HEDGING VOLATILITY RISK

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Abstract

Volatility risk has played a major role in several financial debacles (for example, Barings Bank, Long Term Capital Management). This risk could have been managed using options on volatility which were proposed in the past but were never offered for trading mainly due to the lack of a tradable underlying asset.

The objective of this paper is to introduce a new volatility instrument, an option on a straddle, which can be used to hedge volatility risk. The design and valuation of such an instrument are the basic ingredients of a successful financial product. Unlike the proposed volatility index option, the underlying of this proposed contract is a traded at-the-money-forward straddle, which should be more appealing to potential participants. In order to value these options, we combine the approaches of compound options and stochastic volatility. We use the lognormal process for the underlying asset, the Orenstein-Uhlenbeck process for volatility, and assume that the two Brownian motions are independent. Our numerical results show that the straddle option price is very sensitive to the changes in volatility which means that the proposed contract is indeed a very powerful instrument to hedge volatility risk.

I. INTRODUCTION

Risk management is concerned with various aspects of risk, in particular, price risk and volatility risk. While there are various instruments (and strategies) to deal with price risk, exhibited by the volatility of asset prices, there are practically no instruments to deal with the risk that volatility itself may change. Volatility risk has played a major role in several financial disasters in the past 15 years. Long-Term-Capital-Management (LTCM) is one such example, "In early 1998, Long-Term began to short large amounts of equity volatility." (Lowenstein, R. (2000) p.123)[1]. LTCM was selling volatility on the S&P500 index and other European indexes, by selling options (straddles) on the index. They were exposed to the risk that volatility, as reflected in options premiums, will increase. They did not hedge this risk[2]. Though one can devise a dynamic strategy using options to deal with volatility risk such a strategy may not be practical for most users. There were several attempts to introduce instruments that can be used to hedge volatility risk (e.g., the German DTB launched a futures contract on the DAX volatility index) but those were largely unsuccessful[3].

Given the large and frequent shifts in volatility in the recent past[4] especially in periods like the summer of '97 and the fall of '98, there is a growing need for instruments to hedge volatility risk. Past proposals of such instruments included futures and options on a volatility index. The idea of developing a volatility index was first suggested by

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1 The quote and the information are taken from Roger Lowenstein's book When Genius Failed (2000), Ch.7.

2 Another known example is the volatility trading done by Nick Leeson in '93 and '94 in the Japanese market. His exposure to volatility risk was a major factor in the demise of Barings Bank (see Gapper and Denton (1996)).

3 Volatility swaps have been trading for some time on the OTC market but we have no indication of their success.

4 The volatility of volatility can be observed from the behavior of a volatility index, VIX, provided in Figure 1.

Brenner and Galai (1989). In a follow-up paper, Brenner and Galai (1993) have introduced a volatility index based on implied volatilities from at-the-money options[5]. In 1993 the Chicago Board Options Exchange (CBOE) has introduced a volatility index, named VIX, which is based on implied volatilities from options on the SP100 index. So far there have been no options offered on such an index. The main issue with such derivatives is the lack of a tradable underlying asset which market makers could use to hedge their positions and to price them. Since the underlying is not tradable we cannot replicate the option payoffs and we cannot use the no-arbitrage argument. The first theoretical paper[6] to value options on a volatility index is by Grunbichler and Longstaff (1996). They specify a mean reverting square root diffusion process for volatility similar to that of Stein and Stein (1991) and others. Since volatility is not trading they assume that the premium for volatility risk is proportional to the level of volatility. This approach is in the spirit of the equilibrium approach of Cox, Ingersoll and Ross (1985) and Longstaff and Schwartz (1992). A more recent paper by Detemple and Osakwe (1997) also uses a general equilibrium framework to price European and American style volatility options. They emphasize the mean-reverting in log volatility model.

Since the payoffs of the option proposed here can be replicated by a self-financing portfolio, consisting of the underlying straddle and borrowing, we value the option using a no arbitrage approach. The idea proposed and developed in this paper addresses both related issues: hedging and pricing. The key feature of the straddle option is that the underlying asset is an at-the-money-forward (ATMF) straddle rather than a volatility index. The ATMF straddle is a traded asset priced in the market place and well

5 The same idea is also described in Whaley (1993).

6 Brenner and Galai (1993) use a binomial framework to value such options where tradability is implicitly assumed.

understood by market participants. Since it is ATMF, its relative value (call + put)/stock is mainly affected by volatility. Changes in volatility translate almost linearly into changes in the value of the underlying, the ATMF straddle[7]. Thus options on the ATMF straddle are options on volatility. We believe that such an instrument will be more attractive to market participants, especially to market makers. In the next section we describe in detail the design of the instrument. In section III we derive the value of such an option. Section IV provides the conclusions.

II. The Design of the Straddle Option

To manage the market volatility risk, say of the S&P500 index, we propose a new instrument, a straddle option or STO(K_{STO}, T_1, T_2) with the following specifications. At the maturity date T_1 of this contract, the buyer has the *option* to buy a then at-the-money-forward straddle with a prespecified exercise price K_{STO}. The buyer receives both, a call and a put, with a strike price equal to the forward price, given the index level at time T_1[8]. The straddle matures at time T_2.

Our proposed contract has two main features: first, the value of the contract at maturity depends on the volatility expected in the interval T_1 to T_2

and therefore it is a tool to hedge volatility changes. Second, the underlying asset is a traded straddle. We believe that, unlike the volatility options, this design will greatly

[7] Strictly speaking this is true in a B-S world (See Brenner and Subrahmanyam (1988)) but here, with stochastic volatility, it may include other parameters (e.g. vol. of volatility).

[8] Theoretically there is no difference if the delivered option is a call, a put or a straddle since they are all ATMF. Practically, however, there may be some differences in prices due, for example, to transactions costs. A straddle would provide a less biased hedge vehicle.

enhance its acceptance and use by the investment community. The proposed instrument is conceptually related to two known exotic option contracts: compound options and forward start options[9]. Unlike the conventional compound option our proposed option is an option on a straddle with a strike price, unknown at time 0, to be set at time T_1 to the forward value of the index level. In general, in valuing compound options it is assumed that volatility is constant (see, for example, Geske (1979)). Given that the objective of the instrument proposed here is to manage volatility risk, we need to introduce stochastic volatility.

