Possible Sharing Arrangements in ARMA Supply Chains

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Abstract

We introduce a class of new sharing arrangements in a multi-stage supply chain in which the retailer observes stationary autoregressive moving average demand with Gaussian white noise (shocks). Similar to previous research, we assume each supply chain player constructs its best linear forecast of the leadtime demand and uses it to determine the order quantity via a periodic review myopic order-up-to policy. We demonstrate how a typical supply chain player can create a sequence of partial information shocks (PIS) from its full information shocks FIS and share these with an adjacent upstream player. We go on to show how such a sharing arrangement may be beneficial to the upstream player by characterizing the player's FIS in such a case. Hence, we study how a player can determine its available information under PIS sharing, and use this information to forecast leadtime demand. We characterize the value of FIS sharing for a typical supply chain player. Furthermore, we show conditions under which a player is able to form and share valuable PIS without (i) revealing its historic demand sequence or (ii) revealing its FIS sequence. We also provide a way of comparing various PIS sharing arrangements with each other and with conventional sharing arrangements involving demand sharing or FIS sharing. We show that demand propagates through a supply chain where any player may share nothing or a sequence of PIS shocks with an adjacent upstream player as quasi-ARMA in - quasi-ARMA out.

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1 Introduction

In this paper we consider all possible sharing arrangements that may have value to a player in a multi-stage supply chain in which the retailer observes covariance-stationary autoregressive moving average (ARMA) demand with Gaussian white noise (shocks). This work extends the research found in [Giloni et al., 2012] (hereafter GHS) and [Kovtun et al., 2012] (hereafter KGH), where the only sharing arrangements considered were no sharing, *full information shock* (FIS) sharing and demand sharing between contiguous players. Specifically we consider the possibility of *partial information shock* (PIS) sharing, which at times is equivalent to no sharing, FIS sharing or demand sharing. As we will show, the motivation for studying such arrangements is that they may also provide value that is intermediate to the three previously described sharing arrangements. This is appealing because players would be able to provide valuable information to other players (and be compensated) while also protecting themselves from revealing information that may be part of some confidentiality agreement they have with other players.

KGH described why the assumption of ARMA demand is highly appealing in the context of supply chains. They show that under certain assumptions demand will propagate as quasi-ARMA-in quasi-ARMA-out throughout the supply chain when no sharing, demand sharing, or FIS sharing is allowed to occur between contiguous players. They also show how these sharing arrangements may provide different levels of value. The value of information sharing is measured as a decrease in a player's mean square forecast error (MSFE) with the shared information as opposed to without it.

We make several important contributions to the literature. The first is in describing how PIS shocks may be formed by a player and shared with another player. The second is in showing the value of PIS sharing to a player by considering the player's FIS and MSFE under such an arrangement. Third we create a framework that allows us to compare all possible PIS sharing arrangements as well as comparing them to no sharing, demand sharing, and FIS sharing. Fourth is in showing that demand propagates as quasi-ARMA (QUARMA)-in quasi-ARMA-out when we allow for any PIS sharing arrangement to occur between contiguous players throughout the supply chain. Finally we show how such sharing arrangements can still be valuable while not revealing possibly confidential information.

In this paper, if there is no information sharing, an upstream player receives only an order from the adjacent downstream player. When there is PIS sharing, the downstream player provides its current observed PIS shock in addition to placing its order with the upstream player. As in GHS, demand or FIS sharing refers to the downstream player sharing its current observed demand or FIS in addition to placing its order with the upstream player.

We assume that all supply chain players use a myopic order-up-to inventory policy where negative order quantities are allowed, but the probability of negative demand or negative orders is negligible. As in [Lee et al., 2000] (hereafter LST), it is assumed that the lead time guarantee holds, i.e., if an upstream player does not have enough stock to fill an order from the adjacent downstream player, then the upstream player will meet the shortfall from an alternative source, with additional cost representing the penalty cost to this shortfall. Excess demand at the retailer is backlogged. Similar to previous research (c.f. [Zhang, 2004] (hereafter Zhang), GHS, KGH, and [Aviv, 2001], [Aviv, 2002], [Aviv, 2003], [Aviv, 2007] (hereafter Aviv)), we assume each supply chain player constructs its best linear forecast of leadtime demand and uses it to determine the order quantity via a periodic review myopic order-up-to policy.

With respect to the information structure, we assume that all players know the parameters of the ARMA model generating the retailer's demand and are aware of all the sharing arrangements that occur downstream. It will become apparent within this paper that this assumption implies the information structure assumed by others (including LST), specifically each player knows the form and parameters of the model of its own demand and of the model generating an adjacent downstream player's demand. As a consequence of the assumptions in this paper, we will show that each player in the supply chain will observe demand realizations that follow a quasi-ARMA (QUARMA) model (as defined in GHS) with respect to the player's *full information shocks* (those shocks which generate all of the player's information). The reader should note here that a downstream player's demand realizations, and full information shocks may be private knowledge, and not known to upstream players.

2 Research Problem

In this paper we adapt the framework found in GHS and KGH. Consider a K-stage supply chain where at discrete equally-spaced time periods, the retailer (assumed to be at stage 1) faces external demand $\{D_{1,t}\}$ for a single item with $\{D_{1,t}\}$ following a covariance stationary ARMA process. Although the retailer's demand is ARMA, it will become apparent later that, following the assumptions outlined in this section, a player k will observe demand series $\{D_{k,t}\}$ that is QUARMA (defined below) with respect to a series of observable shocks that contain all the information available to player k ($\{\epsilon_{k,t}\}$). Henceforth $\{D_{k,t}\}$ and $\{\epsilon_{k,t}\}$ will refer to player k's demand series and full information shock (FIS) series, defined below in Definition 2. We will write the observed demand and FIS sequence at time t as $\{D_{k,n}\}_{n=-\infty}^{t}$ and $\{\epsilon_{k,n}\}_{n=-\infty}^{t}$. We will also at times refer to this as *present and past values of* $\{D_{k,t}\}$ and $\{\epsilon_{k,t}\}$. Player k's demand and shock at time t will be $D_{k,t}$ and $\epsilon_{k,t}$.

Definition 1. *$\{D_{k,t}\}$ is QUARMA(p, q_k, J_k) with respect to a Gaussian white noise sequence $\{\epsilon_{k,t}\}$*

if we can write

$$D_{k,t} - \phi_1 D_{k,t-1} - \phi_2 D_{k,t-2} - \ldots - \phi_p D_{k,t-p} = d + \epsilon_{k,t-J_k} - \theta_1 \epsilon_{k,t-J_k-1} - \theta_2 \epsilon_{k,t-J_k-2} - \ldots - \theta_{q_k} \epsilon_{t-J_k-q_k} \quad (1)$$

where d *is a known constant (appearing in the retailer's demand model),* J_k *is a non-negative integer,* $\phi_p \neq 0$*, and* $\theta_{q_k} \neq 0$.

Note that a QUARMA model is very similar to the frequently used ARMA model, except here, the J_k most recent shocks are missing on the right-hand side. In the case $J_k = 0$ we have the equivalence QUARMA$(p, q_k, 0) \equiv$ ARMA(p, q_k). Consider the *backshift operator* B, where $BD_{k,t} = D_{k,t-1}$. We can express (1) in terms of the backshift operator as

$$\phi(B) D_{k,t} = d + B^{J_k} \theta_k(B) \epsilon_{k,t} \quad (2)$$

where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p$ and $\theta_k(B) = 1 - \theta_{k,1} B - \theta_{k,2} B^2 - \ldots - \theta_{k,q_k} B^{q_k}$. Note that $\phi(z)$ and $\theta_k(z)$ can be regarded as polynomials in the complex plane.

