Dynamic Strategic Information Transmission

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March 27, 2013

Abstract

This paper studies strategic information transmission in a finite horizon environment where, each period, a privately informed expert sends a message and a decision-maker takes an action. We show that communication in this dynamic environment is drastically different from in a one-shot game. Our main result is that full information revelation is possible. We provide a constructive method to build such fully revealing equilibria, and show that complicated communication, where far-away types pool together, allows dynamic manipulation of beliefs to enable better information release in the future. If communication is restricted to be monotonic partitional, full revelation is impossible. Finally, we show how conditioning future information release on past actions improves incentives for information revelation.

Keywords: asymmetric information; cheap talk; dynamic strategic communication; full information revelation.

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Golosov and Tsyvinski thank National Science Foundation and EIEF for hospitality. We thank Manuel Amador, Luca Andrelini, Sandeep Baliga, Abhijit Banerjee, Marco Battaglini, Andreas Blume, Adam Brandenburger, Amanda Friedenberg, Johannes Horner, Jerome Mathis, Ronny Razin, Larry Samuelson, and Andy Skrzypacz for inspiring discussions and comments. Skreta gratefully acknowledges financial support from French National Research Agency (A.N.R., project "Regulate to Promote Information Sharing") .

Biased experts impede information transmission, which has serious consequences in many situations: Worse projects are financed, beneficial reforms are blocked, and firms may fail to reward the most productive employees. The seminal analysis of strategic information transmission by Crawford and Sobel (1982) has had a number of applications, ranging from economics and political science, to philosophy and biology.[1] In that paper, a biased and privately informed expert and a decision-maker interact only once. The conflict of interest results in coarse information revelation, and in some cases, in no information revelation at all. There are, however, many environments in which the expert and receiver interact repeatedly and information transmission is dynamic. Many sequential decisions have to take place, and the decision-maker seeks the expert's advice prior to each decision.

We study strategic information transmission in a dynamic, finite-horizon extension of the Crawford and Sobel setup. Each period, the expert sends a message and the decision-maker takes an action. Only the expert knows the state of the world, which remains constant throughout the game. We maintain all other features of the Crawford and Sobel (1982) environment, in particular the conflict of interest between the expert and decision-maker. The goal is to investigate the extent to which conflicts of interest prevent information transmission in multi-period interactions.

Our most surprising and difficult-to-establish finding (Theorem 1) shows that full information revelation is possible. We show this result in a challenging environment where the horizon is finite, and both players are fully patient. The construction of the fully revealing equilibrium relies on two key novel features. The first is the use of what we call "separable groups": the expert employs a signaling rule in which far-apart types pool together initially, but eventually find it optimal to separate and reveal the truth. The second feature is to make advice contingent on actions: the expert promises to reveal the truth later, but only if the decision-maker follows his advice now; this initial advice, in turn, is designed to reward the expert for revealing information. In a nutshell, communication in a multi-period interaction is facilitated via an initial signaling rule that manipulates posteriors (in a way that enables precise information release in the future), initial actions which reward the expert for employing this signaling rule, and trigger strategies which reward the decision-maker for choosing these initial actions. Moving from a one-shot to finitely-repeated game often leaves the qualitative feature of equilibria unchanged; we show here that finite repetition has a drastic impact of the equilibrium nature of strategic communication.

We now explain in more detail our construction of a fully revealing equilibrium. We first show that it is possible to divide all states into separable groups. A separable group is a finite set of states (types) which are sufficiently far apart that each type would rather reveal the truth, than mimic any other type in his group. The expert's initial signaling rule reveals the separable group

[1] For a survey with applications across disciplines see Sobel (2009).

containing the truth; therefore, this creates histories after which it is common knowledge that the decision-maker puts probability one on a particular separable group, at which point the types in this group will find it optimal to separate. The idea of initially pooling together far-away types, who will then later have an incentive to separate, was first proposed in Krishna and Morgan (2004); they demonstrated how this could increase information revelation in dynamic games, and we have pushed the idea further to demonstrate that if the initial groups are finite and chosen in the right way, it is possible for the decision-maker to extract *all* information from a biased expert. The division of all types into separable groups is quite delicate, because, given that there is a continuum of types, we need to form a continuum of such groups. The expert anticipates that once he joins a separable group, he will forgo his informational advantage. For the expert to join the separable group containing his true type, we have to make sure that he does not want to mimic a nearby type by joining some other separable group. This is done via our choice of initial actions, which ensure that any future gain to the expert from mimicking some other type is offset by the initial cost. These expert-incentivizing actions are not myopically optimal for the decision-maker, so we employ trigger strategies: the expert (credibly) threatens to babble in the future if the decision-maker fails to choose the actions that he recommends at the beginning. The final part of the proof then shows that we can design the separable groups and initial actions such that the decision-maker would rather follow the expert's initial advice, knowing that he will then eventually learn the exact truth, than choose the myopically optimal action in the initial periods, knowing that he will then never learn more than the separable group containing the truth.

In a follow-up section (Section 4.1), we adapt our construction to a continuous-time setting, obtaining some more attractive results and generalizations. In particular, Theorem 1 proves that full information revelation is possible when the decision-maker and expert are both perfectly patient with quadratic-loss preferences, but only for some horizons, some priors (held by the decision-maker) over the state space, and when the conflict of interest between expert and decision-maker is small; moreover, the welfare properties of the equilibrium are both difficult to calculate, and in some cases, not very appealing. Proposition 4 shows that with a trivial modification to the timeline, and for the same set of priors covered by Theorem 1, our construction yields also a fully revealing equilibrium for any pair of discount factors, so long as the decision-maker is at least as patient as the expert. Our second main result, Theorem 2, shows that in a continuous-time setting with an impatient expert (positive discount rate), a sufficiently patient decision-maker, and a sufficiently long horizon, our fully revealing equilibrium works for nearly all priors over the state space; the conflict of interest may also be larger than in Theorem 1, and the proofs for the decision-maker do not rely on quadratic-loss preferences. Moreover, the decision-maker's average payoff loss (compared to a full information setting) goes to zero as he becomes perfectly patient.

We emphasize several additional differences between dynamic and static communication games. First, we emphasize that fully revealing equilibria cannot have the monotonic partitional structure from Crawford and Sobel (1982): if attention is restricted to monotonic partition equilibria, learning quickly stops. Moreover, we argue that non-monotonic equilibria can be strictly Pareto superior to all dynamic monotonic equilibria. Welfare properties of equilibria also differ in a dynamic setup. Crawford and Sobel (1982) show that, ex ante, both the expert and the decision-maker will (under typical assumptions) prefer equilibria with more partitions. We provide an example that shows that it is not necessarily the case for dynamic equilibria.[2] We also present an example in which dynamic monotonic partition equilibria can strictly Pareto-dominate the best static equilibrium, and an example showing that non-monotonic equilibria can strictly Pareto dominate the best dynamic monotonic equilibrium.

Our work shows that the nature of dynamic strategic communication is quite distinct from its static counterpart. In the static case, because of the conflict of interest between the decision-maker and the expert, nearby expert types have an incentive to pool together, precluding full information revelation. The single-crossing property also implies that at equilibrium, the action is a monotonic step function of the state. These two forces make complex signaling (even though possible) irrelevant. In the dynamic setup, the key difference is that today's communication sets the stage for tomorrow's communication. Complex signaling helps in the dynamic setup, because it can generate posteriors that put positive probability only on expert types who are so far apart, they have no incentive to mimic each other; this is what enables fully revealing equilibria.

Related Literature

Crawford and Sobel (1982) is the seminal contribution on strategic information transmission. That paper has inspired an enormous amount of theoretical work and myriads of applications. Here we study a dynamic extension. Much of the previous work on dynamic communication has focused on the role of reputation; see, for example, Sobel (1985), Morris (2001), and Ottaviani and Sorensen (2006a, 2006b). Some other dynamic studies allow for multi-round communication protocols, but with a single round of action(s). Aumann and Hart (2003) characterize geometrically the set of equilibrium payoffs when a long conversation is possible. In that paper, two players – one informed and one uninformed – play a finite simultaneous-move game. The state of the world is finite, and players engage in direct (no mediator) communications, with a potentially infinitely long exchange of messages, before simultaneously choosing costly actions. In contrast, in our model, only the informed party sends messages, the uninformed party chooses actions, and the state space is infinite. Krishna and Morgan (2004) add a long communication protocol to Crawford and Sobel

[2] A similar phenomenon occurs when communication is noisy, as shown in an example of the working paper version of Chen, Kartik, and Sobel (2007). In their example, a two-step partition Pareto dominates a three-step partition.

(1982)'s game, and Goltsman, Hőrner, Pavlov and Squintani (2009) characterize such optimal protocols.[3] Forges and Koessler (2008a, 2008b) allow for a long protocol in a setup where messages can be certifiable. In all those papers, once the communication phase is over, the decision-maker chooses one action. In our paper, there are multiple rounds of communication and actions (each expert's message is followed by an action of the decision-maker). The multiple actions correlate incentives in a way that was not possible in these earlier works: the expert is able to condition his advice on the decision-maker's past behavior, and additionally, the decision-maker is able to choose actions which reward the expert appropriately for following a path of advice that ultimately leads to revelation of the true state.

In our setup, the dynamic nature of communication enables full information revelation. In contrast, full information revelation is not possible in the dynamic setup of Anderlini, Gerardi, and Lagunoff (2012), who consider dynamic strategic communication in a dynastic game, and show that if preferences are not fully aligned, "full learning" equilibria do not exist.[4] Renault, Solan, and Vielle (2011) examine dynamic sender-receiver games, and characterize equilibrium payoffs (for quite general preferences) for an infinite-horizon model in which the state space is finite, the state may change each period according to a stationary Markov process, and both players are patient. In contrast, we assume a continuous state space with persistent information, and our focus is on the possibility of full information revelation in finite time.[5]

Our model bears some similarities to models of static strategic communication with multiple receivers. In those models the expert cares also about a sequence of actions, but in contrast to our model, those actions are chosen by different individuals. An important difference is that in our model, the receiver cares about the entire vector of actions chosen; in those models, each receiver cares only about his own action. This enables our use of trigger strategies, which we find is a necessary feature of equilibria with eventual full information revelation. Still, some of the properties of the equilibria that we obtain also appear in the models with multiple receivers. For example, our non-monotonic example presented in Section 3 resembles Example 2 of Goltsman and

[3] They examine the optimal use of a 3rd party, such as a mediator or negotiator, to relay messages. For the expert, our model is equivalent to a one-shot model with a mediator: his expected payoff is the same whether he induces a sequence of actions $(a_t)_{t=1}^T$, or a probability distribution over these actions. For the decision-maker, our model makes things easier in some ways (our expert can condition future advice on the DM's past actions), and more difficult in some ways (in our model, the decision-maker knows for sure that the initial actions he's asked to choose are nowhere near the true state).

[4] In their model, the state space is finite (0 or 1), and there is no perfectly informed player: each receiver gets a signal about the state and a message from his predecessor, and then becomes the imperfectly informed advisor to the next player.

[5] Ivanov (2011) allows for a dynamic communication protocol in a setup where the expert is also initially uninformed, and the decision-maker controls the quality of information available to the expert. He employs separable groups, but in a much different informational setting: His decision-maker has a device that initially reveals (to the expert only) the separable group containing the truth, and contains a built-in threat to only reveal the exact state if the expert reports this information truthfully. Compared to our model, this eliminates all incentive requirements for the decision-maker, and imposes an additional cost on the expert (namely, he will fail to learn the truth himself) if he fails to follow the prescribed strategy, thus weakening the required incentive constraints.

Pavlov (2008). It is also similar to Example 2 in Krishna and Morgan (2004).[6]

Full information revelation is possible in other variations of the Crawford and Sobel (1982) setup: When the decision-maker consults two experts as in Battaglini (2002), Eso and Fong (2008), and Ambrus and Lu (2010); when information is completely or partially certifiable, as in Mathis (2008); and when there are lying costs and the state is unbounded as in Kartik, Ottaviani, and Squintani (2007). In the case of multiple experts, playing one against the other is the main force that supports truthful revelation. In the case of an unbounded state, lying costs become large and support the truth. In the case of certifiable information, one can exploit the fact that messages are state-contingent to induce truth-telling. All these forces are very different from the forces behind our fully revealing construction.

1 Motivating Example: An Impatient Financial Advisor

One of the most stark results of the static strategic communication game is that there is no equilibrium with full information revelation. Although the state can take a continuum of values, the expert sends at most finitely many signals to the decision-maker. That is, a substantial amount of information is not transmitted.

In this example, we show how to construct a fully revealing equilibrium when the expert is myopic, using just two stages. There are two essential ingredients of this example. First, the set of types that pool together in the first period are far enough apart that they can be separated in the second period: that is, each possible first-period message is sent by a separable group of types. Second, each separable group induces the same optimal (for the decision-maker) first-period action. This implies that the expert does not care which group he joins (since a myopic expert cares only about the 1st-period action, which is constant across groups).

Example 1 *Fully revealing equilibrium with impatient experts ($\delta_E = 0$).*

Suppose there is an expert (financial advisor) and a decision-maker (an employee). The expert knows the true state of the world θ, which is drawn from a uniform distribution on $[0,1]$ and remains constant over time. The players' payoffs in period $t \in \{1,2\}$ depend on both the state, θ, and on the action chosen by the decision-maker, y_t. More precisely, payoffs in period t are given by

$$u_t^E(y_t,\theta,b) = -(y_t - \theta - b)^2 \text{ and } u_t^{DM}(y,\theta) = -(y_t - \theta)^2 . \tag{1}$$

where $b > 0$ is the expert's "bias". The expert is myopic, with $\delta_E = 0$; the construction works for any discount factor for the decision-maker.

[6] Equilibria can be non-monotonic also in environments where the decision-maker consults two experts as in Krishna and Morgan (2001).

The expert employs the following signaling rule. In period 1, expert types $\{\frac{1}{8}-\varepsilon; \frac{3}{8}+\varepsilon, \frac{4}{8}+\varepsilon, 1-\varepsilon\}$ pool together and send the message m_ε, for all $\varepsilon \in [0, \frac{1}{8}]$. For all state pairs $\{\frac{1}{8} + \tilde{\varepsilon}, \frac{7}{8} - \tilde{\varepsilon}\}$ with $\tilde{\varepsilon} \in (0, \frac{1}{4})$, the expert sends a message $m_{\tilde{\varepsilon}}$. That is, we have two types of separable groups, indexed by ε and $\tilde{\varepsilon}$. Given this signaling rule, the best response of the decision-maker in period 1 is to choose:

$$\begin{aligned} y_1(m_\varepsilon) &= \frac{\frac{1}{8} - \varepsilon + \frac{3}{8} + \varepsilon + \frac{4}{8} + \varepsilon + 1 - \varepsilon}{4} = 0.5 \text{ for all } \varepsilon \in [0, \frac{1}{8}], \\ y_1(m_{\tilde{\varepsilon}}) &= \frac{\frac{1}{8} + \tilde{\varepsilon} + \frac{7}{8} - \tilde{\varepsilon}}{2} = 0.5 \text{ for all } \tilde{\varepsilon} \in (0, \frac{1}{4}). \end{aligned}$$

In period 2, the expert reveals the truth, and so the decision-maker chooses an action equal to the true state. After any out-of-equilibrium initial message, the decision-maker assigns equal probability to all states, leading to action $y_1^{out} = 0.5$. After any out-of-equilibrium second-period message, the decision-maker assigns probability 1 to the lowest type in his information set (prior to the off-path message), and accordingly chooses an action equal to this type.

We now argue that this is an equilibrium for any $b < \frac{1}{16}$:

First, notice that all messages (even out-of-equilibrium ones) induce the same action in period 1. Hence, the expert is indifferent between all possible first-period messages if he puts zero weight on the future. So, in particular, a myopic expert will find it optimal to send the “right” message, following the strategy outlined above. Now consider, for example, the history following an initial message m_ε. The decision-maker’s posterior beliefs assign probability $\frac{1}{4}$ to each of the types in $\left\{\frac{1}{8} - \varepsilon, \frac{3}{8} + \varepsilon, \frac{4}{8} + \varepsilon, 1 - \varepsilon\right\}$. The expert’s strategy at this stage is to tell the truth: so, if he sends a message that he is type $k \in \left\{\frac{1}{8} - \varepsilon, \frac{3}{8} + \varepsilon, \frac{4}{8} + \varepsilon, 1 - \varepsilon\right\}$, then the decision-maker will believe that k is the true state, and accordingly will choose action k; if the expert deviates to some off-path message, then the decision-maker will assign probability 1 to the lowest type in his information set, $\frac{1}{8} - \varepsilon$, and accordingly choose action $\frac{1}{8} - \varepsilon$. Therefore, to prove that the expert has no incentive to deviate, we need only show that each expert type $k \in \left\{\frac{1}{8} - \varepsilon, \frac{3}{8} + \varepsilon, \frac{4}{8} + \varepsilon, 1 - \varepsilon\right\}$ would rather tell the truth, than mimic any of the other types in his group. Type k prefers action k to k' whenever

$$-(k - k - b)^2 \geq -\left(k' - k - b\right)^2 \Leftrightarrow \left(k' - k\right)\left(k' - k - 2b\right) \geq 0$$

i.e., whenever $k' < k$, or whenever $k' > k + 2b$. So in particular, to make sure that no type in $\left\{\frac{1}{8} - \varepsilon, \frac{3}{8} + \varepsilon, \frac{4}{8} + \varepsilon, 1 - \varepsilon\right\}$ wishes to mimic any other type in this group, it is sufficient to make sure that every pair of types are at least $2b$ apart. Since the closest-together types in the group, $\frac{3}{8}+\varepsilon$ and $\frac{4}{8} + \varepsilon$, are separated by $\frac{1}{8}$, we conclude that the group is separable whenever $\frac{1}{8} > 2b \Leftrightarrow b < \frac{1}{16}$. And similarly after messages $m_{\tilde{\varepsilon}}$.

This construction does not apply with a more patient expert ($\delta_E > 0$), because it does not provide a forward-looking expert with incentives to join the "right" separable group. For example, consider type $\frac{3}{8}$, and suppose that $b = \frac{1}{16}$. The truthful strategy is to reveal group $\{\frac{1}{8}, \frac{3}{8}, \frac{4}{8}, 1\}$ in period 1, and the tell the truth in period 2, inducing actions $(y_1, y_2) = \left(\frac{1}{2}, \frac{3}{8}\right)$. However such strategy cannot be part of an equilibrium if $\delta^E > 0$. The best deviation for $\theta = \frac{3}{8}$ is to mimic type $\frac{3}{8} + \frac{1}{16}$ – initially claiming to be part of the group $\left\{\frac{1}{8} - \frac{1}{16}, \frac{3}{8} + \frac{1}{16}, \frac{4}{8} + \frac{1}{16}, \frac{7}{8} - \frac{1}{16}\right\}$, and then subsequently claiming that the true state is $\frac{3}{8} + \frac{1}{16}$ – thereby inducing actions $(y_1, y_2) = \left(\frac{1}{2}, \frac{3}{8} + \frac{1}{16}\right)$. This deviation then leads to no change in the first-period action, but the 2nd-period action is now equal to type $\frac{3}{8}$'s bliss point, $\frac{3}{8} + \frac{1}{16}$. When $\delta_E > 0$ we need to provide the expert with better incentives to join the "right" separable group: since θ prefers $\theta + b$'s action in the future, he must prefer his own action now. This is much more complex, but in Section 4, we show how to construct such separation-inducing actions.

2 The Model

We extend the classic model of Crawford and Sobel (1982) to a dynamic setting. There are two players, an expert (E) and a decision-maker (DM), who interact for finitely many periods. The expert knows the true state of the world $\theta \in [0, 1]$, which is constant over time and is distributed according to the c.d.f. F, with associated density f. Both players care about their discounted payoff sum: when the state is θ and the decision-maker chooses actions $y^T = (y_1, .., y_T)$ in periods $1, 2, ..., T$, payoffs are given by:

$$\begin{aligned} \text{expert:} \quad & U^E(y^T, \theta, b) = \Sigma_{t=1}^T \delta_E^{t-1} u^E(y_t, \theta, b) \\ \text{DM:} \quad & U^{DM}(y^T, \theta) = \Sigma_{t=1}^T \delta_{DM}^{t-1} u^{DM}(y_t, \theta) \end{aligned}$$

where $b > 0$ is the expert's "bias" and reflects a conflict of interest between the players, and δ_E, δ_{DM} are the players' discount factors. We assume that $u^E(y_t, \theta)$ and $u^{DM}(y_t, \theta, b)$ satisfy the conditions imposed by Crawford and Sobel (1982): for $i = DM, E$, $u^i(\cdot)$ is twice continuously differentiable, $u_1^i(y, \theta) = 0$ for some y and $u_{11}^i(\cdot) < 0$ (so that u^i has a unique maximizer y for each pair (θ, b)), and that $u_{12}^i(\cdot) > 0$ (so that the best action from an informed player's perspective is strictly increasing in θ). Most of our main results will make the more specific assumption that preferences are quadratic, as given by (1).

At the beginning of each period t, the expert sends a (possibly random) message m_t to the decision-maker. The decision-maker then updates his beliefs about the state, and chooses an action $y_t \in \mathbb{R}$ that affects both players' payoffs. Let $y^{DM}(\theta)$ and $y^E(\theta)$ denote, respectively, the decision-maker's and the expert's most preferred actions in state θ; we assume that for all θ, $y^{DM}(\theta) \neq y^E(\theta)$, so that there is a conflict of interest between the players regardless of the state.

The decision-maker observes his payoffs only at the end of the game. (If the decision-maker could observe his payoff each period, the problem would be trivial, as he could simply invert his payoff to determine the true state θ. As usual, we could alternatively assume stochastic payoffs, with sufficient noise that the decision-maker is unable to learn anything about the state from observing his payoff realizations).

A *strategy profile* $\sigma = (\sigma_i)_{i=E,DM}$, specifies a strategy for each player. Let h_t denote a history that contains all the reports submitted by the expert, $m^{t-1} = (m_1, ..., m_{t-1})$, and all actions chosen by the decision-maker, $y^{t-1} = (y_1, ..., y_{t-1})$, up to stage t. The set of all feasible histories at t is denoted by H_t. A behavioral strategy for the expert, σ_E, consists of a sequence of signaling rules that map $[0,1] \times H_t$ to a probability distribution over reports $\mathcal{M}$. Let $q(m\,|\theta, h_t\,)$ denote the probability that the expert reports message m at history h_t when his type is θ. A strategy for the decision-maker, σ_{DM}, is a sequence of maps from H_t to actions. We use $y_t(m\,|h_t\,) \in \mathbb{R}$ to denote the action that the decision-maker chooses at h_t given a report m. A *belief system*, μ, maps H_t to the set of probability distributions over $[0,1]$. Let $\mu(\theta|h_t)$ denote the decision-maker's beliefs about the experts's type after a history h_t. A strategy profile σ and a belief system μ is an assessment. We seek strategy profiles and belief systems that form *Perfect Bayesian Equilibria, (PBE).*

In the paper we use the terminology as follows.

Definition 1 *An equilibrium is called babbling if for all m with $q(m\,|\theta, h_t\,) > 0$, all $\theta \in [0,1]$, all h_t and t, we have that $y_t(m\,|h_t\,) = \hat{y}$.*

In other words, we call an equilibrium babbling if the same action is induced, with probability one, for all states $\theta \in [0,1]$ and all $t \in T$.

Definition 2 *We call a signaling rule q uniform if $q(m\,|\theta, h_t\,)$ is uniform, with support on $[\theta_i, \theta_{i+1}]$ if $\theta \in [\theta_i, \theta_{i+1}]$.*

Definition 3 *A partition equilibrium is one in which, at each period t and history h_t, the expert employs only uniform signaling rules.*

In other words, in a partition equilibrium, the expert follows a pure strategy in which, for any message m, the set of types sending message m is connected (an interval).

Definition 4 *An equilibrium is fully revealing if there exists $\hat{T} \leq T$ such that for all $\theta \in [0,1]$, and all histories along the equilibrium path, the expert reveals the true state with probability one by time $\hat{T}$, and accordingly $y_t(\theta) = y^{DM}(\theta)$ $\forall t \geq \hat{T}$.*

We first briefly summarize the findings of the one-shot strategic information transmission game of Crawford and Sobel (1982), in which uniform signaling rules are the canonical form of communication. We then study properties of uniform signaling in our dynamic setup.

2.1 Uniform Signaling: The Canonical Static Communication

Crawford and Sobel (1982) show that in a one-shot strategic information transmission game, all equilibria are equivalent to partition equilibria: the expert follows a pure strategy in which intervals of types pool together, by sending the same message, inducing actions which are increasing step functions of the state. Communication is then coarse; even though the state θ takes a continuum of values, only finitely many different actions are induced.

The reasons behind this result can be summarized as follows. Fix an equilibrium of the one-shot game and let $y(\theta)$ denote an action induced when the state is θ. The conflict of interest between the expert and the decision-maker implies that at most finitely many actions can be induced at equilibrium. Together with the single-crossing condition and the fact that $u^E(\cdot)$ is strictly concave in y, this implies that equilibrium actions are an increasing step function of the state. Importantly, Crawford and Sobel (1982) show that, without loss of generality, the actions induced at equilibrium can be taken to arise from uniform signaling rules. This result follows from the observation that all messages inducing the same action y can be replaced by a single message. Therefore, more complex signaling rules play no role in the static setup.

