents' assessments of expected losses to the divergence of the rate of interest from its "fairly safe" level. As he put it,

> Unless reasons are believed to exist why future experience will be very different from past experience, a ...rate of interest [much lower than a historical safe rate], leaves more to fear than to hope, and offers, at the same time, a running yield which is only sufficient to offset a very small measure of fear (Keynes [1936, p.202]).

What is remarkable about these passages is that Keynes not only related expected losses to the degree of divergence of the rate of interest from a "fairly safe" benchmark rate based on historical experience, but also postulated that if the current rate of interest is too low (i.e. too far away from the benchmark rate), the potential "fear" of a capital loss will exceed "the running yield". Thus, Keynes surmised that the relationship between a "measure of fear" and "the running yield" is key to understanding the dynamic behavior of interest rates and, more generally, to the returns from speculation in asset markets.

A benchmark is, of course, specific to each asset market. Its value each period is determined by individual agents and so, in general, will differ across agents. However, there are a few general characteristics of a benchmark that are already explicit in Keynes' remarks cited above and that are important for our analysis.

1. Although assessments of the benchmark value at each time t may differ across individual agents, the range over which these individual assessments differ should be smaller (often substantially so) than the range over which the observed asset price varies. For example, this would be the case for individual benchmarks based on averages of historical data.

2. This notion of a benchmark will play an important role in speculative decisions if the historical evidence suggests that: a) asset prices tend to move persistently away from their historical benchmarks for substantial periods of time; and b) asset prices eventually revert, at unpredictable moments in time, back to their historical benchmark values, and often shoot through these levels.[39]

3. Individual speculators recognize the historical record on long swings in asset prices and believe that this evidence is used by other speculators operating in the markets.

Based on the foregoing considerations, we now formulate the gap effect. Let the *expected gap* for an individual agent be defined as the difference between his conditional forecast of the uncertain, payoff-relevant asset price and his assessment of the historical benchmark value for this price. Then, the gap effect can be stated as follows:

- **The Gap Effect**

  As an individual agent increases his assessment of the expected gap, he simultaneously increases (decreases) his assessment of the expected loss if he is a bull (bear).

[39] For evidence that exchange rates exhibit long swings that revolve around historical benchmark levels see the references contained in footnote 54. For the bond market see Bec and Anders (2002) and for the stock market see Campbell and Shiller (1998), and references therein.

In terms of the foreign exchange market, the expected gap is as follows:

$$\widetilde{gap}_t^i = \tilde{s}_{t+1}^i - \tilde{s}_t^{\mathrm{HB}_i} \tag{20}$$

This definition allows us to formulate the gap effect as the following *qualitative* conditions on the revisions of the forecasting and an associated revisions of the expected losses of bulls and bears:

$$\frac{\partial l_t^{i_{\mathrm{L}}}}{\partial \widetilde{gap}_t^{i_{\mathrm{L}}}} < 0 \quad \text{and} \quad \frac{\partial l_t^{i_{\mathrm{S}}}}{\partial \widetilde{gap}_t^{i_{\mathrm{S}}}} > 0 \tag{21}$$

To see the logic behind the gap effect, suppose that $s_t > \tilde{s}_t^{\mathrm{HB}_i}$ for all agents, i.e., all agents believe that the foreign currency is overvalued relative to its perceived historical benchmark level. Also, suppose that all agents revise their assessments of the expected gap up, leading agents to believe in a greater overvaluation. This upward revision in the expected gap can occur despite the assumed belief, on the part of all agents, that the asset price will eventually begin reverting back to its perceived benchmark level.

The forecasting problem all agents face, therefore, is that the asset price may move persistently away from its perceived benchmark over an extended time period prior to "reverting" back. The gap effect implies that as the bulls raise their assessments of the overvaluation, they become less confident that the movement away from the benchmark will continue in the future. This causes them to revise up their assessments of the magnitude of the potential losses, i.e. the $-l_t^{i_{\mathrm{L}}}$'s increase. The bears, on the other hand, are assumed to become more confident that the countermovement will occur, and this causes them to lower their assessments of the magnitude of the potential losses, i.e. the $-l_t^{i_{\mathrm{S}}}$'s fall.[40]

The gap conditions in (21) pre-suppose that the expected loss on a unit position in foreign exchange is a function of the expected gap for all agents. Before we examine the implications of these gap conditions, we need to show that a well-specified function relating l_t^i and $\widetilde{gap}_t^i$, and its derivatives, can be defined.

A revision of the individual forecast $\tilde{r}_{t+1}^i = E_{\mathcal{P}_t^i}(S_{t+1}|x_t^i, \theta_t^i) - s_t - fp_t$ can be viewed formally as involving either one or both of the following two types of changes:

[40] Although the term "benchmark" might suggest a similarity between the gap effect and the house-money effect of Barberis, Huang and Santos (2001 and Barberis and Huang (2001), as is clear from our discussion that the benchmark measure we have in mind is, in fact, very different.

1. A revision of the individual forecasting model, $\mathcal{P}_t^i(S_{t+1}|\mathcal{X}_t^i, \theta_t^i)$, which may involve: a) a revision of the values of the parameter vector θ_t^i, but no change in the composition of variables included in the information set; or b) the dropping of variables from or the adding of variables to the information set, i.e., a change in the composition of $\mathcal{X}_t^i$.

2. Revisions that are triggered by new the realizations of S_t and FP_t and/or the other variables in the information set, and that leave forecasting model *unchanged*.

To construct a function relating the expected loss to the expected gap, define a random variable ϵ_t^i such that

$$\begin{aligned} R_{t+1} &= E_{\mathcal{P}_t^i}(R_{t+1}|\mathcal{X}_t^i, \theta_t^i) + \epsilon_t^i = \tilde{S}_{t+1} - S_t - FP_t + \epsilon_t^i, \text{ where,} \\ \tilde{S}_{t+1} &= E_{\mathcal{P}_t^i}(S_{t+1}|\mathcal{X}_t^i, \theta_t^i) \quad \text{and} \quad E_{\mathcal{P}_t^i}(\epsilon_t^i|\mathcal{X}_t^i, \theta_t^i) = 0 \end{aligned} \tag{22}$$

The analyses for a bull and a bear are analogous, and so we conduct our analysis for a bull only. Evaluating the expectation of R_{t+1} in (22), conditional on a given realization x_t^i yields:

$$\tilde{r}_{t+1} = E_{\mathcal{P}_t^i}(S_{t+1}|x_t^i, \theta_t^i) - s_t - fp_t = \tilde{s}_{t+1}^i - s_t - fp_t \tag{23}$$

Recalling that R_{t+1}^- represents potential losses for a bull, equation (22) readily implies the following relationship between the expected loss on a unit position, l_t^i, and $\tilde{s}_{t+1}^i, x_t^i$ and θ_t^i:

$$\begin{aligned} l_t^i &= l_t^i(\tilde{r}_{t+1}^i, \tilde{s}_t^{\text{HB}_i}, x_t^i, \theta_t^i) = E_{\mathcal{P}_t^i}(R_{t+1}^-|x_t^i, \theta_t^i) = E_{\mathcal{P}_t^i}(R_{t+1}I(R_{t+1} < 0)|x_t^i, \theta_t^i) \\ &= E_{\mathcal{P}_t^i}[(\tilde{r}_{t+1}^i + \epsilon_t^i)I(R_{t+1} < 0)|x_t^i, \theta_t^i] \end{aligned} \tag{24}$$

The realizations of the variables in x_t^i enter the l_t^i function in (24) through its effect on $\tilde{r}_{t+1}^i = \tilde{r}_{t+1}^i(x_t^i, \theta_t^i)$ and moments of ϵ_t^i.