III. Valuation of the Straddle Option

The valuation of the straddle option (STO) will be done in two stages. First, we value the compound option on a straddle assuming deterministic volatility as our benchmark case. In the second stage we use stochastic volatility to value the option and then we relate the two.

A. The Case of Deterministic Volatility

To get a better understanding of the stochastic volatility case we first analyze the case where volatility changes only once and is known at time zero. We assume a constant volatility σ_1 between time 0 and T_1 (expiration date of STO) and a volatility σ_2 between T_1 and T_2 (expiration date of the straddle ST).

[9] Forward start options are paid for now but start at some time T_1 in the future. A forward start option with maturity T_2, as our proposed straddle, can be regarded as a special case of our straddle option in which the strike price K_{STO} is zero.

We first value the straddle at T_1, the day it is delivered. The straddle has the following payoff at maturity T_2

$$\mathrm{ST}(T_2)=\mathrm{C}(T_2)+\mathrm{P}(T_2)=\max(\tilde{S}(T_2)-S(T_1)e^{r(T_2-T_1)},0)$$

$$+\max(S(T_1)e^{r(T_2-T_1)}-\tilde{S}(T_2),0) \qquad (1)$$

where $C(T_2)$ and $P(T_2)$ are the payoff of the call and put respectively, $\tilde{S}(T_2)$ is the stock price at T_2 and $S(T_1)$ is the stock price at T_1. Since the strike price is at-the-money-forward at T_1 $K=S(T_1)e^{r(T_2-T_1)}$.

Assuming that the options are European as is the typical index option and that the Black-Scholes assumptions hold we have

$$ST(t)=C(t)+P(t)=S(t)[2N(d_1)-1]-S(T_1)e^{r(t-T_1)}[2N(d_2)-1] \qquad (2)$$

where $d_1=\dfrac{\ln(S_t/S(T_1))-r(t-T_1)+\frac{1}{2}s_2^2(T_2-t)}{s_2\sqrt{T_2-t}}$ s $\qquad d_2=d_1-s_2\sqrt{T_2-t}$

for the price of the straddle at $T_1\le t\le T_2$.

In particular for t=T_1 we know that (See Brenner and Subrahmanyam (1988))

$$C(T_1)=P(T_1)\approx(1/\sqrt{2p})s_2\sqrt{T_2-T_1}*S(T_1) \qquad (3)$$

Thus

$$ST(T_1)\approx 2\;S(T_1)(1/\sqrt{2p})s_2\sqrt{T_2-T_1} \qquad (4)$$

The straddle is practically linear in volatility. The relative value of the straddle, $ST(T_1)/S(T_1)$ is solely determined by volatility to expiration.

The value of the straddle option (STO) is the value of a compound option where the payoff of STO at expiration (T_1) is

$$\max(ST(T_1) - K_{STO}, 0) = \max[a\, S(T_1) - K_{STO}, 0] \tag{5}$$

where
$$a = 2\frac{1}{\sqrt{2p}} s_2 \sqrt{T_2 - T_1} \tag{6}$$

Equivalently, the payoff can be written as

$$a \max\,[S(T_1) - K_{STO}/a, 0] \tag{7}$$

Thus the price of the straddle, using the B-S model, at any time t, $0 \le t < T_1$ is

$$STO_t = a \cdot S_t \cdot N(d) - K_{STO} e^{-r(T_1 - t)} \cdot N(d - s_1 \sqrt{T_1 - t}) \tag{8}$$

where
$$d = \frac{\ln(a\, S_t / K_{STO} e^{-r(T_1 - t)}) + \frac{1}{2} s_1^2 (T_1 - t)}{s_1 \sqrt{T_1 - t}} \tag{9}$$

Equation (8) gives the value of an option on a straddle[10] which will be delivered at time T_1. This is a compound option that is easy to value since the straddle is at-the-money-forward on the delivery date which reduces the valuation to a univariate like case where the a term includes the parameter s_2.

Using (8) and (9) we can derive all the sensitivities of STO to changes in the various parameters. In particular, we are interested in the sensitivities of STO to the volatility in the first period T_1, called vega$_1$, and in the second period T_2, called vega$_2$. Vega$_1$ is given by

$$vega_1 = \frac{\partial STO_t}{\partial s_1} = S_t \sqrt{T_1 - t} \cdot N'(d) \tag{10}$$

where N'(d) is the standard normal density function, which is a standard result for any option except that d is also determined by a which is in turn determined by s_2, the volatility that will prevail in the second period. Thus, vega in the first period is affected by volatility in the second period which makes sense since the payoff at expiration of STO is determined by the volatility in the subsequent period. This leads to the next question; how does the change in s_2 affect STO_t? This is given by

$$vega_2 = \frac{\partial STO_t}{\partial s_2} = S_t N(d) \frac{\partial a}{\partial s_2} = S_t \cdot N(d) \cdot 2\sqrt{T_2 - T_1 \cdot} N'(d_1(T_1)) \tag{11}$$

where $$d_1(T_1) = \frac{1}{2} s \sqrt{T_2 - T_1}$$

and $N'\ (d_1(T_1))$ is the standard normal density function at T_1, the maturity of STO.

[10] It should be noted that the value of STO is based on an approximation to the value of the ATMF straddle, ST. As argued before this is practically indistinguishable from the theoretical value.

The sensitivity of STO to the volatility during the life of the straddle itself is also a function of the volatility in the current period, not just the volatility of the subsequent period. Since this case is only our benchmark case, we have not derived the other sensitivities, like theta and gamma, etc.

We would like now to turn to the case which is the very reason for offering a straddle option, the stochastic volatility case.

B. The Case of Stochastic Volatility

Several researchers have derived option valuation models assuming stochastic volatility. We are deriving the value of a particular compound option, an option on an ATMF straddle, assuming a diffusion process similar to the one offered by Hull and White (1987), Stein and Stein (1991), and others.

