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Two Ways to Rule Out the Overconsumption Paths in the Ramsey Model with Irreversible Investment

by

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Two Ways to Rule Out the Overconsumption Paths in the Ramsey Model with Irreversible Investment.

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Abstract

In this note I develop two approaches to rule out the overconsumption paths in the Ramsey model with irreversible capital. The first focuses on the multiplier of the irreversible constraint and is applied to the situation where preferences are CES and the production function is Cobb-Douglas. The second, relies on a revealed preference argument and is used to rule out overconsumption paths when the preferences are strictly concave and the initial level of per effective capital is below its steady state level.

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1. Introduction

In the infinite horizon Ramsey model with reversible investment all overconsumption paths of the Ramsey model lead to the consumption of all the capital, at some finite time. This cannot be optimal since the marginal product of capital is

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infinite at $k = 0$ and the neoclassical agent would have been better off by investing the last unit of capital instead of consuming it. In short, overconsumption paths can be ruled out because they violate the Euler equation.[1] This argument does not extend to the situation where capital is irreversible (i.e. $c \leq f(k)$). Then, the locus $c = f(k)$ is reached at some finite time and from then on capital follows the differential equation

$$\dot{k} = -(\delta + n + x)k$$

where δ is the depreciation rate, k is capital per effective unit of labor (i.e. $\frac{K}{LA}$), and x and n are the exogenous rates of technological and population growth. For future reference, c is going to denote consumption per effective unit of labor (i.e. $\frac{C}{LA}$). Note that with this law of motion the $k = 0$ locus is reached only asymptotically and the usual argument cannot be made.

In this note I present two arguments that, under different circumstances, rule out the overconsumption paths in the Ramsey model with irreversible capital. The first approach is applied to the case where the utility function is CES and the production technology is Cobb Douglas and attacks the problem by showing that the multiplier of the irreversible constraint eventually becomes negative at any overconsumption path. Remember that the multiplier of any constraint can be interpreted as the marginal value of relaxing the constraint. Hence the negative multiplier in the irreversible constraint implies that the marginal value of consumption is smaller than the marginal value of saving at some point along the $c = f(k)$ locus, i.e. the agent would be better off by saving more. As a result, overconsumption cannot be optimal. The second technique relies on a revealed preference argument. In particular, if the constraint on the irreversibility of investment is not binding at any instant along the optimal path for the unrestricted problem, and this optimal path is unique, it must also solve the problem once the constraint is introduced. As a corollary of this proposition, when the irreversibility constraint on investment is not binding along the solution to Ramsey problem with reversible investment, overconsumption paths are not optimal in the restricted problem. In order to compare the applicability of the two methods I show that when the initial level of efficiency units of capital is below the steady state level the irreversible constraint on investment does not bind along the optimal path in the traditional Ramsey problem.

[1] See Blanchard and Fisher (1989) or Barro and Sala-i-Martín (1995) among others.

The exercise conducted in this paper is related to two other papers. In his seminal analysis of the Ramsey model, Cass [1965] imposes the constraint on irreversible investment but he does not rule out overconsumption paths. Arrow and Kurz [1970] are interested in understanding when the economies with irreversible capital that converge to the steady state from above consume all the output. Their main conclusion is that, in general, it is very hard to characterize the number of intervals in which the nonnegativity constraint on investment is binding. I am concerned about the optimality of overconsumption paths. My analysis focuses more on economies that converge from below, and for the CES utility function and Cobb Douglas production function case it turns out that, even if the economy converges from above, the constraint on nonnegative investment is never binding.

2. The Ramsey problem with irreversible investment

The social planner enforces the optimal intertemporal allocation of resources given the initial capital (per effective unit) (k_0), the law of motion for capital accumulation (equation 2), the no-Ponzi game restriction on borrowing (inequality 3), and the irreversibility constraints on investment (inequality 4).[2]

(1) $\max_{\{c_t; k_t\}_{t=0}^{\infty}} V = \int_0^{\infty} e^{-(\rho - n - x(1-\theta))t} \left[\frac{c_t^{1-\theta} - e^{-xt}}{1-\theta} \right] dt$

s.t.