2.2 Uniform Signaling: A Special Kind of Dynamic Communication

We now focus on simple partitional communication protocols (uniform signaling) and study their properties in our dynamic setup. We show two results. The first result is that with monotonic partition equilibria, the decision-maker never learns the truth:

Proposition 1 *For all horizons T, there exist no fully revealing monotonic partition equilibria.*

This result follows almost immediately from Crawford and Sobel (1982). A short sketch of the argument is as follows. Suppose, by contradiction, that there exists a fully revealing monotonic partition equilibrium. Then, there exists a period $\hat{T} \leq T$ in which the last subdivision occurs, with $y_t(\theta) = y^{DM}(\theta)$ for all $t \geq \hat{T}$. Then, the incentive constraint at time $\widehat{T}$ for type θ to not mimic type $\theta + \varepsilon$ is

$$\left(1 + \delta + \delta^2 + ... + \delta^{T-\hat{T}-1}\right) u^E\left(y^{DM}(\theta), \theta, b\right) \geq \left(1 + \delta + \delta^2 + ... + \delta^{T-\hat{T}-1}\right) u^E\left(y^{DM}(\theta+\varepsilon), \theta, b\right)$$

and similarly for type $\theta + \varepsilon$. These conditions are equivalent to the static equilibrium conditions in Crawford and Sobel (1982), who proved that they imply that at most finitely many actions can be induced at an equilibrium of a static game, a contradiction to full revelation.

We now proceed to show that if all static equilibria are babbling, then all dynamic monotonic partition equilibria are equivalent to babbling.

Proposition 2 *If all static equilibria are equivalent to the babbling equilibrium, then all dynamic monotonic partition equilibria are equivalent to babbling.*

Proof. See Appendix A. ■

Note that the logic of the arguments used to establish Propositions 1 and 2 applies also to an infinite horizon environment.

Now we move on to show that dynamic monotonic partition equilibrium can Pareto-dominate all equilibria of the one-shot game. In Appendix B, we construct a two-period example in which $\delta_E = \delta_{DM} = 1$, the state θ is uniformly distributed on $[0,1]$, and preferences are given by (1), with $b = \frac{1}{12}$. In the most informative static equilibrium, the state space is partitioned into two pieces, $\left[0, \frac{1}{3}\right] \cup \left[\frac{1}{3}, 1\right]$, inducing actions $\frac{1}{6}$ and $\frac{4}{6}$. On the other hand, there exists a partition equilibrium of the two-period game in which the state space is divided ultimately in three sub-intervals, $[0, 0.25] \cup [0.25, 0.45833] \cup [0.458\,33, 1]$, and which is (ex-ante) strictly Pareto superior to repetition of the static equilibrium.

However, in dynamic settings it is also possible to have equilibria with more partitions that are inferior to ones with less (we present an example of such an equilibrium in Appendix C). This happens because a larger ultimate number of partitions may require extensive pooling earlier on, inducing overall lower welfare. This finding is in contrast to Crawford and Sobel (1982), who show that under their Condition M (essentially a unique equilibrium for each partition size N), equilibria can be easily Pareto ranked: both the expert and the decision-maker prefer (ex ante) the equilibrium with the highest number of partitions.[7] Our findings suggest that Pareto comparisons in dynamic cases are less straightforward, even if we restrict attention to monotonic partitional equilibria.

We proceed to study the role of complex signaling in our dynamic game.

3 An Example with Complex Signaling and Dynamic Information Revelation

In this section, we present an example in which the expert employs a complex signaling rule, which induces non-monotonic actions. In this example, the bias is so severe that in a static setting, all equilibria would be babbling. We show that even in these extreme bias situations, some information can be revealed with just two rounds. This equilibrium has the feature that the decision-maker learns the state quite precisely when the news is either horrific or terrific, but remains agnostic for

[7] The equilibrium with the largest number of partitions is the only equilibrium that satisfies the "no incentive to separate" (NITS) condition (Chen, Kartik, and Sobel (2008)).

intermediate levels. Finally we show that for a range of biases, this non-monotonic equilibrium is Pareto superior to all monotonic ones.

Example 2 *Dynamic equilibria can be non-monotonic*

Consider a two period game where $\delta_E = \delta_{DM} = 1$, types are uniformly distributed on $[0, 1]$ and preferences are given by (1). We will construct an equilibrium with the following "piano teacher" interpretation: a child's parent (the decision-maker) wants the amount of money he spends on lessons to correspond to the child's true talent θ, whereas the piano teacher (expert) wants to inflate this number. In our equilibrium, parents of children who are at either the bottom or top extreme of the talent scale get the same initial message, "you have an interesting child" ($m_{1(1)}$ below), and then find out in the second period whether "interesting" means great ($m_{2(3)}$) or awful ($m_{2(1)}$); parents of average children are told just that in both periods. More precisely, let the expert use the following signaling rule:

In period 1, expert types in $[0, \underline{\theta}) \cup (\bar{\theta}, 1]$ send message $m_{1(1)}$ with probability 1, and types in $[\underline{\theta}, \bar{\theta}]$ send message $m_{1(2)}$ with probability 1. In period 2, the expert adopts the following signaling rule: types in $[0, \underline{\theta})$ send message $m_{2(1)}$, types in $[\underline{\theta}, \bar{\theta}]$ send a message $m_{2(2)}$, and types in $(\bar{\theta}, 1]$ send $m_{2(3)}$ (all with probability 1). With this signaling rule, the optimal actions for the decision-maker in period 1 are $y_{1(1)} = \frac{\underline{\theta}^2 - \bar{\theta}^2 + 1}{2(\underline{\theta} - \bar{\theta} + 1)}$, $y_{1(2)} = \frac{\underline{\theta} + \bar{\theta}}{2}$; in period 2, they are $y_{2(1)} = \frac{\underline{\theta}}{2}$, $y_{2(2)} = \frac{\underline{\theta} + \bar{\theta}}{2}$, $y_{2(3)} = \frac{1 + \bar{\theta}}{2}$. After any out-of-equilibrium message, the decision-maker assigns equal probability to all states in $[\underline{\theta}, \bar{\theta}]$, and so will choose action $y^{out} = \frac{\underline{\theta} + \bar{\theta}}{2}$. With these out-of equilibrium beliefs, no expert type has any incentive to send an out-of-equilibrium message.

In order for this to be an equilibrium, type $\underline{\theta}$ must be indifferent between message sequences $A \equiv \left(m_{1(1)}, m_{2(1)}\right)$ and $B \equiv \left(m_{1(2)}, m_{2(2)}\right)$:

$$-\left(\frac{\underline{\theta}^2 - \bar{\theta}^2 + 1}{2(\underline{\theta} - \bar{\theta} + 1)} - \underline{\theta} - b\right)^2 - \left(\frac{\underline{\theta}}{2} - \underline{\theta} - b\right)^2 = -2\left(\frac{\underline{\theta} + \bar{\theta}}{2} - \underline{\theta} - b\right)^2 \tag{2}$$

and type $\bar{\theta}$ must be indifferent between message sequences B and $C \equiv \left(m_{1(1)}, m_{2(3)}\right)$:

$$-\left(\frac{\underline{\theta}^2 - \bar{\theta}^2 + 1}{2(\underline{\theta} - \bar{\theta} + 1)} - \bar{\theta} - b\right)^2 - \left(\frac{1 + \bar{\theta}}{2} - \bar{\theta} - b\right)^2 = -2\left(\frac{\underline{\theta} + \bar{\theta}}{2} - \bar{\theta} - b\right)^2 . \tag{3}$$

At $t = 2$ it must also be the case that type $\underline{\theta}$ prefers $m_{2(1)}$ to $m_{2(3)}$, and the reverse for type $\bar{\theta}$: that is $-(\frac{\underline{\theta}}{2} - \underline{\theta} - b)^2 \geq -(\frac{1+\bar{\theta}}{2} - \underline{\theta} - b)^2$ and $-(\frac{1+\bar{\theta}}{2} - \bar{\theta} - b)^2 \geq -(\frac{\underline{\theta}}{2} - \bar{\theta} - b)^2$. The global incentive compatibility constraints, requiring that all types $\theta < \underline{\theta}$ prefer sequence A to B and that all types $\theta > \bar{\theta}$ prefer C to B, reduce to a requirement that the average induced action be monotonic, which is implied by indifference constraints (2), (3).[8]

A solution of the system of equations (2) and (3) gives an equilibrium if $0 \leq \underline{\theta} < \bar{\theta} \leq 1$. We solved this system numerically, and found that the highest bias for which it works is $b = 0.256$. Here, the partition cutoffs in our equilibrium are given by $\underline{\theta} = 0.0581$, $\bar{\theta} = 0.9823$. The corresponding optimal actions for period 1 are $y_{1(1)} = 0.253$, $y_{1(2)} = 0.52$, and for period 2 they are $y_{2(1)} = 0.029$, $y_{2(2)} = 0.52$, $y_{2(3)} = 0.991$. Note that while the first period action is non-monotonic, the average action $\bar{y} = \frac{y_1+y_2}{2}$ is still weakly increasing in the state. Ex ante payoffs are -0.275 for the expert, and -0.144 for the decision-maker.

Recall that in a one-shot game with quadratic preferences, the only equilibrium is the babbling one whenever $b > \frac{1}{4}$. Proposition 2 implies that at $b = 0.256$, if we restricted attention to *monotonic* partition equilibria, we would again find only a babbling equilibrium, in which the decision-maker chooses action $y^B = 0.5$ in both periods: this yields ex-ante payoffs of -0.298 to the expert, -0.167 to the decision-maker, strictly worse than in our above construction.

Our example therefore illustrates how allowing for non-monotonic equilibria can both increase the amount of information revelation, and can also strictly Pareto-dominate the best static equilibrium. By pooling together the best and the worst states in period 1, the expert is willing to reveal in period 2 whether the state is very good or very bad. It also has the following immediate implication:

Proposition 3 *There exist non-monotonic equilibria that are Pareto superior to all monotonic partition equilibria.*

We now move on to our first main result, showing that our dynamic setup correlates the incentives of the expert and decision-maker in such a way that *full information revelation* is possible.

4 Learning the Truth when the Expert is Patient

When the expert is patient rather than myopic, getting him to reveal the truth is much more complicated, as we previewed in Section 1. In this section, we construct a fully revealing equilibrium for the quadratic preferences specified in (1). The equilibrium relies on two main tools: separable groups, and trigger strategies. These tools have no leverage in single-round communications, but are powerful in dynamic communications.

[8] Rearranging (3), the LHS is greater than the RHS for type θ (so he prefers C to B) iff $(\theta - \bar{\theta})\left(\frac{y_{1(1)}+y_{2(3)}}{2} - y_{1(2)}\right) > 0$, so we need $\frac{y_{1(1)}+y_{2(3)}}{2} > y_{1(2)}$ for this to hold $\forall \theta > \bar{\theta}$. This is implied by (3) : adding $2\left(\frac{y_{1(1)}+y_{2(3)}}{2} - \bar{\theta} - b\right)^2$ to both sides and factoring yields $\left(y_{1(2)} - \bar{\theta} - b\right)^2 - \left(\frac{y_{1(1)}+y_{2(3)}}{2} - \bar{\theta} - b\right)^2 = \left(\frac{y_{1(1)}-y_{2(3)}}{2}\right)^2 \geq 0$, so we need $\left|y_{1(2)} - \bar{\theta} - b\right| \geq \left|\frac{y_{1(1)}+y_{2(3)}}{2} - \bar{\theta} - b\right|$; since $y_{1(2)} < \bar{\theta} + b$, this implies $y_{1(2)} \leq \frac{y_{1(1)}+y_{2(3)}}{2}$, as desired. And similarly at $\underline{\theta}$.

The equilibrium works as follows: in each period, the expert recommends an action to the decision-maker. Initially, each action is recommended by finitely many (at most four) expert types, who then subdivide themselves further into separable groups of two with an interim recommendation. If the decision-maker chooses all initial actions recommended by the expert, then the expert rewards him by revealing the truth in the final stage of the game, recommending an action $y(\theta) = \theta$. If the decision-maker rejects the expert's early advice, then the expert babbles for the rest of the game, and so the decision-maker never learns more than the separable group containing the truth.

We provide here an outline of how we construct fully revealing equilibria for quadratic preferences. This is followed by the statement of our first main theorem, with full proof details given in Appendix D.

Equilibrium Outline: Separable Groups

Rather than having intervals of types pool together, we construct pairs of far-away types ("partners") who pool together in the initial periods. The advantage is that once the expert joins one of these separable groups, revealing the two possible true states to the decision-maker, we no longer need to worry about him mimicking nearby types: his only options are to tell the truth, or to mimic his partner. Of course, an important part of the proof is to ensure that each expert type wants to join the "right" separable group. For a myopic expert, this is straightforward: if the expert cares only about the first-period action, then to make him join the right group, it is sufficient that the first-period action be constant across groups. (In fact, the myopic expert result relied only on separable groups, without the need for trigger strategies: we were able to group types such that the (constant) action recommended by each group was equal to the average type within the group, i.e. so that it coincided with the decision-maker's myopically optimal choice). For a patient expert, the construction is significantly more involved, as it must take dynamic incentives into account. In particular, the actions induced in the initial stages cannot be flat: if this were the case, then an expert who cares about the future would simply join whichever separable group leads to the best future action. Hence, for a patient expert, we need to construct initial action functions which provide appropriate incentives: if type θ knows that some type θ' will get a more favorable action in the revelation phase, then type θ's group must induce an initial action which is more favorable to type θ than that induced by (θ')'s group.

Equilibrium Outline: Strategies

Before we proceed with the sketch, it is useful to simplify notation and work with a scaled type space by dividing all actions and types by b. When we say that "type $\theta \in [0, \frac{1}{b}]$ recommends $u(\theta)$ in period 1, for disutility $(u(\theta) - \theta - 1)^2$", we mean that (in the unscaled type space) "type θb recommends action $u(\theta)b$, for disutility $(u(\theta)b - \theta b - b)^2 = b^2 (u(\theta) - \theta - 1)^2$".

We first partition the scaled type space $[0, \frac{1}{b}]$ into four intervals, with endpoints $[0, \theta_1, \theta_2, \theta_3, \frac{1}{b}]$. The separable groups are as follows: at time $t = 0$, each type $\theta \in [0, \theta_1]$ pools with a partner $g(\theta) \in [\theta_2, \theta_3]$ to send a sequence of recommendations $(u_1(\theta), u_2(\theta))$, and then reveal the truth at time $t = 2$ iff the decision-maker followed both initial recommendations. Each type $\theta \in [\theta_1, \theta_2]$ initially pools with a partner $h(\theta) \in [\theta_3, \frac{1}{b}]$ to recommend a sequence $(v_1(\theta), v_2(\theta))$, then revealing the truth at time $T - \tau$ ($\tau < T - 2$ a time parameter to be determined) iff the expert followed their advice.[9] For the purpose of this outline, take the endpoints $\theta_1, \theta_2, \theta_3$ as given, along with the partner functions $g : [0, \theta_1] \to [\theta_2, \theta_3]$; $h : [\theta_1, \theta_2] \to [\theta_3, \frac{1}{b}]$, and recommendation functions u_1, u_2, v_1, v_2. In the appendix, we derive the parameters and functions that work, and provide the full details of how to construct fully revealing equilibria.

We now describe the strategy for the expert and for the decision-maker. For notational purposes it is useful to further subdivide the expert types into three groups: I, II, and III.

At time $t = 0$, there are then three groups of experts. Group I consists of types $\theta^I \in [\theta_1, \theta_2]$ with their partners $h\left(\theta^I\right) \in \left[\theta_3, \frac{1}{b}\right]$. Group II consists of all types $\theta^{II} \in [0, \theta_1]$ whose initial recommendation coincides with that of a Group I pair, together with their partners $g(\theta^{II}) \in [\theta_2, \theta_3]$. Group III consists of all remaining types $\theta^{III} \in [0, \theta_1]$ and their partners $g(\theta^{III}) \in [\theta_2, \theta_3]$. In other words, we divided the types in intervals $[0, \theta_1] \cup [\theta_2, \theta_3]$ into two groups, II and III, according to whether or not their initial messages coincide with that of a group I pair.

The timeline of the expert's advice is as follows:

$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
Phase 1 ($t = 0$)	**Phase 2** ($t = 2\alpha_0$)	**Phase 3** ($t = 2\alpha_a$)	**Phase 4** ($t = 2$)	**Phase 5** ($T - \tau$)
Group $I : v_1$	Group I :	Groups II, III :	Groups II, III :	Group I :
Groups $II, III : u_1$	switch to v_2	switch to u_2	reveal the truth	reveal the truth

Figure 1: Timeline

where $0 < \alpha_0 \le \alpha_a < 1$ are specified in the appendix (Section D.2.1).

In words: in the initial phase, types in Group I recommend v_1: $v_1\left(\theta^I\right) = v_1\left(h\left(\theta^I\right)\right)$, while types in Groups II and III recommend the action u_1: $u_1\left(\theta^{II}\right) = u_1\left(g\left(\theta^{II}\right)\right)$. Importantly, the recommendations for the types in Groups I and II coincide (while Group III recommendations do not coincide with those of any Group I pair): for every θ^I in Group I, there exists θ^{II} in Group II with $v_1\left(\theta^I\right) = u_1\left(\theta^{II}\right)$. This is why, for ease of exposition, we have a subdivision into Groups II and III; upon receiving a recommendation $v_1(\theta^I) = u_1(\theta^{II})$, the decision-maker believes that it could have come from any of the types in $\left\{\theta^I, h\left(\theta^I\right), \theta^{II}, g\left(\theta^{II}\right)\right\}$ (see footnote 25 in Section D.2.1 for why this is needed). At time $t = 2\alpha_0$, group I pairs $\{\theta^I, h(\theta^I)\}$ change their recommendation

[9] Note that u_1, u_2, v_1, v_2 are functions of θ, and that in our construction, the expert's messages ("recommendations") are equal to the actions that he wants the decision-maker to take, and the decision-maker can then infer the expert's separable group from the message.

to $v_2(\theta^I) = v_2(h(\theta^I))$, while Groups II and III continue to follow the recommendation function u_1 (that is, they do not yet change their advice). Thus, at this stage, the decision-maker learns whether he is facing a Group I, II or III pair. At time $t = 2\alpha_a \geq 2\alpha_0$, group II and III pairs switch to the recommendation function u_2, where α_a may depend on the specific pair.[10] Group I continues to follow the recommendation function v_2, revealing no further information at this stage. At time $t = 2$, group II and III pairs separate: each type θ^{II} or θ^{III} in $[0, \theta_1]$ sends a message equal to his type (thus revealing the truth for the final $T - 2$ periods), and similarly their partners $g(\theta^{II}), g(\theta^{III})$ send messages equal to their own types. At time $T - \tau > 2$, Group I pairs $\{\theta^I, h(\theta^I)\}$ separate, with type θ^I recommending action θ^I and type $h(\theta^I)$ recommending $h(\theta^I)$ for the final τ periods. It should be noted that the times at which the decision-maker is instructed to change his action $(2\alpha_0, 2\alpha_a, T - \tau)$ are not necessarily integers in our construction. In a continuous-time setting, this clearly poses no problem; in discrete time, we can deal with integer constraints via public randomization and/or scaling up the horizon, as explained in the Appendix (D.2.6).

The decision-maker's strategy is to follow all on-path recommendations. An off-path recommendation at time $t = 0$ is treated as a mistake coming from the pair $\{0, g(0)\}$, and subsequent off-path recommendations are simply ignored as errors (full details at the start of Section D.1 in the Appendix).

To summarize: In the initial phase, separable groups are formed. Each expert type sends a recommendation sequence of the form $\left(\underbrace{v_1(\theta^I)}_{2\alpha_0 \text{ periods}}, \underbrace{v_2(\theta^I)}_{T-\tau-2\alpha_0 \text{ periods}}\right)$ or $\left(\underbrace{u_1(\theta^i)}_{2\alpha_a \text{ periods}}, \underbrace{u_2(\theta^i)}_{2(1-\alpha_a) \text{ periods}}\right)$, with $i \in \{II, III\}$, and such that for all $\theta^I \in [\theta_1, \theta_2]$ there exists $\theta^{II} \in [0, \theta_1]$ with $v_1(\theta^I) = u_1(\theta^{II})$. During these phases, the decision-maker is able to infer the separable group containing the expert's true type, but, rather than choosing the corresponding myopically optimal action, he chooses the actions u_1, u_2, v_1, v_2 recommended by the expert. These action functions are constructed to provide the expert with incentives to join the *right* separable group at time 0. The final phases are the revelation phases: the separable groups themselves separate, revealing the exact truth to the decision-maker, provided that he has followed all of the expert's previous advice; any deviation results in babbling by the expert during the revelation phase.

Incentivizing the Expert

Finally, we briefly explain the construction of the functions (u_1, u_2) and (v_1, v_2), and the corresponding partner functions g, h (and endpoints $\theta_1, \theta_2, \theta_3$), which are given parametrically in the Appendix (see equations $(14), (15)$). For the expert, three sets of constraints must be satisfied:

[10] In Proposition D3 in the appendix, we describe Group II, III types and their recommendations parametrically, as functions of a variable a. Then, in Lemma D7.1, we choose α_a to ensure the desired overlap of the u_1, v_1 recommendation functions.

Expert Local IC:

The first set of constraints can be thought of as local incentive compatibility constraints–that is, those applying within each type θ's interval $[\theta_i, \theta_{i+1}]$. These (dynamic) incentive compatibility constraints ensure that, say, the agent $\theta \in [0, \theta_1]$ prefers to induce actions $u_1(\theta)$ (for $2\alpha_a$ periods), $u_2(\theta)$ (for $2(1-\alpha_a)$ periods), and then reveal his type θ for the final $T-2$ periods, than e.g. to follow the sequence $\left(u_1(\theta'), u_2(\theta'), \theta'\right)$ prescribed for some other type θ' in the same interval $[0, \theta_1]$ (and analogously within each of the other three intervals). For types $\theta \in [0, \theta_1]$, this boils down to a requirement that u_1, u_2 satisfy the following differential equation,

$$2\alpha_a u_1'(\theta)\left(u_1(\theta) - \theta - 1\right) + 2(1-\alpha_a)u_2'(\theta)\left(u_2(\theta) - \theta - 1\right) = T - 2 \tag{4}$$

and that the "average" action, $2\alpha_a u_1(\theta) + 2(1-\alpha_a)u_2(\theta) + (T-2)\theta$, be weakly increasing in θ. We provide a more detailed explanation and solution of this equation in the Appendix, Section D.3.1, and derive similar equations for the other three intervals.

Note that a longer revelation phase (that is, an increase in the RHS term $(T-2)$ in (4)) requires a correspondingly larger distortion in the action functions u_1, u_2 : if the expert anticipates a lengthy phase in which the DM's action will match the true state (whereas the expert's bliss point is to the right of the truth), then it becomes more difficult in the initial phase to provide him with incentives not to mimic the advice of types to his right. This is why a longer horizon does not trivially imply better welfare properties.

Expert Global IC:

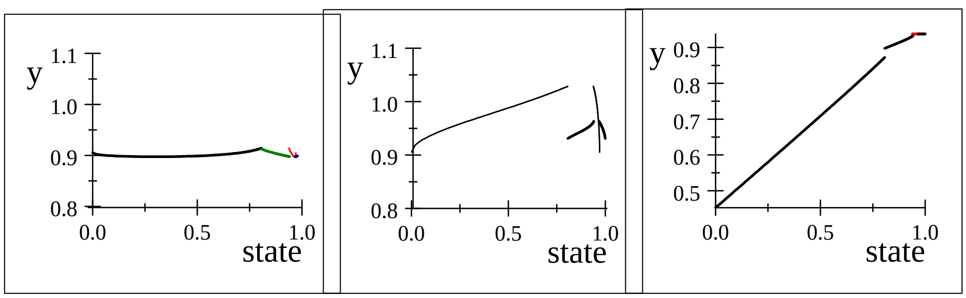
The next set of constraints for the expert can be thought of as "global" incentive compatibility constraints, ensuring that no expert type wishes to mimic any type in any other interval. In the appendix, we show that this boils down to two additional requirements: each endpoint type $\theta_1, \theta_2, \theta_3$ must be indifferent between the two equilibrium sequences prescribed for his type (for example, type θ_1 must be indifferent between sequences $\left(\underbrace{u_1(\theta_1)}_{2\alpha_a}, \underbrace{u_2(\theta_1)}_{2(1-\alpha_a)}, \underbrace{\theta_1}_{T-2}\right)$ and $\left(\underbrace{v_1(\theta_1)}_{2\alpha_0}, \underbrace{u_2(\theta_1)}_{T-\tau-2\alpha_0}, \underbrace{\theta_1}_{\tau}\right)$), and the "average" action must be either continuous or jump up at each endpoint (see Appendix Lemma D.3.2, with further details in Section D.3.1).

Expert Separation:

The final constraint requires that each pair of types indeed be "separable", that is, sufficiently far apart that each type would rather tell the truth than mimic his partner. In our rescaled type space with quadratic preferences, this requires choosing partner functions g, h satisfying $|g(\theta) - \theta| \geq 2$

and $|h(\theta) - \theta| \geq 2$, which we do in the appendix (Section D.1.2). It turns out to be very tricky to satisfy the global incentive compatibility constraints together with the local constraints: it in fact requires a minimum of two distinct actions prior to the revelation phase (this is why e.g. Group III pairs must change their recommendation from u_1 to u_2 at time $2\alpha_a$, even though doing so reveals no further information), and that the type space be partitioned into a minimum of four intervals. Moreover, for any partition into four intervals, there is in fact only *one* partner function $g : [0, \theta_1] \rightarrow [\theta_2, \theta_3]$ that works, and we believe that there is no partition which would allow for expert-incentivizing action functions which are myopically optimal from the decision-maker's perspective. This is why our construction relies on trigger strategies: the expert *only* reveals the truth if the decision-maker follows all of his advice.