We assume that an equity index, S_t, follows the process given by

$$dS_t = rS_t dt + s_t S_t dB_t^1 \tag{12}$$

$$ds_t = d(q - s_t)dt + kdB_t^2 \tag{13}$$

Where r is the riskless rate and s_t is the volatility of S_t. Equation (12) describes the dynamics of the index with a stochastic volatility s_t. Equation (13) describes the dynamics of volatility itself which is reverting to a long run mean q where d is the adjustment rate and k is the volatility of volatility. B_t^1 and B_t^2 are two independent

Brownian motions. To obtain a valuation formula for STO, the option on a straddle, we need to go through a few steps starting from the end payoffs (values). First, to get the index value and the volatility at time T we integrate equations (12) and (13).

$$S_T = S_t \exp\left(\int_t^T \left(r - \frac{1}{2} s_t^{\,2}\right) dt + \int_t^T s_t \, dB_t^1\right)$$

$$s_T = q + (s_t - q) e^{-d(T-t)} + k \int_t^T e^{-d(T-t)} dB_t^2 \qquad (15)$$

The conditional probability density function of S_T is given by

$$f(S_T \mid S_t, s_t; r, T-t, d, q, k) = e^{-r(T-t)} f_o\left(S_T e^{-r(T-t)}\right) \qquad (16)$$

where

$$f_o(S_T) = \frac{1}{2p} \left(\frac{S_t}{S_T}\right)^{3/2} \frac{1}{S_t} \int_{-\infty}^{\infty} I\left(\left(h^2 + \frac{1}{4}\right) \frac{T-t}{2}\right) \cos\left(h \ln \frac{S_T}{S_t}\right) dh$$

where the function $I(l)$ is given by equation (8) of Stein and Stein (1991).

The conditional probability density function of s_T is given by

$$f(s_T \mid s_t; T-t, d, q, k) = \frac{1}{\sqrt{\frac{p k^2}{d}\left(1 - e^{-2d(T-t)}\right)}} e^{-\frac{\left(s_T - q - (s_t - q) e^{-d(T-t)}\right)^2}{\frac{k^2}{d}\left(1 - e^{-2d(T-t)}\right)}} \qquad (17)$$

since s_T is normally distributed with mean $\qquad E(s_T \mid s_t) = q + (s_t - q)e^{-d(T-t)}$

and

variance $\qquad V(s_T \mid s_t) = \dfrac{k^2}{2d}(1 - e^{-2d(T-t)})$

An example of the probability density function of the volatility, s_T, is given in Figure 2. The volatility is normally distributed with a mean 0.2. Theoretically we can have a negative value for volatility but practically the probability is less than 10^{-8} for reasonable parameter values[11].

The joint distribution of S_T and s_T is

$$f(S_T, s_T) = f(S_T)\, f(s_T) \tag{18}$$

since the two Brownian motions are independent. Once we have the joint probability density function, we can price any options written on the asset price and/or the volatility, including straddle options proposed in our paper.

Since STO is a compound option written on a straddle, we have to evaluate the price of the straddle at time T_1 first, then use it as the payoff to evaluate the straddle at time 0. Using risk-neutral valuation the price of the ATMF straddle at time T_1 is

[11] In the example in Figure 2 we use the same parameter values that are used by Stein and Stein (1991).

$$ST_{T_1} = 2e^{-r(T_2-T_1)} \int_{S_{T1}e^{r(T_2-T_1)}}^{\infty} (S_{T_2} - S_{T_1} e^{r(T_2-T_1)}) f(S_{T_2} \mid S_{T_1}) dS_{T_2}$$

$$= 2S_{T1} F(s_{T1}; T_2 - T_1, r, d, q, k) \qquad (19)$$

where the strike price is $S_{T1}e^{r(T_2-T_1)}$

For the constant volatility model where k and d are zero the function $F(s_{T_1})$, derived in the last subsection, can be approximated by

$$F_A(s_{T_1}) = \sqrt{\frac{T_2 - T_1}{2p}} s_{T_1} \qquad (20)$$

which is almost identical with the Black-Scholes values, as mentioned above (equation (3)). In Tables 1, 2ab and 2c we provide the values of the straddle ST computed from the stochastic volatility (SV) model using various parameter values.

Table 1 (and figure 3) provides the values of ST for a combination of initial volatility, s_{T_1} and k (volatility of volatility). The first column provides straddle values using the BS model with a deterministic volatility (k=0). As expected, the value of ST increases as s_{T_1} does and as k does. For low levels of s_{T_1} the effect of k is higher than for high levels of s_{T_1}. For example, when s_{T_1} is 10 percent and k is zero, (i.e. volatility is deterministic), the value of ST is 8.9 which will go up to 9.27 for k=.2 and 11.47 for k=.5. However, for s_{T_1} of 50% the value of ST at k=0 is 19 and it only goes to 20.20 for a high k=.5. In other words, in a high volatility environment the marginal effect of k on the value of a straddle is rather small and the BS model provides values which are indistinguishable from a stochastic volatility model.

Table 2ab shows the effects of the mean reversion parameters. The higher is q, the long-run mean, the higher is the value of ST. The higher is the reversion parameter, d, the lower is the value of the straddle for high initial s_{T_1}, since it converges faster to the lower long-run mean. Table 2c provides the values of ST for different maturity spans of the straddle. The value of the straddle increases with maturity much more at lower initial volatility than at higher volatility, which is expected even when k=0. Stochastic volatility does not change that.

Given the values of the straddle we can now compute the value of the option on the straddle STO.

The price of STO at time t=0 is given by

$$STO_0 = \int_0^\infty G(s_{T1}) f(s_{T1} \mid s_0) d s_{T1} \tag{21}$$

where

$$G(s_{T1}) = 2F(s_{T1}) e^{-rT1} \int_{\frac{K_{STO}}{2F(s_{T_1})}}^{\infty} (S_{T1} - \frac{K_{STO}}{2F(s_{T1})}) f(S_{T1} \mid S_0) dS_{T1}$$