The retailer observes ARMA(p, q_1) demand with respect to Gaussian shocks $\{\epsilon_{1,t}\}$ where $p \geq 0$, $q_1 \geq 0$, i.e.,

$$\phi(B) D_{1,t} = d + \theta_1(B) \epsilon_{1,t} \quad (3)$$

where the polynomial $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p$ has no roots inside or on the unit cirlce and $\theta_1(z) = 1 - \theta_{1,1} z - \theta_{1,2} z^2 - \ldots - \theta_{1,q_1} z^{q_1}$ has no roots inside the unit circle (ie. for any root r_ϕ of $\phi(z)$, $|r_\phi| > 1$ and for any root r_θ of $\theta_1(z)$, $|r_\theta| \geq 1$ where $|\cdot|$ is the modulus of a complex number). This insures that the retailer can recover $\{D_{1,n}\}_{n=-\infty}^{t}$ from $\{\epsilon_{1,n}\}_{n=-\infty}^{t}$ and vice-versa, meaning that both series are observable to the retailer at time t.[1]

As in GHS, we let the replenishment leadtime from the retailer's supplier to the retailer be ℓ_1 periods. Excess demand at the retailer is backlogged. Let the replenishment leadtime from the

[1] Such an ARMA model is said to be *causal* and *invertible* (see [Brockwell and Davis, 1991], pp 83-88). If the ARMA model is not invertible, then at time t, future values in the sequence $\{D_{1,n}\}_{n=t}^{\infty}$ would be required to recover $\{\epsilon_{1,n}\}_{n=-\infty}^{t}$. If the ARMA model is not causal, then at time t, future values in the sequence $\{\epsilon_{1,n}\}_{n=t}^{\infty}$ would be required to recover $\{D_{1,n}\}_{n=-\infty}^{t}$.

player at stage $k+1$ to stage k be ℓ_k periods. We assume that all supply chain players use a myopic order-up-to inventory policy where negative order quantities are allowed, but d is sufficiently large so that the probability of negative demand or negative orders is negligible. Furthermore, h_k and p_k are player k's unit holding and shortage (or backorder) costs per time period. Player k's required service level is given by $c_k = \Phi^{-1}[\frac{p_k}{p_k+h_k}]$, where Φ is the standard Normal cdf. It is assumed that for $k \geq 1$ the ℓ_k period lead time guarantee holds, i.e., if the player at stage $k+1$ does not have enough stock to fill an order from the player at stage k, then the player at stage $k+1$ will meet the shortfall from an alternative source, with additional cost representing the penalty cost to this shortfall. [Gallego and Zipkin, 1999] show how this assumption allows one to decompose a multi-stage system with no alternative source into single-stage systems and to approximate the cost of the system.

Hence, at the end of time period t, after demand $D_{1,t}$ has been observed, the retailer observes the inventory position and places order $D_{2,t}$ with its supplier. The retailer receives the shipment of this order at the beginning of period $t+\ell_1+1$, where $\ell_1 \geq 0$. The sequence of events at all supply chain players is similar. However, it is further assumed that all upstream supply chain players observe their demand, observe their inventory positions and place their orders instantaneously at the end of time period t.

We assume that all players place their orders based on the best linear forecast of their lead-time demand. Thus player k's order will be based on its best linear forecast of the demand it will observe through time period $t+\ell_k+1$ (that is $\sum_{i=1}^{\ell_k+1} D_{k,t+i}$). It is assumed that all upstream supply chain players observe their demand, observe their inventory positions and place their orders instantaneously at the end of every time period t. It is assumed that all players are aware of the retailer's model and all sharing arrangements that occur downstream as in GHS. It follows from the results in this paper that for $k \geq 2$ the form and parameters of the model generating player

$k-1$'s demand and k's demand are known to player k. However player $k-1$'s demand realizations and/or full information shocks may not be observable by player k. In this paper, we assume (unlike KGH) that, at time t, along with placing its order, a player may choose to share PIS $\epsilon^{\star}_{k,t}$, as defined below, with an adjacent upstream player.

We show that no matter how players form their PIS when sharing with adjacent upstream players, we can write $\{D_{k,t}\}$ as QUARMA with respect to $\{\epsilon_{k,t}\}$ for any $k \geq 0$. This is done by mathematical induction on k in Corollary 3 of Section 3. The inductive hypothesis in the proof is that for a particular $k > 1$ we can express player $k-1$'s demand $\{D_{k-1,t}\}$ in terms of $\{\epsilon_{k-1,t}\}$ as

$$\phi(B)D_{k-1,t} = d + B^{J_{k-1}}\theta_{k-1}(B)\epsilon_{k-1,t} \tag{4}$$

We will call this player $k-1$'s *demand equation.*

2.1 Information Sets and Full Information Shocks

Following the notation in GHS, we denote the information set available to player k at time t as $\mathcal{M}_t^k$. GHS show that there exists a white noise sequence $\{\epsilon_{k,t}\}$ such that $\{\epsilon_{k,n}\}_{n=-\infty}^{t}$ is observable to player k and contains all the information available to player k at time t under the assumption of no sharing or FIS sharing. Mathematically, this would mean that any element in $\mathcal{M}_t^k$ can be found as a linear combination of elements from $\{1, \epsilon_{k,t}, \epsilon_{k,t-1}, \epsilon_{k,t-2}, \ldots\}$ (or as the limit of a Cauchy sequence of linear combinations of elements from $\{1, \epsilon_{k,t}, \epsilon_{k,t-1}, \epsilon_{k,t-2}, \ldots\}$) and player k would use this shock sequence to create a best linear forecast of lead-time demand. Thus we define the Hilbert space $\mathcal{M}_t^{\epsilon_k} = \overline{sp}\{1, \epsilon_{k,t}, \epsilon_{k,t-1}, \epsilon_{k,t-2}, \ldots\}$, where $\overline{sp}\{\}$ refers to the *closed linear span.* GHS refer to the shocks $\{\epsilon_{k,t}\}$ as player k's full information shocks according to Definition 2. The space $\mathcal{M}_t^{\epsilon_k}$ is sometimes referred to as the *linear past* of $\{\epsilon_{k,t}\}$.

Definition 2. *Suppose that for $k > 0$ we can represent player k's demand series $\{D_{k,t}\}$ as a QUARMA with respect to a series of shocks $\{\epsilon_{k,t}\}$. We say that $\{\epsilon_{k,t}\}$ are player k's Full Informa-*

tion Shocks (FIS) if $\mathcal{M}_t^k = \mathcal{M}_t^{\epsilon_k}$.

This definition implies two key properties of full information shocks. Player k's information set can be used to find player k's full information shocks. Also, player k's information set can be characterized using player k's full information shocks. In analyzing the case that player $k-1>0$ shares nothing or shares its full information shocks with player k, GHS define two important Hilbert spaces:

$$\begin{aligned}
\mathcal{M}_t^{D_k} &= \overline{sp}\{1, D_{k,t}, D_{k,t-1}, D_{k,t-2}, \ldots\} \\
\mathcal{M}_t^{D_k,\epsilon_{k-1}} &= \overline{sp}\{1, D_{k,t}, \epsilon_{k,t}, D_{k,t-1}, \epsilon_{k,t-1}, \ldots\}
\end{aligned}$$

KGH extends this by considering the case that player $k-1$ shares its demand with player k and therefore defines the information set

$$\mathcal{M}_t^{D_k,D_{k-1}} = \overline{sp}\{1, D_{k,t}, D_{k-1,t}, D_{k,t-1}, D_{k-1,t-1}, \ldots\} \quad (5)$$

In this paper we consider the case that player $k-1$ shares a PIS sequence $\{\epsilon_{k-1}^\star\}$. In this case player k's information set will be $\mathcal{M}_t^k = \mathcal{M}_t^{D_k,\epsilon_{k-1}^\star}$ where

$$\mathcal{M}_t^{D_k,\epsilon_{k-1}^\star} = \overline{sp}\{1, D_{k,t}, \epsilon_{k-1,t}^\star, D_{k,t-1}, \epsilon_{k-1,t-1}^\star, \ldots\} \quad (6)$$

It will turn out that $\mathcal{M}_t^{D_k,\epsilon_{k-1}^\star}$ may be such that

$$\mathcal{M}_t^{D_k,\epsilon_{k-1}^\star} = \mathcal{M}_t^{D_k} \quad (7)$$

in which case PIS sharing is equivalent to no sharing. It can also happen that

$$\mathcal{M}_t^{D_k,\epsilon_{k-1}^\star} = \mathcal{M}_t^{D_k,D_{k-1}} \quad (8)$$

in which case PIS sharing is equivalent to demand sharing. Likewise, its possible that

$$\mathcal{M}_t^{D_k,\epsilon_{k-1}^\star} = \mathcal{M}_t^{\epsilon_{k-1}} \quad (9)$$

in which case PIS sharing is equivalent to FIS sharing. However it may the case that $\mathcal{M}_t^{D_k,\epsilon^\star_{k-1}}$ cannot be described by any of these three sets with $\mathcal{M}_t^{D_k,\epsilon^\star_{k-1}}$ being intermediate to two of them.

We provide conditions for all these cases, but the major contribution of this paper is in showing how the last would occur and analyzing this case.

3 Partial Information Shock Sharing

In this section we will show how player $k-1$ can construct a series of PIS shocks from its FIS shocks $\{\epsilon_{k-1,t}\}$ and share these with player k. We go on to show how player k can determine the form of its FIS $\{\epsilon_{k,t}\}$ and its demand equation under the assumption that player $k-1$ shares its PIS. This will also demonstrate why player $k-1$'s PIS may provide value to player k.

First consider the important definition of the † operator (pronounced "*dagger*").