In this paper, we focus on the relationship between l_t^i and $\widetilde{gap}_t^i$. We assume, therefore, that the effects of changes in the realizations of the variables in x_t^i on l_t^i, other than $\tilde{s}_t^{\text{HB}_i}$, are always dominated by the gap effects through $\tilde{r}_{t+1}^i$.[41] To simplify the exposition, we ignore the effects of changes in x_t^i – through the moments of ϵ_t^i– on l_t^i, other than $\tilde{s}_t^{\text{HB}_i}$. We list $\tilde{s}_t^{\text{HB}_i}$ as a separate

[41] More formally, we assume that the $sign(\frac{\partial l_t^i}{\partial \tilde{r}_{t+1}^i}\frac{d\tilde{r}_{t+1}^i}{dx_t^{ik}} + \frac{\partial l_t^i}{\partial x_t^{ik}}) = sign(\frac{\partial l_t^i}{\partial \tilde{r}_{t+1}^i}\frac{d\tilde{r}_{t+1}^i}{dx_t^{ik}})$ for all $k = 1, ...K_t^i$, where K_t^i is the number of variables in agent i's information set at time t.

argument of l_t^i, because the definition of the gap in (20), and the gap conditions in (21),.imply an additional effect of $\tilde{s}_t^{\text{HB}_i}$ on l_t^i. This allows us to write $l_t^i = l_t^i(\tilde{r}_{t+1}^i, \tilde{s}_t^{\text{HB}_i}, \theta_t^i) = l_t^i(\tilde{s}_{t+1}^i, \tilde{s}_t^{\text{HB}_i}, s_t, fp_t)$.

The formulation in (24) also clarifies, formally, the term "model revision" in the IKF framework: such a revision involves not only an updating of the value of the first moment of $\mathcal{P}_t^i(S_{t+1}|\mathcal{X}_t^i, \theta_t^i)$, i.e. $\tilde{r}_{t+1}^i(x_t^i, \theta_t^i)$ but, in general, it also involves a revision of the other moments of the distribution of ϵ_t^i. In fact, as we shall show next, *the gap effects in (21) necessarily entail revisions of the higher moments of* $P_t^i(.|.)$.

Because $\tilde{s}_t^i$ and $\tilde{s}_t^{\text{HB}_i}$ play a similar role in (20) though with opposite signs, we assume, in this paper, that $\tilde{s}_t^{\text{HB}_i}$ is a constant. This assumption allows us to focus on the gap effects, and their implications, arising from changes in the individual $\tilde{s}_t^i$'s.

To consider both autonomous and endogenous changes in $\tilde{s}_t^i$, we express this conditional forecast as a sum of an autonomous component, $\tilde{s}_{t+1}^{a_i}(z_t^i)$, and an endogenous component, $\tilde{s}_{t+1}^{e_i}(s_t, z_t^i)$, where z_t^i is a subset of x_t^i that does contain s_t, as follows:[42]

$$\tilde{s}_{t+1}^i = \tilde{s}_{t+1}^{a_i} + \tilde{s}_{t+1}^{e_i} \tag{25}$$

To define the derivatives of l_t^i in (24) with respect to $\tilde{s}_{t+1}^i$, s_t and fp_t, without a loss of generality, we consider a revision of the forecasting model, $P_t^i(.|.)$, accompanying an increase in $\tilde{s}_{t+1}^{a_i}$ for a bull. The definition in (23) implies that this revision of the expected gap leads to a one-for-one "rightward shift" of the conditional forecast of the next period's return, $\tilde{r}_{t+1}^i$. Furthermore, we need to check whether such a shift in $\tilde{r}_{t+1}^i$ would be consistent with the gap conditions in (21), which we impose on the revision of the bull's expected loss.[43]

We begin by showing that there is a large class of forecast functions, including the standard REH-based formulations, for which the gap conditions *do not hold.* This straightforward result plays a key role in rendering our

[42] It should be noted, that although z_t^i does not contain s_t, $\mathcal{Z}_t^i$ *does*, in general, contain fundamental variables that may be correlated with the exchange rate. Thus, the IKF framework explicitly allows for the forecasting models that include fundamental variables. For example, see Frydman and Goldberg (2003a) for the IKF-based analysis of long swings relying on fundamentals, in contrast to the non-fundamentals factors, such such sunspots, market psychology, etc.

[43] The authors are grateful to Mike Woodford for insightful comments and suggestions on this point.

model testable: it implies that the gap conditions can be rejected by the data.

The following proposition, which is proved in the appendix, shows that in order for the $l_t^i(\tilde{s}_{t+1}^i, \tilde{s}_t^{\mathrm{HB}_i}, s_t, fp_t)$ functions to satisfy the gap conditions, revisions of $\tilde{s}_{t+1}^i$ *must* be accompanied by simultaneous revisions of the higher moments of the individual conditional distribution of R_{t+1}.

Proposition 1 *Suppose that agent i changes his conditional forecast of the exchange rate, $\tilde{s}_{t+1}^i$, either because he revises the parameter vector θ_t^i or because of new realizations of the information vector x_t^i. Moreover, suppose that while $\tilde{s}_{t+1}^i$ is revised, all other moments of the distribution $P_t^i(S_{t+1}|\mathcal{X}_t^i, \theta_t^i)$ remain* unchanged. *Then the gap conditions in (21) will not hold, irrespective of whether agent i is a bull or a bear.*

Given that the individual forecasting models are represented as individual conditional probability distributions, proposition 1 can be interpreted as follows: in general, for the gap conditions to be satisfied, a revision by an agent of his forecasting model, $\mathcal{P}_t^i(R_{t+1}|\mathcal{X}_t^i, \theta_t^i) = .P_t^i(S_{t+1}|\mathcal{X}_t^i, \theta_t^i) - S_t - FP_t$, would have to involve not just the updating of the mean (i.e. the value of his conditional forecast), but would also have to entail revisions of the other moments of his forecasting model. In particular, the gap conditions would be violated if the forecast revisions were to be triggered *solely* – without changes in the higher moments of an agent's distribution– because of new realizations of the variables in $\mathcal{X}_t^i$.

Proposition 1 also implies that a *ceteris paribus* change in either s_t,or fp_t – affecting *solely* the value of the conditional forecast, $\tilde{r}_{t+1}^i$, while keeping the higher moments unchanged – would violate the gap conditions in (21). Such *ceteris paribus* shifts define the partial derivatives of $l_t^i(\tilde{r}_{t+1}^i, \tilde{s}_t^{\mathrm{HB}_i}, x_t^i, \theta_t^i)$ with respect to s_t and fp_t. The following corollary to proposition 1. collects our results on the partial derivatives of of the function l_t^i.

Corollary 1 *The gap conditions in (21) and proposition 1 imply the following signs on the partial derivatives for a bull:*

$$\frac{\partial l_t^{i_{\mathrm{L}}}}{\partial \tilde{s}_{t+1}^{i_{\mathrm{L}}}} < 0 \quad \frac{\partial l_t^{i_{\mathrm{L}}}}{\partial s_t} < 0 \quad \frac{\partial l_t^{i_{\mathrm{L}}}}{\partial fp_t} < 0 \tag{26}$$

and a bear

$$\frac{\partial l_t^{i_{\mathrm{S}}}}{\partial \tilde{s}_{t+1}^{i_{\mathrm{S}}}} > 0 \quad \frac{\partial l_t^{i_{\mathrm{S}}}}{\partial s_t} > 0 \quad \frac{\partial l_t^{i_{\mathrm{S}}}}{\partial fp_t} > 0 \tag{27}$$

Proposition 1 also implies that as the gap between $\tilde{s}^i_{t+1}$ and the historical benchmark changes, the associated revision of the individual forecasting model should entail changes in the "size" of the "tails" of the subjective distribution. This conclusion, which is proved in the appendix, is summarized in the following proposition. To save space we state this proposition only for an increases in the value of a bull's forecast:

Proposition 2 *Suppose a bull revises upward his forecast of next period's exchange rate (or return), causing an increase in $\widetilde{gap}^{i_L}_{t+1}$. If this updating is consistent with the gap condition in (21), then beyond the updating of the conditional mean, this agent must also update the other moments of his distribution in a way that leads to an increase of his expected loss.*

Proposition 2 has potentially important implications for the modeling of the forecasting process in financial markets. For if the gap effect is an empirically relevant hypothesis concerning the revisions of agents' forecasting models and conditional forecasts, then different approaches to the modeling of expectations could be assessed from the point of view of their compatibility with the *qualitative* restrictions in (21).