The values of STO are computed numerically in Table 3a to 3e using a range of parameter values. Next to the values from the SV model, in 3a, we present the values using the BS model (k=0). As expected, the value of this compound option using the SV model is larger than the value of this option using the BS model. The difference between the two depends on the values of the other parameters in the SV model and the strike price K_{STO}. For relatively low strike prices, K_{STO}, the effect of stochastic volatility is rather small and the values are not that different from a BS value, ignoring stochastic volatility. For higher strike prices, out of the money, the effect of k is much larger. For K_{STO}=11, currently approximately at-the-money, the value of STO at k=.3 is about 90 percent larger than STO at k=.1 (1.75 vs. 0.91) while the BS value is only 0.77. Table 3b shows the effect of initial volatility, σ_0. At low strike prices an increase in initial volatility has a small effect on the values of STO. At high strike prices the value of STO is lower but the marginal effect of σ_0 is much higher. Table 3c shows the effect of θ, the long-run volatility on STO. For low values of θ, the value of STO is declining as we get to the ATM strike. Hedging against changes in volatility in a low volatility environment is not worth much. Table 3d shows the combined effect of volatility and k, volatility of volatility, at the ATM strike of STO. As expected , the value of STO increases in both and is rather monotonic. Stochastic volatility has a relatively bigger effect in a low volatility enviroment. Table 3e provides values of the straddle option for 3 maturities of the straddle. The values are higher for longer maturities since the delivered straddle has longer time to expiration and thus has a higher value. The effect is most pronounced when maturity is one year. The STO has some positive values even for strikes which are way out of the money.

The effect of the various parameters on the value of STO could be discerned from the previous tables but a better understanding of the complex relationships can be obtained from an examination of the various sensitivities given in Tables 4a to 4c. Table

4a provides the sensitivity of STO to changes in volatility, which is the main issue here. Table 4a provides these values at 5 levels of s_0. The values are high at all levels of initial volatility , though they tend to decline as volatility increases , indicating that changes in volatility could be effectively hedged by the straddle option. It becomes less effective as the strike price K_{STO} increases, the option is out-of-the-money. Table 4b provides values for the sensitivity of STO to k, volatility of volatility. The higher is k, the higher is the "vega" of STO. It is most sensitive at intermediate values of the strike price and approaches zero as the strike price increases. Table 4c provides another interesting sensitivity. The sensitivity with respect to the time to maturity of the straddle itself, $T_2 - T_1$. For a maturity of 3 months the sensitivity is higher than for a longer maturity, 6 months or a year, because the incremental value of STO at a shorter maturity is larger than at a longer maturity where the value is already high.

An interesting observation regarding the value of STO emerges. Does STO have a higher value, relative to BS value, in markets with higher volatility? It seems that higher s , for a given k (volatility of volatility), tends to reduce the differences between SV values and BS values since s is the dominant factor in the valuation. However, if higher s is accompanied by higher k STO values will be served little by a stochastic volatility model.

IV. Conclusions

As was evident in several large financial debacles involving derivative securities, like Barings and LTCM, the culprit was the stochastic behavior of volatility which has affected options premiums enough to contribute to their near demise. In this paper we propose a derivative instrument, an option on a straddle that can be used to hedge the risk inherent in stochastic volatility. This option could be traded on exchanges and used for risk management. Since valuation is an integral part of using and trading such an option we derive the value of such an option using a stochastic volatility model. We compare the value of such an option to a benchmark value given by the BS model. We find that the value of such an option is very sensitive to changes in volatility and therefore cannot be approximated by the BS model.

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Appendix: Benchmark Values of ST and STO

Regarding the benchmark values of ST and STO, we would like to set them depending on a mean-reverting deterministic volatility function, i.e.,

$$ds_t = d(q - s_t)dt, \qquad s_T = q + (s_t - q)e^{-d(T-t)}.$$

Then the average volatility over the time period between T_1 and T_2 is given by

$$s_2 = \sqrt{\int_{T_1}^{T_2} s_T{}^2 dT} = \sqrt{\int_{T_1}^{T_2} [q + (s_{T_1} - q)e^{-d(T-T_1)}]^2 dT}$$
$$= \sqrt{q^2 + 2q(s_{T_1} - q)\frac{1-e^{-d(T_2-T_1)}}{d(T_2-T_1)} + (s_{T_1} - q)^2 \frac{1-e^{-2d(T_2-T_1)}}{2d(T_2-T_1)}}.$$

And the average volatility over the time period between $t \in [0, T_1]$ and T_1 is given by

$$s_1 = \sqrt{\int_{t}^{T_1} s_T{}^2 dT} = \sqrt{\int_{t}^{T_1} [q + (s_t - q)e^{-d(T-t)}]^2 dT}$$
$$= \sqrt{q^2 + 2q(s_t - q)\frac{1-e^{-d(T_1-t)}}{d(T_1-t)} + (s_t - q)^2 \frac{1-e^{-2d(T_1-t)}}{2d(T_1-t)}}.$$

Especially for the case $t = 0$, the average volatility between 0 and T_1 is

$$s_1 = \sqrt{q^2 + 2q(s_0 - q)\frac{1-e^{-dT_1}}{dT_1} + (s_0 - q)^2 \frac{1-e^{-2dT_1}}{2dT_1}}.$$

The price of ST at time T_1 is given by equation (4) and the price of STO at time t is given by equation (8) with s_2 and s_1 given by the formulas above. The first columns of table 1, table 3a and table 3d are computed by using these formulas.

Figure 1
S&P 100 Volatility Index (VIX)

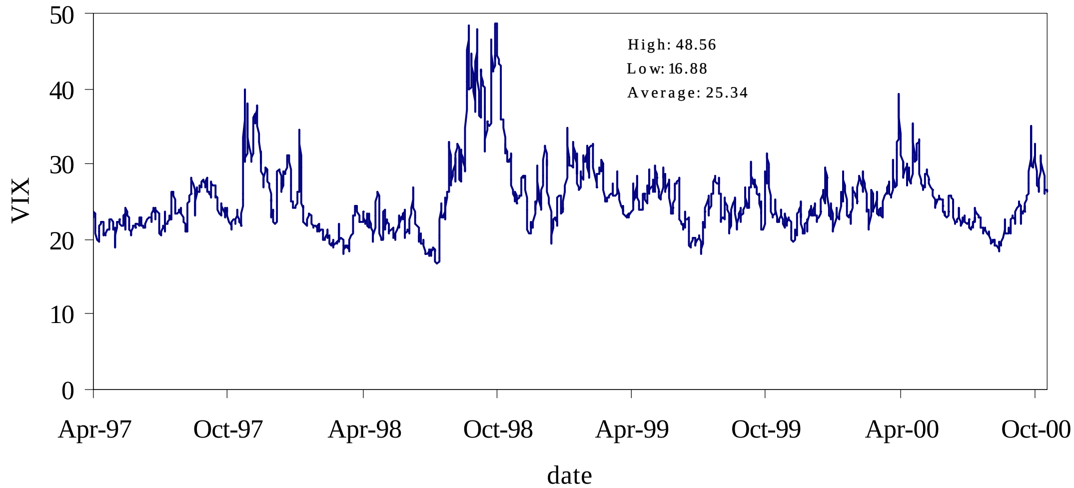

Figure 1. Closing level on the S&P 100 Volatility Index (VIX). The sample period is April 1, 1997 – November 3, 2000. Source: CBOE.