Definition 3. *Suppose a polynomial $P(z)$ factorizes as*

$$P(z) = \prod_{s=1}^{h}(1 - \frac{z}{a_s}) \prod_{s=h+1}^{q}(1 - \frac{z}{a_s})$$

such that $|a_s| < 1$ for $1 \leq s \leq h$ and $|a_s| \geq 1$ for $h+1 \leq s \leq q$.

Define $P^\dagger(z)$ as the polynomial

$$P^\dagger(z) = \prod_{s=1}^{h}(1 - \bar{a}_s z) \prod_{s=h+1}^{q}(1 - \frac{z}{a_s}) \tag{10}$$

where $\bar{a}_s$ is the complex conjugate of a_s

Note that when the † operator is applied to a polynomial, it inverts and conjugates any root of the polynomial that is inside the unit circle. The resulting polynomial will have all its roots outside or on the unit circle.

Consider the inductive hypothesis in Corollary 3 that player $k-1$ observes QUARMA demand $\{D_{k-1,t}\}$ with respect to its full information shocks $\{\epsilon_{k-1,t}\}$,

$$\phi(B)D_{k-1,t} = d + B^{J_{k-1}}\theta_{k-1}(B)\epsilon_{k-1,t} \tag{11}$$

By definition, the shocks $\{\epsilon_{k-1,t}\}$ are observable to player $k-1$. Player $k-1$ can thus form another shock sequence by passing $\{\epsilon_{k-1,t}\}$ through an all-pass filter, while also possibly excluding the most recent $J^\star_{k-1}$ shocks. We will call any shock sequence formed in this manner as player $k-1$'s *partial information shocks.*

Definition 4. *Suppose that for $k>0$ player $k-1$ observes FIS $\{\epsilon_{k-1,t}\}$. Let*

$$\epsilon^\star_{k-1,t} = B^{J^\star_{k-1}} \frac{A_{k-1}(B)}{A^\dagger_{k-1}(B)} \epsilon_{k-1,t} \tag{12}$$

where $A_{k-1}(z)$ is some polynomial having all its roots inside the unit circle and leading coefficient 1, and $J^\star_{k-1}$ is a nonnegative integer. We say that $\{\epsilon_{k-1,t}\}$ are player $k-1$'s partial information shocks.

The term $\frac{A_{k-1}(B)}{A^\dagger_{k-1}(B)}$ is the all-pass filter being used by player $k-1$ to construct $\{\epsilon^\star_{k-1,t}\}$. We will refer to the polynomial $A_{k-1}(z)$ as the polynomial of the all pass filter of player $k-1$. Since $A^\dagger_{k-1}(z)$ has no roots inside or on the unit circle, the right-hand-side of (12) includes no future values of $\{\epsilon_{k-1,t}\}$ and thus the sequence $\{\epsilon^\star_{k-1,t}\}$ is observable to player k. Furthermore the fact that $\frac{A_{k-1}(B)}{A^\dagger_{k-1}(B)}$ is an all-pass filter guarantees that $\{\epsilon^\star_{k-1,t}\}$ are indeed shocks.

Note that we can rewrite (12) as

$$A^\dagger_{k-1}(B)\epsilon^\star_{k-1,t} = B^{J^\star_{k-1}} A_{k-1}(B)\epsilon_{k-1,t} \tag{13}$$

Along with receiving $\{\epsilon^\star_{k-1,t}\}$, we will also assume that player k will know how $\{\epsilon^\star_{k-1,t}\}$ was created. Thus player k will know the polynomial $A_{k-1}(z)$ in (13) and the value of $J^\star_{k-1}$. This is a reasonable assumption after all since player $k-1$ can easily share this information and because the shocks $\{\epsilon^\star_{k-1}\}$ will have little value otherwise. This is also why it does not matter whether or not player $k-1$ includes a constant additive term or multiplicative term in equation (12). Player k could easily remove this term, by subtracting or dividing the shared $\{\epsilon^\star_{k-1}\}$ by an appropriate constant. Since (13) follows immediately from (12) we will refer to (13) as player $k-1$'s sharing equation.

We will show that given $\{\epsilon^\star_{k-1}\}$ along with (13) player k can recover its FIS using equations

$$A^\dagger_{k-1}(B)\epsilon^\star_{k-1,t} = B^{J^\star_{k-1}} A_{k-1}(B)\epsilon_{k-1,t} \tag{14}$$

and

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}_k(B)\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t} \tag{15}$$

where the last equation is player $k-1$'s *order equation* as defined in KGH. The form in (15) was shown to hold in both GHS and KGH as a consequence of how demand propagates according to a myopic order-up-to-policy, and is not related to the sharing arrangement between player $k-1$ and k. The notation of (15) is the exact notation used in this series of papers.

The following lemma is central to finding player k's FIS using equations (14) and (15).

Lemma 1. *Suppose we can represent two sequences $\{X_{1,t}\}$ and $\{X_{2,t}\}$ in terms of a zero-mean stationary process $\{\eta_t\}$ as*

$$A_1(B)X_{1,t} = B^{J_1} A_2(B)(1/\lambda)\eta_t \tag{16}$$

$$\phi(B)X_{2,t} = d + B^{J_2}\Theta(B)\eta_t \tag{17}$$

where $\phi(z)$ and $A_1(z)$ have no roots inside the unit circle, and a leading coefficient 1. $\Theta(z)$ and $A_2(z)$ have no roots at zero and a leading coefficient 1, and λ is a non-zero constant.

There exist functions $\vartheta(z)$ and $\omega(z)$ with one sided Laurent series representations converging in a disk $\mathcal{D}$ that contains the unit circle such that $\vartheta(B)A_1(B)X_{1,t} + \omega(B)\phi(B)X_{2,t} = \omega(1)d + \eta_t$ if and only if the polynomials $z^{J_1}A_2(z)$ and $z^{J_2}\Theta(z)$ have no common common roots inside or on the unit circle.

Lemma 1 is a slight generalization of Lemma 1 of KGH. Here we are not assuming that the polynomials on the left-hand-side are the same, nor that the constant term d appears in both equations. The proof however is very similar to the proof in KGH and is provided in the Appendix. The proof of Theorem 1 will show how Lemma 1 is used so that we can find player k's FIS for all sharing arrangements involving shocks that are formed by passing $\{\epsilon_{k-1,t}\}$ through an all-pass filter.

In order to properly define player k's FIS in such a situation, we adapt the following useful definition from KGH:

Definition 5. *For any $z \in \mathbb{C}$ and polynomial P, if z is a root of P we define $m(z,P)$ as the multiplicity of z in P. If z is not a root of polynomial P we define $m(z,P) = 0$.*

The following theorem gives the form of player k's FIS when player $k-1$ shares $\{\epsilon^{\star}_{k-1,t}\}$.

Theorem 1. *Suppose that player $k-1$ shares shock series $\{\epsilon^{\star}_{k-1,t}\}$ with player k where $\epsilon^{\star}_{k-1,t} = B^{J^{\star}_{k-1}} \dfrac{A_{k-1}(B)}{A^{\dagger}_{k-1}(B)} \epsilon_{k-1,t}$. Then player k's FIS will be $\{\epsilon_{k,t}\}$ where*

$$\epsilon_{k,t} = \frac{\tilde{\theta}_k^{I-CA}(B)}{\tilde{\theta}_k^{\dagger I-CA}(B)} B^{min(\tilde{J}_k, J^{\star}_{k-1})} \lambda_{k,\tilde{J}_k} \epsilon_{k-1,t} \tag{18}$$

where $\tilde{\theta}_k^{I-CA}(z) = \prod_{j=1}^{r_k} (1 - \frac{z}{z_j})^{min(m(z_j,\tilde{\theta}_k), m(z_j, A_{k-1}))}$, $z_1, \ldots, z_{r_k}$ are the r_k distinct roots of $\tilde{\theta}_k(z)$ inside the unit circle, with respective multiplicities $m(z_1, \tilde{\theta}_k), \ldots, m(z_{r_k}, \tilde{\theta}_k)$.

Furthermore player k's demand equation is $\phi(B) D_{k,t} = d + B^{J_k} \theta_k(B) \epsilon_{k,t}$ with $J_k = \tilde{J}_k - min(\tilde{J}_k, J^{\star}_{k-1})$ and $\theta_k(z) = \dfrac{\tilde{\theta}_k(z)}{\tilde{\theta}_k^{I-CA}(z)} \tilde{\theta}_k^{\dagger I-CA}(z)$.