(2) $\dot{k}_t = f(k_t) - c_t - (\delta + n + x)k_t$

(3) $\lim_{T \to \infty} k_T e^{-\int_0^T (f'(k_v) - \delta - n - x)dv} \geq 0$

(4) $f(k_t) \geq c_t, \ \forall t$

given $k_0 > 0$

where $f(k_t)$ is a neoclassical production function.[3]

3. The multiplier method.

If condition (4) is not imposed, the first order conditions of this problem are given by:[4]

[2] Note that for simplicity I write the program with a CES utility function. As will be seen latter, this is only relevant for the first approach.

[3] See Barro and Sala-i-Martín (1995), page 16, for a precise description of the properties that must satisfy a neoclassical production function.

[4] See Barro and Sala-i-Martín (1995), Chapter 2.

(5) $H_c = 0$

(6) $H_k = -\dot{\mu}_t$

(7) $\lim_{T\to\infty} \mu_T k_T = 0$

(8) $\mu_t \geq 0$

where the Hamiltonian (H) is

$$H = e^{-(\rho - n - x(1-\theta))t}\left[\frac{c_t^{1-\theta} - e^{-xt}}{1-\theta}\right] + \mu_t[f(k_t) - c_t - (\delta + n + x)k_t]$$

H_z is the partial derivative of the Hamiltonian with respect to z, μ_t is the Lagrange multiplier associated with the law of motion of capital (equation 2), equation (7) is the transversality condition, and, for future reference, the mutiplier associated with the irreversibility constraint on investment (equation 4), will be denoted by λ_t.[5]

When the constraint on the irreversibility of investment (equation (4)) is binding the first order necessary conditions of the problem are going to be altered in three ways. The marginal utility of consumption is going to exceed (by λ_t) the marginal utility of investment. The other side of this effect is that by increasing the level of capital per effective worker, the representative agent reduces the extent to which her consumption decisions are constrained, by equation (4), in the future. Moreover, the Kuhn-Tucker theorem states that the multiplier of the irreversibility constraint on investment must be non-negative and will be strictly positive only when the constraint is binding. Lemma 3.1 formalizes these intuitions which are proved in the appendix.

Lemma 3.1. The F.O.N.C. of the Ramsey model with irreversible investment and CES utility function are given by

(5') $H_c = \lambda_t$

(6') $H_k = -\dot{\mu}_t - \lambda_t f'(k_t)$

(7') $\lim_{T\to\infty} \mu_T k_T = 0;$

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[5] The solution to this problem is given by the system

$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[f'(k_t) - \delta - \rho - \theta x]$

$\dot{k}_t = f(k_t) - c_t - (\delta + n + x)k_t$

with boundary conditions

$k_t = k_0 > 0$; at $t = 0$:

$\lim_{T\to\infty} k_T e^{-\int_0^T (f'(k_v) - \delta - n - x)dv} = 0$

See Barro and sala-i-Martín (1995), page 71.

(8') $\mu_t \geq 0$

(9.a') $f(k_t) \geq c_t$; (9.b') $\lambda_t \geq 0$; (9.c') $\lambda_t [f(k_t) - c_t] = 0$:

Since the conditions stated in Lemma 1 are necessary for optimality, in order to rule out the overconsumption paths it is sufficient to show that condition (9.b') is eventually violated. By following this strategy, we are able to proof that the marginal utility of consumption at the overconsumption paths is eventually lower than the marginal utility of saving and therefore the agent would be better off by increasing her saving rate along these paths. This result is stated and proved in Proposition 3.2 for the particular case of a Cobb-Douglas production function and CES utility function. The argument used in the proposition does not depend, a priori, on these specifications; however, it involves several first order differential equations and the assumed functional forms are very convenient.