Figure 2
Probability Density of Volatility

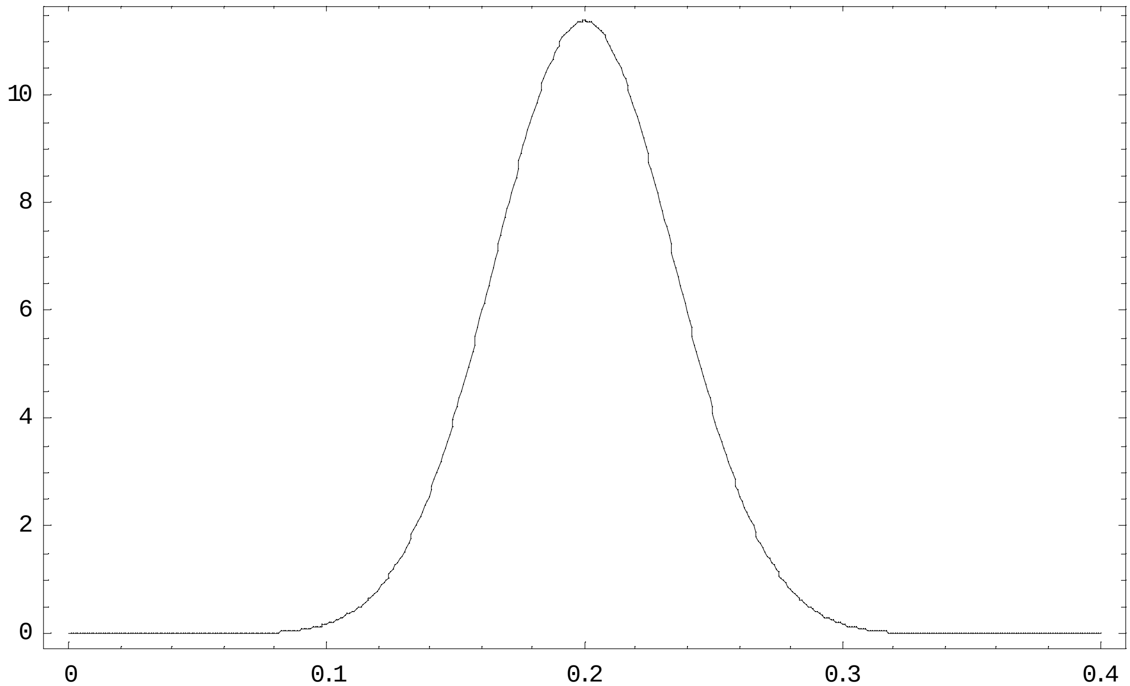

Figure 2 . An example of the probability density function of the **volatility .**
The parameter values are the same as in Stein and Stein (1991).

Table 1
The values of the straddle *ST*

s_{T_1} \ k	0 (BS)	0.10	0.20	0.30	0.40	0.50
0.00	6.9605	7.0657	7.4290	8.0932	9.0286	10.1845
0.10	8.9446	9.0276	9.2782	9.7661	10.5163	11.4783
0.20	11.2744	11.3430	11.5511	11.9298	12.5078	13.2818
0.30	13.7735	13.8323	14.0098	14.3219	14.7879	15.4171
0.40	16.3622	16.4134	16.5679	16.8343	17.2250	17.7497
0.50	19.0014	19.0466	19.1831	19.4157	19.7514	20.1992
0.60	21.6701	21.7104	21.8318	22.0369	22.3328	22.7234
0.70	24.3557	24.3908	24.5009	24.6842	24.9478	25.2938
0.80	27.0506	27.0746	27.1812	27.3459	27.5834	27.8935
0.90	29.7494	29.7642	29.8606	30.0146	30.2305	30.5135
1.00	32.4482	32.4499	32.5246	32.6844	32.8711	33.1440

Table 1: The values of the straddle *ST* for a combination of initial volatility s_{T_1} and volatility of volatility k. S_{T_1} = 100, q = 0.20, d = 4.00, T_2 - T_1 = 0.5 year.

Figure 3
The values of the straddle *ST*

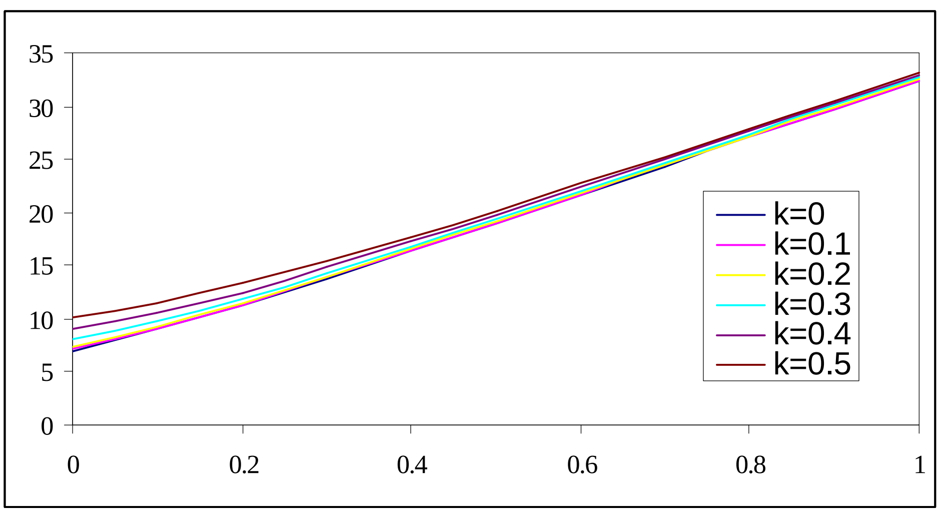

Figure 3: The values of the straddle *ST* for a combination of initial volatility s_{T_1} and volatility of volatility k. S_{T_1} = 100, q = 0.20, d = 4.00, T_2 - T_1 = 0.5 year.