Note here that we do not conclude that player k observes $\{\epsilon_{k-1,t}\}$ and computes its FIS using equation (18). Rather the claim is that the shocks $\{\epsilon_{k,t}\}$ are observable to player k and that $\{\epsilon_{k,t}\}$ is the result of applying (18) to $\{\epsilon_{k-1,t}\}$. The roots of polynomial $\tilde{\theta}_k^{I-CA}(z)$ are those roots of $\tilde{\theta}_k(z)$ inside the unit circle that are also common to $A_{k-1}(z)$ and the multiplicity of each root is the minimum of the multiplicities of the root in $\tilde{\theta}_k(z)$ and $A_{k-1}(z)$. Also note that $\theta_k(z)$ will be a polynomial with leading coefficient one since $\tilde{\theta}_k(z)$ has leading coefficient one and is divisible by $\tilde{\theta}_k^{I-CA}(z)$. The proof of Theorem 1 can be found in the Appendix. We also have the following useful corollaries:

Corollary 1. *Suppose that player $k-1$ shares shock series $\{\epsilon^{\star}_{k-1,t}\}$ with player k where $\epsilon^{\star}_{k-1,t} = B^{J^{\star}_{k-1}} \dfrac{A_{k-1}(B)}{A^{\dagger}_{k-1}(B)} \epsilon_{k-1,t}$. If $z^{\tilde{J}_k} \tilde{\theta}_k(z)$ and $z^{J^{\star}_{k-1}} A_{k-1}(z)$ have no common roots inside the unit circle, then player k's FIS are given by*

$$\epsilon_{k,t} = \lambda_{k,\tilde{J}_k} \epsilon_{k-1,t} \tag{19}$$

Furthermore $J_k = \tilde{J}_k$ *and* $\theta_k(z) = \tilde{\theta}_k(z)$*.*

Note that in this case player k can figure out the sequence $\{\epsilon_{k-1,t}\}$ by observing $\{\epsilon_{k,t}\}$ and dividing this sequence by $\lambda_{k,\tilde{J}_k}$.

Corollary 2. *Suppose that player* $k-1$ *shares shock series* $\{\epsilon^\star_{k-1,t}\}$ *with player* k *where* $\epsilon^\star_{k-1,t} = B^{J^\star_{k-1}} \dfrac{A_{k-1}(B)}{A^\dagger_{k-1}(B)} \epsilon_{k-1,t}$*. If* $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ *and* $z^{J^\star_{k-1}} A_{k-1}(z)$ *have at least one common root inside the unit circle, then at time* t*, player* k *cannot recover the sequence* $\{\epsilon_{k-1,n}\}_{n=-\infty}^{t}$*.*

Proof of Corollary 1. If $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ and $z^{J^\star_{k-1}} A_{k-1}(z)$ have no common roots inside the unit circle, then $\tilde{\theta}_k^{I-CA}(z) \equiv 1$, $\tilde{\theta}_k^{\dagger I-CA}(z) \equiv 1$, $\tilde{\theta}_k^{NC}(z) \equiv \tilde{\theta}_k(z)$ and $min(\tilde{J}_k, J^\star_{k-1}) = 0$ in (18) giving the desired result.

□

Proof of Corollary 2. If $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ and $z^{J^\star_{k-1}} A_{k-1}(z)$ have a common root inside the unit circle, then either $min(\tilde{J}_k, J^\star_{k-1}) > 0$ or $\tilde{\theta}_k^{I-CA}(z) \not\equiv 1$ in (18). Rewriting (18) as

$$\epsilon_{k-1,t} = \frac{\tilde{\theta}_k^{\dagger I-CA}(B)}{\tilde{\theta}_k^{I-CA}(B)} B^{-min(\tilde{J}_k, J^\star_{k-1})} (1/\lambda_{k,\tilde{J}_k}) \epsilon_{k,t} \tag{20}$$

we see that if the former is true then, at time t, player k would need to know $\epsilon_{k,t+min(\tilde{J}_k, J^\star_{k-1})}$ to recover $\epsilon_{k-1,t}$ in the sequence $\{\epsilon_{k-1,n}\}_{n=-\infty}^{t}$. But $\epsilon_{k,t+min(\tilde{J}_k, J^\star_{k-1})}$ is not known at time t if $min(\tilde{J}_k, J^\star_{k-1}) > 0$. If the latter is true then the polynomial in the denominator of (20) has a root inside the unit circle, so that at time t one would need to know future values in the sequence $\{\epsilon_{k-1,n}\}_{n=t}^{\infty}$ to recover $\{\epsilon_{k-1,n}\}_{n=-\infty}^{t}$. □

As mentioned, Theorem 1 cannot be used as a way to find player k's FIS. This is because the sequence appearing on the right-hand-side of (18) may not be observable to player k as shown by Corollary 2. The following is a brief discussion on how player k would recover its FIS in practice.

Consider player $k-1$'s sharing and order equations (whose forms are known to player k).

$$A^\dagger_{k-1}(B)\epsilon^\star_{k-1,t} = B^{J^\star_{k-1}} A_{k-1}(B)\epsilon_{k-1,t}$$

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}_k(B)\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$$

We can rewrite these as

$$A^{\dagger}_{k-1}(B)\epsilon^{\star}_{k-1,t} = B^{J^{\star}_{k-1}}A^{OUT}_{k-1}(B)A^{IN}_{k-1}(B)A^{ON}_{k-1}(B)\epsilon_{k-1,t} \tag{21}$$

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}^{OUT}_k(B)\tilde{\theta}^{IN}_k(B)\tilde{\theta}^{ON}_k(B)\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t} \tag{22}$$

where the superscript "OUT" indicates that the factors included in this polynomial have roots outside the unit circle, "IN" indicates that the polynomial has all roots inside the unit circle, and "ON" indicates that all the roots are on the unit circle. It is possible that the polynomial $\tilde{\theta}_k(z)$ has no roots outside the unit circle, in which case we take $\tilde{\theta}^{OUT}_k(z) \equiv 1$. Similar convention holds for the rest.

As in the proof of Lemma 1 we can apply the Euclidean Algorithm to obtain polynomials $Q_1(z)$ and $Q_2(z)$ such that

$$z^{J^{\star}_{k-1}}Q_1(z)A_{k-1}(z) + z^{\tilde{J}_k}Q_2(z)\tilde{\theta}_k(z)\lambda_{k,\tilde{J}_k} = z^{min(J^{\star}_{k-1},\tilde{J}_k)}\tilde{\theta}^{I-CA}_k(z)\tilde{\theta}^{OUT-CA}_k(z)\tilde{\theta}^{ON-CA}_k(z)$$

where the subscript "I-CA" indicates that this polynomial has all the roots common to $\tilde{\theta}^{IN}_k(z)$ and $A^{IN}_{k-1}(z)$, "OUT-CA" indicates that the polynomial has all the roots common to $\tilde{\theta}^{OUT}_k(z)$ and $A^{OUT}_{k-1}(z)$, and "ON-CA" indicates that the polynomial has all the roots common to $\tilde{\theta}^{ON}_k(z)$ and $A^{ON}_{k-1}(z)$. We consider a root common to two polynomials if it is a root of both polynomials. We take the multiplicity of such a root to be the minimum of the multiplicities of the root in the two polynomials.

We can multiply (21) and (22) by $Q_1(z)$ and $Q_2(z)$ and add them together to get

$$Q_1(B)\phi(B)A^{\dagger}_{k-1}(B)\epsilon^{\star}_{k-1,t} + Q_2(B)\phi(B)D_{k,t} - C = B^{min(J^{\star}_{k-1},\tilde{J}_k)}\tilde{\theta}^{OUT-CA}_k(B)\tilde{\theta}^{I-CA}_k(B)\tilde{\theta}^{ON-CA}_k(B)\epsilon_{k-1,t}$$

where $C = Q_2(B)d$. Dividing both sides by $\tilde{\theta}^{OUT-C}_k(B)\tilde{\theta}^{ON-C}_k(B)\tilde{\theta}^{\dagger I-C}_k(B)$ and multiplying by $\lambda_{k,\tilde{J}_k}$ we have

$$\frac{\lambda_{k,\tilde{J}_k}}{\tilde{\theta}^{OUT-CA}_k(B)\tilde{\theta}^{ON-CA}_k(B)\tilde{\theta}^{\dagger I-CA}_k(B)}\Big(Q_1(B)A^{\dagger}_{k-1}(B)\epsilon^{\star}_{k-1,t}+Q_2(B)\phi(B)D_{k,t}-C\Big) = \lambda_{k,\tilde{J}_k}B^{min(J^{\star}_{k-1},\tilde{J}_k)}\frac{\tilde{\theta}^{I-CA}_k(B)}{\tilde{\theta}^{\dagger I-CA}_k(B)}\epsilon_{k-1,t}$$

Note that the right-hand side of this equation is the right-hand-side of (18). The polynomials in the denominator on the left-hand-side contain no roots inside the unit circle. Therefore the expression can be computed without using any future values of $\{\epsilon^\star_{k-1,t}\}$ and $\{D_{k,t}\}$ and can therefore be found from historical data.