Proposition 3.2. (Ruling out overconsumption paths) Suppose that the preferences are CES, and the production function is Cobb-Douglas. Then, the overconsumption paths in the Ramsey model with irreversible investment are not optimal.

Proof. Figure 3.1 illustrates two possible overconsumption paths drawn in the $(k; c)$ space. Overconsumption paths in an infinite horizon world are characterized by reaching in a finite time the $c_t = f(k_t)$ locus. Let's denote by t^* this instant. Inspecting the phase diagram we can see that the $c_t = f(k_t)$ locus is above the $\dot{k} = 0$ locus and therefore an economy on an overconsumption path will eventually reach the Northern quadrants. Once the economy is in this region the dynamics implied by the F.O.N.C. tend to reduce the capital stock per effective unit of labor. From t^* to infinity, the representative agent will consume $c_t = f(k_t)$: We can use the law of motion for capital together with this restriction to pin down the path of capital starting at t^*: In particular, $c_t = f(k_t)$ for $t \geq t^*$ together with equation (2) imply that

$$\frac{\dot{k}}{k} = -(\delta + x + n); \ \forall t \geq t^*$$

and therefore

(10) $k_t = k_{t^*} e^{-(\delta+n+x)t}$; for all $t \geq t^*$:

A Cobb-Douglas production function takes the form $f(k_t) = A k_t^{\alpha}$: Hence for $t \geq t^*$; $c_t = A k_{t^*}^{\alpha} e^{-(\delta+n+x)\alpha t}$:

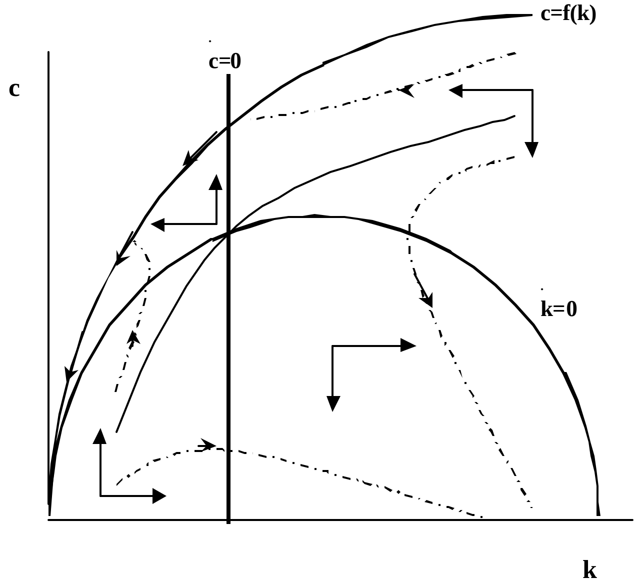

Figure 3.1: Phase Diagram

Now we can rewrite conditions (5') to (9') using the definition of the Hamiltonian and the expressions of c_t and k_t: In particular, for $t \geq t^*$

(11) $(Ak_{t^*}^{\alpha})^{-\theta} e^{-(\rho - n(1+\alpha\theta) - x(1-\theta+\alpha\theta) - \alpha\theta\delta)t} = \mu_t + \lambda_t$

(12) $(\mu_t + \lambda_t)\alpha A k_{t^*}^{\alpha-1} - \mu_t(\delta + n + x) = -\dot{\mu}_t$

(13) $\lim_{T\to\infty} \mu_T k_{t^*} e^{-(\delta+n+x)T} = 0;$

(8') $\mu_t \geq 0$

(14.a) $c_t = Ak_{t^*}^{\alpha} e^{-(\delta+n+x)\alpha t}$; (14.b) $\lambda_t \geq 0$:

Only three steps remain to conclude the proof. First I derive the evolution of μ_t and then I use the TVC to restrict the parameters. Finally, I reach a contradiction by showing that under this condition inequality (14.b) cannot hold.