Consider the REH approach from this point of view. Recall that aside from infrequent changes in policy, the updating of REH forecasts arises as a consequence of the arrival of new information. Proposition 2 implies that for the updating of the REH-based forecasts to be consistent with the gap effect, the higher moments of the so-called "objective" probability distribution would have to be specific functions of its conditional mean. A discussion of the compatibility of the REH with our gap effect is outside of the scope of this paper.[44]

In the next subsection, we employ the gap effect to examine the movement of the equilibrium uncertainty premium, up_t, over time. Our analysis of this movement will demonstrate that qualitative conditions on individual forecasting do lead to testable implications from IKF-based models.

[44]However, even before the compatibilty of the REH with a gap effect can be considered, the prior question of the very meaning of the "objective distribution" needs to be addressed. See the concluding remarks.

4.2 The Behavior of the Equilibrium Uncertainty Premium Over Time

In this subsection, we derive the main testable implication of our model, that the equilibrium uncertainty premium varies positively with an aggregate measure of the expected gap, denoted by $\widetilde{gap}_t$, i.e., $\frac{dup_t}{d\widetilde{gap}_t} > 0$. We conduct our analysis for a given fp_t.[45]

The UAUIP condition in (19) shows that up_t is a weighted average of the individual expected losses and degrees of loss aversion computed over all agents, where in (21) we have assumed these individual expected losses to be functions of the perceived expected gaps. If we were prepared to assume homogeneity of preferences and forecasting models then these individual gap effects would carry over to the aggregate relationship between up_t and the aggregate (or market) expected gap, i.e., a rise in the market expected gap would necessarily lead to a rise in up_t. In this case, all speculative positions would be zero in equilibrium (i.e., $f_t^{\mathrm{O}i} = 0$ for all agents), as $\tilde{r}_{t+1}$ would exactly equal the uncertainty premium required by each agent, up_t.[46]

Thus, to obtain an equilibrium in which speculators do hold non-zero open positions (i.e., in which bulls and bears exist), the homogeneity assumption must be dropped. But with heterogeneity, the individual gap effects, by themselves, no longer ensure a gap effect in the aggregate. In this section we derive sufficient conditions on the aggregation under which the individual gap effects do imply an aggregate gap effect. We find that these conditions are rather mild.

The equilibrium uncertainty premium in (19), and the aggregate gap, can be written as follows:

$$up_t = up_t = \sum_{i}^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i I(type^i)\left(1-\lambda_1^i\right) l_t^i(\tilde{s}_{t+1}^i, s_t, \tilde{s}_t^{\mathrm{HB}}) \tag{28}$$

$$\widetilde{gap}_t = \sum_{i}^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i \widetilde{gap}_t^i = \sum_{i}^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i \tilde{s}_{t+1}^i - \sum_{i}^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i \tilde{s}_t^{\mathrm{HB}i} = \tilde{s}_{t+1} - \tilde{s}_t^{\mathrm{HB}} \tag{29}$$

[45] We examine changes in fp_t in Frydman and Goldberg (2002), where we find that our IKF-based model of the uncertainty premium sheds new light on the forward-premium anomaly.

[46] This no-trade result in equilibirum is a common feature of all asset market models with homogeneous expectations.

where the weights are the same as in (19).

To derive the implication of a change in the aggregate expected gap on the equilibrium uncertainty premium, we compute the total differential of equation (28). To this end, we use (25) and obtain:

$$d\tilde{s}_{t+1}^{i} = d\tilde{s}_{t+1}^{a_i} + \beta_t^i ds_t \tag{30}$$

where $\beta_t^i = \frac{\partial \tilde{s}_{t+1}^{e_i}}{\partial s_t}$. Aggregating (30) yields the following:

$$d\tilde{s}_{t+1} = \sum_{i}^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i d\tilde{s}_{t+1}^{a_i} + \sum_{i}^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i \beta_t^i ds_t = d\tilde{s}_{t+1}^{a} + \beta_t ds_t \tag{31}$$

The total differential of up_t in (28) results in:

$$dup_t = \sum_{i}^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i I(type^i)\left(1-\lambda_1^i\right)\left[\frac{\partial l_t^i}{\partial \tilde{s}_t^i} d\tilde{s}_{t+1}^{a_i} + \left(1-\beta_t\right)\xi_t^i ds_t\right] \tag{32}$$

where $\xi_t^i = \frac{\left(\frac{\partial l_t^i}{\partial s_t} + \frac{\partial l_t^i}{\partial \tilde{s}_t^i}\beta_t^i\right)}{(1-\beta_t)}$. The UAUIP condition in (19) implies the following equilibrium movements:

$$ds_t = \frac{1}{(1-\beta_t)}\left(d\tilde{s}_{t+1}^{a} - dup_t\right) \tag{33}$$

where we have used the aggregate expression in (31). Plugging equation (33) into equation (32) yields the expression for the equilibrium movement of up_t:

$$dup_t = \frac{\sum_{i}^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i I(type^i)\left(1-\lambda_1^i\right)\left[\frac{\partial l_t^i}{\partial \tilde{s}_t^i} d\tilde{s}_{t+1}^{a_i} + \xi_t^i d\tilde{s}_{t+1}^{a}\right]}{1 + \sum_{i}^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i I(type^i)\left(1-\lambda_1^i\right)\xi_t^i} \tag{34}$$

To determine the sign of $\frac{dup_t}{d\widehat{gap}_t}$, we examine the sign of $\frac{dup_t}{d\tilde{s}_{t+1}^{a}}$. Note, that except for the β_t^i's and β_t, the signs of the parameters of (34) all follow from our gap conditions and proposition (1) in (26) and (27), and from the assumption of dynamic loss aversion. The β_t^i's capture the endogenous influence of changes in s_t on the one-period ahead conditional forecasts, $\tilde{s}_{t+1}^{i}$, and, through an indirect gap effect, on the higher moments. In general, an individual β_t^i can be positive or negative.

There is, however, much evidence in the behavioral economics literature suggesting that, at *short horizons*, and at the aggregate level, β_t is positive.

For example, the use of technical trading rules, or "charts," – many of which extrapolate past price trends – is widely known to be prevalent in all financial markets. In a study of the London foreign exchange market, Allen and Taylor (1990) and Taylor and Allen (1992) find that approximately ninety percent of respondents reported using some technical rules at the short horizons when forming expectations.[47] There is also much experimental evidence that speculators in simulated markets react positively to price trends (e.g., see Andreassen and Kraus, 1990, and De Bondt, 1993). Moreover, the many studies on investor underreaction and overreaction to news also indicate that trend-following behavior at the shorter horizons is widespread in speculative markets.[48]

The foregoing discussion suggests rather strongly that, although a negative β_t^i may characterize the behavior of some speculators, setting $\beta_t > 0$ is a reasonable assumption. Moreover, we assume that if $\beta_t^i < 0$ for some agents, then this negative effect on $\tilde{s}_{t+1}^i$ is limited, so that $I(type^i)\left(\frac{\partial l_t^i}{\partial s_t} + \frac{\partial l_t^i}{\partial \tilde{s}_t^i}\beta_t^i\right) < 0$ for all agents. We call these two assumptions on β_t and the β_t^i's an *average opinion effect*, because it is reminiscent of Keynes' arguments that changing positions in a way that is consistent with the average opinion is pervasive in speculative markets.[49]

In what follows, we also assume that $\beta_t < 1$, which is required for the stability of the model.[50]. This assumption of stability, together with an average opinion effect, implies that the denominator in (34) is positive and $I(type^i)\xi^i < 0$ for all agents. Thus, the direction of change of the equilibrium uncertainty premium, given a revision in $\tilde{s}_{t+1}^a$ (and thus revisions in the $\tilde{s}_{t+1}^{a_i}$'s), will be determined by the expression in the numerator of (34). Note, that under $\beta_t > 0$, if $\frac{dup_t}{d\tilde{s}_{t+1}^a} > 0$, the equilibrium movements of up_t and $\widetilde{gap}_t$ are such that $\frac{dup_t}{d\widetilde{gap}_t} > 0$, i.e., if the expected gap increases initially because of a rise in $\tilde{s}_{t+1}^a$, then the endogenous increase in $\tilde{s}_{t+1}$ will preserve the positive

[47]For early studies on the use of technical trading systems and trend-following behavior and their implications for asset price dynamics, see Schulmeister (1987) and Soros (1987). See also the study of the Hong Kong foreign exchange market in Lui and Mole (1998).