Using the results developed we can show that demand propagates according to the QUARMA-in QUARMA-out property of GHS. We will do this by induction. We have already used the inductive hypothesis of the following corollary to show the expression of player k's FIS in Theorem 1). We will now show that demand propagates as QUARMA-in QUARMA-out as a consequence of this expression.

Corollary 3. *Suppose the retailer observes causal and invertible ARMA demand*

$$\phi(B)D_{1,t} = d + \theta_1(B)\epsilon_{1,t}$$

and for any $k > 0$, player $k-1$ can share shocks $\{\epsilon^\star_{k-1,t}\}$ with an adjacent upstream player where $\epsilon_{k-1,t} = B^{J^\star_{k-1}}\dfrac{A_{k-1}(B)}{A^\dagger_{k-1}(B)}\epsilon_{k-1,t}$ for some polynomial $A_{k-1}(z)$ having all its roots inside the unit circle (but not at zero) and leading coefficient 1.

Then for any $k > 0$ we can express player k's demand as QUARMA with respect to player k's full information shocks:

$$\phi(B)D_{k,t} = d + B^{J_k}\theta_k(B)\epsilon_{k,t} \tag{23}$$

where $\theta_k(z)$ has a leading coefficient 1 and no roots at zero.

Proof of Corollary 3. The proof follows by induction immediately from Theorem 1.

For $k = 1$, we have the retailer observing (23) with $J_1 = 0$.

Now assume that (23) holds for some $k > 1$ where player $k-1$'s demand is QUARMA with respect to its FIS:

$$\phi(B)D_{k-1,t} = d + B^{J_{k-1}}\theta_{k-1}(B)\epsilon_{k-1,t}$$

Suppose player $k-1$ shares shocks $\{\epsilon^\star_{k-1,t}\}$ with player k.

By Theorem 1, we have that player k observes QUARMA demand with respect to its FIS:

$$\phi(B)D_{k,t} = d + B^{J_k}\theta_k(B)\epsilon_{k,t}$$

where J_k and $\theta_k(z)$ are provided by the theorem. □

4 Comparing Various Sharing Arrangements

We saw in the previous section that player $k-1$ can share many different shock sequences with player k. This section will be devoted to a discussion on the potential of these sharing arrangements to have different value to player k. We will see that we can compare the different sharing arrangements by comparing player k's FIS under the various sharing arrangements as well as the resulting MSFE. We will compare the value of player $k-$'s PIS that it shares with player k with other possible PIS. We will also compare the value of player $k-1$'s PIS to player k with the value of no sharing, demand sharing, and full information shock sharing discussed in KGH. We will see that there are situations where player $k-1$ can construct PIS that will provide more value to player k than no sharing, but less value to player k than demand sharing. Furthermore, as we will discuss, in these situations player k would be unable to recover player $k-1$'s demand sequence as a result. We will specify a whole class of all-pass filters player $k-1$ could use to achieve this result. Likewise, we will find instances where player $k-1$ can construct PIS that will provide more value to player k than demand sharing, but less value to player k than full information shock sharing. We will discuss the class of all-pass filters that would achieve this result as well. By the end of this section, for any conceivable order equation (15) of player $k-1$, we would be able to specify the various sharing arrangements from player $k-1$ to k that would lead to different information levels for player k. Furthermore we would instantly be able to tell the most profitable sharing arrangements for player k.

As we will see, the value of player $k-1$'s PIS to player k centers around the $\tilde{\theta}_k(z)$ polynomial appearing in (15). Since we will refer to this equation frequently in this section, we rewrite this equation here.

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}_k(B)\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t} \tag{24}$$

Using Theorem 1 (or Corollary 1) and Proposition 2 of KGH, we have the following proposition.

Proposition 1. *For some $k > 0$ if $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ in (24) has no roots inside the unit circle, then player k's FIS will be $\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$ regardless of the sharing arrangement between player $k-1$ and k.*

This proposition implies that there is no value to information sharing between player $k-1$ and k when $\tilde{J}_k = 0$ and $\tilde{\theta}_k(z)$ has no roots inside the unit circle. Thus no sequence of PIS will have value to player k. The gain in value from PIS sharing will only occur when either or both of these conditions fail. The following remark shows that the variance of player k's FIS is proportional to the variance player $k-1$'s FIS, and depends on the roots of $\tilde{\theta}_k^{I-CA}(z)$.

Remark 1. *The variance of player k's FIS ($\sigma^2_{\epsilon_k,A_{k-1}}$) when player $k-1$ shares $\{\epsilon^\star_{k-1,t}\}$ where $\epsilon^\star_{k-1,t} = B^{J^\star_{k-1}}\dfrac{A_{k-1}(B)}{A^\dagger_{k-1}(B)}\epsilon_{k-1,t}$ is given by*

$$\sigma^2_{\epsilon_k,A_{k-1}} = \lambda^2_{k,\tilde{J}_k}\sigma^2_{\epsilon_{k-1}}\prod_{j=1}^{n_c}\frac{1}{|r_j|^2}$$

where $r_1, ..., r_{n_c}$ are the roots of $\tilde{\theta}_k^{I-CA}(z)$.

Note that this remark gives the variance of player k's FIS given by Theorem 1. Furthermore, each root r_j that is common to $\tilde{\theta}_k(z)$ and $A_{k-1}(z)$ inside the unit circle will increase the variance of player k's FIS since $\frac{1}{|r_j|} > 1$. The lower the product of the modulus of roots of $\tilde{\theta}_k^{I-CA}(z)$, the lower the innovation variance of player k's FIS. The remark follows immediately from the all-pass filter appearing in (18) that creates $\epsilon_{k,t}$ from $\{\epsilon_{k-1,t}\}$ (see [Brockwell and Davis, 1991] pp. 127-129).

A vital question to ask is how many different FIS can player k observe having different innovation variances under all possible PIS sharing arrangements with player $k-1$. Looking at the previous remark we see that the variance of player k's FIS is a product of $\lambda^2_{k,\tilde{J}_k}$, $\sigma^2_{\epsilon_{k-1}}$ and the modulus of the inverse of the squared-modulus of all the roots of $\tilde{\theta}_k^{I-CA}(z)$. Of these, only the latter will change based on the sequence of PIS. Thus, the all-pass filter used by player $k-1$ to create the PIS $\{\epsilon^{\star}_{k-1}\}$ will impact the term $\tilde{\theta}_k^{I-CA}(z)$ and hence the variance of player k's FIS. Looking at how this polynomial is constructed, we see that its roots can be any combination of (possibly non-distinct) roots of $\tilde{\theta}_k(z)$ inside the unit circle. This combination of roots depends on whether the roots are also roots of $A_{k-1}(z)$.

Suppose $\tilde{\theta}_k(z)$ has $\tilde{q}_{k,IN}$ roots inside the unit circle. A suitable $A_{k-1}(z)$ can be picked so that any combination of the $\tilde{q}_{k,IN}$ roots are also roots of $\tilde{\theta}_k^{I-CA}(z)$ with the exception that if a complex root of $\tilde{\theta}_k(z)$ is in the combination, so is its complex conjugate (which is guaranteed to be a root of $\tilde{\theta}_k(z)$ as well). This is to insure that the polynomial $A_{k-1}(z)$ has real coefficients. This leads to the following remark.

Remark 2. *Suppose that $\tilde{\theta}_k(z)$ has r real roots inside the unit circle and $\tilde{q}_{k,IN}-r$ complex roots inside the unit circle. Then there are $2^{\frac{\tilde{q}_{k,IN}+r}{2}}$ possible combinations of roots that $\tilde{\theta}_k^{I-CA}(z)$ may have and $2^{\frac{\tilde{q}_{k,IN}+r}{2}}$ possible full information shock sequences that player k may have, possibly containing different innovation variances, based on different sharing arrangements.*

Note that $\tilde{q}_{k,IN}-r$ will be even. The way to compare the innovation variance resulting from two different sharing arrangements is given by the following proposition.