Table 2ab: The values of the straddle *ST* for a combination of initial volatility s_{T_1} and the mean-reverting parameters (q, d) of volatility. S_{T_1} = 100, k = 0.2, T_2 - T_1 = 0.5 year.

s_{T_1}	$\theta = 0.10$ $\delta = 4.00$	$\theta = 0.20$ $\delta = 4.00$	$\theta = 0.30$ $\delta = 4.00$	$\theta = 0.40$ $\delta = 4.00$	$\theta = 0.20$ $\delta = 4.00$	$\theta = 0.20$ $\delta = 8.00$	$\theta = 0.20$ $\delta = 16.0$
0.00	4.7148	7.4290	10.7019	14.0967	7.4290	9.2234	10.3086
0.10	6.3157	9.2782	12.5645	15.9299	9.2782	10.2161	10.7792
0.20	8.6366	11.5511	14.7267	18.0004	11.5511	11.4747	11.4057
0.30	11.2007	14.0098	17.0700	20.2523	14.0098	12.9187	12.1637
0.40	13.8507	16.5679	19.5199	22.5892	16.5679	14.4909	13.0299
0.50	16.5427	19.1831	22.0478	25.0447	19.1831	16.1523	13.9838
0.60	19.2517	21.8318	24.6187	27.5357	21.8318	17.8788	15.0083
0.70	21.9585	24.5009	27.2092	30.0273	24.5009	19.6506	16.0898
0.80	24.7000	27.1812	29.7943	32.4909	27.1812	21.4566	17.2171
0.90	27.4138	29.8606	32.3373	34.8785	29.8606	23.2875	18.3814
1.00	30.0962	32.5246	34.7911	37.1382	32.5246	25.1349	19.5760

Table 2c: The values of *ST* for a combination of initial volatility s_{T_1} and different maturity spans of the straddle. S_{T_1} =100, k = 0.20, q = 0.20, d = 8.00.

T_2-T_1 s_{T_1}	0.25	0.5	1.0
0.00	5.1472	9.2234	14.7622
0.10	6.4605	10.2161	15.4186
0.20	8.0782	11.4747	16.2945
0.30	9.8316	12.9187	17.3547
0.40	11.6538	14.4909	18.5667
0.50	13.5110	16.1523	19.9018
0.60	15.4040	17.8788	21.3358
0.70	17.3080	19.6506	22.8498
0.80	19.2215	21.4566	24.4262
0.90	21.1411	23.2875	26.0516
1.00	23.0644	25.1349	27.7129

Table 3a: The value of *STO* at $t = 0$ for a combination of strike price K_{STO} and volatility of volatility k. S_0 =100, r = 0, s_0 = 0.20, q = 0.20, d = 4.00, T_1 =0.5, T_2 = 1.0.

k / K_{STO}	0 (BS)	0.10	0.20	0.30	0.40	0.50
0	11.2744	11.3521	11.5802	11.8411	12.1468	12.5649
1	10.2744	10.3522	10.5832	10.8747	11.2317	11.6998
2	9.2744	9.3522	9.5857	9.9047	10.3111	10.8294
3	8.2744	8.3522	8.5879	8.9335	9.3885	9.9570
4	7.2744	7.3522	7.5900	7.9621	8.4653	9.0839
5	6.2744	6.3522	6.5922	6.9906	7.5421	8.2108
6	5.2744	5.3522	5.5949	6.0200	6.6196	7.3381
7	4.2745	4.3527	4.6018	5.0548	5.7004	6.4673
8	3.2778	3.3607	3.6291	4.1110	4.7933	5.6028
9	2.3080	2.4080	2.7139	3.2222	3.9196	4.7545
10	1.4398	1.5648	1.9074	2.4283	3.1137	3.9424
11	0.7745	0.9086	1.2542	1.7579	2.4064	3.1956
12	0.3559	0.4700	0.7718	1.2234	1.8121	2.5386
13	0.1405	0.2181	0.4468	0.8203	1.3318	1.9812
14	0.0484	0.0920	0.2453	0.5319	0.9572	1.5218
15	0.0148	0.0358	0.1290	0.3351	0.6743	1.1523
16	0.0041	0.0131	0.0657	0.2063	0.4668	0.8614
17	0.0010	0.0045	0.0328	0.1248	0.3186	0.6368
18	0.0002	0.0015	0.0161	0.0746	0.2151	0.4665
19	0.0001	0.0005	0.0079	0.0444	0.1441	0.3392
20	0.0000	0.0002	0.0039	0.0263	0.0961	0.2454

Table 3b: The value of *STO* at t = 0 for a combination of strike price K_{STO} and the initial volatility s_0. S_0 =100, r = 0, k = 0.20, q = 0.20, d = 4.00, T_1 =0.5, T_2 = 1.0.

s_0 / K_{STO}	0.10	0.20	0.30	0.40	0.50
0	11.2316	11.5802	11.9050	12.2228	12.5358
1	10.2513	10.5832	10.9091	11.2339	11.5587
2	9.2599	9.5857	9.9111	10.2375	10.5653
3	8.2645	8.5879	8.9124	9.2386	9.5667
4	7.2682	7.5900	7.9136	8.2392	8.5671
5	6.2720	6.5922	6.9148	7.2400	7.5679
6	5.2761	5.5949	5.9170	6.2425	6.5715
7	4.2831	4.6018	4.9253	5.2537	5.5869
8	3.3084	3.6291	3.9570	4.2916	4.6329
9	2.3948	2.7139	3.0453	3.3869	3.7372
10	1.6027	1.9074	2.2333	2.5752	2.9295
11	0.9836	1.2542	1.5577	1.8859	2.2328
12	0.5527	0.7718	1.0348	1.3327	1.6578
13	0.2860	0.4468	0.6574	0.9118	1.2023
14	0.1378	0.2453	0.4018	0.6066	0.8545
15	0.0628	0.1290	0.2379	0.3942	0.5970
16	0.0275	0.0657	0.1374	0.2515	0.4114
17	0.0118	0.0328	0.0779	0.1581	0.2804
18	0.0050	0.0161	0.0436	0.0983	0.1895
19	0.0021	0.0079	0.0242	0.0607	0.1273
20	0.0009	0.0039	0.0134	0.0373	0.0852