[48]For studies on underreaction see Cutler, Poterba and Summers (1991) and Chan, Jegadeesh and Lakonishok (1997), and references therein, and for studies on overreaction see De Bondt and Thaler (1985,1987). See also Shiller (2000).

[49]See, for example, the passages in Keynes (1936), p. 156.

[50]It can be easily shown that if s_t is displaced from its momentary equilibium point, then $\beta_t < 1$ is required for the momentary equilibirum to be reestablished.

change in $\widetilde{gap}_t$.

4.2.1 An Aggregate Gap Effect

To gain insight into the conditions needed for an aggregate gap effect, consider a one-time increase in $\tilde{s}^a_{t+1}$, which arises due one-time changes in some or all of the $\tilde{s}^{a_i}_{t+1}$'s. Since the analysis for a decrease in $\tilde{s}^a_{t+1}$ is analogous, we conduct our analysis for an increase only. The following lemma is useful in uncovering the difficulty posed by aggregation:

Lemma 1 *If the increase in $\tilde{s}^a_{t+1}$ arises solely because a subset of the bulls and bears raise their forecasts of S_{t+1} (i.e., no agent lowers his forecast), then the equilibrium uncertainty premium necessarily increases, given the assumptions on individual updating in (26) and (27) and an average opinion effect.*

Proof. With $d\tilde{s}^{a_i}_{t+1} > 0$ for all agents, the assumptions on individual updating in (26) and (27) and the average opinion effect imply that for bulls $\sum_i^{n^{\text{L}}} w^{i_{\text{L}}}\left(1-\lambda_1^{i_{\text{L}}}\right)\left[\frac{\partial l_t^{i_{\text{L}}}}{\partial \tilde{s}_t^{i_{\text{L}}}} d\tilde{s}^{a_{i_{\text{L}}}}_{t+1} + \xi_t^{i_{\text{L}}} d\tilde{s}^a_{t+1}\right] = dup_t^{\text{L}} > 0$, and bears $\sum_i^{n^{\text{S}}} w^{i_{\text{S}}}\left(1-\lambda_1^{i_{\text{S}}}\right)$ $\left[-\frac{\partial l_t^{i_{\text{S}}}}{\partial \tilde{s}_t^{i_{\text{S}}}} d\tilde{s}^{a_{i_{\text{S}}}}_{t+1} - \xi_t^{i_{\text{S}}} d\tilde{s}^a_{t+1}\right] = -dup_t^{\text{S}} > 0$. Since $up_t = up_t^{\text{L}} - up_t^{\text{S}}$, we have $\frac{dup_t}{d\tilde{s}^a_{t+1}} > 0$ and a positive relationship between the equilibrium uncertainty premium and the market expected gap. ■

The intuition behind Lemma 1 is straightforward. As the bulls (bears) raise their assessments of the expected gap, causing $\tilde{s}^a_{t+1}$ to rise, they become more (less) nervous and simultaneously raise (lower) their assessments of the potential losses, i.e., the $\left(1-\lambda_1^{i_{\text{L}}}\right)\frac{\partial l_t^{i_{\text{L}}}}{\partial \tilde{s}_t^{i_{\text{L}}}} d\tilde{s}^{a_{i_{\text{L}}}}_{t+1}$ and $-\left(1-\lambda_1^{i_{\text{S}}}\right)\frac{\partial l_t^{i_{\text{S}}}}{\partial \tilde{s}_t^{i_{\text{S}}}} d\tilde{s}^{a_{i_{\text{S}}}}_{t+1}$ terms are all positive. The increase in $\tilde{s}^a_{t+1}$ also results in an increase in s_t, and this works to raise (lower) the uncertainty premia of bulls (bears), i.e., the $\xi_t^{i_{\text{L}}} d\tilde{s}^a_{t+1}$ and $-\xi_t^{i_{\text{S}}} d\tilde{s}^a_{t+1}$ terms are all positive.[51] Thus, the rise in $\widetilde{gap}_t$ necessarily leads to an increase in up_t.

[51] It is possible for s_t to fall when $\tilde{s}_{t+1}$ rises, if the increase in $\tilde{s}_{t+1}$ leads to a larger increase in up_t, i.e., bulls, for example, become so much more nervous about potential losses when they increase their forecasts that they reduce their long positions. But in this case note that up_t increases, thereby giving rise to an even larger gap effect than if s_t had risen. In what follows we ignore this case and assume that although agents' speculation is tempered by their greater sensitivity to potential losses, the increase in bull (bear) i's uncertainty premium resulting from a revision of $\tilde{s}^i_{t+1}$ up (down) is not so

The difficulty emerges when, in general, the forecasts of some of the bulls and bears increase while for others their forecasts decrease. It is clear from Lemma 1 that the uncertainty premia of the bulls (bears) who revise down their forecasts may fall (rise) in this case. This would occur if the updating of the expected loss due to the fall in $\tilde{s}^{a_i}_{t+1}$, which works to lower $up_t^{i_{\rm L}}$ and $-up_t^{i_{\rm S}}$, outweighed the positive effect due to the rise in s_t. And, the λ_1^i's and $\frac{\partial l_t^{i_{\rm S}}}{\partial \tilde{s}_t^{i_{\rm S}}}$'s for the group of bulls and bears who revise down their forecasts may be so much larger than for the group who revise their forecasts up, that in the aggregate up_t falls when $\tilde{s}^a_{t+1}$ rises.

The following lemma, which is proved in the appendix, is useful in illuminating sufficient conditions that ensure an aggregate gap effect:

Lemma 2 *Let* $I(type^i)\frac{(1-\lambda_1^i)}{C}\frac{\partial l_t^i}{\partial \tilde{s}_t^i} = \rho^i > 0$ *and* $I(type^i)\frac{(1-\lambda_1^i)}{C}\xi_t^i = \gamma^i > 0$*, where* C *denotes the expression in the denominator of (34). Then if* $\rho^i = \rho$ *for all agents, then* $\frac{dup_t}{d\tilde{s}^a_{t+1}} > 0$*.*

The intuition behind lemma 2 is that with $\rho^i = \rho$ for all agents, the weights placed on a given change in $d\tilde{s}^{a_i}_{t+1}$ for the group of bulls and bears who revise their forecasts up (call this group, group 1) are necessarily larger then for the group of bulls and bears who revise their forecasts down (call this group, group 2), i.e., $\left(\rho + \gamma^{i_1}\frac{d\tilde{s}^a_{t+1}}{d\tilde{s}^{a_{i_1}}_{t+1}}\right) > \left(\rho + \gamma^{i_2}\frac{d\tilde{s}^a_{t+1}}{d\tilde{s}^{a_{i_2}}_{t+1}}\right)$, where i_1 and i_2 denote values for groups 1 and 2, respectively. This follows because with group 1 (group 2) speculators raising (lowering) their forecasts, $\frac{d\tilde{s}^a_{t+1}}{d\tilde{s}^{a_{i_1}}_{t+1}} > 0$ and $\frac{d\tilde{s}^a_{t+1}}{d\tilde{s}^{a_{i_2}}_{t+1}} < 0$, i.e., the influence on individual expected losses from the endogenous change in $\tilde{s}_{t+1}$ always works to increase the weights for group 1 agents and lower the weights for group 2 agents.