Proposition 2. *Suppose for $k>0$, player $k-1$'s order equation is given by (24). Consider two all-pass filters $\dfrac{A_{k-1,1}(B)}{A^{\dagger}_{k-1,1}(B)}$ and $\dfrac{A_{k-1,2}(z)}{A^{\dagger}_{k-1,2}(z)}$ where $A_{k-1,1}(z)$ has distinct roots $a_{1,1},...,a_{1,n_{A_{k-1,1}}}$ inside the unit circle and $A_{k-1,2}(z)$ has distinct roots $a_{2,1},...,a_{2,n_{A_{k-1,2}}}$ inside the unit circle. Let $\sigma^2_{\epsilon_k,1}$ be the variance of player k's FIS if player $k-1$ were to share shocks that were formed by passing $\{\epsilon_{k-1,t}\}$ through the all-pass filter $\dfrac{A_{k-1,1}(B)}{A^{\dagger}_{k-1,1}(B)}$. Let $\sigma^2_{\epsilon_k,2}$ be the variance of player k's FIS if player $k-1$ were to share shocks that were formed by passing $\{\epsilon_{k-1,t}\}$ through the all-pass filter $\dfrac{A_{k-1,2}(B)}{A^{\dagger}_{k-1,2}(B)}$.*

Then

$$\sigma^2_{\epsilon_k,1} = \frac{\prod_{j=1}^{n_{A_{k-1,2}}} |a_{2,j}|^{2\cdot min[m(a_{2,j},A_{k-1,2}),m(a_{2,j},\tilde{\theta}_k)]}}{\prod_{j=1}^{n_{A_{k-1,1}}} |a_{1,j}|^{2\cdot min[m(a_{1,j},A_{k-1,1}),m(a_{1,j},\tilde{\theta}_k)]}} \cdot \sigma^2_{\epsilon_k,2}$$

In KGH it was assumed that each player could share nothing, its demand, or its full information shocks with an adjacent upstream player. The case of demand sharing could easily be related to the framework we have here. Namely there is a sequence of shocks, that could be formed by passing $\{\epsilon_{k-1,t}\}$ through an all-pass filter, that contain the same amount of information as player $k-1$'s demand series $\{D_{k-1,t}\}$. We can see this by applying player $k-1$'s demand equation given by

$$\phi(B)D_{k-1,t} = d + B^{J_{k-1}}\theta_{k-1}(B)\epsilon_{k-1,t} \tag{25}$$

which can be rewritten as

$$\frac{\phi(B)}{\theta^{OUT}_{k-1}}D_{k-1,t} = \frac{1}{\theta^{OUT}_{k-1}(B)}d + B^{J_{k-1}}\theta^{IN}_{k-1}(B)\epsilon_{k-1,t} \tag{26}$$

Proposition 3. *The sequence of shocks* $\{\epsilon^*_{k-1,t}\}$ *given by*

$$\epsilon^*_{k-1,t} = B^{J_{k-1}}\frac{\theta^{IN}_{k-1}(B)}{\theta^{\dagger IN}_{k-1}(B)}\epsilon_{k-1,t} \tag{27}$$

contains the same amount of information as $D_{k-1,t}$.

Proof. To see this, rewrite (26) in terms of $\epsilon^*_{k-1,t}$ given by

$$\frac{\phi(B)}{\theta^{OUT}_{k-1}}D_{k-1,t} = \frac{1}{\theta^{OUT}_{k-1}(B)}d + \theta^{\dagger IN}_{k-1}(B)\epsilon^*_{k-1,t} \tag{28}$$

The polynomials on the left and right hand sides of the equation have no roots inside the unit circle and therefore $\mathcal{M}^{D_{k-1}}_t \equiv \mathcal{M}^{\epsilon^*_{k-1}}_t$. □

Proposition 2 and (27) allow us to compare the variances of player k's FIS when receiving PIS $\{\epsilon^\star_{k-1}\}$, formed by passing $\{\epsilon_{k-1}\}$ through some all-pass filter, with the variance of player k's FIS when receiving player $k-1$'s demand $\{D_{k-1,t}\}$. If we assume that $J_{k-1}=0$, we can classify the shock sequence as being less valuable to player k than demand sharing, more valuable to player k than demand sharing, or equally valuable to player k than demand sharing.

Remark 3. *Suppose player $k-1$ shares a series of shocks $\{\epsilon^{\star}_{k-1,t}\}$ with player k which are formed by passing $\{\epsilon_{k-1,t}\}$ through the all-pass filter $\frac{A_{k-1}}{A^{\dagger}_{k-1}}$ with A_{k-1} having roots $a_1,...,a_{n_{A_{k-1}}}$ inside the unit circle. Suppose further that $J_{k-1}=0$.*

If $\prod_{j=1}^{n_{A_{k-1}}} |a_j|^{2\cdot min[m(a_j,A_{k-1}),m(a_j,\tilde{\theta}_k)]} > \prod_{j=1}^{n_{\theta^{IN}_{k-1}}} |a_j|^{2\cdot min[m(a_j,\theta^{IN}_{k-1}),m(a_j,\tilde{\theta}_k)]}$ *then $\{\epsilon^{\star}_{k-1,t}\}$ is less valuable to player k than demand sharing.*

If $\prod_{j=1}^{n_{A_{k-1}}} |a_j|^{2\cdot min[m(a_j,A_{k-1}),m(a_j,\tilde{\theta}_k)]} < \prod_{j=1}^{n_{\theta^{IN}_{k-1}}} |a_j|^{2\cdot min[m(a_j,\theta^{IN}_{k-1}),m(a_j,\tilde{\theta}_k)]}$ *then $\{\epsilon^{\star}_{k-1,t}\}$ is more valuable to player k than demand sharing.*

If $\prod_{j=1}^{n_{A_{k-1}}} |a_j|^{2\cdot min[m(a_j,A_{k-1}),m(a_j,\tilde{\theta}_k)]} = \prod_{j=1}^{n_{\theta^{IN}_{k-1}}} |a_j|^{2\cdot min[m(a_j,\theta^{IN}_{k-1}),m(a_j,\tilde{\theta}_k)]}$ *then $\{\epsilon^{\star}_{k-1,t}\}$ is as valuable to player k as demand sharing.*

4.1 When Can Player k Recover $\{D_{k-1,t}\}$

Player k can recover player $k-1$'s demand series $\{D_{k-1,t}\}$ when if it can write player $k-1$ demand at time t as a linear combination of FIS observed up to time t. Recall Theorem 1 which tells us that player k's FIS are given by

$$\epsilon_{k,t} = \frac{\tilde{\theta}^{I-CA}_k(B)}{\tilde{\theta}^{\dagger I-CA}_k(B)} B^{min(\tilde{J}_k,J^{\star}_{k-1})} \lambda_{k,\tilde{J}_k} \epsilon_{k-1,t}.$$

This can be rewritten as

$$\epsilon_{k-1,t} = \frac{\tilde{\theta}^{\dagger I-CA}_k(B)}{\tilde{\theta}^{I-CA}_k(B)} B^{-min(\tilde{J}_k,J^{\star}_{k-1})} (1/\lambda_{k,\tilde{J}_k}) \epsilon_{k,t}. \tag{29}$$

Now recall player $k-1$'s demand equation (25),

$$\phi(B) D_{k-1,t} = d + B^{J_{k-1}} \theta_{k-1}(B) \epsilon_{k-1,t}$$

Substituting the expression for $\epsilon_{k-1,t}$ we have

$$\phi(B)D_{k-1,t} = d + B^{J_{k-1}-min(\tilde{J}_k,J^\star_{k-1})}\theta_{k-1}(B)\frac{\tilde{\theta}_k^{\dagger I-CA}(B)}{\tilde{\theta}_k^{I-CA}(B)}(1/\lambda_{k,\tilde{J}_k})\epsilon_{k,t} \quad (30)$$

Note that there is a polynomial in the denominator on the RHS of this equation. If $\tilde{\theta}_k^{I-CA}(z) \not\equiv 1$ then this polynomial will have roots inside the unit circle. This leads to the following Proposition:

Proposition 4. *Player k can recover $\{D_{k-1,t}\}$ using present and past values of $\{\epsilon_{k,t}\}$ if and only if the roots of $\tilde{\theta}_k^{I-CA}(z)$ are also roots of $\theta_{k-1}(z)$ and $J_{k-1} \geq min(\tilde{J}_k, J^\star_{k-1})$.*

From this proposition we see that player k cannot recover $\{D_{k-1,t}\}$ whenever $\tilde{\theta}_k^{I-CA}(z)$ has a root that is not a root of $\theta_{k-1}(z)$. Thus player $k-1$ can form and share a PIS sequence with player k without revealing its historic demand as long as the polynomial $A(z)$ it uses in its all-pass filter has a common root inside the unit circle with $\tilde{\theta}_k(z)$, which is not also common with $\theta_{k-1}(z)$. Note that this sharing arrangement will be valuable as long as $\tilde{\theta}_k(z)$ has a root inside the unit circle not common with $A(z)$. Another way to say this is that the variance of player k FIS will decrease when player $k-1$ shares $\{\epsilon^\star_{k-1,t}\}$ as long as $\tilde{\theta}_k^{I-CA}(z) \not\equiv \tilde{\theta}_k(z)$. If this is not the case, player k's forecast may still improve as long as $J^\star_{k-1} < \ell_k \leq \tilde{J}_k$.

The question of when can player k recover $\{\epsilon_{k-1,t}\}$ is much simpler. This is in fact already provided by Corollaries 1 and 2. We combine these two into the following proposition.