We graph the equilibrium actions u_1, v_1 in the left-most graph, the u_2, v_2 in the middle graph, and the average action for $b = \frac{1}{60.885}$ and $T = 4$:



Actions u_1, v_1 Actions u_2, v_2 Average Action

Incentivizing the decision-maker:

Suppose that the expert recommends an action $u_1(\theta)$, which the decision-maker believes could only have come from types $\theta, g(\theta)$. If the decision-maker follows the recommendation, then he expects the expert to switch his recommendation to $u_2(\theta)$ at time $2\alpha_a$, and then recommend the true state θ for the final $T-2$ periods. If the decision-maker assigns probabilities $p_\theta, 1-p_\theta$ to types $\theta, g(\theta)$, then this yields an expected disutility of

$$p_\theta \left(2\alpha_a \left(u_1(\theta) - \theta\right)^2 + 2(1-\alpha_a)\left(u_2(\theta) - \theta\right)^2\right) + (1-p_\theta)\left(2\alpha_a \left(u_1(\theta) - g(\theta)\right)^2 + 2(1-\alpha_a)\left(u_2(\theta) - g(\theta)\right)^2\right)$$

(noting that disutility in the final $T-2$ periods is zero). The problem is that the initial recommendations $u_1(\theta), u_2(\theta)$ do not coincide with the decision-maker's myopically optimal action, $y^*(\theta) \equiv p_\theta\theta + (1-p_\theta)g(\theta)$. We therefore employ *trigger strategies*: the expert only reveals the truth in the final stage if the decision-maker follows his recommendations at the beginning of the game. If the decision-maker ever rejects his advice, then the expert babbles for the rest of the game, and so the decision-maker's disutility is at best

$$T \cdot \left[p_\theta \cdot \left(\underbrace{p_\theta \theta + (1-p_\theta) g(\theta)}_{y^*(\theta)} - \theta \right)^2 + (1-p_\theta) \cdot \left(\underbrace{p_\theta \theta + (1-p_\theta) g(\theta)}_{y^*(\theta)} - g(\theta) \right)^2 \right]$$

So, for the equilibrium to work for the decision-maker, we need to make sure that the benefit to learning the exact state, rather than just the separable group containing it, is large enough to compensate him for the cost of following the expert's initial recommendations, rather than deviating to the myopically optimal actions. This is what limits the priors for which our construction works, and imposes the upper bound $b \cong \frac{1}{61}$ on the bias (see Appendix D.3.2, end of first paragraph). The construction works for the expert $\forall b < \frac{1}{16}$ (see appendix, end of proof of Proposition D2 in Section D.1.2).

Beliefs

We assume that the decision-maker is Bayesian: if he believes that the expert's first-period messages are given by a function $M : [0,1] \to \mathbb{R}$, with the property that

$$M(x) = M(p(x))$$

for all x in some interval $[\underline{x}, \overline{x}]$ and $p : [\underline{x}, \overline{x}] \to [0,1] \backslash [\underline{x}, \overline{x}]$ some continuous differentiable function (i.e., types x and $p(x)$ are "partners" who follow the same messaging strategy), then, after receiving the message $m = M(x) = M(p(x))$, the decision-maker's beliefs satisfy

$$\frac{\Pr(x|m)}{\Pr(p(x)|m)} = \lim_{\Delta \to 0} \frac{F(x+\Delta) - F(x-\Delta)}{F(p(x+\Delta)) - F(p(x-\Delta))} = \frac{f(x)}{f(p(x))} \left| \frac{1}{p'(x)} \right| \tag{5}$$

This says that the likelihood of type x relative to $p(x)$ is equal to the unconditional likelihood ratio (determined by the prior F), times a term which depends on the shape of the p-function, in particular due to its influence on the size of the interval of p-types compared to their partner interval, $[\underline{x}, \overline{x}]$.[11]

We now state our main result:

Theorem 1 *Suppose that $\delta_E = \delta_{DM} = 1$ and that the preferences of the expert and of the decision-maker are given by* (1) *. For any bias $b \leq \frac{1}{61}$, there is an open set of priors F,*[12] *and a horizon T^*, for which a fully revealing equilibrium exists whenever $T \geq T^*$.*

[11] To understand this formula, consider an example in which F is uniform and $p(\cdot)$ is linear, say $p(x) = \alpha + \beta x$. In this case, the interval $[p(\underline{x}), p(\overline{x})]$ is β times as large as the interval $[\underline{x}, \overline{x}]$, so intuitively, it is as if the message sent by type x is sent by β "copies" of type $p(x)$: therefore, the decision-maker's beliefs assign β times as much weight to type $p(x)$ as to type x, which is precisely what our formula says. Beliefs are assigned analogously after period 1.

[12] This is slightly strengthened from previous versions of the paper, which claimed only an infinite (rather than open) set of priors.

The details of the construction can be found in the Appendix.

Substantively, this Theorem establishes an unexpected finding: even with a forward-looking expert and an infinite state space, there are equilibria in which the truth is revealed in finite time. We initially expected to prove the opposite result. Technically, the construction involves several innovative ideas that we expect to be useful in analyzing many dynamic games with persistent asymmetric information.

Discussion

The true state is revealed at either time 2 or time $T-\tau$, where $T-\tau$ can be chosen to be at most 5 (specified at start of Appendix D.2.1). Thus, the decision-maker chooses his best possible action, equal to the true state, in all but the first few periods. It is tempting to conclude that a long horizon means an equilibrium approaching the first-best, but unfortunately this is not true when the decision-maker and expert are equally patient. As explained after equation (4), a long horizon also makes it difficult to incentivize the expert, requiring a proportionally larger distortion in the initial recommendation functions, and thereby imposing a proportionally larger cost to the decision-maker (from having to follow such bad early advice in order to learn the truth). We do, however, show in the next subsection that if the decision-maker is more patient than the expert, our fully revealing equilibrium has more attractive welfare properties, and works for a much larger set of decision-maker preferences and beliefs: If the expert does not care much about the future, it becomes easy to incentivize him to join the right separable group, which, in turn, implies little need to distort the initial recommendations, and therefore little cost to the decision-maker from following bad advice in the first couple of periods. The benefit to following this advice – knowing the exact optimal action in all but the first few periods – will then outweigh this cost for a patient decision-maker. (Section 1 illustrated this in the extreme case $\delta_E = 0$, where the decision-maker learned the exact truth with no distortion in the expert's initial advice).

Remark 1 *If we look at situations where the decision-maker cares only about the ultimate decision, it is easy to see that our construction works for any prior (for all* $b < \frac{1}{16}$*, the bound from the proof of Proposition D2 required for the expert), and yields the best possible outcome for the decision-maker.*

Remark 2 *If the decision-maker is not Bayesian, and his posterior beliefs (following any history) simply assign equal probability to each type in his information set, then our construction yields fully revealing equilibrium for* any *prior on* $[0,1]$ *if* $b \leq \frac{1}{61}$.[13]

[13] Our proof shows (not explicitly stated in this verison of the paper) that the DM's incentive compatibility constraints are satisfied if his posterior beliefs, after each expert recommendation, assign sufficiently high probability to each type in his information set. (We then show that for a Bayesian DM, there is an open set of priors generating such posteriors).

4.1 Information Revelation in Continuous Time

The equilibrium we constructed to prove Theorem 1 can be easily modified to yield a fully revealing equilibrium in a continuous-time setting with arbitrary discount rates, so long as the decision-maker is at least as patient as the expert.[14] In particular, suppose that actions and recommendations may be made at any time up until the end of the game, and that the decision-maker and the expert discount the future at rates r^{DM}, r^{E}, respectively. We then obtain the following result:

Proposition 4 *Suppose that preferences are given by* (1)*. For any bias* $b \leq \frac{1}{61}$ *and prior* F *for which Theorem 1 holds, any horizon* $\widehat{T}$*, any expert discount rate* $r^E > 0$*, and any decision-maker discount rate* $r^{DM} \leq r^E$*, a fully revealing equilibrium exists.*

Proof:

Leave all action functions and specifications from the proof of Theorem 1 unchanged, except for the timeline shown in Figure 1: now, let Group I pairs recommend v_1 up to time $t_1(\alpha_0)$, then v_2 up to time t_4, and then reveal the truth, and let Group II, III pairs now recommend u_1 up to time $t_2(\alpha_a)$, u_2 up to time t_3, then reveal the truth, where

$$t_1(\alpha_0) = \frac{\ln\left(1 - 2\phi\alpha_0 r^E\right)}{-r^E}, t_2(\alpha_a) = \frac{\ln\left(1 - 2\phi\alpha_a r^E\right)}{-r^E}, t_3 = \frac{\ln\left(1 - 2\phi r^E\right)}{-r^E}, t_4 = \frac{\ln\left(1 - (T-\tau)\,\phi r^E\right)}{-r^E} \tag{6}$$

with $\phi = \frac{1-e^{-r^E\widehat{T}}}{Tr^E}$ ($\widehat{T}$ is the (freely specified) horizon in the statement of the Proposition, and the T is the horizon used in our original construction, see appendix Section D.2.1).

By construction, this simply multiplies the expert's payoffs from our original construction by a constant, ϕ. The disutility to expert type θ from following the strategy of a Group II or III pair – say, recommending $u_1(\theta')$ up to time $t_2(\alpha_a)$, $u_2(\theta')$ up to time t_3, and θ' up to time $\widehat{T}$ – is

$$\begin{aligned} & \int_0^{t_2(\alpha_a)} e^{-r^E t}\left(u_1(\theta') - \theta - b\right)^2 dt + \int_{t_2(\alpha_a)}^{t_3} e^{-r^E t}\left(u_2(\theta') - \theta - b\right)^2 dt + \int_{t_3}^{\widehat{T}} e^{-r^E t}\left(\theta' - \theta - b\right)^2 dt \\ = \;& \phi\left[2\alpha_a\left(u_1(\theta') - \theta - b\right)^2 + 2(1-\alpha_a)\left(u_2(\theta') - \theta - b\right)^2 + (T-2)\left(\theta' - \theta - b\right)^2\right] \end{aligned}$$

(the second line simply evaluates the integrals using (6) : for example, $\int_0^{t_1(\alpha_0)} e^{-r^E t} dt = \frac{1-e^{-r^E t_1(\alpha_0)}}{r^E} = 2\phi\alpha_0$). This is precisely ϕ times the payoff, from our original construction, to an expert of type $\theta \in [0, \theta_1]$ from following the strategy prescribed for type $\theta' \in [0, \theta_1]$ (see (25) and (29) in appendix,

[14] We switch here to continuous time for convenience. The proof of Theorem 1 is also essentially written for continuous time – in our initial equilibrium construction, the times at which the DM is instruced to change his action are not necessarily integers – but we show at the end of the proof how to modify the construction for a discrete-time setting, via public randomization and/or scaling up the horizon. Something similar could of course be done here, but as our construction requires fairly exact ratios on the (discounted values of the) durations of each action, this is much more convenient in continuous time.

Section D.1.4). Similarly, the disutility to expert type θ from following the strategy of a Group I pair – say, recommending $v_1(\theta')$ up to time $t_1(\alpha_0)$, $v_2(\theta')$ up to time t_4, and θ' up to time $\widehat{T}$ – is exactly ϕ times the payoff, from our original construction, to a perfectly patient expert who recommends $v_1(\theta')$ up to time $2\alpha_0$, $v_2(\theta')$ up to time $T-\tau-2\alpha_0$, and the truth up to time T. So, since the expert's payoffs are exactly the same as before, for each possible true type θ and each possible type θ' he could choose to mimic, it follows that if the expert finds it optimal to tell the truth in our original construction (with discrete time and discount factor $\delta_E = 1$), then an expert with continuous-time discount rate r^E will likewise find it optimal to tell the truth, given our modified timeline.

For the DM: if $r^{DM} = r^E$, then we likewise obtain that in continuous time, with discount rate r^{DM} and our modified timeline, all payoffs are identical to those in our construction used to prove (1) . If $r^{DM} < r^E$, so that the DM is more patient than the expert, then things only become easier. As discussed in the appendix (Observation D4 of Section D.2), we need only show that the DM cannot gain by deviating at time $t = 2\alpha_0$ (which is now time $t = t_1(\alpha_0)$ with our modified timeline) if he receives a recommendation $v_2(\theta^I)$ from a Group I pair $\{\theta^I, h(\theta^I)\}$, or at time $t = 0$ (when he may get either a recommendation $u_1(\theta^{II}) = v_1(\theta^I)$ which could have been sent by any of the 4 types in $\{\theta^I, h(\theta^I), \theta^{II}, g(\theta^{II})\}$, or a recommendation $u_1(\theta^{III})$, which could only come from a Group III pair $\{\theta^{III}, g(\theta^{III})\}$. Let us first consider deviations at time $t = t_1(\alpha_0)$: in our modified timeline, if he assigns probabilities $p, 1-p$ to the types $\theta^I, h(\theta^I)$ in his information set, then he expects to earn flow disutility $\left(p\left(v_2(\theta^I) - \theta^I\right)^2 + (1-p)\left(v_2(\theta^I) - h(\theta^I)\right)^2\right)$ from time $t_1(\alpha_0)$ to time t_4, at which point the expert should reveal the truth (so disutility drops to zero for the rest of the game). If he instead deviates to the best myopically optimal action, $x^* \equiv p\theta^I + (1-p)h(\theta^I)$, then from time $t_1(\alpha_0)$ to $\widehat{T}$ he will earn expected flow disutility

$$p\left(x^* - \theta^I\right)^2 + (1-p)\left(x^* - h(\theta^I)\right)^2 = p(1-p)\left(h(\theta^I) - \theta^I\right)^2$$

We need the equilibrium disutility to be smaller than the disutility from deviating, which rearranges to the following condition:

$$\left(\frac{\int_{t_1(\alpha_0)}^{t_4} e^{-r^{DM}t}dt}{\int_{t_1(\alpha_0)}^{\widehat{T}} e^{-r^{DM}t}dt}\right)\left(\frac{p\left(v_2(\theta^I) - \theta^I\right)^2 + (1-p)\left(v_2(\theta^I) - h(\theta^I)\right)^2}{p(1-p)\left(h(\theta^I) - \theta^I\right)^2}\right) \leq 1 \tag{7}$$

Similarly, at a time $t = 0$ information set of the form $\{\theta^{III}, g(\theta^{III})\}$, the gain from deviating is smaller than the cost whenever the following expression is ≤ 1 :

$$\frac{\int_0^{t_2(\alpha_a)} e^{-r^{DM}t}dt}{\int_0^{\widehat{T}} e^{-r^{DM}t}dt}\left(\frac{E[\text{flow disutility from } u_1 \mid \theta\in\{\theta,g(\theta)\}}{E[\text{flow disutility from best deviation}]}\right)+\frac{\int_{t_2(\alpha_a)}^{t_3} e^{-r^{DM}t}dt}{\int_0^{\widehat{T}} e^{-r^{DM}t}dt}\left(\frac{E[\text{flow disutility from action } u_2 \mid \theta\in\{\theta,g(\theta)\}}{E[\text{flow disutility from best deviation}}\right) \tag{8}$$

And at a time $t = 0$ information set containing both a Group I pair and a Group II pair, $\{\theta^I, h(\theta^I), \theta^{II}, g(\theta^{II})\}$, letting p_1, p_2, p_3, p_4 denote the respective probabilities on the four types, an upper bound on the ratio of equilibrium disutility, to disutility from the best possible deviation, is[15]

$$(p_1+p_2)\left[\begin{array}{c} \frac{\int_0^{t_1(\alpha_0)} e^{-r^{DM}t}dt}{\int_0^{\widehat{T}} e^{-r^{DM}t}dt}\left(\frac{E[\text{flow disutility from } v_1(\theta^I)|\{\theta^I,h(\theta^I)\}]}{E[\text{flow disutility from } \frac{p_1\theta^I+p_2h(\theta^I)}{p_1+p_2}|\{\theta^I,h(\theta^I)\}}\right) \\ +\frac{\int_{t_1(\alpha_0)}^{t_4} e^{-r^{DM}t}dt}{\int_0^{\widehat{T}} e^{-r^{DM}t}dt}\left(\frac{E[\text{flow disutility from } v_2(\theta^I)|\{\theta^I,h(\theta^I)\}]}{E[\text{flow disutility from } \frac{p_1\theta^I+p_2h(\theta^I)}{p_1+p_2}|\{\theta^I,h(\theta^I)\}}\right) \end{array}\right]$$
$$(p_3+p_4)\left[\begin{array}{c} \frac{\int_0^{t_2(\alpha_a)} e^{-r^{DM}t}dt}{\int_0^{\widehat{T}} e^{-r^{DM}t}dt}\left(\frac{E[\text{flow disutility from } u_1(\theta^{II}) \mid \{\theta^{II},g(\theta^{II})\}}{E[\text{flow disutility from } \frac{p_3\theta^{II}+p_4g(\theta^{II})}{p_3+p_4}|\{\theta^{II},g(\theta^{II})\}}\right) \\ +\frac{\int_{t_2(\alpha_a)}^{t_3} e^{-r^{DM}t}dt}{\int_0^{\widehat{T}} e^{-r^{DM}t}dt}\left(\frac{E[\text{flow disutility from } u_2(\theta^{II}) \mid \{\theta^{II},g(\theta^{II})\}}{E[\text{flow disutility from } \frac{p_3\theta^{II}+p_4g(\theta^{II})}{p_3+p_4}|\{\theta^{II},g(\theta^{II})\}}\right) \end{array}\right] \quad (9)$$

We have an equilibrium if (7) holds for every Group I pair $\{\theta^I, h(\theta^I)\}$, the expression in (8) is weakly below 1 for every Group III pair $\{\theta^{III}, g(\theta^{III})\}$, and the expression in (9) is weakly below 1 for every information set of the form $\{\theta^I, h(\theta^I), \theta^{II}, g(\theta^{II})\}$. At $r^{DM} = r^E$, these reduce to precisely the inequalities proven to hold in Section D4. (For example, consider (7) : at $r^{DM} = r^E$, using (6), the first "time ratio"term reduces to $\frac{T-\tau-2\alpha_0}{T-2\alpha_0}$. And since we have not made any modifications to the action functions or information sets, the constraint then simply says that the length of the v_2-recommendation phase in our original construction $(T - \tau - 2\alpha_0)$, times the flow disutility from choosing v_2, must be smaller than the length of the remaining game $(T - 2\alpha_0)$, times the flow disutility from choosing the myopically optimal action).

We complete the proof in the Appendix (Section E), showing that the constraints in (9), (8), and (7) become more relaxed as r^{DM} decreases. ■

As r^{DM} approaches zero and the horizon increases, we can push the result further, obtaining an equilibrium with attractive welfare properties for a large range of biases and priors:

Theorem 2 *If r^E is bounded above zero and preferences are given by (1) with $b < \frac{1}{16}$, then, for any prior F with a density that is everywhere bounded away from zero and infinity, there is a horizon T^* and discount rate r^* such that a fully revealing equilibrium exists whenever $r^{DM} < r^*$ and $\widehat{T} > T^*$. In this equilibrium, the decision-maker's average disutility goes to zero as $\widehat{T} \to \infty$ and $r^{DM} \to 0$.*

[15] In the "disutility ratio" terms, the denominator supposes that whenever $\theta \in \{\theta^I, h(\theta^I)\}$, the DM chooses the myopically optimal action conditional on his information set; and that when $\theta \in \{\theta^{II}, g(\theta^{II})\}$, he chooses the corresponding myopically optimal action. In fact, at time 0, he knows only that $\theta \in \{\theta^I, h(\theta^I), \theta^{II}, g(\theta^{II})\}$, so he will typically do worse than if he knew the true pair: therefore, our expression gives an upper bound on the ratio of equilibrium to deviation disutilty. Section D.2 (see outline, specifically the paragraph referring to Proposition D6) proves that this upper bound is less than 1 (for a perfectly patient DM and our original timeline, at all such information sets), which is sufficient to establish that the DM cannot gain by deviating.

Proof: Again use the timeline (6), together with all action and strategy specifications of our original construction; as above in Proposition 4, this leaves all utility expressions and analysis for the expert unchanged from our original construction; and as noted in the appendix, Proposition D2, our equilibrium works for the expert for all $b < \frac{1}{16}$. For the decision-maker, we just need to check the incentive constraints in (9), (8), and (7). We can conclude, from the fact that they held in our original construction, that the equilibrium flow disutilities (numerators in the second term of each expression) must be bounded. Our original construction also specifies "partner" functions which have positive and finite derivatives, which implies (see (5)) that for any prior which has a positive bounded density, the decision-maker's posteriors over all information sets assign a strictly positive probability to each type; this implies that the myopically optimal action at each information set is bounded away from the true state, and therefore the flow disutility to the decision-maker if he deviates to the myopically optimal action is bounded away from zero. We conclude that the second "flow disutility" ratios in (9), (8), and (7) are all finite. However, the first "time ratio" terms in these expressions go to zero as $r^{DM} \to 0$ and $\widehat{T} \to \infty$, noting that (6) and r^E bounded above zero imply that $t_1(\alpha_0), t_2(\alpha_a), t_3, t_4$ are all finite. Therefore, the ratio of the DM's equilibrium disutility, compared to his disutility from the best deviation, goes to zero as $r^{DM} \to 0$ and $\widehat{T} \to \infty$, implying that we have an equilibrium. Moreover, since the DM's equilibrium flow disutility is bounded up to time t_4 and zero thereafter, with t_4 finite, it follows that as $r^{DM} \to 0$, the DM's average expected payoff goes to zero, thus completing the proof. ■

Remark 3 *Compared to Theorem 1, this result guarantees a fully revealing equilibrium for nearly all priors over the state space, and for a much larger set of biases ($b < \frac{1}{16}$, rather than $b < \frac{1}{61}$). Note also that the argument does not rely on quadratic preferences for the decision-maker.*

5 Concluding Remarks

This paper shows that dynamic strategic communication differs from its static counterpart. Our most striking result is that fully revealing equilibria exist. The equilibria are admittedly complex, and we do not suggest that they resemble any communication schemes currently in practice. This was not our goal; rather, we wished to determine whether it is *possible* for a decision-maker to design a questions-and-incentives scheme to elicit the precise truth out of a biased expert, such that the expert would be willing to commit to and follow the proposed scheme. Our construction proves that it is indeed possible, explains exactly how to do so when the expert has quadratic-loss preferences[16] and the true state is constant,[17] and highlights the conditions under which he would indeed desire to do so. In particular, we have shown that the proposed communication scheme can be of great benefit to the decision-maker if he is either more patient than the expert, or if he can hire the expert on a short-term basis. (This may provide one additional rationale for hiring consultants rather than permanent advisors).

The main novel ingredient of our model is that there are multiple rounds of communication, with a new action chosen after each round. The dynamic incentive considerations for the expert allow us to group together types that are far apart, forming "separable groups", which is the key to obtaining greater information revelation. Our dynamic setup also allows for future communication to be conditioned on past actions (trigger strategies), and we show how information revelation can be facilitated through this channel.

The forces that we identify may be present in many dynamic environments with asymmetric information. Think, for example, of a dynamic contracting environment with limited commitment, or more generally, of a dynamic mechanism problem. In these models as well, past behavior sets the stage for future behavior. And, in contrast to the vast majority of the recent literature on dynamic mechanism design,[18] one needs to worry about both global and local incentive constraints, even with simple stage payoffs that satisfy the single-crossing property.

Lastly, given the important insights from cheap talk literature which have been widely applied in both economics and political science, we hope and expect that the novel aspects of strategic communication emphasized in our analysis will help shed light on many interesting dynamic problems.

[16] It would be interesting to understand more generally the types of expert preferences for which this is possible, but this is beyond the scope of the current paper. The general question is difficult to analyze, given the large class of possible equilibrium structures: in principle, one might need large finite separable groups in the first stage (instead of our groups of two), gradually subdividing via long action sequences (instead of our sequences of two initial actions).

[17] One could presumably apply our construction in a model where the state evolves slowly over time, for example by restricting how frequently the expert can observe state changes, and playing our equilibrium within each "block" between state observations. If the probability of a state change between observations is small, this would lead to an equilibrium where the decision-maker knows the true state most of the time.

[18] In recent years, motivated by the large number of important applications, there has been substantial work on dynamic mechanism design. See, for example, the survey of Bergemann and Said (2011) and the references therein, or Pavan, Segal, and Toikka (2011).

A Proof of Proposition 2

When we restrict attention to monotonic partition equilibria, there will be some point in the game at which the last subdivision of an interval occurs, say period $\widehat{T} \leq T$. Assume (without loss of generality) that one interval is partitioned into two, inducing actions y_1 and y_2, and let $\widehat{\theta}$ be the expert type who is indifferent between y_1, y_2. Since no subdivision occurs after period $\widehat{T}$, it follows that type $\widehat{\theta}$'s indifference condition in period $\widehat{T}$ is

$$\left(1+\delta+..+\delta^{T-\widehat{T}-1}\right)u^E\left(y_1,\hat{\theta},b\right) \geq \left(1+\delta+..+\delta^{T-\widehat{T}-1}\right)u^E\left(y_2,\hat{\theta},b\right),$$

which reduces to the static indifference condition. But then, if this subdivision is possible, it cannot be the case that all static equilibria are equivalent babbling. This follows by Corollary 1 of Crawford and Sobel (1982).