This reasoning indicates that an aggregate gap effect will follow even if the ρ^i's differ across agents, as long as the degree of this heterogeneity remains within some bounds that depend on the magnitudes of the γ^i's. This leads to the following proposition, which states a sufficient condition for an aggregate gap (see the appendix for the proof):

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large as to prevent agent i from increasing his open position. Formally, $\frac{df_t^{{\rm O}_i}}{d\tilde{s}^i_{t+1}} = I(type) + \left(\lambda_1^i - 1\right)\frac{\partial l_t^i}{\partial \tilde{s}^i_{t+1}} > 0$ for all i.

Proposition 3 *If the following condition holds*

$$\max(\rho^i) - \min(\rho^i) < \min(\gamma^i) \min_1 \left(\frac{d\tilde{s}^a_{t+1}}{d\tilde{s}^{a_i}_{t+1}} \right) - \max(\gamma^i) \max_2 \left(\frac{d\tilde{s}^a_{t+1}}{d\tilde{s}^{a_i}_{t+1}} \right) \quad (35)$$

then $\frac{dup_t}{d\tilde{s}^a_{t+1}} > 0$*, where* $\max(.)$ *and* $\min(.)$ *denote the maximum and minimum values of the respective parameters over all agents and* $\min_1$ *and* $\max_2$ *refer to the maximum and minimum over agents in groups* 1 *and* 2*.*

Proposition 3 shows that an aggregate gap effect necessarily follows from the assumptions on individual expected losses in (26) and (27), and the average opinion effect, as long as the dispersion of the individual gap effects remain within some bounds that depend on the magnitudes of the γ^i's. Note that $\min(\gamma^i) \min_1 \left(\frac{d\tilde{s}^a_{t+1}}{d\tilde{s}^{a_i}_{t+1}} \right) > 0$ and $-\max(\gamma^i) \max_2 \left(\frac{d\tilde{s}^a_{t+1}}{d\tilde{s}^{a_i}_{t+1}} \right) > 0$, implying that an increase in the individual γ^i's allows for a greater dispersion of the ρ^i's. The reasoning here follows directly from Lemma 2: greater γ^i values imply greater effects on the individual expected losses from changes in s_t, which work to increase $up_t^{i_L}$ and $-up_t^{i_S}$ irrespective of whether the bull or bear is in group 1 or 2.

Proposition 3 shows that the set of individual parameter values implying an aggregate gap effect is non-empty and potentially quite large. Thus, the condition in Proposition 3 is compatible with heterogeneity in how individual agents revise their forecasting models. Whether the distributions of the ρ^i's and γ^i's are in fact consistent with this condition is an empirical question. Our empirical findings in the next section, which provide evidence of an aggregate gap effect, indicate that our IKE-based model of the uncertainty premium with the aggregation condition in (35) is consistent with the data.

This analysis leads to the following conclusion:

Conclusion 3 *If the gap between the aggregate forecast and the perceived historical benchmark increases (decreases) – and the gap conditions in (26) and (27), the heterogeneity condition in (35) and an average opinion effect hold – then the equilibrium uncertainty premium necessarily increases (decreases).*

4.3 The Behavior of the Equilibrium Premium under a House-Money Effect

In this subsection we examine the implications of the house-money effect used in Barberis, Huang and Santos (2001) and Barberis and Huang (2001) for our

IKE-based model of returns. To this end, we formalize a house-money effect in the context of our model of foreign exchange speculation. We show that a house money effect does not lead to limits to speculation and a well defined equilibrium in a model based solely on prospect theory. We must continue to assume, therefore, dynamic loss aversion on the individual level. As such, the resulting formulation of our model possesses dynamic preferences from two sources, i.e., changes in position size and reference dependence.

As before, it is the revisions of forecasts that gives rise to the dynamics in the model and movements in the momentary equilibrium. But to highlight the implications of a house-money effect, we drop the assumption of an individual gap effect. We find that while a house-money effect generates additional volatility in returns, it also implies a *negative* relationship between the *individual* premia on foreign exchange speculation and the expected gap. This negative relationship stands in sharp contrast to the positive relationship obtained in the model when the assumption of a gap effect is imposed.

To formalize a house-money effect, we adopt a simple formulation of the utility function, analogous to (9), that incorporates the idea that the degree of loss aversion depends on prior investment performance. To this end, let cpr_t^i denote cumulative portfolio gains or losses for an agent between some reference point in time, chosen by the agent at time t. The following utility function embodies a house-money effect:

$$V_t^i = g_t^i f_t^i + (\delta_0^i - \delta_1^i cpr_t^i) l_t^i f_t^i \tag{36}$$

where $1 < \delta_0^i < \lambda^{\max}$ and, analogously to section 3, $\delta_1^i > 0$ is bounded from above in such a way as to insure the degree of loss aversion lies within a finite range.

We note that since V_t^i in (36) is linear in f_t^i, a house-money effect alone is not sufficient to limit the size of agents' positions under heterogenous expectations. Thus, we must incorporate our assumption of dynamic loss aversion into the utility function. In a way analogous to (11) in section 3, we modify (36) as follows:

$$V_t^i = g_t^i f_t^i + (\delta_0^i - \delta_1^i cpr_t^i) l_t^i f_t^i - \lambda_2^i (f_t^i)^2 \tag{37}$$

Setting the derivative of (37) with respect to f_t^i equal to zero yields:

$$f_t^{\text{o}_i} = \frac{1}{2\lambda_2^i}\left[I(i)\tilde{r}_{t+1}^i - pr_t^i\right] \tag{38}$$

where

$$pr_t^i = (1 - \delta_0^i + \delta_1^i cpr_t^i) l_t^i \tag{39}$$

denotes the expected premium on foreign exchange.

To analyze the movements of pr_t^i over time – using the qualitative methodology of the IKF framework – we must specify the qualitative conditions on updating for the case in which the gap conditions in (21) do not hold. The violations of these conditions and partial derivatives of l_t^{L} with respect to s_t, in the corollary 1 result in

$$\frac{\partial l_t^{i_{\text{L}}}}{\partial \tilde{s}_{t+1}^{i_{\text{L}}}} > 0, \quad \frac{\partial l_t^{i_{\text{L}}}}{\partial s_t} < 0 \tag{40}$$

$$\frac{\partial l_t^{i_{\text{S}}}}{\partial \tilde{s}_{t+1}^{i_{\text{S}}}} < 0, \quad \frac{\partial l_t^{i_{\text{S}}}}{\partial s_t} > 0 \tag{41}$$

Moreover, proposition 1 implies that for both bulls and bears:

$$\frac{\partial l_t^i}{\partial \tilde{s}_{t+1}^i} = -\frac{\partial l_t^i}{\partial s_t} \tag{42}$$

Totally differentiating (39) and using (42) yields:

$$dpr_t^i = (1 - \delta_0^i + \delta_1^i cpr_t^i) \frac{\partial l_t^i}{\partial \widetilde{gap}_{t+1}^i} [d\tilde{s}_{t+1}^i - ds] + \delta_1^i d(cpr_t^i) l_t^i \tag{43}$$

Let $b_t^i > 0$ and $a_t^i > 0$ be defined as follows:

$$\delta_1^i \frac{d(cpr_t^i)}{ds} l_t^i = -I(i) b_t^i \quad \text{and} \quad (1 - \delta_0^i + \delta_1^i cpr_t^i) \frac{\partial l_t^i}{\partial \widetilde{gap}_{t+1}^i} = -I(i) a_t^i$$

and note that if agent i is a bull (bear) $\frac{d(cpr_t^i)}{ds}$ is positive (negative). We use this notation to rewrite pr_t^i as follows:

$$pr_t^i = I(i) a_t^i (s_t - \tilde{s}_{t+1}^i) - I(i) b_t^i s_t \tag{44}$$

To determine the equilibrium s_t under the representative-agent assumption, we omit the superscript i and use the UAUIP condition in (19):

$$\tilde{r}_{t+1} = \tilde{s}_{t+1} - s_t - fp_t = pr_t \tag{45}$$

Without loss of generality, suppose expectations are such that $\tilde{r}_{t+1} > 0$. Equations (44) and (45) imply the following expression for the equilibrium exchange rate:

$$s_t = \frac{1+a_t}{1+a_t-b_t}\tilde{s}_{t+1} - \frac{1}{1+a_t-b_t}fp_t \tag{46}$$

Totally differentiating (46) and setting $dfp_t = 0$, implies that, for our example of an increase in $\tilde{s}_{t+1}$, $ds_t > 0$. This, in turn, implies the following equilibrium movement in pr_t:

$$dpr_t = d\tilde{s}_{t+1} - ds_t < 0 \Rightarrow \frac{dpr_t}{\partial\widetilde{s}_{t+1}} = \frac{dpr_t}{\partial\widetilde{gap}_{t+1}} < 0 \tag{47}$$

Equation (47) reveals a negative relationship between the representative agent's premium and the expected gap when an agent's preferences embody a house-money effect.[52] Thus, a house-money effect and a gap effect lead to opposite conclusions concerning the relationship between the premium on foreign exchange speculation and the expected gap. In the next section we present empirical evidence of a positive relationship, thereby lending support to our IKF model with a gap effect.