Proposition 5. *Player k can recover $\{\epsilon_{k-1,t}\}$ using present and past values of $\{\epsilon_{k,t}\}$ if and only if $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ and $z^{J^\star_{k-1}}A_{k-1}(z)$ have no common roots inside the unit circle.*

5 Summary and Conclusion

In this paper we describe demand propagation under sharing arrangements involving a player passing its shocks through an all-pass-filter and sharing the resulting sequence. We create a framework under which we can compare all such arrangements and compare these to conventional sharing arrangements of no sharing, demand sharing, and shock sharing. We also describe how demand will propagate up the supply chain when any such sharing arrangement is allowed to occur between contiguous players.

We show that it is possible for a player to form a partial shock sequence that will be valuable to an adjacent player, but may yet protect its interests by keeping the player from recovering its demand sequence. This is important because it may not want its demand sequence revealed due to an agreement with other players or because of privacy issues.

We make several important contributions to the literature. The first is in describing how PIS shocks may be formed by a player and shared with another player. The second is in showing the value of PIS sharing to a player by considering the player's FIS and MSFE under such an arrangement. Third we create a framework that allows us to compare all possible PIS sharing arrangements as well as comparing them to no sharing, demand sharing, and FIS sharing. Fourth is in showing that demand propagates as quasi-ARMA (QUARMA)-in quasi-ARMA-out when we allow for any PIS sharing arrangement to occur between contiguous players throughout the supply chain. Finally we show how such sharing arrangements can still be valuable while not revealing possibly confidential information.

6 Appendix

Proof of Lemma 1. We can rewrite (16) and (17) as

$$\frac{A_1(B)}{A_2^{OUT}(B)} X_{1,t} = B^{J_1} A_2^{IN}(B) A_2^{ON}(B)(1/\lambda)\eta_t \tag{31}$$

$$\frac{\phi(B)}{\Theta^{OUT}(B)} X_{2,t} = \frac{d}{\Theta^{OUT}(1)} + B^{J_2}\Theta^{IN}(B)\Theta^{ON}(B)\eta_t \tag{32}$$

where the superscript "OUT" indicates that the factors included in this polynomial have roots outside the unit circle, "IN" indicates that the polynomial has all roots inside the unit circle, and "ON" indicates that the factors included in this polynomial have roots on the unit circle. It is possible that the polynomial $\Theta(z)$ has no roots outside the unit circle, in which case we take $\tilde{\theta}_k^{OUT}(z) \equiv 1$. Similar convention holds for the rest.

Consider the polynomials $P_1(z) = (1/\lambda)z^{J_1}A_2^{IN}(z)A_2^{ON}(z)$ and $P_2(z) = z^{J_2}\Theta^{IN}(z)\Theta^{ON}(z)$. Suppose $P_2(z)$ has r_2 distinct non-zero roots $b_1, ..., b_{r_2}$.

Define

$$GCD(P_1, P_2) := z^{min(J_1,J_2)} \prod_{j=1}^{r_2} (1 - \frac{z}{b_j})^{min[m(b_j,P_1),m(b_j,P_2)]}$$

The roots of $GCD(P_1, P_2)$ are those roots that are common to both P_1 and P_2. Furthermore the multiplicity of each root is the minimum of the multiplicities of the root in P_1 and P_2. By construction, the coefficient in front of the lowest power of z of $GCD(P_1, P_2)$ is 1.

By the Euclidean Algorithm for polynomials (cf. Koblitz (1998) pg 63) we know that there exist polynomials $Q_1(z)$ and $Q_2(z)$ such that

$$Q_1P_1 + Q_2P_2 = GCD(P_1, P_2) \tag{33}$$

Suppose $A_2^{ON}(z)$ has r_{on} distinct roots $b_1, ..., b_{r_{on}}$. Define A_2^{ON-C} as

$$A_2^{ON-C} := \prod_{j=1}^{r_{on}} (1 - \frac{z}{b_j})^{min[m(b_j,A_2^{ON}),m(b_j,\Theta^{ON})]}$$

Note that if $A_2^{ON}(z)$ and $\Theta^{ON}(z)$ have no common roots, then $A_2^{ON-C} \equiv 1$.

Noting that $GCD((1/\lambda)z^{J_1}A_2^{IN}(z)A_2^{ON}(z), z^{J_2}\Theta^{IN}(z)\Theta^{ON}(z)) = z^{min(J_1,J_2)}A_2^{I-C}(z)A_2^{ON-C}(z)$ the Euclidean Algorithm tells us how to find polynomials $Q_1(z)$ and $Q_2(z)$ such that

$$(1/\lambda)z^{J_1}Q_1(z)A_2^{IN}(z) + z^{J_2}Q_2(z)\Theta^{IN}(z) = z^{min(J_1,J_2)}A_2^{I-C}(z)A_2^{ON-C}(z) \tag{34}$$

Therefore multiplying (31) and (32) by $Q_1(B)$ and $Q_2(B)$ and summing we get

$$\frac{1}{A_2^{OUT}(B)}A_1(B)Q_1(B)X_{1,t} + \frac{1}{\Theta^{OUT}(B)}\phi(B)Q_2(B)X_{2,t} = C + B^{min(J_1,J_2)}A_2^{I-C}(B)A_2^{ON-C}\eta_t \tag{35}$$

where $C = \frac{Q_2(1)d}{\Theta^{OUT}(1)}$ is a constant.

If $B^{J_1}A_2^{IN}(z)A_2^{ON}(z)$ and $B^{J_2}\Theta^{IN}(z)\Theta^{ON}(z)$ have no common roots then $A_2^{I-C}(z) \equiv 1$, $A_2^{ON-C}(z) \equiv 1$ (and $min(J_1, J_2) = 0$) in (35) and therefore we can take $\vartheta(z) = \frac{Q_1(z)}{A_2^{OUT}(z)}$ and $\omega(z) = \frac{Q_2(z)}{\Theta^{OUT}(z)}$ to get

$$\vartheta(B)A_1(B)X_{1,t} + \omega(z)\phi(B)X_{2,t} = C + \eta_t$$

Furthermore since $A_2^{OUT}(z)$ and $\Theta^{OUT}(z)$ have no roots inside or on the unit circle by construction, their reciprocals have one-sided Laurent series representations that converge in a disk $\mathcal{D}$ that contains the unit circle. Therefore the constructed $\vartheta(z)$ and $\omega(z)$ have one-sided Laurent Series Representations that converge for all $z \in \mathcal{D}$. Note that $C = \omega(1)d$.

Now suppose that there exist functions $\vartheta(z)$ and $\omega(z)$ with one sided Laurent Series Representations that converge in $\mathcal{D}$ such that $\vartheta(B)A_1(B)X_{1,t} + \omega(B)\phi(B)X_{2,t} = \omega(1)d + \eta_t$.

From (16) and (17) we can rewrite this as

$$\omega(1)d + B^{J_1}\vartheta(B)A_2(B)(1/\lambda)\eta_t + B^{J_2}\omega(B)\Theta(B)\eta_t = \omega(1)d + \eta_t$$

which simplifies to

$$B^{J_1}\vartheta(B)A_2(B)(1/\lambda)\eta_t + B^{J_2}\omega(B)\Theta(B)\eta_t = \eta_t \tag{36}$$

Define $L(z) := z^{J_1}\vartheta(z)A_2(z)(1/\lambda) + z^{J_2}\omega(z)\Theta(z) - 1$. Note that (36) implies that for all $\mu \in [-\pi, \pi], L(e^{-i\mu}) \equiv 0$. Consider the Laurent series expansion of $L(z)$ for $z \in \mathcal{D}$,

$$L(z) = \sum_{k=-\infty}^{\infty} g_k z^k = z^{J_1}\vartheta(z)A_2(z)(1/\lambda) + z^{J_2}\omega(z)\Theta(z) - 1 \tag{37}$$

The Laurent Series (37) must have the same coefficients g_k as the Fourier series expansion

$$L(e^{-i\mu}) = \sum_{k=-\infty}^{\infty} g_k e^{-i\mu k} = e^{-i\mu J_1}\vartheta(e^{-i\mu})A_2(e^{-i\mu})(1/\lambda) + e^{-i\mu J_2}\omega(e^{-i\mu})\Theta(e^{-i\mu}) - 1 \tag{38}$$

Since $L(e^{-i\mu}) \equiv 0, g_k \equiv 0$ for all k in (38) and therefore in (37). This shows that $L(z) \equiv 0$ for all $z \in \mathcal{D}$.