Observe that all the arguments in this proof go through even if we allow for trigger strategies. This is because at the point where the last subdivision occurs, it is impossible to incentivize the decision-maker to choose anything other than his myopic best response: he knows that no further information will be revealed, and so he knows that he cannot be rewarded in the future for choosing a suboptimal action now. So, the above argument applies.

B Monotonic partition equilibria with more partitions

Suppose that $\delta_E = \delta_{DM} = 1$, types are uniformly distributed on $[0,1]$ and preferences satisfy (1), with bias $b = \frac{1}{12}$. Using the standard arguments, one can establish that game has only two equilibria:[19] a babbling equilibrium, and an equilibrium with two partitions, $\left[0,\frac{1}{3}\right]\cup\left[\frac{1}{3},1\right]$, inducing actions $\frac{1}{6}$ and $\frac{4}{6}$. Now we show that when $T=2$, there exists a monotonic partition equilibrium where the state space is ultimately divided into three sub-intervals.

We look for an equilibrium with the following signaling rule:

$$\begin{aligned}
&\text{types in } [0,\theta_1] \text{ send message sequence } A=(m_{1(1)},m_{2(1)}),\\
&\text{types in } [\theta_1,\theta_2] \text{ send message sequence } B=(m_{1(2)},m_{2(2)}),\\
&\text{types in } [\theta_2,1] \text{ send message sequence } C=(m_{1(2)},m_{2(3)}).
\end{aligned}$$

With this signaling rule, in the first period the interval $[0,1]$ is partitioned into $[0,\theta_1]$ and $[\theta_1,1]$. The indifference condition for type θ_2 in period 2 yields

$$\left(\frac{\theta_1+\theta_2}{2}-\theta_2-b\right)^2 = \left(\frac{1+\theta_2}{2}-\theta_2-b\right)^2 \Rightarrow \theta_2 = \frac{1}{3}+\frac{1}{2}\theta_1 \tag{12}$$

The second-period actions induced are $y_{2(1)} = \frac{\theta_1}{2}, y_{2(2)} = \frac{3}{4}\theta_1+\frac{1}{6}$ and $y_{2(3)} = \frac{1}{4}\theta_1+\frac{2}{3}$, and the first-period actions are $y_{1(1)} = \frac{\theta_1}{2}$ and $y_{1(2)} = \frac{1+\theta_1}{2}$.

[19] The largest number of subintervals that the type space can be divided into is the largest integer that satisfies

$$-2bp^2+2bp+1>0, \tag{10}$$

whose solution is

$$\left\langle -\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{2}{b}}\right\rangle, \tag{11}$$

and where $\langle x\rangle$ denotes the smallest integer greater than or equal to x.

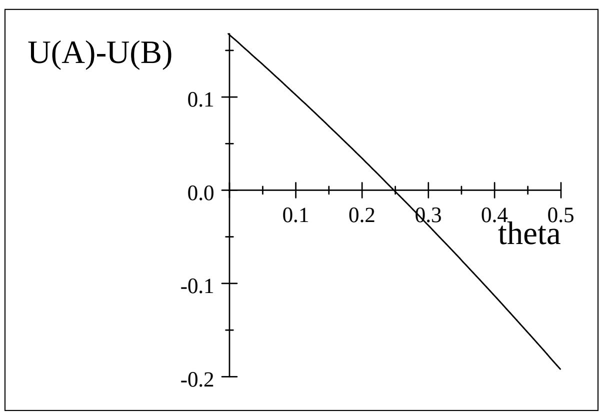
After any out-of-equilibrium message the decision-maker assigns probability one to the state belonging in $[0, \theta_1]$ inducing $y^{out} = \frac{\theta_1}{2}$. With these out-of equilibrium beliefs it is immediate to see that no type has an incentive to send an out-of-equilibrium message.

At equilibrium, θ_1 must satisfy the following indifference condition:

$$\left(\underbrace{\frac{1+\theta_1}{2}}_{y_{1(2)}} - \theta_1 - \frac{1}{12}\right)^2 + \left(\underbrace{\frac{3}{4}\theta_1 + \frac{1}{6}}_{y_{2(2)}} - \theta_1 - \frac{1}{12}\right)^2 = 2\left(\underbrace{\frac{\theta_1}{2}}_{y_{1(1)}=y_{2(1)}} - \theta_1 - \frac{1}{12}\right)^2$$

which is solved by $\theta_1 = 0.2482$; together with (12) , we then obtain three final partitions, with cutoffs $\theta_1 = 0.2482$, $\theta_2 = 0.457\,43$; with this, the actions become $y_{1(1)} = y_{2(1)} = 0.124\,1$, $y_{1(2)} = 0.624\,1$, $y_{2(2)} = 0.352\,8$, and $y_{2(3)} = 0.728\,7$.

In constructing this strategy profile, we imposed only local incentive compatibility constraints, requiring that type θ_1 is indifferent in period 1 between inducing action sequence $\left(y_{1(1)}, y_{2(1)}\right)$ and $\left(y_{1(2)}, y_{2(2)}\right)$, and that type θ_2 is indifferent in period 2 between inducing actions $y_{2(2)}$ and $y_{2(3)}$. Now we want to verify that these conditions are sufficient for global incentive compatibility. At $t = 2$ the game is isomorphic to the static one, where the fact that θ_2 is indifferent between $y_{2(2)}$ and $y_{2(3)}$ implies that all types above θ_2 prefer $y_{2(3)}$ and all types below θ_2 prefer $y_{2(2)}$. To verify that types below θ_1 prefer message sequence A and types above θ_1 prefer message sequence B, we plot the difference $U(A, \theta) - U(B, \theta)$ and show that it is positive for all $\theta < \theta_1$ and negative for $\theta > \theta_1$:



In our dynamic equilibrium, the expert's (ex ante) payoff is -0.0659 and the decision-maker's (ex ante) payoff is -0.052. If the most informative static equilibrium is played in both periods, payoffs are -0.069 to the expert, -0.055 to the decision-maker, both strictly worse than in our dynamic monotonic partition equilibrium.

C Pareto comparisons of dynamic cheap talk equilibria

The following example demonstrate that equilibria with more partitions can be Pareto inferior to the equilibria with fewer partitions

Take $\delta_E = \delta_{DM} = 1$ and $b = 0.08$, and consider the most informative static partition equilibrium where the number of partitions is $p = 3$. At this equilibrium the state space is divided into $[0, 0.013]$, $[0.013, 0.347]$ and $[0.347, 1]$. The corresponding optimal actions of the decision-maker are given by

$$y_1 = 0.0067 \quad y_2 = 0.18 \quad y_3 = 0.673,$$

from which we can calculate the ex-ante expected utility levels for the expert -0.032 and for the decision-maker -0.0263. Then, at the equilibrium of the dynamic game where the most informative static equilibrium is played at $t = 1$ and babbling thereafter, the total expected utility is -0.065 for the expert, and -0.053 for the decision-maker.

We now construct a dynamic equilibrium where the type space is subdivided into more subintervals, but both players' ex-ante expected payoffs are lower. We look for an equilibrium with the following signaling rule:

$$\begin{aligned}
&\text{types in } [0, \theta_1] \text{ send message sequence } (m_{1(1)}, m_{2(1)})\\
&\text{types in } [\theta_1, \theta_2] \text{ send message sequence } (m_{1(2)}, m_{2(2)})\\
&\text{types in } [\theta_2, \theta_3] \text{ send message sequence } (m_{1(2)}, m_{2(3)})\\
&\text{types in } [\theta_3, 1] \text{ send message sequence } (m_{1(3)}, m_{2(4)}).
\end{aligned}$$

So types are partitioned into four intervals in stage 2, but in stage 1, the types in $[\theta_1, \theta_2]$ and $[\theta_2, \theta_3]$ pool together to send the same message $m_{1(2)}$. Since the signaling rule does not depend on the decision-maker's action at stage 1, the decision-maker will choose the following myopically optimal actions:

$$\begin{aligned}
y_{1(1)} &= y_{2(1)} = \frac{\theta_1}{2},\\
y_{1(2)} &= \frac{\theta_1 + \theta_3}{2}, \; y_{2(2)} = \frac{\theta_1 + \theta_2}{2}, \; y_{2(3)} = \frac{\theta_2 + \theta_3}{2},\\
y_{1(3)} &= y_{2(4)} = \frac{1 + \theta_3}{2}.
\end{aligned}$$

After any out-of-equilibrium message the decision-maker assigns probability one to the state belonging in $[0, \theta_1]$ inducing $y^{out} = \frac{\theta_1}{2}$. With these out-of-equilibrium beliefs it is immediate to see that no type has an incentive to deviate.

In equilibrium, type θ_1 is indifferent between action sequences $\{y_{1(1)}, y_{2(1)}\}$ and $\{y_{1(2)}, y_{2(2)}\}$, type θ_2 is indifferent between 2nd-period actions $y_{2(2)}$ and $y_{2(3)}$, and type θ_3 is indifferent between action sequences $\{y_{1(2)}, y_{2(3)}\}$ and $\{y_{1(3)}, y_{2(4)}\}$. Therefore, equilibrium cutoffs are the solution to the following system of equations:[20]

$$\begin{aligned}
2\left(\frac{\theta_1}{2} - \theta_1 - b\right)^2 - \left(\frac{\theta_1 + \theta_3}{2} - b - \theta_1\right)^2 - \left(\frac{\theta_1 + \theta_2}{2} - b - \theta_1\right)^2 &= 0,\\
\left(\frac{\theta_1 + \theta_2}{2} - b - \theta_2\right)^2 - \left(\frac{\theta_2 + \theta_3}{2} - b - \theta_2\right)^2 &= 0,\\
2\left(\frac{1 + \theta_3}{2} - b - \theta_3\right)^2 - \left(\frac{\theta_1 + \theta_3}{2} - b - \theta_3\right)^2 - \left(\frac{\theta_2 + \theta_3}{2} - b - \theta_3\right)^2 &= 0.
\end{aligned}$$

At $b = 0.08$, the only solution that gives numbers in $[0, 1]$ is $\theta_1 = 0.0056, \theta_2 = 0.015, \theta_3 = 0.345$, and the actions induced for $t = 1$ and for $t = 2$ are respectively given by $y_{1(1)} = y_{2(1)} = 0.00278$, $y_{1(2)} = 0.175$, $y_{2(2)} = 0.0105$, $y_{2(3)} = 0.18$ and $y_{1(3)} = y_{2(4)} = 0.673$. This implies the following total ex-ante expected utility for the expert -0.066, which is lower than $2(-0.033) = -0.0656$. The utility for the decision-maker is -0.053 which is lower than $2(-0.026) = 0.052$.

This example illustrates that although the interval is divided into more subintervals here, both players strictly worse off compared to the one where the most informative static equilibrium is played in the first period and babbling thereafter. The feature that less partitions lead to higher ex-ante welfare for both players also appears in example 1 of Blume, Board, and Kawamura (2007).

[20] It is trivial to check exactly as we did in previous examples that these indifference conditions suffice for global incentive compatibility.

D Proof of Theorem 1

We will prove by construction that a fully revealing equilibrium exists. We first choose the endpoints $\theta_1, \theta_2, \theta_3$ described in the proof outline: for any bias $b < \frac{1}{61}$, define $a_\gamma < 0$ by

$$(a_\gamma - 2 + 2e^{-a_\gamma})e^2 - a_\gamma = \frac{1}{b} \tag{13}$$

and then set

$$\theta_3 = \frac{1}{b} + a_\gamma,\ \theta_2 = \theta_3 - 2,\ \theta_1 = \theta_2 - \theta_3 e^{-2} \tag{14}$$

It will be convenient to describe types parametrically, via functions $x : [-2, 0] \to [0, \theta_1]$, $g : [-2, 0] \to [\theta_2, \theta_3]$, $z : [a_\gamma, 0] \to [\theta_1, \theta_2]$, and $h : [a_\gamma, 0] \to [\theta_3, \frac{1}{b}]$. Then, let $u_1(a, \alpha_a)$, $u_2(a, \alpha_a)$ denote the first, second recommendations of types $x(a), g(a)$ (for all $a \in [-2, 0]$), and let $v_1(a, \alpha_0), v_2(a, \alpha_0)$ denote the first, second recommendations of types $(z(a), h(a))$ (for all $a \in [a_\gamma, 0]$). With this notation, Groups I,II, III described in the text are as follows:

$$\begin{aligned}
\text{Group } I &= \{z(a), h(a) \mid a \in [a_\gamma, 0]\} \\
\text{Group } II &= \{x(a), g(a) \mid a \in [-2, 0],\ \text{and } \exists\ a' \in [a_\gamma, 0] \text{ with } v_1(a', \alpha_0) = u_1(a, \alpha_a)\} \\
\text{Group } III &= \{x(a), g(a) |\ a \in [-2, 0],\ \text{and } x(a), g(a) \notin \text{Group } II\}
\end{aligned}$$

In our proposed equilibrium construction, each Group I pair $\{z(a), h(a)\}$ recommends $v_1(a, \alpha_0)$ for $2\alpha_0$ periods, then $v_2(a, \alpha_0)$ for $T - \tau - 2\alpha_0$ periods, then reveals the truth at time $T - \tau$; each Group II pair $\{x(a), g(a)\}$ recommends $u_1(a, \alpha_a)$ for $2\alpha_a$ periods, then $u_2(a, \alpha_a)$ for $2(1 - \alpha_a)$ periods, then separates and reveals the truth for the final $T - 2$ periods; and moreover, the recommendation $u_1(a, \alpha_a)$ coincides with the recommendation $v_1(a', \alpha_0)$ of some Group I pair $\{z(a'), h(a')\}$. Group III is identical to Group II, except that their recommendations *do not* coincide with those of any Group I pair.

We also specify the following off-path strategy for the expert: if the decision-maker ever deviates, by rejecting a recommendation that the expert made, then (i) if the expert himself has *not* previously deviated: send no further recommendations (equivalently, repeat the current recommendation in all subsequent periods). And (ii) if the expert *has* observably deviated in the past, behave as if the deviation did not occur. (For example, if he sends the initial recommendation $u_1(0, \alpha_0)$ prescribed for types $\{x(0), g(0)\}$, but then follows this with anything other than recommendation $u_2(0, \alpha_0)$ at time $2\alpha_0$, subsequently behave as if the deviation never occurred and he indeed sent $u_2(0, \alpha_0)$ at time $2\alpha_0$).

D.1 Optimality for the Expert

We prove that the expert wishes to follow the prescribed recommendation strategy via three propositions. Proposition D1 specifies strategies and beliefs for the decision-maker such that the expert has no incentive to send an out-of-equilibrium recommendation sequence, so we need only make sure that he does not wish to mimic any other type. Proposition D2 shows that in the prescribed revelation phase, the expert indeed finds it optimal to reveal the truth, provided that there have been no previous deviations. It remains only to show that the expert has no incentive to deviate prior to the prescribed revelation phase - by mimicking the initial recommendations of some other type - which we show in Proposition D3.

We specify the following strategy and beliefs for the decision-maker:

If there are no detectable deviations by the expert (i.e., he sends the equilibrium recommendation sequence for some type $\theta \in [0, \frac{1}{b}]$), then follow all recommendations, using Bayes' rule to assign beliefs at each information set. Following deviations: (i) If the expert observably deviates at time 0 (sending an off-path initial recommendation), subsequently adopt the strategy/beliefs that would follow if the expert had instead sent the recommendation $u_1(0, \alpha_0)$ prescribed for types $\{x(0), g(0)\}$; (ii) If the expert observably deviates on his 2nd recommendation (i.e., if an initial recommendation

$u_1(a, \alpha_a)$ (or $v_1(a, \alpha_0)$) is followed by something *other* than $u_2(a, \alpha_a)$ (or $v_2(a, \alpha_0)$), ignore it as an error, and subsequently adopt the strategy/beliefs that would follow had the deviation not occurred; (iii) If the expert deviates observably in the revelation phase, ignore it as an error, assigning probability 1 to the lowest type in the current information set, and accordingly choosing this as the myopically optimal action; (iv) And finally, if the decision-maker himself deviates, rejecting some recommendation by the expert, then he subsequently maintains the current (at time of deviation) beliefs, anticipating that the expert will subsequently repeat the current (at time of deviation) recommendation, and ignoring any other recommendations as errors.

D.1.1 Expert Optimality: Off-Path Behavior

Proposition D1: Under the above strategy and beliefs prescribed for the decision-maker, the expert has no incentive to choose an off-path recommendation sequence.

Proof of Proposition D1: Follows trivially from the specified strategy and beliefs for the decision-maker: (i) a deviation at time zero is equivalent to mimicking type $x(0)$ (who recommends $u_1(0, \alpha_0)$ at time $t = 0$); (ii) a deviation on the 2nd recommendation has no effect, since the decision-maker ignores it; (iii) a deviation in the revelation phase, if there have been no previous deviations, is equivalent to mimicking the strategy of the lowest type in the decision-maker's current (pre-revelation) information set; and (iv) if the decision-maker *has* previously deviated, then (by point (iv) of the above strategy-belief specification) he will chose whichever action was myopically optimal at the time of deviation, regardless of the expert's message; therefore, babbling is optimal for the expert, since his message has no effect on the decision-maker's action. ■

D.1.2 Expert Optimality: Truth Revelation Phase

Proposition D2: In the prescribed revelation phase, (i) if there have been no previous deviations by the decision-maker, then the expert finds it optimal to reveal the truth; (ii) if the decision-maker has ever deviated, then the expert finds it optimal to babble (e.g. by remaining silent).

Proof of Proposition D2: Part (ii) follows immediately from Proposition D1 (iv). For part (i): our specification of the expert strategy is such that at time $2\alpha_0$, the decision-maker's information set contains at most two types: either a pair $\{x(a), g(a)\}$ (in which case the truth should be revealed at time 2, and the DM plans to choose $g(a)$ if the expert recommends $g(a)$, $x(a)$ otherwise), or a pair $\{z(a), h(a)\}$ (in which case the truth should be revealed at time $T - \tau$, and the DM plans to choose $h(a)$ if the expert recommends it, $z(a)$ otherwise). So, it suffices to show that each type would rather tell the truth than mimic his partner: in our rescaled type space, this requires simply that all paired types be at least 2 units apart (so that $(\theta - \theta - 1)^2 \le (p(\theta) - \theta - 1)^2$ for any pair $\{\theta, p(\theta)\}$). By (15) we have

$$\begin{aligned} \min_{a \in [-2,0]} |g(a) - x(a)| &= \theta_2 - \theta_1 \\ \min_{a \in [a_\gamma,0]} |h(a) - z(a)| &= \theta_3 - \theta_2 \end{aligned}$$

And by (14), $\theta_3 - \theta_2 = 2$, and $\theta_2 - \theta_1 = (a_\gamma - 2 + 2e^{-a_\gamma})$, which is greater than 2 whenever $a_\gamma < -.8951 \Leftrightarrow b < \frac{1}{15.67}$ (using (13)). This is in fact all that is needed for the construction to work for the expert, but we specify $b < \frac{1}{61}$ in (13) to make the construction work for the decision-maker. ■

D.1.3 Expert Optimality: Initial Recommendations

Propositions D1,D2 imply that once the expert has sent the initial recommendation (u_1 or v_1) prescribed for some type θ, it is optimal to follow also the continuation recommendations prescribed for that type. So, the only time when it could possibly be profitable to deviate is at time $t = 0$: we need to make sure that each type θ prefers to send the proposed equilibrium sequence of recommendations, rather than the sequence prescribed for any other type θ'.[21] We now choose parametrizations of functions x, g, z, h, along with action function u_1, u_2, v_1, v_2, which guarantee that the expert indeed finds it optimal to send the prescribed initial recommendation:

Proposition D3: Let the action functions and type parametrizations be as follows:

$$x(a) = \theta_3 + a - \theta_3 e^a, g(a) = \theta_3 + a,\ z(a) = \frac{1}{b} + a - 2e^{a-a_\gamma},\ h(a) = \frac{1}{b} + a \tag{15}$$

$$u_1(a, \alpha_a) = \theta_3 + K - \frac{T-2}{2}a - \sqrt{\frac{1-\alpha_a}{\alpha_a}}\sqrt{T-2}\sqrt{C_u + a\left(K - \frac{T}{4}a\right)} \tag{16}$$

$$u_2(a, \alpha_a) = \theta_3 + K - \frac{T-2}{2}a + \sqrt{\frac{\alpha_a}{1-\alpha_a}}\sqrt{T-2}\sqrt{C_u + a\left(K - \frac{T}{4}a\right)} \tag{17}$$

$$v_1(a, \alpha_0) = \theta_3 + \frac{2K - \tau(a-a_\gamma)}{T-\tau} - \frac{\sqrt{\frac{\tau(T-\tau-2\alpha_0)}{\alpha_0}}\sqrt{\frac{(T-\tau)(T-2)}{\tau}C_u + \left(\frac{T-\tau-2}{\tau}\right)K^2 + 2K(a-a_\gamma) - \frac{T}{2}(a-a_\gamma)^2}}{T-\tau} \tag{18}$$

$$v_2(a, \alpha_0) = \theta_3 + \frac{2K - \tau(a-a_\gamma)}{T-\tau} + \frac{\sqrt{\frac{4\tau\alpha_0}{T-\tau-2\alpha_0}}\sqrt{\frac{(T-\tau)(T-2)}{\tau}C_u + \left(\frac{T-\tau-2}{\tau}\right)K^2 + 2K(a-a_\gamma) - \frac{T}{2}(a-a_\gamma)^2}}{T-\tau} \tag{19}$$

for constants C_u, K, and for now taking T, α_0, α_a as given (T is the horizon, and α_a, α_0 relate to the duration of recommendations u_1, v_1 as described in the strategies above). Also set τ (length of the revelation phase for types in Group I) according to

$$\frac{\tau}{T-2} = \beta \equiv \frac{(\theta_2 - \theta_1)(\theta_2 - \theta_1 - 2)}{(\frac{1}{b} - \theta_1)(\frac{1}{b} - \theta_1 - 2)} \tag{20}$$

Then, for all types $\theta, \theta' \in [0, \frac{1}{b}]$, expert type θ prefers his equilibrium recommendation sequence to that sent by type θ', and in particular has no incentive to deviate at time $t = 0$.

Proof of Proposition D3:

Let $D_u(\theta'|\theta)$ denote the disutility to type θ from following the recommendation sequence prescribed for a type $\theta' \in [0, \theta_1] \cup [\theta_2, \theta_3]$, and let $D_v(\theta'|\theta)$ denote the disutility to type θ from following the strategy prescribed for a type $\theta' \in [\theta_1, \theta_2] \cup [\theta_3, \frac{1}{b}]$. The proof proceeds through two main Lemmas. Lemma D3.1 proves that the expert strategy is locally incentive compatible: for each interval $[\theta_i, \theta_{i+1}]$, $i \in \{0, 1, 2, 3\}$, no expert type $\theta \in [\theta_i, \theta_{i+1}]$ wishes to mimic any other type $\theta' \in [\theta_i, \theta_{i+1}]$ from the same interval. Lemma D3.2 proves that the expert strategy is also globally incentive compatible: no expert type wishes to mimic any type θ' from any other interval. The proofs will use calculations obtained below in Lemmas D3.3 and D3.4.

Lemma D3.1 (Local IC): For each interval $[\theta_i, \theta_{i+1}]$, with $i = 0, 1, 2, 3$, and any pair of types $\theta, \theta' \in [\theta_i, \theta_{i+1}]$, the disutility to type θ from mimicking type θ' is (weakly) *increasing* in $\left|\theta' - \theta\right|$, thus minimized when $\left|\theta' - \theta\right| = 0$. Therefore, for each $\theta \in [\theta_i, \theta_{i+1}]$, truth-telling is (weakly) better than mimicking any other type in the interval.

Proof of Lemma D3.1:

Differentiating disutility expressions (25), (26) , (27) , and (28) (obtained below in Lemma D3.3)

[21] This is what the text refers to as "providing incentives to join the right separable group". We need to make sure, for example, that type $\theta = 0$ prefers to induce the action sequence $(u_1(0, \alpha_0), u_2(0, \alpha_0), 0)$, rather than e.g. the sequence that type $\theta' \neq 0$ is supposed to send; by Propositions D1,D2, the choice to follow a different recommendation sequence can only be made at time $t = 0$.

gives

$$\begin{aligned}
\frac{dD_u(x(a)|\theta)}{dx(a)} &= \frac{dD_u(g(a)|\theta)}{dx(a)} + \frac{2(T-2)\,(x(a)-\theta-1)\,x'(a) - 2(T-2)\,(g(a)-\theta-1)\,g'(a)}{x'(a)} \\
&= 0 + 2(T-2)\left(\theta_3 + a - \theta_3 e^a - \theta - 1 - \frac{\theta_3 + a - \theta - 1}{1-\theta_3 e^a}\right) \text{ (by (15))} \\
&= 2(T-2)\left(\frac{\theta_3 e^a}{\theta_3 e^a - 1}\right)(x(a)-\theta) \qquad (21) \\
\frac{dD_v(z(a)|\theta)}{dz(a)} &= \frac{dD_v(h(a)|\theta)/da}{z'(a)} + \frac{2\tau\,(z(a)-\theta-1)\,z'(a) - 2\tau\,(h(a)-\theta-1)\,h'(a)}{z'(a)} \\
&= 2\tau\left(\frac{2e^{a-a_\gamma}}{2e^{a-a_\gamma}-1}\right)(z(a)-\theta) \qquad (22) \\
\frac{dD_u(g(a)|\theta)}{dg(a)} &= 0 \qquad (23) \\
\frac{dD_v(h(a)|\theta)}{dh(a)} &= 0 \qquad (24)
\end{aligned}$$

Consider first a type $\theta \in [0, \theta_1]$. By (21) , noting that $\frac{\theta_3 e^a}{\theta_3 e^a - 1} > 0$ (since $\theta_3 e^a \geq \theta_3 e^{-2} = \theta_2 - \theta_1 \geq 8$, by Proposition D2), we see that $\frac{dD_u(x(a)|\theta)}{dx(a)}$ has the same sign as $(x(a) - \theta)$. So if $x(a) - \theta > 0$, then $D_u(x(a)|\theta)$ is *increasing* in $x(a)$, thus increasing in $(x(a) - \theta)$; while if $x(a) - \theta < 0$, then $D_u(x(a)|\theta)$ is increasing in $(-x(a))$, thus increasing in $\theta - x(a)$. Combined, these establish that $D_u(x(a)|\theta)$ is strictly increasing in $|x(a) - \theta|$, as desired.