5 Some Empirical Evidence

The preceding two sections developed a model of the expected excess return on foreign exchange based on IKE and dynamic prospect theory. One of the main implications of our model is that movements in the equilibrium uncertainty premium should be positively related to movements in the expected gap. In this section we present some empirical evidence in support of this prediction. To this end we utilize the survey data from MMSI, which allows us to construct monthly observations of the median four-week forecast of the German mark-U.S. dollar exchange rate from a group of foreign exchange market participants.[53] Our data on spot and forward rates, which

[52] Having shown this on the individual level, we do not examine the aggregation problem. Even if the individual effect were to be overturned through aggregation, this aggregation result would not serve as a sensible foundation for the development of macroeconomic relationships based on well-specified microfoundations.

[53] For a more detailed description of the data see Frankel and Froot (1987). From January 1983 through December 1984, MMSI provides only two-week forecasts.

Table 1: OLS Regression
Uncertainty Premium and the Expected Gap

Constant	**0.857**
	(0.954)
$\widetilde{gap}_t$	**1.334*****
	(0.222)
Adjusted R^2=.394	DW Statistic=2.15

Standard errors (adjusted for heteroskedasticity) are in parentheses
$***$denotes significance with a p-value of .01

are bid-asked averages, comes from the WEFA data base.[54]

Table 1 reports the results of an OLS regression of up_t on $\widetilde{gap}_t$. To address the problem of serial correlation and possible stochastic trends we added to the regression lags of the dependent and independent variables (these coefficients are not shown).[55] Diagnostic tests revealed that two lags were sufficient. The results show that the expected gap is highly significant and positively related to the expected excess return on foreign exchange (in this case the U.S. dollar), as predicted by our IKE-based model of up_t.

The results reported in Table 1 should be treated with some caution. This is because we would expect the relationship between up_t and $\widetilde{gap}_t$ to be nonlinear: agents most likely place little weight on small deviations from PPP values, whereas large deviations will be given a relatively large weight. Also, with IKE, we would expect the OLS regression to be temporally unstable, as the correlation between up_t and $\widetilde{gap}_t$ shifts over time. One way

[54] See footnote 24 for our measure of $\widetilde{gap}_t$. For evidence that PPP does serve as a historical benchmark for the foreign exchange market, see the cointegration studies of Juselius (1995) and Cheung and Lai (1993), among others, as well as studies on the long-horizon predictability of PPP and monetary fundamentals (e.g., Mark (1995) and Mark and Sul (2001). See also Obstfeld [1995] and references therein. We note, however, that although it may be plausible to use PPP as a proxy for the benchmark level for many exchange rates, this *does not* imply that the PPP level is a long-run equilibrium in the sense of being the rate at which the foreign exchange market "settles".

[55] See Juselius and Hendry (2000) on the validity on using OLS standard errors for inference with unit-root variables when lagged values of the dependent and independent variables are included in the regression.

Table 2
Contingency Table Analysis

	$\triangle up_t > 0$	$\triangle up_t < 0$
$\triangle \widetilde{gap}_t > 0$	43	25
$\triangle \widetilde{gap}_t < 0$	30	51

to handle the changing nature of the relationship between up_t and $\widetilde{gap}_t$ is to make use of a non-parametric procedure such as contingency-table analysis. Contingency-table analysis provides a way to test the qualitative relationship, while allowing for the exact form of the quantitative relationship to change over time.

Table 2 presents the contingency table results. The diagonal (off-diagonal) cells in the table denote the number of observations for which the changes in up_t and $\widetilde{gap}_t$ (denoted $\triangle up_t$ and $\triangle \widetilde{gap}_t$, respectively) were in the same (opposite) direction. Since the number of observations along the diagonal are larger than the off-diagonal, the results show that the relationship between up_t and $\widetilde{gap}_t$ is a positive one. A χ^2 statistic of 10.15 indicates that this positive relationship is significant at less than the .01 level.

6 Concluding Remarks: IKF vs. REH

The preceding section presented evidence that our IKF-based model of the uncertainty premium with a gap effect is consistent with the empirical record. The numerous and largely unsuccessful attempts under the REH to explain the behavior of excess returns on foreign exchange as the equilibrium compensation for risk, suggests that our IKF framework provides a superior alternative to the REH approach from a purely empirical, positivist standpoint (Friedman, 1953). Moreover, because quantitative rules, such as those under the REH, are in general inconsistent with individual rationality, the IKF framework's replacement of such rules with qualitative assumptions on the modeling of agents' forecasting behavior, speaks in favor of IKF on theoretical grounds (Frydman and Goldberg, 2003a, and references therein). Finally, the assumption that agents must formulate and revise their forecasting models on the basis of imperfect knowledge seems to be uncontroversial on behavioral grounds.

Despite serious empirical, theoretical and behavioral problems besetting the REH approach, we surmise that many of our colleagues will be reluctant to consider, let alone embrace, our view that the modeling of the forecasting process on the basis of imperfect knowledge is required in order to understand the dynamics of exchange rates as well as other asset prices. This reluctance may stem from the acceptance by most macroeconomists of Lucas's methodological dictum to "beware of theorists bearing free parameters (Sargent, 1999, p. 73)." At a minimum, our IKF approach insists that macroeconomic models should contain "free" parameters arising from agents' forecasting models. Furthermore, despite the usual claims that "resolving empirical difficulties by adding new parameters always works (Lucas, 2003a, p.8)," our empirical testing of one IKF model of the premium with a gap effect against another IKF model with a house-money effect clearly shows that models with free parameters can be rejected by the data. Although both models relied on the qualitative updating of forecasting models to generate dynamics, they produced unambiguous and opposite predictions concerning the qualitative relationship between the equilibrium uncertainty premium and the expected gap.

In contrast to the IKF, the REH approach presumes that the economist who models agents' behavior can specify precisely the individual forecasting models. This assumption begs *the* question as to which of the potential (extant and *yet to be invented*) models of macroeconomic phenomena the economist should attribute to the agents. The standard REH application implicitly assumes that agents forecast on the basis of the model formulated by the economist who happens to be analyzing the behavior of macroeconomic aggregates. This procedure rigidly ties agents' forecasting models to the economist's own model and, *by assumption*, eliminates free parameters arising from the forecasting process in macroeconomic analysis.

By banning expectations from playing an active role in macroeconomic analysis, the REH has severely hampered the ability of macroeconomists to provide explanations of the dynamics of asset markets and other phenomena in which expectations play an active and important role. Macroeconomic fundamentals included in economists' models vary much less than asset returns; thus, models that rigidly tie expectations to these fundamentals will continue to stumble in explaining the observed behavior of asset prices.[56] More generally, *empirically relevant* models of macroeconomic phenomena in which expectations play a key role are very likely to involve free parameters arising from agents' forecasting models.[57]

Paradoxically, the rigidity of the REH approach, something commonly viewed as its key virtue, has forced economists working in the REH tradition to introduce new forms of utility functions to explain anomalies in asset markets. However, models based on new forms of utility functions, for example the house-money effect, may appear to be empirically relevant when based on the narrow criteria of matching moments (see Barberis, Huang and Santos, 2002), and yet as we showed in section 5, may be inconsistent with the actual behavior of asset returns over time.[58]

Beyond the difficulties in providing empirically relevant explanations, the REH approach suffers from fundamental problems on purely theoretical grounds. The key issue is its compatibility with the basic postulate of rationality in economics: Under what conditions would it be *rational,* in the

[56] A related point concerns the observed temporal instability of empirical models of asset price dynamics (e.g., in the foreign exchange market see Golderg and Frydman, 1996a,b,2001). Since this instability is too pervasive to be attributed to changes in policy rules, the updating of expectations needs to be introduced into the analysis.