If $A_2(z)$ and $\Theta(z)$ had a common root z_0 inside or on the unit circle then we would have $L(z_0) = -1$ in (37) which is a contradiction. □

Proof of Theorem 1. We would like to show that $\mathcal{M}_t^{D_k, \epsilon^\star_{k-1}} = \mathcal{M}_t^{\epsilon_k}$ and that we can represent $\{D_{k,t}\}$ as QUARMA with respect to $\{\epsilon_{k,t}\}$. Consider player $k-1$'s sharing and order equations

$$A^\dagger_{k-1}(B)\epsilon^\star_{k-1,t} = B^{J^\star_{k-1}}A_{k-1}(B)\epsilon_{k-1,t} \tag{39}$$

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}_k(B)\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t} \tag{40}$$

We can use (18) to rewrite these in terms of $\epsilon_{k,t}$ as

$$A^{\dagger}_{k-1}(B)\epsilon^{\star}_{k-1,t} = B^{J^{\star}_{k-1}}A_{k-1}(B)\frac{\tilde{\theta}^{\dagger I-CA}_{k}(B)}{\tilde{\theta}^{I-CA}_{k}(B)}B^{-min(\tilde{J}_k,J^{\star}_{k-1})}(1/\lambda_{k,\tilde{J}_k})\epsilon_{k,t}$$

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}_k(B)\lambda_{k,\tilde{J}_k}\frac{\tilde{\theta}^{\dagger I-CA}_{k}(B)}{\tilde{\theta}^{I-CA}_{k}(B)}B^{-min(\tilde{J}_k,J^{\star}_{k-1})}(1/\lambda_{k,\tilde{J}_k})\epsilon_{k,t}$$

Using the fact that the roots of $\tilde{\theta}^{I-CA}_{k}(z)$ are roots of $A_{k-1}(z)$ and $\tilde{\theta}_k(z)$ by definition, these simplify to

$$A^{\dagger}_{k-1}(B)\epsilon^{\star}_{k-1,t} = B^{J^{\star}_{k-1}-min(\tilde{J}_k,J^{\star}_{k-1})}A^{NC}_{k-1}(B)\tilde{\theta}^{\dagger I-CA}_{k}(B)(1/\lambda_{k,\tilde{J}_k})\epsilon_{k,t} \tag{41}$$

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k-min(\tilde{J}_k,J^{\star}_{k-1})}\tilde{\theta}^{NC}_{k}(B)\tilde{\theta}^{\dagger I-CA}_{k}(B)\epsilon_{k,t} \tag{42}$$

where $A^{NC}_{k-1}(z) = \dfrac{A_{k-1}(z)}{\tilde{\theta}^{I-CA}_{k}(z)}$ and $\tilde{\theta}^{NC}_{k}(z) = \dfrac{\tilde{\theta}_k(B)}{\tilde{\theta}^{I-CA}_{k}(z)}$ are polynomials. Furthermore $J^{\star}_{k-1} - min(\tilde{J}_k, J^{\star}_{k-1}) \geq 0$ and $\tilde{J}_k - min(\tilde{J}_k, J^{\star}_{k-1}) \geq 0$ so that no future values from $\{\epsilon_{k,n}\}^{\infty}_{n=t}$ appear in (41) and (42).

Since $A^{\dagger}_{k-1}(z)$ and $\phi(z)$ have no roots inside or on the unit circle, there exist one-sided Laurent series representations of $\dfrac{1}{A^{\dagger}_{k-1}(z)}$ and $\dfrac{1}{\phi(z)}$, called $L_{A^{\dagger}}(z)$ and $L_{\phi}(z)$. Thus we can rewrite (41) and (42) as

$$\epsilon^{\star}_{k-1,t} = B^{J^{\star}_{k-1}-min(\tilde{J}_k,J^{\star}_{k-1})}L^{A^{\dagger}}(B)A^{NC}_{k-1}(B)\tilde{\theta}^{\dagger I-CA}_{k}(B)(1/\lambda_{k,\tilde{J}_k})\epsilon_{k,t}$$

and

$$D_{k,t} = d + B^{\tilde{J}_k-min(\tilde{J}_k,J^{\star}_{k-1})}L^{\phi}(z)\tilde{\theta}^{NC}_{k}(B)\tilde{\theta}^{\dagger I-CA}_{k}(B)\epsilon_{k,t}$$

Since the right-had-side of either of these equations contains no future values from $\{\epsilon_{k,n}\}^{\infty}_{n=t}$, it is clear that $\mathcal{M}^{D_k,\epsilon^{\star}_{k-1}}_{t} \subset \mathcal{M}^{\epsilon_k}_{t}$.

Now consider (41) and (42) rewritten here

$$A^{\dagger}_{k-1}(B)\epsilon^{\star}_{k-1,t} = B^{J^{\star}_{k-1}-min(\tilde{J}_k,J^{\star}_{k-1})}A^{NC}_{k-1}(B)\tilde{\theta}^{\dagger I-CA}_{k}(B)(1/\lambda_{k,\tilde{J}_k})\epsilon_{k,t}$$

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k-min(\tilde{J}_k,J^{\star}_{k-1})}\tilde{\theta}^{NC}_{k}(B)\tilde{\theta}^{\dagger I-CA}_{k}(B)\epsilon_{k,t}$$

Suppose $\tilde{\theta}_k^{NC}(B)$ has no roots on the unit circle. Then the assumptions of Lemma 1 hold and there exist functions $\vartheta(z)$ and $\omega(z)$ with one sided Laurent series representations converging in a disk $\mathcal{D}$ that contains the unit circle such that $\vartheta(B)A_{k-1}^{\dagger}(B)\epsilon_{k-1,t}^{\star} + \omega(B)\phi(B)D_{k,t} = \omega(1)d + \epsilon_{k,t}$. Thus here we have that $\mathcal{M}_t^{\epsilon_k} \subset \mathcal{M}_t^{D_k,\epsilon_{k-1}^{\star}}$.

Now suppose $\tilde{\theta}_k^{NC}(B)$ has $h > 0$ distinct roots on the unit circle $b_1, ..., b_h$. Let $\theta^{ON-CA}(z) = \prod_{j=1}^{h}(1 - \frac{z}{b_j})^{min(m(b_j,\tilde{\theta}_k^{NC}),m(b_j,A_{k-1}^{NC}))}$. Furthermore let

$$\gamma_{k,t} = \theta^{ON-CA}(z)\epsilon_{k,t} \tag{43}$$

We can rewrite (41) and (42) in terms of $\gamma_{k,t}$ as

$$A_{k-1}^{\dagger}(B)\epsilon_{k-1,t}^{\star} = B^{J_{k-1}^{\star}-min(\tilde{J}_k,J_{k-1}^{\star})}A_{k-1}^{*}(B)\tilde{\theta}_k^{\dagger I-CA}(B)(1/\lambda_{k,\tilde{J}_k})\gamma_{k,t}$$

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k-min(\tilde{J}_k,J_{k-1}^{\star})}\tilde{\theta}_k^{*}(B)\tilde{\theta}_k^{\dagger I-CA}(B)\gamma_{k,t}$$

where $A_{k-1}^{*}(z) = \dfrac{A_{k-1}^{NC}(z)}{\theta^{ON-CA}(z)}$ and $\tilde{\theta}_k^{*}(z) = \dfrac{\tilde{\theta}_k^{NC}(B)}{\theta^{ON-CA}(z)}$.

By Lemma 1 there exist functions $\vartheta(z)$ and $\omega(z)$ with one sided Laurent series representations converging in a disk $\mathcal{D}$ that contains the unit circle such that $\vartheta(B)A_{k-1}^{\dagger}(B)\epsilon_{k-1,t}^{\star} + \omega(B)\phi(B)D_{k,t} = \omega(1)d + \gamma_{k,t}$. Thus here we have that $\mathcal{M}_t^{\gamma_k} \subset \mathcal{M}_t^{D_k,\epsilon_{k-1}^{\star}}$. But applying Brockwell & Davis (1991)Proposition 4.4.1 to (43), we see that $\mathcal{M}_t^{\epsilon_k} \subset \mathcal{M}_t^{\gamma_k}$. Therefore $\mathcal{M}_t^{\epsilon_k} \subset \mathcal{M}_t^{D_k,\epsilon_{k-1}^{\star}}$ in this case as well. Thus we have that $\mathcal{M}_t^{\epsilon_k} = \mathcal{M}_t^{D_k,\epsilon_{k-1}^{\star}}$.

Finally we must confirm that player k's demand $\{D_{k,t}\}$ can be written as QUARMA with respect to $\epsilon_{k,t}$. Indeed, looking at equation (42), we see that

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k-min(\tilde{J}_k,J_{k-1}^{\star})}\tilde{\theta}_k^{NC}(B)\tilde{\theta}_k^{\dagger I-CA}(B)\epsilon_{k,t}$$

where we take $J_k = \tilde{J}_k - min(\tilde{J}_k, J_{k-1}^{\star})$ and $\theta_k(z) = \tilde{\theta}_k^{NC}(z)\tilde{\theta}_k^{\dagger I-CA}(z)$.

□

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