Next consider a type $\theta \in [\theta_1, \theta_2]$. By (22) , noting that $\left(\frac{2e^{a-a_\gamma}}{2e^{a-a_\gamma}-1}\right) > 0$ (since $a \in [a_\gamma, 0]$ implies $2e^{a-a_\gamma} \geq 2$), we see that $\frac{dD_v(z(a)|\theta)}{dz(a)}$ has the same sign as $z(a) - \theta$, and is thus positive (disutility increasing in $z(a) - \theta$) if $z(a) > \theta$, and negative (disutility increasing in $\theta - z(a)$) if $z(a) < \theta$. Combined, these establish that $D_u(z(a)|\theta)$ is strictly increasing in $|z(a) - \theta|$, as desired.

By (23) and (24) , the disutility to type θ from mimicking a type $g(a) \in [\theta_2, \theta_3]$ or $h(a) \in [\theta_3, \frac{1}{b}]$ is independent of the particular type $g(a), h(a)$ chosen. Thus, $D_u(g(a)|\theta)$ is weakly increasing (in fact constant) in $|g(a) - \theta|$, and $D_u(h(a)|\theta)$ is weakly increasing (constant) in $|h(a) - \theta|$, completing the proof. ■

Lemma D3.2: For every interval $[\theta_i, \theta_{i+1}]$ $(i = 0, 1, 2, 3)$, and every $\theta \in [\theta_i, \theta_{i+1}]$, following the prescribed (truthful) recommendation sequence is better than mimicking any type θ' drawn from any *other* interval $[\theta_j, \theta_{j+1}]$ with $j \neq i$.

Proof of Lemma D3.2:

Consider first a type $\theta \in [0, \theta_1]$. By Lemma D3.1, truth-telling is better than mimicking any other type $\theta' \in [0, \theta_1]$, in particular type $\theta_1 = x(-2)$. By Lemma D3.4 (i) (below), type $\theta \in [0, \theta_1]$ prefers type $x(-2)$'s sequence to type $z(0)$'s sequence; and by Lemma D3.1, it is better to mimic type $z(0) = \theta_1$, than any other type $z(a) \in (\theta_1, \theta_2]$ (since $z(a) > \theta$ implies that $D_v(z(a)|\theta)$ is increasing in $z(a) - \theta$); together, these establish that mimicking a type $\theta' \in [\theta_1, \theta_2]$ is not optimal. By Lemma D3.4 (ii), type $\theta \leq \theta_2$ prefers $z(a_\gamma)$'s sequence (right endpoint of $[\theta_1, \theta_2]$) to $g(-2)$'s sequence (left endpoint of $[\theta_2, \theta_3]$); and by Lemma D3.1, disutility to type θ from mimicking type $g(a) \in [\theta_2, \theta_3]$ is independent of a; together, this implies that type θ also does not want to mimic any type $g(a) \in [\theta_2, \theta_3]$. And finally, by Lemma D3.4 (iii), type $\theta \leq \theta_2$ prefers the sequence prescribed for type $g(0)$ (right endpoint of $[\theta_2, \theta_3]$) to that prescribed for type $h(a_\gamma)$ (left endpoint of $[\theta_3, \frac{1}{b}]$), which (by Lemma D3.1) yields the same utility as mimicking any other type $h(a) \in [\theta_3, \frac{1}{b}]$, thus it is not optimal to mimic any type $h(a) \in [\theta_3, \frac{1}{b}]$. This establishes that type $\theta \in [0, \theta_1]$ does not wish to mimic any type θ' from any other interval.

Next consider type $\theta \in [\theta_1, \theta_2]$. By Lemma D3.1, truth-telling is better than mimicking any other type $z(a) \in [\theta_1, \theta_2]$, in particular type $z(0) = \theta_1$; by Lemma D3.4 (i), type $\theta \geq \theta_1$ prefers the sequence prescribed for type $z(0)$, to that prescribed for type $x(-2)$; and by Lemma D3.1, it is better to mimic $x(-2)$ (right endpoint of $[0,\theta_1]$) than any other type $x(a) \in [0,\theta_1]$, since $D_u(x(a)|\theta)$ is increasing in $|\theta - x(a)|$ and we have here $\theta > x(a)$; together, this implies that type θ does not wish to mimic any type $\theta' \in [0,\theta_1]$. The proof that he doesn't wish to mimic any type $g(a) \in [\theta_2,\theta_3]$ or $h(a) \in [\theta_3, \frac{1}{b}]$ is identical to the one given in the previous paragraph.

Now consider a type $\theta \in [\theta_2,\theta_3]$. As explained in the previous two paragraphs, following the truthful recommendation sequence yields the same utility as mimicking any other type $g(a) \in [\theta_2,\theta_3]$ or $h(a) \in [\theta_3,\frac{1}{b}]$, so we just need to make sure that it is not optimal to mimic types $\theta' \in [0,\theta_1] \cup [\theta_1,\theta_2]$. By Lemma D3.4 (ii), type $\theta \geq \theta_2$ prefers type $g(-2)$'s sequence (left endpoint of $[\theta_2,\theta_3]$) to type $z(a_\gamma)$'s sequence (right endpoint of $[\theta_1,\theta_2]$); by Lemma D3.1, such a type $\theta \geq \theta_2$ also prefers type $z(a_\gamma)$'s sequence to the one prescribed for any other (further-away) type $z(a) \in [\theta_1,\theta_2]$; combined, this establishes that mimicking a type $z(a) \in [\theta_1,\theta_2]$ is not optimal. By Lemma D3.4 (i), it is better to mimic type $z(0)$'s sequence than $x(-2)$'s sequence, which in turn is better (by Lemma D3.1) than any other type $x(a)$'s sequence. Thus, it is not optimal to mimic any type $x(a) \in [0,\theta_1]$, completing the proof for types $\theta \in [\theta_2,\theta_3]$.

The argument that types $\theta \in [\theta_3,\frac{1}{b}]$ don't wish to mimic types from other intervals is identical to the proof in the previous paragraph (for types $\theta \in [\theta_2,\theta_3]$).

This completes the proof of Lemma D3.2. ■

D.1.4 Expert Optimality: Preliminary Calculations

Lemma D3.3: Given the type parametrizations and action functions given in Proposition D3, disutility expressions $D_u(\theta'|\theta)$, $D_v(\theta'|\theta)$ are given by

$$D_u(x(a)|\theta) = D_u(g(a)|\theta) + (T-2)(x(a)-\theta-1)^2 - (T-2)(g(a)-\theta-1)^2 \tag{25}$$

$$D_u(g(a)|\theta) = T(\theta_3-\theta-1)^2 + 4K(\theta_3-\theta-1) + 2K^2 + 2(T-2)C_u \tag{26}$$

$$D_v(z(a)|\theta) = D_v(h(a)|\theta) - 2\tau(h(a)-z(a))\left(\frac{h(a)+z(a)}{2} - \theta - 1\right) \tag{27}$$

$$D_v(h(a)|\theta) = 2K^2 + 2(T-2)C_u + 4(\theta_3-\theta-1)K + T(\theta-\theta_3+1)^2 \tag{28}$$

Proof of Lemma D3.3:

The disutility $D_u(g(a)|\theta)$ to expert type θ from following the strategy prescribed for type $g(a) \in [\theta_2,\theta_3]$, using (16), (17), is

$$\begin{aligned}
& 2\alpha_a(u_1(a,\alpha_a)-\theta-1)^2 + 2(1-\alpha_a)(u_2(a,\alpha_a)-\theta-1)^2 + (T-2)(g(a)-\theta-1)^2 \qquad (29)\\
= \; & 2(1-\alpha_a)\left(\theta_3 + K - \frac{T-2}{2}a - \theta - 1 + \sqrt{\frac{\alpha_a}{1-\alpha_a}}\sqrt{T-2}\sqrt{C_u + a\left(K - \frac{T}{4}a\right)}\right)\\
& +2\alpha_a\left(\theta_3 + K - \frac{T-2}{2}a - \theta - 1 - \sqrt{\frac{1-\alpha_a}{\alpha_a}}\sqrt{T-2}\sqrt{C_u + a\left(K - \frac{T}{4}a\right)}\right)^2\\
& +(T-2)(x(a)-\theta-1)^2
\end{aligned}$$

Expanding gives [22]

$$D_u(g(a)|\theta) = 2\left(\theta_3 - \theta - 1 + K - \frac{T-2}{2}a\right)^2 + 2(T-2)\left(C_u + a\left(K - \frac{T}{4}a\right)\right) + (T-2)(\theta_3 + a - \theta - 1)^2$$

If we now expand this expression, the coefficients on a^2, a reduce to zero (this is due to our choice

[22] Note that the coefficients on the square roots were chosen to make this independent of α_a, as mentioned in Appendix D3.1 (following (89)).

$g(a) = \theta_3 + a$), leaving

$$D_u(g(a)|\theta) = 2(\theta_3 - \theta - 1 + K)^2 + 2(T-2)C_u + (T-2)(\theta_3 - \theta - 1)^2$$

which rearranges to expression (26).

The disutility to type θ from following the strategy prescribed for type $x(a) \in [0, \theta_1]$, $D_u(x(a)|\theta)$, is given by (29), just replacing $g(a)$ with $x(a)$: this gives the desired expression (25).

The disutility to type θ from following the strategy prescribed for type $h(a) \in [\theta_3, \frac{1}{b}]$ is

$$D_v(h(a)|\theta) = 2\alpha_0 (v_1(a, \alpha_0) - \theta - 1)^2 + (T - \tau - 2\alpha_0)(v_2(a, \alpha_0) - \theta - 1)^2 + \tau (h(a) - \theta - 1)^2$$

Again, the coefficients on the square root terms in v_1, v_2 were chosen to make both disutility and average action independent of α_0 : substituting (18), (19) into the above expression and expanding, we get

$$\begin{aligned} D_v(h(a)|\theta) &= (T-\tau)\left(\theta_3 + \frac{2K + \tau a_\gamma}{T-\tau} - \frac{\tau}{T-\tau} a - \theta - 1\right)^2 + \tau (h(a) - \theta - 1)^2 \\ &\quad + 2\tau \left(\frac{(T-2)}{\tau} C_u + \frac{\left(\frac{T-\tau-2}{\tau}\right) K^2 + 2K(a - a_\gamma) - \frac{T}{2}(a - a_\gamma)^2}{T-\tau}\right). \end{aligned}$$

Substituting in $h(a) = \frac{1}{b} + a$, using $\theta_3 = \frac{1}{b} + a_\gamma$, and expanding, we find (this is due to our choice $h'(a) = 1$) that the coefficients on both a^2, a reduce to zero, so that our expression simplifies further to (28). Finally, using the fact that the strategies for types $h(a), z(a)$ differ only in the revelation phase, so

$$D_v(h(a)|\theta) - D_v(z(a)|\theta) = \tau (h(a) - \theta - 1)^2 - \tau (z(a) - \theta - 1)^2$$

we obtain (27). This completes the proof. ■

Lemma D3.4: (utility at the endpoints)

Under the expressions given in Proposition D3, we have that (i) endpoint $\theta_1 = x(-2) = z(0)$: type θ (weakly) prefers type $x(-2)$'s recommendation sequence to $z(0)$'s sequence iff $\theta \in [0, \theta_1]$; (ii) endpoint $\theta_2 = z(a_\gamma) = g(-2)$: type θ prefers $z(a_\gamma)$'s sequence to $g(-2)$'s sequence iff $\theta \in [0, \theta_2]$; and (iii) endpoint $\theta_3 = g(0) = h(a_\gamma)$: all types are indifferent between the sequences sent by types $g(0), h(a_\gamma)$.[23]

Proof of Lemma D3.4:

At $\theta_1 = x(-2) = z(0)$, we have (using the expressions in Lemma D3.3 and simplifying) that $D_v(z(0)|\theta) - D_u(x(-2)|\theta)$ equals

$$(T-2)(\theta_2 - \theta_1)(\theta_1 + \theta_2 - 2\theta - 2) - \tau \left(\frac{1}{b} - \theta_1\right)\left(\frac{1}{b} + \theta_1 - 2\theta - 2\right)$$

Using $\tau(\frac{1}{b} - \theta_1) = (T-2)\frac{(\theta_2 - \theta_1)(\theta_2 - \theta_1 - 2)}{(\frac{1}{b} - \theta_1 - 2)}$ (by (20)), this simplifies to

$$D_v(z(0)|\theta) - D_u(x(-2)|\theta) = 2(T-2)(\theta_2 - \theta_1)(\frac{1}{b} - \theta_2)\left(\frac{\theta_1 - \theta}{\frac{1}{b} - \theta_1 - 2}\right) \tag{30}$$

This is negative, meaning that type θ prefers $z(0)$'s strategy to $x(-2)$'s strategy, iff $\theta > \theta_1$, thus

[23] For example, consider part (i). In our construction, type θ_1 is both the right endpoint $x(-2)$ of the interval $[0, \theta_1]$, and the left endpoint $z(0)$ of the interval $[\theta_1, \theta_2]$: part (i) says that type θ_1 is indifferent between the two sequences prescribed for his type, and that everyone below θ_1 prefers the strategy of type $x(-2)$, everyone above θ_1 prefers the strategy of type $z(0)$.

establishing part (i).

At $\theta_2 = g(-2) = z(a_\gamma)$, we have (by (27) and (26)

$$\begin{aligned} D_u(g(-2)|\theta) - D_v(z(a_\gamma)|\theta) &= \tau(\theta_3 - \theta_2)(\theta_3 + \theta_2 - 2\theta - 2) \\ &= 4\tau(\theta_2 - \theta) \text{ (using } \theta_3 - \theta_2 = 2) \end{aligned} \tag{31}$$

This is negative, meaning that type θ prefers $g(-2)$'s strategy to $z(a_\gamma)$'s strategy, iff $\theta > \theta_2$, proving part (ii).

At θ_3, we have (by (26) and (28)),

$$D_u(g(0)|\theta) - D_v(h(a_\gamma)|\theta) = 0 \tag{32}$$

so that all types are indifferent between the strategies prescribed for type $g(0) = \theta_3$, $h(a_\gamma) = \theta_3$, as desired to complete the proof. ■

D.2 Optimality for the decision-maker

Let the expert strategy be as specified in the previous subsection, using the action functions and parametrizations from Proposition D3, with $\tau = \beta(T-2)$ as in (20). Recall that we had the following free parameters: constants K, C_u, the horizon T, a number $\alpha_0 \in [0,1]$, and numbers $\alpha_a \in [0,1]$ $\forall a \in [-2,0]$. We wish to show that the specified strategies constitute a fully revealing PBE: since we established expert optimality in the previous section, and since the beliefs and off-path strategies specified for the decision-maker (see Proposition D1) trivially satisfy all PBE requirements, all that remains is to prove that the decision-maker's on-path strategy is optimal.

Recall the timeline presented in Figure 1 (Section 4). It is immediately clear that during the revelation phase, when the expert's recommendation is equal (with probability 1) to the true state, the decision-maker indeed finds it optimal to follow the recommendation. In between time $2\alpha_0$ (when Group I separates from Group II by switching to v_2) and the revelation phase, no new information is revealed, but any failure by the decision-maker to follow the expert's recommendations will result in the expert subsequently babbling, rather than revealing the truth. So, the best possible deviation is to choose the myopically optimal action in all subsequent periods, and the strongest incentive to do so occurs at the *earliest* time that new information is revealed (when the "reward phase", revelation of the truth, is furthest away). So to prove decision-maker optimality, we need only show that he does not want to deviate to the myopically optimal action either at time $t = 0$, or at time $t = 2\alpha_0$ if he learns that he is in fact facing a Group I pair. We summarize this as:

Observation D4: If the decision-maker cannot gain by deviating at time $t \in \{0, 2\alpha_0\}$, then the prescribed strategy is optimal.

D.2.1 Optimality for the decision-maker: Outline and Parameter Choices

Given T, α_0 (and with $\tau = \beta(T-2)$ as specified by (20), we set the constants C_u, K according to

$$C_u = \frac{1-\alpha_0}{\alpha_0}\frac{K^2}{T-2} \tag{33}$$

$$K = \frac{\alpha_0 \tau a_\gamma \left(1 + \sqrt{\frac{(T-2\alpha_0)(T-\tau)}{2\tau\alpha_0}}\right)}{(T - \tau - 2\alpha_0)} \tag{34}$$

And choose a horizon $T \in [T_{\min}, T_{\max}]$, where[24]

$$T_{\min} = \begin{cases} 7 & \text{if } \beta \in [0.4173, 0.50102) \\ \frac{5-2\beta}{1-\beta} & \text{if } \beta \in [0.50102, 0.79202) \\ \frac{5.4748\beta}{2.7374\beta - 1.7374} & \text{if } \beta \in [0.79202, 0.95203) \\ 6 & \text{if } \beta \geq 0.95203 \end{cases}, \quad T_{\max} = \begin{cases} 7 & \text{if } \beta \in [0.4173, 0.50102) \\ \frac{8-2\beta}{1-\beta} & \text{if } \beta \in [0.50102, 0.79202) \\ \frac{4-2\beta}{1-\beta} & \text{if } \beta \in [0.79202, 0.90913) \\ \frac{12.005\beta}{6.0025\beta - 5.0025} & \text{if } \beta \geq 0.90913 \end{cases} \tag{35}$$

All proofs use α_0 near 1 when $\beta a_\gamma^2 < 8$, and α_0 near 0 when $\beta a_\gamma^2 > 8$. The parameter α_a (relating to the time $2\alpha_a$ at which Group II and III pairs $\{x(a), g(a)\}$ switch from u_1 to u_2) may depend on the specific pair $\{x(a), g(a)\}$, but is chosen in Lemma D7.1 to satisfy $\alpha_0 \leq \alpha_a \leq 1\ \forall a$. In particular, we prove in Lemma D7.1 that our parameter choices guarantee that all action functions are real-valued, and that every recommendation $v_1(a, \alpha_0)$ sent by a Group I pair, is also sent by some Group II pair $\{x(a), g(a)\}$, for at least as long. The need for this overlap of u_1, v_1 is as follows: the decision-maker's gain to following the expert's advice is large at information sets containing only a Group II or III pair, but would be negative at time $t = 0$, for *all* priors, if his information set contained only a Group I pair $\{z(a), h(a)\}$ close to the endpoint pair $\{z(a_\gamma), h(a_\gamma)\} = \{\theta_2, \theta_3\}$.[25] So, for the equilibrium to work, we need to make sure that each Group I pair's initial message coincides with that of a Group II pair, and then ensure (via the prior and construction details) that the weight the decision-maker places on the Group II pair is high enough to make him want to follow the recommendation. .

Proposition D6 shows that for a range of priors, the decision-maker's gain to deviating at time $t = 0$ (or later) is strictly negative at any information set containing only a group II or III pair $\{x(a), g(a)\}$, so long as his posterior assigns a probability to type $x(a)$ which lies within ε of some number $p_a^* \in (0, 1)$ (a sufficient condition is that he assigns a probability between 0.3 and 0.7 to each type). This then implies also that at time $t = 0$, if he gets a message $v_1(a, \alpha_0)$ which could have been sent by either a Group II pair (in which case he wants to follow the advice) or a Group I pair (in which case he might want to reject the advice), he will find it optimal to follow the recommendation as long as his posterior beliefs assign a high enough weight to the Group II pair, so we conclude that there exist beliefs for which the decision-maker has no incentive to deviate at time $t = 0$. It is also Proposition D6 that places an upper bound on the biases b for which the equilibrium works.

Proposition D5 shows that if the expert sends a message $v_2(a, \alpha_0)$ at time $t = 2\alpha_0$, thus revealing to the decision-maker that he is facing a Group I pair $\{z(a), h(a)\}$, then there exists an interval of posteriors on each type for which the decision-maker will find it optimal to choose the action $v_2(a, \alpha_0)$.

Proposition D7 completes the proof, by proving that there exists an open set of probability distributions over the state space which generate the posteriors needed in Propositions D5, D6.

Before proceeding with the proof, we briefly comment on the timeline. First, note that Theorem 1 places only a lower bound on the horizon T^*, whereas the constraint (35) in fact also places an *upper* bound on the horizon. However, the construction may trivially be extended for larger horizons in two ways: (i) add a babbling phase at the beginning; or (ii) scale everything up. All that matters is the ratios – the duration of each action phase relative to the horizon– so e.g. the analysis for a T-period equilibrium (where Groups II, III reveal the truth at time 2, Group I at time $T - \tau$) is identical to the analysis for a $T\lambda$-period equilibrium (with Groups II, III revealing the truth at time 2λ, Group I at time $(T - \tau)\lambda$, for λ any positive number).[26] It should also be noted that, while (35) can always be satisfied by an integer T, the times at which the DM is instructed to

[24] The previous version of the paper used the same bounds for $\beta > 0.79202$, but used the exact horizon $T = \frac{4-2\beta}{1-\beta}$ on the range $\beta \in [0.4173, 0.79202]$. We have modified the horizon, to guarantee that (i) T may be chosen to be an integer; and (ii) to expand the set of priors (from "infinite" to "open") for which the construction works.

[25] It may in fact be shown (see Section D.3.2) that this is necessarily true of any fully revealing construction: the expert's local + global IC constraints imply a sufficiently large distortion in some interval of types' (and their partners') recommendations that, if the DM were certain that he was facing one of these pairs, he would rather forego learning the exact truth than follow their advice.

[26] This follows immediately from the derivations in Appendix D3.

change his action – namely, times $2\alpha_0, 2\alpha_a, T-\tau$ – are not necessarily integers in our construction. In a continuous-time setting, where action changes can be made frequently, this clearly poses no problem. At the end of this proof (Section D.2.6), we explain how to handle integer constraints if time is discrete.

D.2.2 Optimality for the decision-maker: Deviations at time $t = 2\alpha_0$

Proposition D5: Fix $a_\gamma \leq -1.773$, choose parameters C_u, K, T as specified by $(33), (34)$, and (35), and define $\Delta \equiv \frac{T-\tau}{2} \Leftrightarrow_{\text{via (20)}} T = \frac{2(\Delta-\beta)}{1-\beta}$. Suppose that the decision-maker receives recommendation $v_2(a,\alpha_0)$ at time $t = 2\alpha_0$ for some $a \in [a_\gamma, 0]$, and assigns probabilities $q_a, 1-q_a$ to the two types $z(a), h(a)$ in his information set. Then: (i) if $\beta a_\gamma^2 > 8$, there exist numbers $\underline{\alpha}_0 < 1$ and $\varepsilon \geq 0.25$, and a continuous function $q_a^* : [a_\gamma, 0] \to (0.3, 0.6)$, such that the DM's gain to deviating is strictly negative whenever $\alpha_0 \geq \underline{\alpha}_0$ and $q_a \in (q_a^* - \varepsilon, q_a^* + \varepsilon)$; (ii) if $\beta a_\gamma^2 \leq 8$, there exist numbers $\overline{\alpha}_0 < 1$ and $\varepsilon \geq 0.145$, and a continuous function $q_a^* : [a_\gamma, 0] \to (0.2, 0.7)$ such that the DM's gain to deviating is strictly negative whenever $\alpha_0 \leq \overline{\alpha}_0$ and $q_a \in (q_a^* - \varepsilon, q_a^* + \varepsilon)$.