Table 3c: The value of *STO* at t = 0 for a combination of strike price K_{STO} and the mean-reverting level q of volatility. S_0 =100, r = 0, k = 0.20, s_0 = 0.2, q = 0.20, d = 4.00, T_1 =0.5, T_2 = 1.0.

q / K_{STO}	0.10	0.20	0.30	0.4
0	6.5127	11.5331	16.5661	21.9330
1	5.5732	10.5675	15.6756	20.9542
2	4.6270	9.5817	14.7327	19.9653
3	3.6795	8.5872	13.7590	18.9704
4	2.7329	7.5900	12.7693	17.9726
5	1.8161	6.5922	11.7725	16.9732
6	1.0610	5.5949	10.7732	15.9733
7	0.5478	4.6018	9.7733	14.9733
8	0.2488	3.6291	8.7737	13.9734
9	0.1002	2.7139	7.7756	12.9735
10	0.0362	1.9074	6.7825	11.9743
11	0.0119	1.2542	5.8031	10.9768
12	0.0036	0.7718	4.8525	9.9834
13	0.0011	0.4468	3.9528	8.9985
14	0.0004	0.2453	3.1296	8.0288
15	0.0001	0.1290	2.4058	7.0837
16	0.0000	0.0657	1.7963	6.1749
17	0.0000	0.0328	1.3049	5.3148
18	0.0000	0.0161	0.9246	4.5155
19	0.0000	0.0079	0.6411	3.7869
20	0.0000	0.0039	0.4366	2.1360

Table 3d: The value of *STO* at t = 0 for a combination of the initial volatility s_0 and the volatility of volatility k. S_0 =100, r = 0, q = 0.20, d = 4.00, T_1 =0.5, T_2 = 1.0, K_{STO} = 11.5.

k / s_0	0.00	0.10	0.20	0.30	0.40	0.50
0.00	0.0951	0.2205	0.5477	1.0228	1.6334	2.3890
0.10	0.2729	0.4082	0.7453	1.2298	1.8503	2.6106
0.20	0.5354	0.6628	0.9918	1.4736	2.0947	2.8540
0.30	0.8493	0.9649	1.2771	1.7479	2.3637	3.1169
0.40	1.1939	1.2988	1.5922	2.0479	2.6535	3.3967
0.50	1.5582	1.6543	1.9299	2.3688	2.9611	3.6916
0.60	1.9365	2.0255	2.2851	2.7067	3.2842	4.0003
0.70	2.3252	2.4082	2.6540	3.0590	3.6207	4.3214
0.80	2.7224	2.8003	3.0340	3.4233	3.9690	4.6538
0.90	3.1267	3.2001	3.4229	3.7977	4.3277	4.9966
1.00	3.5372	3.6063	3.8184	4.1810	4.6947	5.3488

Table 3e: The value of *STO* at t = 0 for a combination of strike price K_{STO} and and different maturity spans of the straddle. S_0 =100, r = 0, k = 0.20, s_0 = 0.20, q = 0.20, d = 8.00, T_1 =0.5.

$T_2 - T_1$ K_{STO}	0.25	0.5	1.0
0	8.0720	11.4667	16.2738
1	7.0894	10.4872	15.2968
2	6.0932	9.4949	14.3088
3	5.0936	8.4970	13.3143
4	4.0936	7.4973	12.3164
5	3.0952	6.4974	11.3171
6	2.1197	5.4974	10.3172
7	1.2541	4.4979	9.3172
8	0.6212	3.5038	8.3173
9	0.2575	2.5394	7.3173
10	0.0916	1.6684	6.3178
11	0.0289	0.9753	5.3208
12	0.0084	0.5057	4.3342
13	0.0023	0.2354	3.3787
14	0.0006	0.1004	2.4926
15	0.0002	0.0401	1.7246
16	0.0000	0.0153	1.1143
17	0.0000	0.0057	0.6733
18	0.0000	0.0021	0.3828
19	0.0000	0.0008	0.2067
20	0.0000	0.0003	0.1071

Table 4a: The sensitivity of *STO* with respect to s_0. S_0 =100, r = 0, k = 0.20, q = 0.20, d = 4.00, T_1 = 0.5, T_2 = 1.0.

s_0	0.10	0.20	0.30	0.40	0.50
K_{STO}					
0	3.84	3.28	3.21	3.14	3.15
1	3.41	3.27	3.26	3.25	3.26
2	3.27	3.25	3.26	3.27	3.29
3	3.23	3.24	3.25	3.27	3.30
4	3.21	3.23	3.24	3.27	3.30
5	3.19	3.22	3.24	3.26	3.29
6	3.17	3.20	3.23	3.27	3.31
7	3.16	3.21	3.26	3.31	3.36
8	3.18	3.25	3.32	3.39	3.45
9	3.12	3.26	3.37	3.46	3.55
10	2.94	3.18	3.36	3.49	3.61
11	2.54	2.90	3.18	3.40	3.55
12	2.03	2.45	2.83	3.14	3.37
13	1.38	1.89	2.36	2.76	3.07
14	0.87	1.34	1.84	2.30	2.68
15	0.50	0.89	1.35	1.83	2.27
16	0.27	0.55	0.94	1.39	1.85
17	0.13	0.32	0.63	1.02	1.47
18	0.16	0.19	0.41	0.74	1.14
19	0.03	0.10	0.26	0.52	0.87
20	0.01	0.05	0.16	0.36	0.65