[57] It is worth noting that Lucas has recently commented on difficulties experienced by REH-based macroeconomics. See Lucas (2003a,b). For example, commenting on progress since the REH was added to the seminal macroeconomic models in Phelps (1969), Lucas (2003b, p.140) acknowledged that,

> New frameworks—contracts, monopolistic compretition—are introduced, motivated by the inability of earlier theory to resolve this difficulty [of explaining persistent real responses to nominal shocks], but the problem of persistence has proved to be persistent itself...Ever since the January, 1969, conference that Ned Phelps invited us to, the 14 authors of the Phelps volume have been apologetic about the fact that we couldn't resolve these issues. After watching so many of our talented colleagues struggling with them over the 30 years since, maybe we shouldn't feel so bad.

[58] Another important difficulty with the REH is that it often leads to tight connections between relationships in asset and other markets. As Lucas (2003a, p.8) points out, formulating new utility functions to resolve the equity premium puzzle "often [raises] more problems " for explanations in other markets. In contrast, the IKF approach does not suffer from such inherent difficulties. By according the forecasting process an active role, the IKF framework substantially weakens the link between the asset and other markets. We build on this point in Frydman, Goldberg and Juselius (2003), where we show that the IKF approach leads to a resolution of the PPP-puzzle (Rogoff, 1996).

sense of not passing up profit opportunities, for an agent to use the model written down by the economist, as is assumed under the conventional use of the REH? Summarizing earlier arguments concerning epistemological problems with the REH (see footnote 3 for references), Sargent (1993) has stated *the* answer to this question in a particularly striking way. As he put it,

> rational expectations is an equilibrium concept that at best describes how the system might eventually behave if the system will ever settle down to a situation in which all of the agents have solved their "scientific problems" (Sargent, 1993, p. 23).

Therefore, in a world in which the scientific problem has not been solved, *rational* agents in pursuit of profit opportunities will, in general, search and adopt forecasting models and methods that differ from the quantitative model that the REH theorist attributes to agents.

This conclusion has an immediate implication for the compatibility of the basic rationality postulate in economics with the REH-based representative agent approach. This approach abstracts from both the individual differences in preferences and the diversity of individual forecasting models. Homogeneous preferences might be rationalized as a necessary approximation to make the macroeconomic models tractable; this assumption does not, in general, conflict with economic rationality. However, even if we acknowledge considerable difficulties in solving aggregation problems, we *cannot* appeal to tractability to justify the REH as an approximation of the forecasting models used by heterogenous agents *limited by imperfect knowledge*. Therefore, unless the macroeconomist is prepared to abandon the basic economic rationality postulate, the REH *cannot* be treated as an approximate representation of the expectations formed by *rational* agents in a world in which "agents have [not] solved their scientific problems."

Of course, if one were prepared to assume that the "scientific problem" has been solved, the incompatibility of the REH approach with the rationality postulate in economics would disappear. Such attribution of "perfect knowledge" to individual agents operating in the real world markets might be behind a general belief that the REH is *the* solution to the problem of modeling forecasting process of rational agents. The appeal of the REH has been also based on behavioral considerations. As Sargent put it

> The idea of rational expectations is sometimes explained informally by saying that it reflects a process in which individuals

> are inspecting and altering their forecasting records in ways to eliminate systematic forecast errors. It is also sometimes said that to embody the idea that economists and the agents they are modelling should be placed on equal footing: the agents in the model should be able to forecast and profit-maximize and utility-maximize as well as...the econometrician who constructed the model (Sargent [1993], p.21).

Sargent argues, however, that "these ways of explaining things are suggestive, but misleading, because they make rational expectations sound less restrictive and more behavioral than it really is.(Sargent [1993, p.21])."

While the compatibility of the REH with the basic rationality postulate in economics awaits the ultimate discovery of *the true* model of aggregate behavior, or the so-called "objective"probability distribution, the IKF framework offers an approach to the modeling of forecasting models, and their revisions, by *rational* agents in a world in which the "scientific problem" has *not* been solved.

In view of its recognition that the *creative process* governing the acquisition of knowledge can at best be characterized in a *qualitative* as opposed to a quantitative manner, the IKF framework imposes *only qualitative conditions* on the form and updating of individual forecasting models. Moreover, by *not* restricting the forecasting models and the revision process of agents to a specific set of *quantitative rules*, such a qualitative approach to the modeling of the forecasting process is compatible with the postulate of individual rationality.

Before we conclude this paper, we should acknowledge that the foregoing arguments concerning the fundamental theoretical difficulties of the REH draw on many of the great debates in social sciences. Insistence on banning free parameters arising from agents' expectations from macroeconomic analysis is reminiscent of the perennial attempts to discover the secret mechanism behind the apparent contingencies of historical events. This has been pursued by Hegel, Marx and many other thinkers and has by now been generally abandoned.[59] In particular, this kind of approach leaves no room for human spontaneity and innovation which drive not only history, but also the dynamism of capitalist market economies and economic development in

[59] For two prominent examples of analyses of this point and related issues, see Popper (1964) and Hayek (1978).

general.[60]

The inherent impossibility of quantifying the creative process governing the acquisition of knowledge by *rational* agents in modern, capitalist economies, leads us to the conclusion that macroeconomists will need to continue to work with aggregate relationships containing free parameters arising from individual forecasting models. Because the IKF framework makes only qualitative assumptions about these models, it does not have to *prespecify* quantitative models used by individual agents and, thus, in contrast to the REH, is compatible with individual rationality in a world of imperfect knowledge.[61] Moreover, as we have shown in this paper, models based on the qualitative IKF approach can explain the behavior of asset price dynamics that the REH models deem anomalous.

[60] See Frydman, Gray, Hessel and Rapaczynski (1999) for empirical evidence on the importance of this point for understanding the role of private ownership. Phelps (2003) puts forth many compelling reasons for the importance of according expectations an active role in economic analysis and for explaining macroeconomic behavior of capitalist economies.

[61] The qualitative IKF approach should also be contrasted with some recent attempts by leading scholars working in the REH tradition to introduce diversity of forecasting models into the analysis of expectations. For example, Hansen and Sargent (2001a) have expressed genuine concern about the epistemological and behavioral foundations of the REH. Nevertheless, after acknowledging the importance of letting "agents inside the economist's model share his doubts about the model specification (p. 3)," they move on to restrict the set of "misspecified models" so as to preserve the basic features of the REH approach. See also Hansen and Sargent (2001b).