Proof of Proposition D5:

If the decision-maker follows recommendation $v_2(a,\alpha_0)$ (expecting to choose this action until time $T-\tau$, then learn the truth), his expected disutility is

$$(T-\tau-2\alpha_0)\left(q_a\left(v_2(a,\alpha_0)-z(a)\right)^2 + (1-q_a)\left(v_2(a,\alpha_0)-h(a)\right)^2\right) + \tau(0)$$

The best possible deviation is to instead choose myopically optimal action $q_a z(a) + (1-q_a)h(a)$ in all remaining $T - 2\alpha_0$ periods, for disutility

$$\begin{aligned}
& (T-2\alpha_a)\left(q_a\left(q_a z(a) + (1-q_a)h(a) - z(a)\right)^2 + (1-q_a)\left(q_a z(a) + (1-q_a)h(a) - h(a)\right)^2\right) \\
= \; & (T-2\alpha) q_a (1-q_a)\left(h(a)-z(a)\right)^2
\end{aligned}$$

So, the gain to deviating is negative at any belief q_a satisfying the following inequality:

$$\begin{aligned}
0 \; > \; & \left(q_a\left(v_2(a,\alpha_0)-z(a)\right)^2 + (1-q_a)\left(v_2(a,\alpha_0)-h(a)\right)^2\right) - \frac{(T-2\alpha_0)}{(T-\tau-2\alpha_0)} q_a(1-q_a)\left(h(a)-z(a)\right)^2 \\
= \; & q_a\left(2\left(\frac{v_2(a,\alpha_0)-h(a)}{h(a)-z(a)}\right)+1\right) + \left(\frac{v_2(a,\alpha_0)-h(a)}{h(a)-z(a)}\right)^2 - \phi^2 q_a(1-q_a), \qquad (36)
\end{aligned}$$

$$\text{where } \phi^2 \equiv \frac{T-2\alpha_0}{T-\tau-2\alpha_0} \qquad (37)$$

Solving, we need $q_a \in (q_a^* - \varepsilon_a, q_a^* + \varepsilon_a)$, where

$$q_a^* = \frac{\phi^2 - 1 - 2\left(\frac{v_2(a,\alpha_0)-h(a)}{h(a)-z(a)}\right)}{2\phi^2} \qquad (38)$$

$$\varepsilon_a = \frac{\sqrt{\phi^2-1}}{2\phi^2}\sqrt{\left(\phi - 1 - 2\left(\frac{v_2(a,\alpha_0)-h(a)}{h(a)-z(a)}\right)\right)\left(\phi + 1 + 2\left(\frac{v_2(a,\alpha_0)-h(a)}{h(a)-z(a)}\right)\right)} \qquad (39)$$

By (19), (15), and (37), we have that $\frac{v_2(a,\alpha_0)-h(a)}{h(a)-z(a)}$ is continuous in both a and α_0, and ϕ is continuous in α_0; this establishes the desired continuity of q_a^* in a, and also implies that q_a^*, ε_a are both continuous in α_0. Then, to complete the proof, it is sufficient to show that (i) if $\beta a_\gamma^2 > 8$, then, in the limit as $\alpha_0 \to 0$, the value q_a^* in (38) lies in $(0.3, 0.6)$ $\forall a \in [a_\gamma, 0]$, and the value ε_a in (39) is greater than 0.25 $\forall a \in [a_\gamma, 0]$; (ii) if $\beta a_\gamma^2 \leq 8$, then, in the limit as $\alpha_0 \to 1$, the value q_a^* in

(38) lies in $(0.2, 0.7)$ $\forall a \in [a_\gamma, 0]$, and the value ε_a in (39) is greater than 0.145.

To this end, we first calculate bounds on $2\left(\frac{v_2(a,\alpha_0)-h(a)}{h(a)-z(a)}\right)$. Substituting $C_u = \frac{1-\alpha_0}{\alpha_0}\frac{K^2}{T-2}$ (from (33)) into (19), we obtain that $v_2(a,\alpha_0) - h(a)$ equals

$$\theta_3 + \frac{2K-\tau(a-a_\gamma)}{T-\tau} + \frac{\sqrt{\frac{4\tau\alpha_0}{T-\tau-2\alpha_0}}\sqrt{\frac{(T-\tau)(T-2)}{\tau}\frac{1-\alpha_0}{\alpha_0}\frac{K^2}{T-2}+\left(\frac{T-\tau-2}{\tau}\right)K^2+2K(a-a_\gamma)-\frac{T}{2}(a-a_\gamma)^2}}{T-\tau} - h(a)$$

$$= \frac{2K - T(a-a_\gamma)}{T-\tau} + \sqrt{\left(\frac{2K}{T-\tau}\right)^2 + \frac{2\tau\alpha_0}{(T-\tau)(T-\tau-2\alpha_0)}\left(\frac{4K}{T-\tau}(a-a_\gamma) - \frac{T}{T-\tau}(a-a_\gamma)^2\right)} \tag{40}$$

(second line uses $\theta_3 - h(a) = a_\gamma - a$ (from (15) and (14)) and simplifies the square root term). Setting $k \equiv \frac{2K}{T-\tau}$, $t \equiv \frac{T}{T-\tau}$, and $y \equiv a - a_\gamma$, noting (using (37)) that $\frac{2\tau\alpha_0}{(T-\tau)(T-\tau-2\alpha_0)} = \phi^2 - t$, and multiplying by $\frac{2}{h(a)-z(a)} = \frac{1}{e^y}$ (by (15) with $y = a - a_\gamma$), we can simplify further to:

$$2\left(\frac{v_2(a,\alpha_0)-h(a)}{h(a)-z(a)}\right) = \frac{k - ty + \sqrt{k^2 + \left(\phi^2 - t\right)\left(2ky - ty^2\right)}}{e^y} \equiv \frac{\xi(y)}{e^y} \tag{41}$$

So we wish to obtain upper and lower bounds on the expression $\frac{\xi(y)}{e^y}$ in (41), for $a \in [a_\gamma, 0] \Leftrightarrow y \in [0, -a_\gamma]$. By construction, the value of K specified in (34) sets the square root portion of v_1, v_2 equal to zero at $a = 0 \Leftrightarrow y = -a_\gamma$ (see Lemma D7.1), so we have

$$k = a_\gamma\left(\phi^2 - t + \phi\sqrt{\phi^2 - t}\right) \tag{42}$$

Next, observe that

$$\begin{aligned} \xi'(y) &= -t + \frac{\left(\phi^2 - t\right)(k - ty)}{\sqrt{k^2 + \left(\phi^2 - t\right)\left(2ky - ty^2\right)}} \\ \xi''(y) &= \frac{-k^2\phi^2(\phi^2 - t)}{\left(k - ty + \sqrt{k^2 + \left(\phi^2 - t\right)\left(2ky - ty^2\right)}\right)^{\frac{3}{2}}} \end{aligned}$$

both strictly negative, by $\phi^2 > t$, $k < 0$, and $y \geq 0$. Therefore, $\xi(y)$ reaches a maximum over the interval $y \in [0, -a_\gamma]$ at $y = 0$, and lies above the straight line connecting the points $(0, \xi(0))$ and $(-a_\gamma, \xi(-a_\gamma))$: since we have $\xi(-a_\gamma) = k + ta_\gamma$ and $\xi(0) = k + \sqrt{k^2} = 0$, this line $\widetilde{\xi}$ is given by

$$\widetilde{\xi}(y) - \widetilde{\xi}(0) = \frac{\widetilde{\xi}(-a_\gamma) - \widetilde{\xi}(0)}{-a_\gamma}(y - 0) \Rightarrow \widetilde{\xi}(y) = \frac{k + ta_\gamma}{-a_\gamma}y$$

Substituting in (42), we conclude that

$$\begin{aligned} \min_{y\in[0,-a_\gamma]} \frac{\xi(y)}{e^y} &\geq \min_{y\in[0,-a_\gamma]} \frac{\widetilde{\xi}(y)}{e^y} = \left(-\phi^2 - \phi\sqrt{\phi^2 - t}\right)\left(\max_{y\in[0,-a_\gamma]} \frac{y}{e^y}\right) = \frac{-\phi^2 - \phi\sqrt{\phi^2 - t}}{e} \\ \max_{y\in[0,-a_\gamma]} \frac{\xi(y)}{e^y} &\leq \frac{\max_{y\in[0,-a_\gamma]} \xi(y)}{\min_{y\in[0,-a_\gamma]} e^y} = \frac{\xi(0)}{e^0} = 0 \end{aligned}$$

And finally, substituting $2\left(\frac{v_2(a,\alpha_0)-h(a)}{h(a)-z(a)}\right) \in \left[\frac{-\phi^2-\phi\sqrt{\phi^2-t}}{e}, 0\right]$ into (39) and (38), we obtain

$$q_a^* \in \left[\frac{\phi^2-1}{2\phi^2}, \frac{\phi^2-1+\frac{\phi^2+\phi\sqrt{\phi^2-t}}{e}}{2\phi^2}\right] \tag{43}$$

$$\min_{a\in[a_\gamma,0]} \varepsilon_a \geq \frac{\sqrt{\phi^2-1}}{2\phi^2}\sqrt{(\phi-1-0)\left(\phi+1-\frac{\phi^2+\phi\sqrt{\phi^2-t}}{e}\right)} \tag{44}$$

We now complete the proof for $\beta a_\gamma^2 > 8 \Leftrightarrow \beta > 0.79202$. Consider the limit as $\alpha_0 \to 0$, in which case $\phi^2 \to \frac{T}{T-\tau} = t$; substituting $t = \phi^2$ into (44), we obtain

$$\min_{a\in[a_\gamma,0]} \varepsilon_a \geq \frac{\sqrt{\phi^2-1}}{2\phi^2}\sqrt{(\phi-1)\left(\phi+1-\frac{\phi^2}{e}\right)}$$

This exceeds $\frac{1}{4}$ whenever $\phi \in (1.6545, 2.45)$, in which case (43) yields $q_a^* \in (0.317\,34, 0.600\,64) \subseteq (0.3, 0.6)$, the desired bounds. So, to complete the proof, we just need to show that (35) indeed yields $\lim_{\alpha_0\to 0}\phi \in (1.6545, 2.45)$. For this, recall that $\lim_{\alpha_0\to 0}\phi^2 \equiv \frac{T}{2\Delta} = \frac{1-\frac{\beta}{\Delta}}{1-\beta}$, so that

$$\phi > 1.6545 \Leftrightarrow \frac{1-\frac{\beta}{\Delta}}{1-\beta} > (1.6545)^2 \Leftrightarrow \Delta > \frac{\beta}{2.737\,4\beta-1.737\,4} \tag{45}$$

$$\phi < 2.45 \Leftrightarrow (6.002\,5\beta-5.002\,5)\,\Delta < \beta \tag{46}$$

Substituting (45) into the equation $T = \frac{2(\Delta-\beta)}{1-\beta}$, we obtain the horizon constraint $T > \left(\frac{5.474\,8\beta}{2.7374\beta-1.7374}\right)$, which is implied by the bound $T > T_{\min}$ in (35) (noting that $\frac{5.474\,8\beta}{2.7374\beta-1.7374} < 6$ whenever $\beta > 0.95203$). The inequality in (46) is trivially satisfied by any horizon if $\beta \leq \frac{5.0025}{6.0025} \cong 0.833\,4$; for $\beta > 0.8334$, we need $\Delta < \frac{\beta}{6.0025\beta-5.0025} \Leftrightarrow T < \frac{12.005\beta}{6.0025\beta-5.0025}$, which is implied by the bound $T < T_{\max}$ in (35) (noting that $\frac{12.005\beta}{6.0025\beta-5.0025} > \frac{4-2\beta}{1-\beta}$ whenever $\beta < 0.90913$). As desired, this establishes that $q_a^* \in (0.3, 0.6)$ and $\varepsilon_a > 0.25$, for any horizon T satisfying (35) and α_0 sufficiently close to zero.

Finally, we complete the proof for $\beta a_\gamma^2 < 8$, in which case we consider the limit as $\alpha_0 \to 1$. Then, $\phi^2 \to \frac{T-2}{T-\tau-2} = \frac{1}{1-\beta}$ (using (20), in particular $\tau = \beta(T-2)$), and $t = \frac{T}{T-\tau} = \frac{1-\frac{\beta}{\Delta}}{1-\beta}$; substituting into (43) and (44), we obtain

$$q_a^* \in \left[\frac{\frac{1}{1-\beta}-1}{\frac{2}{1-\beta}}, \frac{\frac{1}{1-\beta}-1+\frac{\frac{1}{1-\beta}+\sqrt{\frac{1}{1-\beta}}\sqrt{\frac{1}{1-\beta}-\frac{1-\frac{\beta}{\Delta}}{1-\beta}}}{e}}{\frac{2}{1-\beta}}\right] = \left[\frac{\beta}{2}, \frac{\beta+\frac{1+\sqrt{\frac{\beta}{\Delta}}}{e}}{2}\right]$$

$$\min_{a\in[a_\gamma,0]} \varepsilon_a \geq \frac{\sqrt{\beta}}{2}\sqrt{\beta-\left(\frac{1-\sqrt{1-\beta}}{\sqrt{1-\beta}}\right)\left(\frac{1+\sqrt{\frac{\beta}{\Delta}}}{e}\right)}$$

For the range $\beta \in [0.4173, 0.50102)$, (35) specifies $T = 7 \Leftrightarrow \Delta = \frac{7-5\beta}{2}$; in this case, it may easily be verified numerically that our lower bound on ε_a reaches a minimum (at $\beta = 0.4172$) of 0.163, our lower bound on q_a^* is at least $\frac{\beta}{2} \geq \frac{0.4172}{2}$, and our upper bound on q_a is at most $\max_{\beta \in [0.4172, 0.50102]} \left(\frac{\beta}{2} + \frac{1+\sqrt{\frac{2\beta}{7-5\beta}}}{2e} \right) =$ $0.521\,30$. For the range $\beta \in [0.50102, 0.79202)$, (35) specifies $\Delta \in [2.5, 4]$; over this range, it may easily be verified numerically that our lower bound on ε_a is minimized at $\beta = 0.79202$, and is increasing in Δ, with a minimum value (at $\beta = 0.79202, \Delta = 2.5$) of 0.14505 (any $\Delta \in [3.4]$ guarantees $\varepsilon_a > 0.15$); our lower bound on q_a is at least $\frac{0.50102}{2} > 0.25$, and our upper bound on q_a is at most $\max_{\beta \in [0.50102, 0.79202]} \left(\frac{\beta}{2} + \frac{1+\sqrt{\frac{\beta}{2.5}}}{2e} \right) = 0.7$. As desired, this establishes that if we choose a horizon T satisfying (35) and take α_0 sufficiently close to 1, then $q_a^* \in (0.2, 0.7)$ and $\varepsilon_a > 0.145$. ■

D.2.3 Optimality for the decision-maker: Deviations at time $t = 0$

Proposition D6: Fix $a_\gamma \leq -1.773 \Rightarrow b < \frac{1}{61}$, and choose parameters C_u, K, T satisfying (33), (34), and (35). There exists a continuous function $p_a^* : [-2, 0] \to (0, 1)$, a number $\varepsilon > 0$, and numbers $0 < \alpha' < \alpha'' < 1$ such that if the DM receives recommendation $u_1(a, \alpha_a)$ at time $t \geq 0$ for some $a \in [-2, 0]$, believes he is facing either type $x(a)$ or $g(a)$, and assigns probability p_a to type $x(a)$, then (i) if $\beta a_\gamma^2 > 8$, his gain to deviating is strictly negative whenever $p_a \in (p_a^* - \varepsilon, p_a^* + \varepsilon)$ and $\alpha_0 < \alpha'$; (ii) if $\beta a_\gamma^2 \leq 8$, his gain to deviating is strictly negative whenever $p_a \in (p_a^* - \varepsilon, p_a^* + \varepsilon)$ and $\alpha_0 > \alpha''$.

Proof of Proposition D6: As explained in Observation D4, it suffices to prove that the gain to deviating is negative at time $t = 0$. Substituting $x(a) = \theta_3 + a - \theta_3 e^a$ and $g(a) = \theta_3 + a$ into (94), we obtain that the decision-maker's gain to deviating at time 0 at information set $\{x(a), g(a)\}$, if he assigns probability p_a to type $x(a)$, is

$$= 2K^2 + 4K\left(p_a \theta_3 e^a - a\right) + T\left(p_a \theta_3 e^a - a\right)^2 - (T-2)p_a(\theta_3 e^a)^2 + 2(T-2)C_u$$

Substituting in $(T-2)C_u = \frac{1-\alpha_0}{\alpha_0} K^2$ from (33) and simplifying, this becomes

$$\frac{2K^2}{\alpha_0} + 4K\left(p_a \theta_3 e^a - a\right) + T\left(p_a \theta_3 e^a - a\right)^2 - (T-2)p_a(\theta_3 e^a)^2$$

Setting this expression to be negative and solving for p_a, we find that deviations are unprofitable so long as $p_a \in (p_a^* - \varepsilon_a, p_a^* + \varepsilon_a)$, where

$$p_a^* = \frac{1}{2} + \frac{a}{\theta_3 e^a} - \frac{1 + \frac{2K}{\theta_3 e^a}}{T} \tag{47}$$

$$\varepsilon_a = \sqrt{\left(\frac{T-2}{T}\right)\left(\frac{1}{4} + \frac{a}{\theta_3 e^a} - \frac{\left(1 + \frac{2K}{\theta_3 e^a}\right)^2}{2T}\right) - \left(\frac{1-\alpha_0}{\alpha_0}\right)\frac{2K^2}{T\left(\theta_3 e^a\right)^2}} \tag{48}$$

So, noting the continuity in a, α_0, it suffices to take the limits of (47), (48) as $\alpha_0 \to \begin{cases} 1 & \text{if } \beta a_\gamma^2 \leq 8 \\ 0 & \text{if } \beta a_\gamma^2 > 8 \end{cases}$, and to show that the expressions for p_a^*, ε_a in (47), (48) satisfy $p_a^* \in (0, 1)$ $\forall \in [-2, 0]$, and $\min_{a \in [-2,0]} \varepsilon_a > 0$. In what follows, it will be useful to recall the relationships $\theta_3 e^{-2} = a_\gamma - 2 + 2e^{-a_\gamma}$ (from (14)), and $\Delta = \frac{T-\tau}{2} \Leftrightarrow T = \frac{2(\Delta - \beta)}{1-\beta}$ (using (20)).

If $\beta a_\gamma^2 > 8 \Leftrightarrow a_\gamma < -3.18$, $\beta > 0.79202$, then consider the limit as $\alpha_0 \to 0$, and let $6 \leq T \leq \frac{4-2\beta}{1-\beta}$ (implied by (35)). Note that in this range, $\min_{a \in [-2,0]} \theta_3 e^a = \theta_3 e^{-2} = a_\gamma - 2 + 2e^{-a_\gamma} > 40$. By (34), we have $\lim_{\alpha_0\to 0}\left(\frac{1-\alpha_0}{\alpha_0}\right)K^2 = \frac{T\tau a_\gamma^2}{2(T-\tau)}$ and $\lim_{\alpha_0\to 0} K = 0$; substituting into (48), we obtain

$$\lim_{\alpha_0\to 0}\varepsilon_a = \sqrt{\left(\frac{T-2}{T}\right)\left(\frac{1}{4}+\frac{a}{\theta_3 e^a}-\frac{1}{2T}\right)-\frac{\tau a_\gamma^2}{T-\tau}\left(\frac{1}{\theta_3 e^a}\right)^2}$$

This is increasing in a, and therefore reaches a minimum at $a=-2 \Rightarrow \theta_3 e^a = a_\gamma - 2 + 2e^{-a_\gamma}$: so, we have

$$\min_{a\in[-2,0]}\left(\lim_{\alpha_0\to 0}\varepsilon_a\right) = \sqrt{\left(\frac{T-2}{2T}\right)^2\left(1-\left(\frac{8T}{T-2}\right)\left(\frac{1}{a_\gamma-2+2e^{-a_\gamma}}\right)\right)-\frac{\tau}{T-\tau}\left(\frac{a_\gamma}{a_\gamma-2+2e^{-a_\gamma}}\right)^2} \tag{49}$$

We first consider the final subtracted term in (49) : using the relationship $\Delta = \frac{T-\tau}{2} \Leftrightarrow T = \frac{2(\Delta-\beta)}{1-\beta}$ and our horizon restriction $T \le \frac{4-2\beta}{1-\beta} \Leftrightarrow \Delta \le 2$, we obtain $\frac{\tau}{T-\tau} = \left(\frac{\beta}{1-\beta}\right)\left(\frac{\Delta-1}{\Delta}\right) \le \frac{\beta}{2(1-\beta)}$; this is less than $\frac{\left(a_\gamma-2+2e^{-a_\gamma}\right)^2}{2(2-a_\gamma)\left(a_\gamma-4+4e^{-a_\gamma}\right)}$ by (20), so that $\frac{\tau}{T-\tau}\left(\frac{a_\gamma}{a_\gamma-2+2e^{-a_\gamma}}\right)^2 < \frac{a_\gamma^2}{2(2-a_\gamma)\left(a_\gamma-4+4e^{-a_\gamma}\right)}$. Substituting this into (49), noting that the resulting expression is strictly increasing in T, and using our horizon restriction $T \ge 6$, we then have:

$$\min_{a\in[-2,0]}\left(\lim_{\alpha_0\to 0}\varepsilon_a\right) > \sqrt{\frac{1}{9}-\frac{4}{3}\left(\frac{1}{a_\gamma-2+2e^{-a_\gamma}}\right)-\frac{a_\gamma^2}{2\left(2-a_\gamma\right)\left(a_\gamma-4+4e^{-a_\gamma}\right)}}$$

It may easily be verified graphically that this expression is increasing in $-a_\gamma$, with a lower bound, at $-a_\gamma = 3.18$, of approximately 0.263. On the other hand, (47) yields

$$\lim_{\alpha_0\to 0} p_a^* = \frac{T-2}{2T}+\frac{a}{\theta_3 e^a}$$

Since $\max_{a\in[-2,0]}\left(\lim_{\alpha_0\to 0} p_a^*\right) = \frac{T-2}{2T} < \frac{1}{2}\ \forall T$, and $\min_{a\in[-2,0]}\left(\lim_{\alpha_0\to 0} p_a^*\right) = \frac{T-2}{2T} - \frac{2}{\theta_3 e^{-2}} > \frac{4}{12} - \frac{2}{40} = \frac{17}{60}$ (by horizon restriction $T \ge 6$ and the fact that $a_\gamma < -3.18 \Rightarrow \theta_3 e^{-2} = a_\gamma - 2 + 2e^{-a_\gamma} > 40$), this establishes the desired result, with $\varepsilon_a > 0.25$ and $p_a^* \in \left[\frac{17}{60}, \frac{1}{2}\right]$.

If $a_\gamma \in [-3.18, -2) \Leftrightarrow \beta \in (.50102, .79202]$, then consider the limit as $\alpha_0 \to 1$: in this case, we have

$$\lim_{\alpha_0\to 1}\varepsilon_a = \sqrt{\left(\frac{T-2}{T}\right)\left(\frac{1}{4}+\frac{a}{\theta_3 e^a}-\frac{\left(1+\frac{2K}{\theta_3 e^a}\right)^2}{2T}\right)} \tag{50}$$

$$\text{with } K = \frac{\beta a_\gamma}{1-\beta}\left(1+\sqrt{\frac{\Delta}{\beta}}\right),\ T = \frac{2(\Delta-\beta)}{1-\beta}\ \text{(using (34), (20) at } \alpha_0 = 1) \tag{51}$$

Recall that in this range, (35) specifies $\Delta \in [2.5, 4] \Leftrightarrow T \in \left[\frac{5-2\beta}{1-\beta}, \frac{8-2\beta}{1-\beta}\right]$ and implies $T > 6$. To prove that $\min_{a\in[-2,0]}\varepsilon_a > 0$, it suffices to show that for any $\Delta \in [2.5, 4]$ and $a_\gamma \in [-3.18, -2]$, the following two results hold:

$$\text{(i)} \quad : \quad \left(1+\frac{2K}{\theta_3 e^a}\right)^2 \leq 1-\varepsilon',\ \text{for some } \varepsilon'>0 \tag{52}$$

$$\text{(ii)} \quad : \quad \frac{1}{2T} \leq \frac{1}{4}-\frac{2}{\theta_3 e^{-2}} \tag{53}$$

To see this, substitute (52) and (53) into (50) , to obtain

$$\begin{aligned}
\lim_{\alpha_0\to 1}\varepsilon_a &> \sqrt{\left(\frac{T-2}{T}\right)\left(\frac{1}{4}-\frac{2}{\theta_3 e^{-2}}-\frac{1-\varepsilon'}{2T}\right)}\ \text{(by (52) and } \frac{d}{da}\frac{a}{\theta_3 e^a}>0\text{)}\\
&> \sqrt{\left(\frac{T-2}{T}\right)\left(\frac{\varepsilon'}{2T}\right)}\ \text{(by (53))}
\end{aligned}$$

Strictly positive as desired, since $\varepsilon' > 0$ and $T > 6$. For $p_a^* \in (0,1)$, note that (52) implies $\left(1+\frac{2K}{\theta_3 e^a}\right) \in (-1,1)$; substituting this into (47) , we obtain

$$\begin{aligned}
\min_{a\in[-2,0]}\left(\frac{1}{2}+\frac{a}{\theta_3 e^a}-\frac{1}{T}\right) &\leq \lim_{\alpha_0\to 1} p_a^* \leq \max_{a\in[-2,0]}\left(\frac{1}{2}+\frac{a}{\theta_3 e^a}-\frac{-1}{T}\right)\\
&\Leftrightarrow\ p_a^* \in \left(2\left(\frac{1}{4}-\frac{1}{\theta_3 e^{-2}}-\frac{1}{2T}\right),\frac{1}{2}+\frac{1}{T}\right)
\end{aligned}$$

By (53) (for the lower bound) and $T \geq 6$ (for the upper bound), this implies $p_a^* \in (0,1)$, as desired.