Plugging equation (11) into (12) we obtain the differential equation

$$((Ak_{t^*}^{\alpha})^{-\theta} e^{-(\rho - n(1+\alpha\theta) - x(1-\theta+\alpha\theta) - \alpha\theta\delta)t})\alpha A k_{t^*}^{\alpha-1} - \mu_t(\delta + n + x) = -\dot{\mu}_t;\ \forall t \geq t^*$$

The solution to this differential equation takes the form

$$\mu_t = -\alpha A^{1-\theta} k_{t^*}^{\alpha(1-\theta)-1} e^{(\delta+n+x)t} \left[\frac{e^{\xi t}}{\xi} + b \right];\ \forall t \geq t^* \tag{15}$$

where b is an integration constant which must be pin down by some boundary condition, and $\xi \equiv -\rho + n + x(1-\theta) - \alpha(1-\theta)(\delta + n + x)$.

Using this expression for μ; and the expression for the level of physical capital; the transversality condition (13) can be written as

$$\lim_{T\to\infty}\left[-\alpha A^{1-\theta}k_{t^*}^{\alpha(1-\theta)}\frac{e^{\xi T}}{\xi} + b\right] = 0 \tag{16}$$

Since both b and ξ are finite constants, the only possibility for the TVC to hold is that $\xi < 0$; and $b = 0$:

Finally, using these restrictions and equations (11) and (15) I can obtain an expression for the multiplier of the irreversibility constraint.

$$\lambda_t = e^{\xi t}\left(\underbrace{(Ak_{t^*}^{\alpha})^{-\theta}}_{>0} + \underbrace{\frac{\alpha A^{1-\theta}k_{t^*}^{\alpha(1-\theta)-1}}{\xi}e^{(\delta+n+x)t}}_{<0}\right);\ \forall t \geq t^* \tag{17}$$

Note that since $(\delta+n+x) > 0$ the second term in the parenthesis will eventually dominate the first and therefore $\exists \hat{T}$ s.t. $\lambda_t < 0;\ \forall t > \hat{T}$: $*$

The intuition behind this result comes from equation (11). When λ_t is negative, the marginal utility of investment (μ_t) exceeds the marginal utility of consumption ($(Ak_{t^*}^{\alpha})^{-\theta}e^{-(\rho-n(1+\alpha\theta)-x(1-\theta+\alpha\theta)-\alpha\theta\delta)t}$) and therefore the agent would be better of by reducing her consumption (i.e. not following an overconsumption path).

4. The revealed preference argument

A second approach consists in contrasting the solution to the traditional Ramsey model with the Ramsey model with irreversible investment. If the constraint does not bind along the optimal path, and this is unique,[6] the unrestricted solution coincides with the solution to the problem once constraint (4) is incorporated. Next I state precisely and prove this result.

Theorem 4.1. Whenever the initial level of capital per effective unit of labor does not exceed the steady state level (i.e. $k_0 \leq k^{s:s}$) and the instantaneous utility

[6] A sufficient condition for uniqueness of the optimal consumption path is that the instantaneous utility function is strictly concave.

function is strictly concave, the solution to the Ramsey model with irreversible investment is exactly the same as the solution to the traditional Ramsey model.

Proof. Let $c(t)$ be the optimal path of consumption in the problem defined by (1) to (3) with a given k_0; and let $c^R(t)$ be the optimal path for the consumption level per unit of effective worker of the representative agent when the irreversibility constraint on investment (4) is added to the problem. Since the set of possible paths for consumption for the unrestricted problem (C) is a superset of the set of possible paths for the restricted problem (C^R);[7] the utility attained by $c^R(t)$ cannot exceed the utility of an agent that consumes according to $c(t)$, i.e. $V(c(t)) \geq V(c^R(t))$:

Next I use the law of motion for capital , equation (2), to show that the irreversibility constraint on investment (4) does not bind along the optimal path for the unrestricted problem. We know from the analysis of the reversible Ramsey model that if $k_0 \leq k^{ss}$; then $\dot{k}_t \geq 0; \forall t$: But, from equation (2), $\dot{k}_t \geq 0$ implies that $f(k_t) \geq c_t$;and therefore there is no need to consume the existing capital along the optimal path of the unconstrained problem.