Figure 4a
The Sensitivity of *STO*

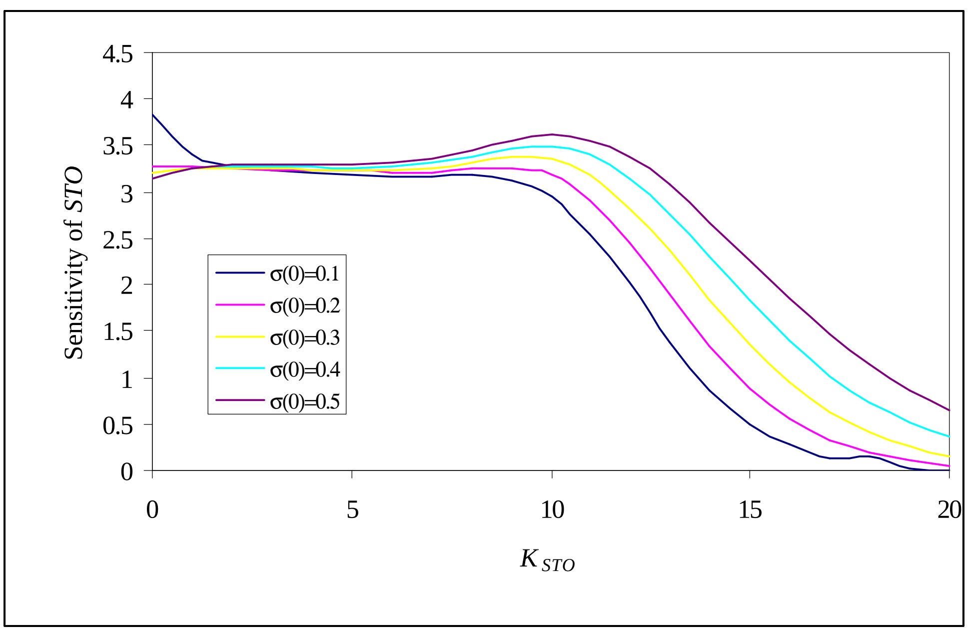

Table 4b: The sensitivity of *STO* with respect to *k*. S_0 =100, r = 0, s_0 = 0.20, q = 0.20, d = 4.00, T_1 = 0.5, T_2 = 1.0.

K_{STO} \ k	0.10	0.20	0.30	0.40	0.50
0	1.61	2.78	2.82	3.52	4.85
1	1.61	2.93	3.29	4.04	5.30
2	1.61	3.04	3.74	4.57	5.77
3	1.61	3.16	4.17	5.09	6.25
4	1.61	3.27	4.58	5.60	6.72
5	1.61	3.38	5.01	6.12	7.19
6	1.61	3.50	5.44	6.63	7.66
7	1.64	3.65	5.83	7.13	8.07
8	1.76	3.89	6.12	7.55	8.55
9	2.10	4.21	6.22	7.76	8.87
10	2.50	4.46	6.12	7.65	8.93
11	2.60	4.37	5.80	7.26	8.64
12	2.24	3.89	5.23	6.67	8.07
13	1.60	3.10	4.46	5.92	7.33
14	0.98	2.24	3.60	5.08	6.49
15	0.53	1.50	2.76	4.21	5.62
16	0.26	0.94	2.02	3.38	4.77
17	0.12	0.56	1.42	2.64	3.97
18	0.05	0.33	0.98	2.01	3.24
19	0.02	0.18	0.65	1.50	2.60
20	0.00	0.10	0.43	1.10	2.06

Figure 4b
The Sensitivity of STO

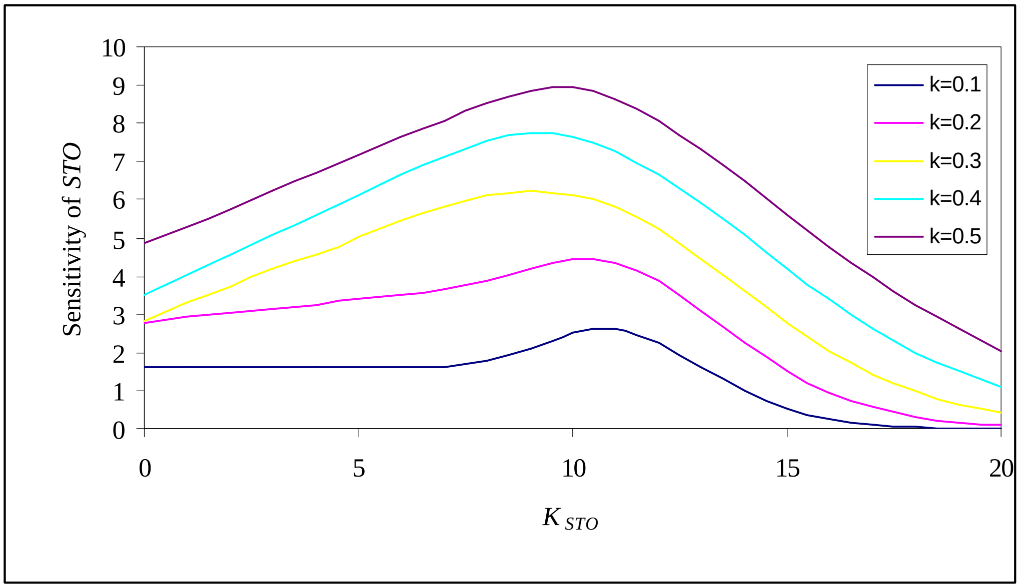

Table 4c: The sensitivity of *STO* with respect to T_1. S_0 =100, r = 0, k = 0.2, s_0 = 0.20, q = 0.20, d = 8.00, T_2 - T_1 = 0.5.

K_{STO} \ T_1	0.25	0.50	1.00
0	0.002	0.000	0.000
1	0.002	0.000	0.000
2	0.002	0.000	0.000
3	0.003	0.000	0.000
4	0.003	0.000	0.000
5	0.003	0.000	0.001
6	0.003	0.001	0.004
7	0.004	0.007	0.028
8	0.018	0.045	0.093
9	0.114	0.171	0.207
10	0.394	0.381	0.336
11	0.708	0.559	0.424
12	0.741	0.585	0.444
13	0.511	0.472	0.401
14	0.258	0.313	0.323
15	0.105	0.180	0.239
16	0.037	0.094	0.165
17	0.012	0.046	0.109
18	0.003	0.021	0.069
19	0.001	0.009	0.043
20	0.000	0.004	0.025

Figure 4c
Sensitivity of STO