So, to complete the range for $a_\gamma \in [-3.18,-2)$, we just need to prove that $\Delta \in [2.5,4]$ implies (52) and (53) . To prove (52) , it suffices to set $\varepsilon' = \left(\frac{-4K}{\theta_3}\right)\left(1+\frac{K}{\theta_3 e^{-2}}\right)$, and to prove that $\Delta \in [2.5,4]$ and $a_\gamma \in [-3.18,-2] \Rightarrow 1+\frac{K}{\theta_3 e^{-2}} > 0$: then, since it is immediate from (51) that $\frac{-4K}{\theta_3} > 0$, we'll have

$$0<\varepsilon' \equiv \left(\frac{-4K}{\theta_3}\right)\left(1+\frac{K}{\theta_3 e^{-2}}\right) < \min_{a\in[-2,0]}\left(\left(\frac{-4K}{\theta_3 e^a}\right)\left(1+\frac{K}{\theta_3 e^a}\right)\right) = \min_{a\in[-2,0]}\left(1-\left(1+\frac{2K}{\theta_3 e^a}\right)^2\right)$$

which rearranges to yield $\left(1+\frac{2K}{\theta_3 e^a}\right)^2 < 1-\varepsilon'\ \forall a \in [-2,0]$, as desired. To this end, we first solve for the values of Δ that guarantee $1+\frac{K}{\theta_3 e^{-2}} > 0$: substituting (51) into this inequality, along with the relationship $\theta_3 e^{-2} = a_\gamma - 2 + 2e^{-a_\gamma}$, and then solving for Δ, we obtain

$$\Delta < \frac{1}{\beta}\left(\frac{2(1-\beta)(e^{-a_\gamma}-1)+a_\gamma}{-a_\gamma}\right)^2 \tag{54}$$

To prove that this is satisfied by any $\Delta \in [2.5,4]$, it suffices to prove that the upper bound in (54) exceeds 4. For this, it is sufficient (by $\beta < 1 \Rightarrow \frac{1}{\beta} > 1$) that

$$\frac{2(1-\beta)(e^{-a_\gamma}-1)+a_\gamma}{-a_\gamma} > 2 \Leftrightarrow 2(1-\beta)\left(e^{-a_\gamma}-1\right)+3a_\gamma > 0$$

Substituting $1-\beta = \frac{(2-a_\gamma)\left(a_\gamma-4+4e^{-a_\gamma}\right)}{4e^{-a_\gamma}\left(e^{-a_\gamma}-1\right)}$ (from (20)) into this inequality and multiplying through

by $2e^{-a_\gamma}$, it becomes
$$(2-a_\gamma)\left(a_\gamma-4+4e^{-a_\gamma}\right)+6a_\gamma e^{-a_\gamma}>0$$

The LHS of this expression is strictly concave (2nd derivative w.r.t. a_γ is $2(2+a_\gamma)e^{-a_\gamma}-2$, which is strictly negative by $a_\gamma \leq -2$), and therefore it reaches a minimum over the interval $a_\gamma \in [-3.18,-2]$ at either $a_\gamma=-3.18$, of 5.5562, or at $a_\gamma=-2$, of 2.2443; we can therefore conclude that the RHS of (54) is at least 4, and so (54) is indeed satisfied by any $\Delta \in [2.5,4]$, as needed to complete the proof of (52). For (53), we need to show that $\Delta \in [2.5,4]$ implies that

$$\frac{1}{2T} \leq \frac{1}{4}-\frac{2}{\theta_3 e^{-2}} \Leftrightarrow T \geq 2\frac{(a_\gamma-2+2e^{-a_\gamma})}{(a_\gamma-2+2e^{-a_\gamma})-8}$$

Substituting $T=\frac{2\Delta-2\beta}{1-\beta}$ (from $\Delta \equiv \frac{T-\tau}{2}$ and (20), in particular $\tau=\beta(T-2)$) into this inequality, it becomes
$$\frac{\Delta-\beta}{1-\beta} \geq \frac{(a_\gamma-2+2e^{-a_\gamma})}{(a_\gamma-2+2e^{-a_\gamma})-8} \Leftrightarrow \Delta \geq \left(\frac{a_\gamma-2+2e^{-a_\gamma}-8\beta}{a_\gamma-2+2e^{-a_\gamma}-8}\right)$$

To complete the proof, it then suffices to show that

$$\begin{aligned} a_\gamma &\in [-3.18,-2) \Rightarrow \left(\frac{a_\gamma-2+2e^{-a_\gamma}-8\beta}{a_\gamma-2+2e^{-a_\gamma}-8}\right) \leq 2.5 \\ &\Leftrightarrow_{\text{rearranging}} 3\left(a_\gamma-2+2e^{-a_\gamma}\right)+16\beta \geq 40 \end{aligned}$$

Using (20), the LHS is strictly decreasing in a_γ, with a minimum value, at $a_\gamma=-2$, of 40.331, thus satisfying the inequality. This completes the proof of (53), and hence of Proposition D6 for $a_\gamma \in [-3.18,-2)$.

Finally, we prove Proposition D6 for $a_\gamma \in [-2,-1.773]$. Again consider the limit as $\alpha_0 \to 1$, so that (50), (47), (51) continue to hold. First, we note that (35) implies $K+T \geq 0$:

$$K+T \geq 0 \Leftrightarrow \frac{\beta a_\gamma}{1-\beta}\left(1+\sqrt{\frac{\Delta}{\beta}}\right)+\frac{2(\Delta-\beta)}{1-\beta} \geq 0 \Leftrightarrow \Delta \geq \beta\left(1-\frac{a_\gamma}{2}\right)^2 \tag{55}$$

Since $a_\gamma \geq -2 \Rightarrow \beta\left(1-\frac{a_\gamma}{2}\right)^2 \leq 4\beta$, this holds for any $\Delta \geq 4\beta$; by (35), we have $T=7 \Rightarrow \Delta=\frac{7-5\beta}{2}$; and since $a_\gamma \geq -2 \Rightarrow \beta \leq 0.50102 < \frac{7}{13} \Rightarrow \frac{7-5\beta}{2} > 4\beta$, we conclude that (55) indeed holds, and so $K+T \geq 0$. Next, we note that $K+T \geq 0$ implies that $\lim_{\alpha_0 \to 1} \varepsilon_a$ is minimized (over our interval $a \in [-2,0]$) at either $a=-2$ or $a=0$: for this, we just need to show that $\frac{d^2\left(\lim_{\alpha_0 \to 1}\varepsilon_a\right)}{da^2}<0$, which follows from differentiating (50) to obtain

$$\frac{d^2\varepsilon_a}{da^2}=\frac{\left(-8K^2+Ta\theta_3 e^a-2(K+T)\theta_3 e^a\right)}{T\left(\theta_3 e^a\right)^2}<0 \text{ (by } a<0 \text{ and } K+T \geq 0)$$

This is negative, by $a<0$ and $K+T \geq 0$. So, to prove that $\lim_{\alpha_0 \to 1}\varepsilon_a > 0\ \forall a \in [-2,0]$, we just need to show that $\lim_{\alpha_0 \to 1}\varepsilon_a > 0$ at $a \in \{-2,0\}$. For $\lim_{\alpha_0 \to 1}\varepsilon_0 > 0$: since $a_\gamma \leq -1.773 \Rightarrow \theta_3=(a_\gamma-2+2e^{-a_\gamma})e^2 > 59$, and $0 > K \geq -T=-7$, we have $1 > 1+\frac{2K}{\theta_3} > 1-\frac{14}{59}$, so that $\left(1+\frac{2K}{\theta_3}\right)^2<1$; substituting this into (50) evaluated at $a=0$, we obtain

$$\lim_{\alpha_0 \to 1}\varepsilon_0=\frac{1}{4}-\frac{\left(1+\frac{2K}{\theta_3}\right)^2}{2T} > \frac{1}{4}-\frac{1}{2(7)}$$

Strictly positive, as desired. For $\lim_{\alpha_0 \to 1} \varepsilon_{-2} > 0$, evaluate (50) at $a = -2$ and $T = 7$, recalling from (14) that $\theta_3 e^{-2} = a_\gamma - 2 + 2e^{-a_\gamma}$, to obtain

$$\begin{aligned}\varepsilon_{-2} &= \frac{1}{4} - \frac{2}{a_\gamma - 2 + 2e^{-a_\gamma}} - \frac{\left(1 + \frac{2K}{a_\gamma - 2 + 2e^{-a_\gamma}}\right)^2}{14} \\ &> 0 \text{ whenever } \left(1 + \frac{2K}{a_\gamma - 2 + 2e^{-a_\gamma}}\right)^2 < 14\left(\frac{1}{4} - \frac{2}{a_\gamma - 2 + 2e^{-a_\gamma}}\right) \end{aligned} \tag{56}$$

Substituting $T = 7 \Rightarrow \Delta = \frac{7-5\beta}{2}$ and (20) into (51) to obtain K as a function of a_γ, it may easily be verified numerically that (56) holds iff $a_\gamma > -1.7743$, implied for the range under consideration. This completes the proof that $\varepsilon_{-2} > 0$, and hence that $\min_{a\in[-2,0]} \varepsilon_a > 0$. Finally, to prove that $p_a^* \in (0,1)$: first, $a_\gamma \in [-2, -1.773]$ implies that the RHS expression in (56) is between 0 and 1, and therefore (56) implies that $\left(1 + \frac{2K}{\theta_3 e^{-2}}\right)^2 < 1$; using this, together with $K < 0$ and the fact that $\frac{2K}{\theta_3 e^a}$ is increasing in a, we then have

$$1 > 1 + \frac{2K}{\theta_3} \geq 1 + \frac{2K}{\theta_3 e^a} \geq 1 + \frac{2K}{\theta_3 e^{-2}} > -1 \tag{57}$$

Substituting $\left(1 + \frac{2K}{\theta_3 e^a}\right) \in (-1, 1)$ into (47) , together with $\frac{a}{\theta_3 e^a} \in [\frac{-2}{\theta_3 e^{-2}}, 0]$ and $T = 7$, we obtain $p_a^* \in \left[\frac{1}{2} - \frac{2}{\theta_3 e^{-2}} - \frac{1}{7}, \frac{1}{2} + \frac{1}{7}\right]$; the upper bound is clearly below 1, and the lower bound is at least $\frac{3}{28}$ by $a_\gamma \geq -1.773 \Rightarrow \theta_3 e^{-2} > 8$, so $p_a^* \in (0, 1)$, as desired.

D.2.4 Optimality for the decision-maker: Completing the Proof

As explained at the beginning of this section, it remains only to prove that there is an open set of priors generating posteriors which satisfy the conditions in Propositions D5, D6, which we prove here.

Proposition D7: For any $0 < \underline{\lambda} < \overline{\lambda} < 1$ and continuous functions $p : [-2, 0] \to [\underline{\lambda}, 1]$, $q : [a_\gamma, 0] \to [\underline{\lambda}, 1]$, and $r : [a_\gamma, 0] \to [0, \overline{\lambda}]$, there exists a density f over the state space such that, in our construction, a Bayesian decision-maker will hold the following posterior beliefs: (i) $\Pr(x(a)|\{x(a), g(a)\}) = p(a)$; (ii) $\Pr(z(a)|\{z(a), h(a)\}) = q(a)$; (iii) $\Pr(\{z(a), h(a)\} \mid \{z(a), h(a), x(\widehat{a}), g(\widehat{a})\} = r(a)$. Then, since we proved that the equilibrium works for the DM whenever $p(a) \in (p_a^* - \varepsilon, p_a^* + \varepsilon)$ (for some continuous $p_a^* : [-2, 0] \to (0, 1)$ and $\varepsilon > 0$, see Proposition D6), $q(a) \in (q^* - \varepsilon, q^* + \varepsilon)$ (for some continuous $q_a^* : [a_\gamma, 0] \to (0, 1)$ and $\varepsilon > 0$, see Proposition D5), and $r(a) < \varepsilon'$ (for some $\varepsilon' > 0$, see section D.2.1), with p_a^*, q_a^*, $1 - r_a^*$ bounded away from zero, it follows we can find some $\varepsilon'' > 0$ such that our construction works also for any perturbation of the prior which does not change the density at any point by more than ε''. As desired, this constructs an open set of prior distributions for which our construction constitutes a fully revealing equilibrium.

Proof of Proposition D7: Bayesian beliefs satisfy

$$\frac{\Pr(x(a)|\{x(a), g(a)\})}{\Pr(g(a)|\{x(a), g(a)\})} = \frac{f(x(a))}{f(g(a))}(\theta_3 e^a - 1) \tag{58a}$$

$$\frac{\Pr(z(a)|\{z(a), h(a)\})}{\Pr(h(a)|\{z(a), h(a)\})} = \frac{f(z(a))}{f(h(a))}\left(2e^{a-a_\gamma} - 1\right) \tag{58b}$$

$$\frac{\Pr(\{z(a), h(a)\})}{\Pr(\{x(\widehat{a}), g(\widehat{a})\})} = \frac{f(z(a))}{f(\widehat{x}(a))}\left(\frac{2e^{a-a_\gamma} - 1}{\theta_3 e^{\widehat{a}(a)} - 1}\right)\left(\frac{1}{|\widehat{a}'(a)|}\right)\left(\frac{p(\widehat{a})}{q(a)}\right) \tag{58c}$$

where $\widehat{a}(a) = u_1^{-1}(v_1(a'))$ (as explained in Section D.3). We want the expression in (58a) to equal

$\frac{p(a)}{1-p(a)}$, the expression in $(58b)$ to equal $\frac{q(a)}{1-q(a)}$, and the expression in $(58c)$ to equal $\frac{r(a)}{1-r(a)}$. It is straightforward to construct such a density f : for example, for each $a \in [-2,0]$, set $f(x(a)) = \frac{1}{M}$, with M a constant to be determined (this assigns a density for types $x(a) \in [0, \theta_1]$). Then, assign probabilities to types $g(a) \in [\theta_2, \theta_3]$ by setting the RHS of $(58a)$ equal to $\frac{p(a)}{1-p(a)}$, substituting in $f(x(a)) = \frac{1}{M}$, and solving for $f(g(a))$, to obtain

$$f(g(a)) = f(x(a))(\theta_3 e^a - 1)\left(\frac{1-p(a)}{p(a)}\right) = \frac{(\theta_3 e^a - 1)}{M}\frac{1-p(a)}{p(a)}$$

Next, for each $a \in [a_\gamma, 0]$, set the RHS of $(58c)$ equal to $\frac{r(a)}{1-r(a)}$, replace the final RHS term of (5) with $\frac{p(\widehat{a})}{q(a)}$, sub in $f(x(\widehat{a})) = \frac{1}{M}$, and solve for $f(z(a))$, to obtain

$$f(z(a)) = \frac{\frac{q(a)}{p(\widehat{a})}\frac{\theta_3 e^{\widehat{a}(a)}-1}{2e^{a-a_\gamma}-1}\cdot|\widehat{a}'(a)|}{M}\frac{r(a)}{1-r(a)}$$

(This assigns a prior for types $z(a) \in [\theta_1, \theta_2]$). And similarly, use this and $(58b)$ to assign beliefs to types $h(a) \in [\theta_3, \frac{1}{b}]$, obtaining

$$f(h(a)) = \frac{\frac{q(a)}{p(\widehat{a}(a)))}\left(\theta_3 e^{\widehat{a}(a)} - 1\right)\cdot|\widehat{a}'(a)|}{M}\left(\frac{r(a)}{1-r(a)}\right)\left(\frac{1-q(a)}{q(a)}\right)$$

Finally, choose M so that the total measure of the type space integrates to 1 (This is possible since $(16), (18)$ imply that u_1, v_1 and their derivatives w.r.t. a are finite and non-zero except perhaps at a single point a, from which it follows that $|\widehat{a}'(a)|$ is bounded; since we also have $p(a) \geq \underline{\lambda}$, $q(a) \geq \underline{\lambda}$, and $1-r(a) \geq 1-\overline{\lambda}$, it follows that all of the specified densities $f(x(a)), f(g(a)), f(z(a)), f(h(a))$ are finite numbers divided by a number M. So, integrating over the state space yields a finite number divided by M; choose M so that this equals 1). ■

D.2.5 Optimality for the decision-maker: Preliminary Calculations

Lemma D7.1: Let C_u, K, and T be given by (33), (34), and (35), and define $\Delta \equiv \frac{T-\tau}{2}$ (so that, using $\tau = \beta(T-2)$, we have $T = \frac{2(\Delta-\beta)}{1-\beta}$). Then: (i) if $\beta a_\gamma^2 > 8$, there exists $\overline{\alpha_0} > 0$ such that whenever $\alpha_0 < \overline{\alpha_0}$, the functions u_1, u_2, v_1, v_2 specified in (16)-(19) are real-valued; moreover, there exists a continuous decreasing function $\alpha_a : [-2,0] \to [\alpha_0, 1)$ such that $\forall a \in [a_\gamma, 0]$, there exists $\widetilde{a} \in [-2,0]$ with $v_1(a, \alpha_0) = u_1(\widetilde{a}, \alpha(\widetilde{a}))$; (ii) if $\beta a_\gamma^2 \leq 8$, there exists $\underline{\alpha_0} < 1$ such that whenever $\alpha_0 > \underline{\alpha_0}$, the functions u_1, u_2, v_1, v_2 specified in (16)-(19) are real-valued; moreover, $v_1(a, \alpha_0) \in [u_1(0, \alpha_0), u_1(-2, \alpha_0)]$ $\forall a \in [a_\gamma, 0]$; (iii) (modification to (i), needed only for the next subsection "Integer Constraints"): if $\beta a_\gamma^2 > 8$ and C_u, K are as given by (33), (34), then part (i) holds also at $T = 6$, $\alpha_0 = 0.1$.
Proof of Lemma D7.1:

Substituting (33) into (16), (17), (18), (19), our action functions become:

$$u_1(a,\alpha_a) = \theta_3 + K - \frac{T-2}{2}a - \sqrt{\frac{1-\alpha_a}{\alpha_a}}\sqrt{\frac{1-\alpha_0}{\alpha_0}K^2 + (T-2)a\left(K - \frac{T}{4}a\right)} \tag{59}$$

$$u_2(a,\alpha_a) = \theta_3 + K - \frac{T-2}{2}a + \sqrt{\frac{1-\alpha_a}{\alpha_a}}\sqrt{\frac{1-\alpha_0}{\alpha_0}K^2 + (T-2)a\left(K - \frac{T}{4}a\right)} \tag{60}$$

$$v_1(a,\alpha_0) = \theta_3 + \frac{2K-\tau(a-a_\gamma)}{T-\tau} - \frac{\sqrt{\frac{\tau(T-\tau-2\alpha_0)}{\alpha_0}}\sqrt{\left(\frac{T-\tau-2\alpha_0}{\tau}\right)\frac{K^2}{\alpha_0}+2K(a-a_\gamma)-\frac{T}{2}(a-a_\gamma)^2}}{T-\tau} \tag{61}$$

$$v_2(a,\alpha_0) = \theta_3 + \frac{2K-\tau(a-a_\gamma)}{T-\tau} + \frac{\sqrt{\frac{4\tau\alpha_0}{T-\tau-2\alpha_0}}\sqrt{\left(\frac{T-\tau-2\alpha_0}{\tau}\right)\frac{K^2}{\alpha_0}+2K(a-a_\gamma)-\frac{T}{2}(a-a_\gamma)^2}}{T-\tau} \tag{62}$$

We first prove that v_1, v_2 are real-valued $\forall\alpha_0$. For this, we need to show that the following expression is non-negative, for all $a \in [a_\gamma, 0]$:

$$\left(\frac{T-\tau-2\alpha_0}{\tau}\right)\frac{K^2}{\alpha_0} + 2K(a-a_\gamma) - \frac{T}{2}(a-a_\gamma)^2 \tag{63}$$

For this, observe that the expression in (63) is decreasing in a (the derivative w.r.t. a is $2K - T(a - a_\gamma)$, which is strictly negative, since (34) implies $K < 0$); therefore, the expression in (63) reaches a minimum over the interval $a \in [a_\gamma, 0]$ at $a = 0$, and so it is sufficient to prove that this minimum value is non-negative: that is, we need

$$\left(\frac{T-\tau-2\alpha_0}{\tau}\right)\frac{K^2}{\alpha_0} - 2Ka_\gamma - \frac{T}{2}a_\gamma^2 \geq 0$$

The value for K specified in (34) is precisely the negative root of this equation, and so we conclude that v_1, v_2 are real-valued for all values of a, α_0.

Next, we prove that u_1, u_2 are real-valued for α_0 near zero when $\beta a_\gamma^2 > 8$, and for α_0 near 1 when $\beta a_\gamma^2 \leq 8$. To this end: for u_1, u_2 to be real-valued, we need to choose α_0 such that

$$\min_{a\in[-2,0]}\left(\frac{1-\alpha_0}{\alpha_0}K^2 + (T-2)a\left(K - \frac{T}{4}a\right)\right) \geq 0 \tag{64}$$

The second derivative of the bracketed expression w.r.t. a is $-\frac{T(T-2)}{2} < 0$, implying that the minimum value over the interval $a \in [-2, 0]$ is attained at one of the two endpoints. If at $a = 0$, then (64) becomes $\frac{1-\alpha_0}{\alpha_0}K^2 \geq 0$, trivially satisfied $\forall\alpha_0 \in [0, 1]$; if at $a = -2$, then we need

$$\frac{1-\alpha_0}{\alpha_0}K^2 - (T-2)(2K+T) \geq 0 \tag{65}$$

Suppose first that $\beta a_\gamma^2 > 8$, and consider the limit as $\alpha_0 \to 0$: by (34), we have $\lim_{\alpha_0\to 0} K = 0$, and $\lim_{\alpha_0\to 0}\frac{1-\alpha_0}{\alpha_0}K^2 = \frac{T(T-2)\beta a_\gamma^2}{2(T-\tau)}$, so we have

$$\lim_{\alpha_0\to 0}\left(\frac{1-\alpha_0}{\alpha_0}K^2 - (T-2)(2K+T)\right) = T(T-2)\left(\frac{\beta a_\gamma^2}{2(T-\tau)} - 1\right)$$

This is strictly positive, since $\beta a_\gamma^2 > 8$, and (35) specifies $(T-\tau) \leq 4$ in this range; therefore, (65) is satisfied with strict inequality in the limit as $\alpha_0 \to 0$, and therefore, by continuity, holds also for α_0 sufficiently close to zero. If $\beta a_\gamma^2 \leq 8$, then consider the limit as $\alpha_0 \to 1$. For (64), it is sufficient to prove that

$$2K + T \leq 0 \Leftrightarrow -K \geq \frac{T}{2}$$

Substituting in (34), taking limits as $\alpha_0 \to 1$, and using $\frac{T}{2} = \frac{\Delta-\beta}{1-\beta}$, this becomes

$$\frac{\beta(-a_\gamma)}{1-\beta}\left(1+\sqrt{\frac{\Delta}{\beta}}\right) \geq \frac{\Delta-\beta}{1-\beta} \Leftrightarrow (-a_\gamma)\left(1+\sqrt{\frac{\Delta}{\beta}}\right) \geq \frac{\Delta}{\beta} - 1$$

This is satisfied by any $\sqrt{\frac{\Delta}{\beta}} \in [0, 1-a_\gamma] \Leftrightarrow 0 \leq \Delta \leq \beta(1-a_\gamma)^2$, which is implied by (35) : in the range $a_\gamma \in [-3.18, -2) \Leftrightarrow \beta \in (.50102, .79202]$, we have $\beta(1-a_\gamma)^2 \geq .50102(1+2)^2 > 4$, while (35) specifies $\Delta \in [2.5, 4]$; and in the range $a_\gamma \in [-2, -1.773) \Leftrightarrow \beta \in (.4172, .50102]$, we have $\beta(1-a_\gamma)^2 \geq .4172(1+1.773)^2 > 3$, while (35) specifies $T = 7 \Leftrightarrow \Delta = \frac{7-5\beta}{2} < 3$. As desired, this establishes that u_1, u_2 are real-valued under the Lemma conditions.

Finally, we prove the desired overlap of functions u_1, v_1. We begin by computing the range of v_1 : differentiating (61), we obtain

$$\begin{aligned}
\frac{T-\tau}{\tau}\frac{\partial v_1(a,\alpha_0)}{\partial a} &= -1 + \frac{\frac{T}{2}(a-a_\gamma) - K}{\sqrt{K^2 + \frac{\tau\alpha_0}{T-\tau-2\alpha_0}\left(2K(a-a_\gamma) - \frac{T}{2}(a-a_\gamma)^2\right)}} \\
\frac{T-\tau}{\tau}\frac{\partial^2 v_1(a,\alpha_0)}{\partial a^2} &= \sqrt{\frac{T-\tau-2\alpha_0}{\alpha_0}}\frac{\frac{1}{2}\frac{K^2}{\alpha_0}(T-\tau)(T-2\alpha_0)}{\sqrt{\frac{T-\tau-2\alpha_0}{\alpha_0}K^2 + \tau\left(2K(a-a_\gamma) - \frac{T}{2}(a-a_\gamma)^2\right)}^3}
\end{aligned}$$

Since $\frac{\partial^2 v_1(a,\alpha_0)}{\partial a^2}$ is strictly positive, we conclude that $\frac{T-\tau}{\tau}\frac{\partial v_1(a,\alpha_0)}{\partial a}$ is strictly increasing over the interval $a \in [a_\gamma, 0]$, and therefore reaches a minimum value, at $a = a_\gamma$, of

$$\min_a \left(\frac{T-\tau}{\tau}\frac{\partial v_1(a,\alpha_0)}{\partial a}\right) = -1 + \frac{-K}{\sqrt{K^2}} = 0 \text{ (since } K < 0 \text{ by (34))}$$

So v_1 is increasing on $[a_\gamma, 0]$, and so we have

$$\min_{a\in[a_\gamma,0]} v_1(a,\alpha_0) = v_1(a_\gamma,\alpha_0) = \theta_3 + \frac{2K}{T-\tau} - \frac{\sqrt{\frac{\tau(T-\tau-2\alpha_0)}{\alpha_0}}\sqrt{\left(\frac{T-\tau-2\alpha_0}{\tau\alpha_0}\right)K^2}}{T-\tau} = \theta_3 + \frac{K}{\alpha_0} \tag{66}$$

$$\max_{a\in[a_\gamma,0]} v_1(a,\alpha_0) = v_1(0,\alpha_0) = \theta_3 + \frac{2K+\tau a_\gamma}{T-\tau} \tag{67}$$

(The final equality uses the fact that, by construction, (34) sets the square rooted portion of $v_1(0,\alpha_0)$ equal to zero). So to complete the proof, by continuity in α_a, α_0, a, it suffices to prove that (i) if $\beta a_\gamma^2 > 8$ and α_0 is sufficiently close to zero, then $u_1(0,\alpha_0) \leq \min_a v_1(a,\alpha_0)$, and $u_1(-2,1) > \max_a v_1(a,\alpha_0)$;[27] and (ii) if $\beta a_\gamma^2 \leq 8$ and α_0 is sufficiently close to 1, then $u_1(0,\alpha_0) \leq \min_a v_1(a,\alpha_0)$, and $u_1(-2,\alpha_0) \geq \max_a v_1(a,\alpha_0)$ (so the range of the function $v_1(a,\alpha_0)$ is completely contained in the range of $u_1(a,\alpha_0)$, so the desired overlap follows).