Hence $c(t) \in C^R$;and since $V(c(t)) \geq V(c^R(t))$; and the optimal path is unique (by assumption) then it must be the case that $c(t) = c^R(t)$: ■

But since overconsumption is suboptimal in the Ramsey problem with reversible capital so is when capital is irreversible.

Corollary 4.2. The overconsumption paths in the infinite horizon Ramsey model with irresversible investment are not optimal when $k_0 < k_{ss}$.

5. Conclusions

In this note, I have presented two approaches to rule out overconsumption paths in the Ramsey model with irreversible investment. The first is based in showing that along any overconsumption path the multiplier of the irreversible constraint is eventually negative and therefore the marginal utility of saving exceeds the marginal utility of consumption. The second, relies on a revealed preference argument. In particular, if the solution to the unconstrained Ramsey problem is unique and along the optimal path there is no need to consume the existing capital, the optimal path of the constrained and unconstrained problem must coincide.

[7] i.e. $C \supseteq C^R$:

Both proofs are complementary. I use the ¯rst to rule out overconsumption paths when preferences are CES and the production technology is Cobb-Douglas; however, there is nothing speci¯c to these functional forms, apart from being convenient to conduct the prove with pencil and paper, that is necessary to use the multiplier argument. The second method just requires that the initial level of capital is below its steady state level. In this sense the multiplier method might be more general than the revealed preference argument.

References

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[2] Barro, R. and X. Sala-i-Mart¶ın Economic Growth, Advanced Series in Economics. New York; London and Montreal: McGraw-Hill, 1995.

[3] Blanchard, O. and S. Fisher Lectures on Macroeconomics, Cambridge, Mass. and London: MIT Press, 1989.

[4] Cass, D. (1965), Optimum Growth in an Aggregative Model of Capital Accumulation" Review of Economic Studies XXXII (3), No. 91 (July), pp. 233-40.

Appendix

In this appendix I prove Lemma 3.1.

Lemma 3.1 The F.O.N.C. of the Ramsey model with irreversible investment and CES utility function are given by

(5') $H_c = \lambda_t$
(6') $H_k = -\dot{\mu}_t - \lambda_t f'(k_t)$
(7') $\lim_{T \to \infty} \mu_T k_T = 0;$
(8') $\mu_t \geq 0$
(9.a') $f(k_t) \geq c_t$; (9.b') $\lambda_t \geq 0$; (9.c') $\lambda_t [f(k_t) - c_t] = 0.$

Proof. In the proof I follow the perturbation method used by Barro and Sala-i-Martín (1995) (Appendix 1.3).

In particular, I start with the finite horizon problem

(A.1) $\max_{\{c_t;k_t\}_{t=0}^{T}} V = \int_0^T e^{-(\rho - n - x(1-\theta))t} \left[\frac{c_t^{1-\theta} - e^{-xt}}{1-\theta}\right] dt$

s.t.

(A.2) $\dot{k}_t = f(k_t) - c_t - (\delta + n + x)k_t$
(A.3) $k_T e^{-\int_0^T (f'(k_v) - \delta - n - x)dv} \geq 0$
(A.4) $f(k_t) \geq c_t$, $t \in [0;T]$
given $k_0 > 0$

Next I define the Lagrangian (L)

$$\begin{aligned} L = & \int_0^T e^{-(\rho - n - x(1-\theta))t} \left(\frac{c_t^{1-\theta} - e^{-xt}}{1-\theta}\right) dt + \\ & + \int_0^T \left[\mu_t (f(k_t) - c_t - (\delta + n + x)k_t - \dot{k}_t)\right] dt + \\ & + \nu_T k_T e^{\int_0^T (-f'(k_\tau) + \delta + n + x)d\tau} + \int_0^T \lambda_t (f(k_t) - c_t)\, dt \end{aligned} \quad (A.5)$$