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Appendix

Proof of Proposition 1 in Section 4.1: Since the proof for a bear is analogous, we provide the proof for a bull only. Moreover, we consider increases in $\tilde{s}^i_{t+1}$ only, as the proof for decreases is identical. Thus, we will show that if $\Delta\tilde{s}^i_{t+1} > 0$, then $\Delta l^i_t > 0$, thereby violating (21), irrespective of whether the higher forecast stems from the new realizations of x^i_t or the updating of the parameter vector θ^i_t. Thus, suppose for some reason that $\tilde{s}^i_{t+1}$increases to some higher value, say, $\tilde{s}^{i_h}_{t+1}$.[62] Rewriting (24) implies that:

$$\Delta l^i_t = l^i_t(\tilde{s}^{i_h}_{t+1}, x^i_t, \theta^i_t) - l^i_t(\tilde{s}^i_{t+1}, x^i_t, \theta^i_t) \tag{48}$$

$$= \tilde{r}^{i_h}_{t+1}\mathcal{P}^i_t[\epsilon^i_t < -\tilde{r}^{i_h}_{t+1}|x^{i_h}_t, \theta^{i_h}_t] - \tilde{r}^i_{t+1}\mathcal{P}^i_t[\epsilon^i_t < -\tilde{r}^i_{t+1}|x^i_t, \theta^i_t]$$
$$- E_{\mathcal{P}^i_t}[\epsilon^i_t I(-\tilde{r}^{i_h}_{t+1} < \epsilon^i_t < -\tilde{r}^i_{t+1})|x^i_t, \theta^i_t, x^{i_h}_t, \theta^{i_h}_t]$$

where to collapse the difference in expectations we used the assumption that except for the conditional mean $\tilde{s}^i_{t+1}$, all other moments of the distribution have remained unchanged, i.e.:

$$\begin{aligned} E_{\mathcal{P}^i_t}[\epsilon^i_t I(\epsilon^i_t &< -\tilde{r}^{i_h}_{t+1})|x^{i_h}_t, \theta^{i_h}_t] - E_{\mathcal{P}^i_t}[\epsilon^i_t I(\epsilon^i_t < -\tilde{r}^i_{t+1})|x^i_t, \theta^i_t] = \\ &= -E_{\mathcal{P}^i_t}[\epsilon^i_t I(-\tilde{r}^{i_h}_{t+1} < \epsilon^i_t < -\tilde{r}^i_{t+1})|x^i_t, \theta^i_t, x^{i_h}_t, \theta^{i_h}_t] \end{aligned} \tag{49}$$

Furthermore we note that:

$$\begin{aligned} &- E_{\mathcal{P}^i_t}[\epsilon^i_t I(-\tilde{r}^{i_h}_{t+1} < \epsilon^i_t < -\tilde{r}^i_{t+1})|x^i_t, \theta^i_t, x^{i_h}_t, \theta^{i_h}_t] \geq \\ &\geq \tilde{r}^i_{t+1}\mathcal{P}^i_t[\epsilon^i_t < -\tilde{r}^i_{t+1}|x^i_t, \theta^i_t] - \tilde{r}^i_{t+1}\mathcal{P}^i_t[\epsilon^i_t < -\tilde{r}^{i_h}_{t+1}|x^{i_h}_t, \theta^{i_h}_t] \end{aligned} \tag{50}$$

Using (50) in (48) and (49), leads us to a lower bound for Δl^i_t i.e

$$\Delta l^i_t \geq (\tilde{r}^{i_h}_{t+1} - \tilde{r}^i_{t+1})\mathcal{P}^i_t[\epsilon^i_t < -\tilde{r}^{i_h}_{t+1}|x^{i_h}_t, \theta^{i_h}_t] > 0$$

This completes the proof of the lemma.

[62]Strictly speaking we analyze here a jump in $\tilde{s}^i_{t+1}$. But the same argument would work for continuous changes occuring within the infinite decimal interval $(t, t+h)$.Then, the derivatives would be defined in a completely standard way by letting $h \to 0$. Our analysis can be easily seen as applying to both jumps and continuous changes, but analyzing jumps is notationally simpler.

Proof of Proposition 2 in Section 4.1: In contrast to the proof proposition 1, the agent's expectation cannot be collapsed as in (49) because the distribution of ϵ_t^i after the updating of $\tilde{s}_{t+1}^i$ is different than before. Thus we denote the post-updating random variable by $\epsilon_t^{i_h}$. It readily follows from (48) and (49) that if

$$\left\{\tilde{r}_{t+1}^{i_h}\mathcal{P}_t^i[\epsilon_t^{i_h} < -\tilde{r}_{t+1}^{i_h}|x_t^{i_h},\theta_t^{i_h}] - \tilde{r}_{t+1}^i\mathcal{P}_t^i[\epsilon_t^i < -\tilde{r}_{t+1}^i|x_t^i,\theta_t^i]\right\} < 0 \text{ and} \tag{51}$$

$$E_{\mathcal{P}_t^i}[\epsilon_t^{i_h}{}_t^i I(\epsilon_t^{i_h} < -\tilde{r}_{t+1}^{i_h})|x_t^{i_h},\theta_t^{i_h}] - E_{\mathcal{P}_t^i}[\epsilon_t^i I(\epsilon_t^i < -\tilde{r}_{t+1}^i)|x_t^i,\theta_t^i] < 0 \tag{52}$$

$$\text{then } \ \Delta l_t^i < 0$$

implying that there exist conditions on an updating of the higher moments of agent's subjective distribution, i.e. (51) and (52), such that the updating of the expected loss part of the distribution is consistent with the gap-effect.

Proof of Lemma 2 in Section 4.2.1: With $\rho^i = \rho$ for all agents, (34) can be written as follows:

$$\begin{aligned} dup_t = \sum_i^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i I(type^i) dup_t^i &= \sum_i^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i \left(\rho^i + \gamma^i \frac{d\tilde{s}_{t+1}^a}{d\tilde{s}_{t+1}^{a_i}}\right) d\tilde{s}_{t+1}^{a_i} \\ &= \rho \sum_i^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i d\tilde{s}_{t+1}^{a_i} + d\tilde{s}_{t+1} \sum_i^{n^{\mathrm{L}}+n^{\mathrm{S}}} w^i \gamma^i = (\rho+\gamma)\, d\tilde{s}_{t+1}^a \end{aligned} \tag{53}$$

where γ denotes the weighted average of the γ^i's. Since $(\rho+\gamma) > 0$, $\frac{dup_t}{d\tilde{s}_{t+1}^a} > 0$.

Proof of Proposition 3 in section 4.2.1: Given the following:

$$d\tilde{s}_{t+1}^a = \sum_i^{n_1} w_i d\tilde{s}_{t+1}^{a_{i_1}} + \sum_i^{n_2} w_i d\tilde{s}_{t+1}^{a_{i_2}} > 0 \tag{54}$$

$$dup_t^{i_1} = \frac{1}{\alpha^{i_1}} I(type^i) d\tilde{s}_{t+1}^{a_{i_1}} \ \text{ where } \ \frac{1}{\alpha^{i_1}} = \rho^{i_1} + \gamma^{i_1}\frac{d\tilde{s}_{t+1}^a}{d\tilde{s}_{t+1}^{a_{i_1}}} \tag{55}$$

$$dup_t^{i_2} = \frac{1}{\alpha^{i_2}} I(type^i) d\tilde{s}_{t+1}^{a_{i_2}} \ \text{ where } \ \frac{1}{\alpha^{i_2}} = \rho^{i_2} + \gamma^{i_2}\frac{d\tilde{s}_{t+1}^a}{d\tilde{s}_{t+1}^{a_{i_2}}} \tag{56}$$

we know

$$\sum_{i}^{n_1} \alpha^{i_1} w_i I(type^i) dup_t^{i_1} + \sum_{i}^{n_2} \alpha^{i_2} w_i I(type^i) dup_t^{i_2} > 0 \tag{57}$$

Given that condition (35) can be written as $\max_1(\alpha^i) < \min_2(\alpha^i)$, we know that

$$\begin{aligned} 0 &< \sum_{i}^{n_1} \alpha^{i_1} w_i I(type^i) dup_t^{i_1} + \sum_{i}^{n_2} \alpha^{i_2} w_i I(type^i) dup_t^{i_2} \\ &< \max_1(\alpha^i) \sum_{i}^{n_1} w_i I(type^i) dup_t^{i_1} + \min_2(\alpha^i) \sum_{i}^{n_2} w_i I(type^i) dup_t^{i_2} \\ &< \max_1(\alpha^i) \left(\sum_{i}^{n_1} w_i I(type^i) dup_t^{i_1} + \sum_{i}^{n_2} w_i I(type^i) dup_t^{i_2} \right) = \max_1(\alpha^i) dup_t \end{aligned} \tag{58}$$

Since $\max_1(\alpha^i) > 0$, $dup_t > 0$.