Integration by parts of the term $\int_0^T \mu_t \dot{k}_t dt$ gives

$$\int_0^T \mu_t \dot{k}_t dt = [\mu_t k_t]_0^T - \int_0^T \dot{\mu}_t k_t dt \quad (A.6)$$

and by plugging it into (A.5) the Lagrangian can be written as

$$\begin{aligned} L &= \int_0^T \left(H(c_t, k_t, t) + \dot{\mu}_t k_t\right) dt + \mu_0 k_0 - \mu_T k_T \\ &\quad + v_T k_T e^{\int_0^T (-f'(k_\tau)+\delta+n+x)d\tau} + \int_0^T \lambda_t \left(f(k_t) - c_t\right) dt \end{aligned} \tag{A.7}$$

where

$$H(c_t, k_t, t) = e^{-(\rho-n-x(1-\theta))t}\left[\frac{c_t^{1-\theta} - e^{-xt}}{1-\theta}\right] + \mu_t\left(f(k_t) - c_t - (\delta+n+x)k_t\right)$$

Let c^*_t and k^*_t be the optimal time paths for consumption and capital. If we perturb the optimal path c^*_t by an arbitrary perturbation function, p_{1t}, then we can generate a neighboring path for consumption,

$$c_t = c^*_t + \varepsilon p_{1t}$$

When c_t is perturbed, there must be a corresponding perturbation of k_t and k_T so as to satisfy the budget constraint:

$$\begin{aligned} k_t &= k^*_t + \varepsilon p_{2t} \\ k_T &= k^*_T + \varepsilon dk_T \end{aligned}$$

If the initial paths are optimal then $\frac{\partial L}{\partial \varepsilon} = 0$ when L is evaluated at the optimal paths. Let's rewrite the Lagrangian as a function of ε by substituting c_t, k_t and k_T into (A.7).

$$\begin{aligned} L &= \int_0^T \left(H(c^*_t + \varepsilon p_{1t}, k^*_t + \varepsilon p_{2t}, t) + \dot{\mu}_t (k^*_t + \varepsilon p_{2t})\right) dt + \mu_0 k_0 \\ &\quad + (k^*_T + \varepsilon dk_T)\left(v_T e^{\int_0^T (-f'(k_\tau)+\delta+n+x)d\tau} - \mu_T\right) + \int_0^T \lambda_t \left(f(k^*_t + \varepsilon p_{2t}) - (c^*_t + \varepsilon p_{1t})\right) dt \end{aligned}$$

$\frac{\partial L}{\partial \varepsilon}(0) = 0$ implies that

$$\int_0^T \left[(H_c - \lambda_t)p_{1t} + (H_k + \dot{\mu}_t + \lambda_t f'(k_t)p_{2t})\right] dt + dk_T\left(v_T e^{\int_0^T (-f'(k_\tau)+\delta+n+x)d\tau} - \mu_T\right) = 0 \tag{A.8}$$

Equation (A.8) can hold for all perturbation paths described by p_{1t}; p_{2t} and dk_T if each of the components in the equation vanishes, i.e.

$$H_c = \lambda_t \tag{A.9}$$

$$H_k = -\dot{\mu}_t - \lambda_t f'(k_t) \tag{A.10}$$

$$v_T e^{\int_0^T (-f'(k_\tau)+\delta+n+x)d\tau} = \mu_T \tag{A.11}$$

Using (A.11), (A.3) and the complementary slackness condition associated with (A.3) we can derive that

$$\mu_T k_T = 0$$

If $T \to \infty$ this condition takes the form (7'). Conditions (8') and (9') are necessary conditions from the Kuhn Tucker Theorem. $\square$