[27] So, if we set $\alpha_a(0) = \alpha_0$, and $\alpha_a(-2)$ close to 1, and then choose $\alpha_a(\cdot)$ to be any continuous function connecting these two points, then we will have $\min_a u_1(a,\alpha_a(a)) \leq \min v_1$, and $\max_a u_1(a,\alpha_a(a)) \geq \max v_1$, with $u_1(a,\alpha_a(a))$ continuous in a; since the range of v_1 is completely contained in the range of u_1, the desired overlap then follows.

To this end, evaluate (59) at $(a = 0, \alpha_a = \alpha_0)$ and $(a = -2, \alpha_a)$, to obtain

$$u_1(0,\alpha_0) = \theta_3 + K - \sqrt{\frac{1-\alpha_0}{\alpha_0}}\sqrt{\frac{1-\alpha_0}{\alpha_0}K^2} = \theta_3 + \frac{K}{\alpha_0} \tag{68}$$

$$u_1(-2,\alpha_a) = \theta_3 + K + T - 2 - \sqrt{\frac{1-\alpha_a}{\alpha_a}}\sqrt{\frac{1-\alpha_0}{\alpha_0}K^2 - (T-2)(2K+T)} \tag{69}$$

Comparing (68) and (66), we immediately have $u_1(0,\alpha_0) \le \min_a v_1(a,\alpha_0)$, as desired. So, it remains to prove that (i) if $\beta a_\gamma^2 \le 8$, then $\lim_{\alpha_0\to 1}\left(u_1(-2,\alpha_0) - \left(\theta_3 + \frac{2K+\tau a_\gamma}{T-\tau}\right)\right) > 0$, and (ii) if $\beta a_\gamma^2 > 8$, then $\lim_{\alpha_0\to 0}\left(u_1(-2,1) - \left(\theta_3 + \frac{2K+\tau a_\gamma}{T-\tau}\right)\right) > 0$. For (i), suppose $\beta a_\gamma^2 > 8$, and consider the limit as $\alpha_0 \to 0$: then, using $\lim_{\alpha_0\to 0} K = 0$ and $\lim_{\alpha_0\to 0}\frac{1-\alpha_0}{\alpha_0}K^2 = \frac{T(T-2)\beta a_\gamma^2}{2(T-\tau)}$, we have

$$\lim_{\alpha_0\to 0} u_1(-2,\alpha_a) = \theta_3 + T - 2 - \sqrt{\frac{1-\alpha_a}{\alpha_a}}\sqrt{T(T-2)\left(\frac{\beta a_\gamma^2}{2(T-\tau)} - 1\right)}$$

At $\alpha_a = 1$, we then have

$$\lim_{\alpha_0\to 0}\left(u_1(-2,\alpha_a) - \max_{a\in[a_\gamma,0]} v_1(a,\alpha_0)\right) = (\theta_3 + T - 2 - 0) - \left(\theta_3 + \frac{\tau a_\gamma}{T-\tau}\right) = (T-2)\left(1 - \frac{\beta a_\gamma}{T-\tau}\right)$$

Strictly positive, as desired, by $a_\gamma < 0$.

For (ii), let $\beta a_\gamma^2 \le 8$, and use (69) and (67) to obtain

$$u_1(-2,\alpha_0) - \max_{a\in[a_\gamma,0]} v_1(a,\alpha_0) = K + T - 2 - \sqrt{\frac{1-\alpha_0}{\alpha_0}}\sqrt{\frac{1-\alpha_0}{\alpha_0}K^2 - (T-2)(2K+T)} - \tfrac{2K+\tau a_\gamma}{T-\tau}$$

Using $\tau = \beta(T-2)$ and $\Delta = \frac{T-\tau}{2}$, and taking limits as $\alpha_0 \to 1$, we then wish to show that the following expression is positive:

$$\begin{aligned}\lim_{\alpha_0\to 1} \quad & \left(\frac{\Delta-1}{\Delta}K + (T-2)\left(1 - \frac{\beta a_\gamma}{2\Delta}\right) - \sqrt{\frac{1-\alpha_0}{\alpha_0}}\sqrt{\frac{1-\alpha_0}{\alpha_0}K^2 - (T-2)(2K+T)}\right)\\ = \quad & \left(\frac{\Delta-1}{1-\beta}\right)\left(2 - \sqrt{\frac{\beta a_\gamma^2}{\Delta}}\right)\end{aligned}$$

Since $\beta a_\gamma^2 \le 8$, this is strictly positive, as desired, for any $\Delta > 2$, which is implied by (35).

Finally, for (iii), let $\beta a_\gamma^2 > 8 \Leftrightarrow a_\gamma < -3.18$, $\beta > .79202$, and set $T = 6$, $\alpha_0 = .1$. Here, by (34), we have $K = \frac{2\beta a_\gamma}{29-20\beta}\left(1 + \sqrt{29\frac{3-2\beta}{2\beta}}\right) < -2.3317$ (this is the value at $a_\gamma = -3.18$ and $\beta = .79202$), which guarantees that (65) is satisfied (so u_1, u_2 are real-valued). For the overlap of u_1, v_1, we can construct the function $\alpha_a : [-2,0] \to [\alpha_0, 1]$ by setting $\alpha_a(0) = \alpha_0 = 0.1$ (so that $u_1(0,\alpha_a(0)) = \min_{a\in[a_\gamma,0]} v_1(a,\alpha_0)$, as shown after (68)), and then setting $\alpha_a(-2) = \max\{\alpha_0, \alpha^*\}$, where α^* satisfies $u_1(-2,\alpha^*) = \max_{a\in[a_\gamma,0]} v_1(a,\alpha_0)$; by (69), (67), evaluated at $T = 6$, $\alpha_0 = 0.1$, this requires

$$\sqrt{\frac{\alpha^*}{1-\alpha^*}} = \frac{\sqrt{9K^2 - 4(2K+6)}}{4 + \frac{2}{3-2\beta}\left((1-\beta)K - \beta a_\gamma\right)} \tag{70}$$

Note that this can always be satisfied by some $\alpha^* \in (0,1)$: the LHS can take on any positive value, the RHS numerator is guaranteed to be real-valued by our proof that u_1, u_2 are real-valued, and the RHS denominator rearranges, at $T=6, \alpha_0=0.1$, to

$$\left(\underbrace{\frac{2\beta a_\gamma}{3-2\beta}}_{<0 \text{ by } a_\gamma<0}\right)\left(\underbrace{\left(\frac{1-\beta}{29-20\beta}\right)\left(1+\sqrt{29\frac{3-2\beta}{2\beta}}\right)-1}_{\text{decreasing in } \beta, \text{ max value } -0.90373 \text{ at } \beta=.79202} + \underbrace{\frac{6-4\beta}{\beta a_\gamma}}_{<0 \text{ by } a_\gamma<0,\ \beta<1}\right) > 0$$

Then: if $\max\{\alpha_0, \alpha^*\} = \alpha_0$, implying that $u_1(-2, \alpha_0) > \max_{a\in[a_\gamma,0]} v_1(a, \alpha_0)$, set $\alpha_a(a) = \alpha_0$ $\forall a \in [-2, 0]$; if $\max\{\alpha_a, \alpha^*\} = \alpha^*$, then let $\alpha_a : [-2, 0] \to [\alpha_0, 1]$ be any continuous strictly decreasing function with $\alpha_a(0) = \alpha_0$, $\alpha_a(-2) = \alpha^* > \alpha_0$; either way, $u_1(a, \alpha_a(a))$ will then be continuous in a, with $[u_1(0, \alpha_0), u_1(-2, \alpha_a(-2)] \supseteq [\min_{a\in[a_\gamma,0]} v_1(a, \alpha_0), \max_{a\in[a_\gamma,0]} v_1(a, \alpha_0)]$, implying the desired overlap: any recommendation $v_1(a, \alpha_0)$ sent by pair $\{z(a), h(a)\}$ with $a \in [a_\gamma, 0]$, coincides with the recommendation $u_1(\widetilde{a}, \alpha_a(\widetilde{a}))$ of some pair $\{x(\widetilde{a}), g(\widetilde{a})\}$ with $\widetilde{a} \in [-2, 0]$, with u_1 recommended for at least as long. ■

D.2.6 Integer Constraints

If time is discrete, so that there is an integer constraint on the times at which the expert change his advice, then our construction is most easily modified via a combination of public randomization and scaling up.

If $\beta a_\gamma^2 \leq 8$, the modification is straightforward. We first can show that all results of Lemma D7.1 and Propositions D5, D6 hold also at $\alpha_0 = 0.9$ (our original proof uses α_0 near 1 in this range).[28] Next, choose an integer T satisfying (35), and then scale everything up by a factor of 5: that is, actions u_1, v_1 are now recommended for $5(2\alpha_0) = 9$ periods, u_2 for $5(2(1-\alpha_0)) = 1$ period (after which Group II, III types reveal the truth), and the game now lasts $5T$ periods. The only difficulty is the time at which Group I pairs reveal the truth, originally $T - \tau$, with (by (20)) $\tau = \beta(T-2)$, $\beta \in (0, 1)$ a continuous function of the bias: for most biases, if $5T$ is an integer, then $5(T-\tau)$ is not. The easiest solution is to choose the two integers $t_1 < 5(T-\tau) < t_2$ closest to $5(T-\tau)$, and then use a public randomization device to determine whether the expert should reveal the truth at t_1, t_2, with probabilities chosen such that the expected revelation time is $5(T-\tau)$. So long as this randomization does not take place until time t_1, all expected payoffs are unchanged for both players, at all times when they must decide whether or not to follow the prescribed strategies. So, we again have a fully revealing equilibrium.

If $\beta a_\gamma^2 > 8$, then we can show that all results of Propositions D5, D6 continue to hold at $T = 6$ and $\alpha_0 = 0.1$, and that all actions are real-valued at these parameters. Now, again scale all time parameters up by a factor of 5, and employ public randomization as above to deal with an integer constraint on the time at which Group I pairs reveal the truth. The new complication for this range is the time at which pairs $\{x(a), g(a)\}$ (with $a \in [-2, 0]$) switch from recommendation $u_1(a, \alpha_a)$ to $u_2(a, \alpha_a)$: in expectation, we now want this to occur at time $5(2\alpha_a) = 10\alpha_a$, where $\alpha_a : [-2, 0] \to [\alpha_0, 1]$ is a (necessarily) continuous function chosen in Lemma D7.1 to guarantee overlap of u_1, v_1 : for these parameters, $\alpha_a(\cdot)$ takes on a minimum value, at $a = 0$, of $\alpha_a(0) = \alpha_0 = 0.1$, and a maximum value, at $a = -2$, of $\alpha_a(-2) = \alpha^*$, with $\alpha^* \in (0, 1)$ defined by (70). Since $10\alpha_a$ potentially takes on all values in the interval $[10\alpha_0, 10] = [1, 10]$, we clearly cannot just "scale up" to get integers. So, consider using public randomization: for each $a \in [-2, 0]$ and the corresponding value $\alpha_a(a) \in [\alpha_0, 1]$, choose the integer $n \in \{1, 2, ..., 9\}$ satisfying the condition

[28] For brevity of exposition, we do not include the proof of this claim here, but it is available on request. If the reader does not wish to take our word for it: our original proof (Theorem 1 without integer constraints) shows, for this range, that there exists $\alpha^* > 0$ such that we have an equilibrium whenever $\alpha_0 < \alpha^*$. So, one could apply our argument here by simply choosing any $\alpha_0 \in (0, \alpha^*)$ (rather than $\alpha_0 = .1$), and then scaling up by $\frac{1}{2\alpha_0}$ (rather than by 5). We describe the integer modification here using $\alpha_0 = 0.1$ simply because we have verified that it works, and we wish to illustrate that one need not scale up the horizon by an astronomical amount to obtain an equilibrium in discrete time.

$10\alpha_a(a) \in [n, n+1]$; if $10\alpha_a(a) \in (n, n+1)$, then use public randomization (in period n) to determine whether the expert should switch to u_2 after n or $n+1$ periods, such that the expected duration of $u_1(a, \alpha_a(a))$ is $10\alpha_a(a)$. In expectation, nothing changes for the expert when he makes his first recommendation (recall that all subsequent expert deviations are deterred in our construction simply via the DM's off-path beliefs), and nothing changes for the DM *except* for the period n in which the outcome of the public randomization is determined: here, the DM must now prefer both (i) choosing u_1 for one more period, then u_2 for $9-n$ more periods, then learning the truth; and (ii) choosing u_2 for $10-n$ periods, then learning the truth; to deviating to the myopically optimal action in all remaining $5T-n$ periods. Letting $D_1(a,\alpha_a), D_2(a,\alpha_a), D^*(a)$ denote (respectively) the DM's expected per-period disutility from choosing $u_1(a,\alpha_a)$, $u_2(a,\alpha_a)$, and the myopically optimal action, given posterior beliefs that assign probabilities $p(a)$, $1-p(a)$ to the two types $x(a), g(a)$ in his information set, we can write these interim IC constraints as:

$$D_1(a,\alpha_a) + (9-n)\, D_2(a,\alpha_a) \quad \leq \quad (5T-n) D^*(a) \tag{71}$$

$$(10-n) D_2(a,\alpha_a) \quad \leq \quad (5T-n) D^*(a) \tag{72}$$

Our original construction proved that there was an interval of values for $p(a)$ (which by Proposition D7 could be generated by an open set of priors) for which the following ex ante incentive constraint was satisfied, $\forall a \in [-2, 0]$:

$$2\alpha_a D_1(a,\alpha_a) + 2(1-\alpha_a) D_2(a,\alpha_a) \leq T D^*(a) \tag{73}$$

Constraint (71) is implied by (73): to see this, we just need to show that

$$\begin{aligned} \frac{D_1(a,\alpha_a) + (9-n) D_2(a,\alpha_a) + n D^*(a)}{5} &\leq 2\alpha_a D_1(a,\alpha_a) + 2(1-\alpha_a) D_2(a,\alpha_a) \\ \Leftrightarrow \quad D^*(a) &\leq \left(\frac{10\alpha_a - 1}{n}\right) D_1(a,\alpha_a) + \left(\frac{n+1-10\alpha_a}{n}\right) D_2(a,\alpha_a) \end{aligned}$$

This follows since the RHS is a weighted average of $D_1(a,\alpha_a)$ and $D_2(a,\alpha_a)$, which are both strictly greater than $D^*(a)$, by definition of $D^*(a)$ as the DM's lowest possible per-period disutility. Constraint (72), however, represents a new constraint, not guaranteed to hold in our original construction.[29] So, to prove that our modified construction constitutes an equilibrium, it remains to prove that $\forall a \in [-2, 0]$, and for an interval of values for $p(a)$, constraint (72) is satisfied.

For this: by definition of $D_2(a,\alpha_a)$ and $D^*(a)$, , we have

$$\frac{D_2(a,\alpha_a)}{D^*(a)} = \frac{p(a)\,(u_2(a,\alpha_a) - x(a))^2 + (1-p(a))\,(u_2(a,\alpha_a) - g(a))^2}{p(a) \cdot (p(a)x(a) + (1-p(a))g(a) - x(a))^2 + (1-p(a))\,(p(a)x(a) + (1-p(a))g(a) - g(a))^2}$$

Using (60) and (15), this simplifies to

$$\begin{aligned} \frac{D_2(a,\alpha_a)}{D^*(a)} \quad &= \quad \frac{p(a) + 2p(a)X(a,\alpha_a) + (X(a,\alpha_a))^2}{p(a)\,(1-p(a))} \\ \text{where } X(a,\alpha_a) \quad &\equiv \quad \frac{K - \frac{T}{2}a + \sqrt{\frac{\alpha_a}{1-\alpha_a}}\sqrt{\frac{1-\alpha_0}{\alpha_0}K^2 + (T-2)a\left(K - \frac{T}{4}a\right)}}{\theta_3 e^a} \end{aligned}$$

[29] The DM dislikes u_2 more than u_1, so is not willing to choose it for too much longer than the expected duration. This is one reason we have scaled everything up, rather than employing public randomization alone: to create a finer "time grid", so realized and expected times can be made close. The other reason is that the expert's expected payoff, at the time when he makes his initial recommendation, must be the same as in our original construction. Without "scaling up", this would require that he expect his first recommendation to last $2\alpha_0$ when he makes it: this is problematic when $2\alpha_0 < 1$.

Substituting this into (72), which rearranges as $\frac{D_2(a,\alpha_a)}{D^*(a)} \leq \frac{T-\frac{n}{5}}{2-\frac{n}{5}}$, setting $p(a) \equiv p$ for notational convenience, and evaluating at $T = 6$, $\alpha_0 = 0.1$, we can solve (72) as follows:

$$X(a,\alpha_a) \in \left[-p - \sqrt{\frac{20p(1-p)}{10-n}}, -p + \sqrt{\frac{20p(1-p)}{10-n}}\right] \ \forall a \in [-2,0] \tag{74}$$

$$\text{with } X(a,\alpha_a) = \frac{K - 3a + \sqrt{\frac{\alpha_a}{1-\alpha_a}}\sqrt{9K^2 + 4a\left(K - \frac{3}{2}a\right)}}{\theta_3 e^a} \tag{75}$$

So to complete the proof, it suffices to prove existence of a value $p \in (0,1)$ such that the following two inequalities hold:

$$\max_{a\in[-2,0],\ \alpha_a\in[\alpha_0,1)} X(a,\alpha_a) < -p + \sqrt{\frac{20p(1-p)}{10-n}} \tag{76}$$

$$\min_{a\in[-2,0],\ \alpha_a\in[\alpha_0,1)} X(a,\alpha_a) > -p - \sqrt{\frac{20p(1-p)}{10-n}} \tag{77}$$

By continuity, it will then follow that $\forall a \in [-2,0]$ and $\forall \alpha_a \in [a_0, 1)$, there is an interval of values for $p(a)$ such that (74), hence (72), is satisfied.

To this end, we first note that $X(a,\alpha_a)$ is strictly decreasing in a, for any fixed α_a : to show this, it is sufficient by (75) to prove that $0 > \frac{d}{da}\frac{\sqrt{9K^2+4a\left(K-\frac{3}{2}a\right)}}{e^a}$, that is,

$$0 > \frac{2K - 6a - \left(9K^2 + 4a\left(K - \frac{3}{2}a\right)\right)}{\sqrt{9K^2 + 4a\left(K - \frac{3}{2}a\right)}e^a} \tag{78}$$

The numerator in (78) is strictly convex, and therefore it reaches a maximum value at either $a = 0$, of $2K - 9K^2 < 0$ (by $K < 0$), or at $a = -2$, where it is again negative $\forall K < -1.5202$, which is implied by (34) for the range under consideration ($T = 6$, $\alpha_0 = 0.1$, and $\beta a_\gamma^2 > 8 \Rightarrow K < -2.3317$). We conclude that (78) holds, and so $X(a,\alpha_a)$ is maximized (given α_a) at $a = -2$, minimized at $a = 0$; it is immediate that $X(a,\alpha_a)$ is increasing in α_a, and we have from Lemma D7.1 (iii) that $\alpha_a \geq 0.1$, and $\alpha_a \leq \alpha^*$ (α^* as given by (70)). Substituting these values into (75), and using (14) to write $\theta_3 e^{-2} = a_\gamma - 2 + 2e^{-a_\gamma}$, we then have:

$$\min_{a\in[-2,0],\alpha_a\in[\alpha_0.1)} X(a,\alpha_a) = X(0,\alpha_0) = \frac{K + \sqrt{\frac{.1}{1-.1}}\sqrt{9K^2}}{\theta_3 e^a} = 0 \tag{79}$$

$$\max_{a\in[-2,0],\ \alpha_a\in[\alpha_0,1)} X(a,\alpha_a) = X(-2,\alpha^*) = \frac{K + 6 + \frac{9K^2-8K-24}{4+\frac{2}{3-2\beta}((1-\beta)K-\beta a_\gamma)}}{a_\gamma - 2 + 2e^{-a_\gamma}} \tag{80}$$

Then, we trivially have that (77) is satisfied for any $p \in (0,1)$, since the RHS is strictly negative, while the LHS is at least zero by (79). So, it remains to prove that there exists $p \in (0,1)$ for which

(76) is satisfied, $\forall a_\gamma < -3.18$; substituting (80) into (76), it becomes

$$\begin{aligned} \frac{K+6+\frac{9K^2-8K-24}{4+\frac{2}{3-2\beta}((1-\beta)K-\beta a_\gamma)}}{a_\gamma-2+2e^{-a_\gamma}} &< \max_{p\in(0,1)}\left(-p+\sqrt{\frac{20p(1-p)}{10-n}}\right),\ \forall n\in\{1,2,..,9\} \quad (81) \\ \text{where } K &= \frac{2\beta a_\gamma\left(1+\sqrt{29\left(\frac{3}{2\beta}-1\right)}\right)}{29-20\beta} \quad (82) \end{aligned}$$

(2nd line obtained by evaluating (34) at $T=6$, $\alpha_0=0.1$). The RHS of (81) reaches a maximum value, at $p=\frac{1}{2}\left(1-\sqrt{\frac{2-\frac{n}{5}}{6-\frac{n}{5}}}\right)\in(0,\frac{1}{2})$, of $\frac{1}{2}\left(\sqrt{\frac{6-\frac{n}{5}}{2-\frac{n}{5}}}-1\right)$: this is strictly increasing in n, and therefore at least $\frac{1}{2}\left(\sqrt{\frac{6-\frac{1}{5}}{2-\frac{1}{5}}}-1\right)\cong 0.397\,53$ for $n\in\{0,1,...,9\}$. To show that the LHS of (81) is below this for any $a_\gamma<-3.18$, note that (34) implies that $((1-\beta)K-\beta a_\gamma)>0$ (for all values of β, a_γ in the range under consideration), and that $9K^2-8K-24>0$ (by $a_\gamma<-3.18\Rightarrow K<-2.3317$); then, the LHS of (81) satisfies

$$\frac{K+6+\frac{9K^2-8K-24}{4+\frac{2}{3-2\beta}((1-\beta)K-\beta a_\gamma)}}{a_\gamma-2+2e^{-a_\gamma}} < \frac{K+6+\frac{9K^2-8K-24}{4}}{a_\gamma-2+2e^{-a_\gamma}} = \frac{\left(\frac{3K}{2}\right)^2-K}{a_\gamma-2+2e^{-a_\gamma}}$$

Using (82) and (20), it may easily be shown that this expression $\left(\frac{\left(\frac{3K}{2}\right)^2-K}{a_\gamma-2+2e^{-a_\gamma}}\right)$ is strictly increasing in a_γ (going to zero as $a_\gamma\to-\infty$), with a maximum value, at $a_\gamma=-3.18\Rightarrow K=-2.3317$, of 0.33919; therefore our sufficient condition (81) is indeed satisfied $\forall a_\gamma<-3.18$ and $\forall n\in\{1,...,9\}$, as desired to complete the proof.

D.3 Derivations

In this supplementary section, we explain how the functions and parameters in our fully revealing construction were chosen.

D.3.1 For the Expert:

Suppose we wanted an equilibrium in which each type $\theta\in[0,\theta_1]$ pools with a type $g(\theta)\in[\theta_2,\theta_3]$, to recommend an action $u_1(\theta)$ in period 1, $u_2(\theta)$ in period 2, and then reveal the truth for the final $T-2$ periods. The disutilities to types $\theta, g(\theta)$ from following (respectively) the strategies prescribed for types $\theta', g(\theta')$ are then

$$\begin{aligned} D_u(\theta'|\theta) &= \left(u_1(\theta')-\theta-1\right)^2+\left(u_2(\theta')-\theta-1\right)^2+(T-2)\left(\theta'-\theta-1\right)^2 \quad (83) \\ D_u(g(\theta')|g(\theta)) &= \left(u_1(\theta')-g(\theta)-1\right)^2+\left(u_2(\theta')-g(\theta)-1\right)^2+(T-2)\left(g(\theta')-g(\theta)-1\right)^2 \end{aligned}$$

In order for this to be an equilibrium, it must be that $D_u(\theta'|\theta)$ reaches a minimum over $[0,\theta_1]$ at $\theta'=\theta$ (so that type θ earns a lower disutility by telling the truth than by mimicking any other type θ' in the interval $[0,\theta_1]$), and that $D_u(g(\theta')|g(\theta))$ reaches a minimum at $g(\theta')=g(\theta)$. We can do this by simply choosing functions that satisfy the corresponding first- and second-order conditions:

beginning with the F.O.C.'s, we need

$$0 = \frac{1}{2}\left.\frac{dD_u(\theta'|\theta)}{d\theta'}\right|_{\theta'=\theta} = u_1'(\theta)\left(u_1(\theta)-\theta-1\right)) + u_2'(\theta)\left(u_2(\theta)-\theta-1\right)-(T-2) \tag{84}$$

$$0 = \frac{1}{2}\left.\frac{dD_u(g(\theta')|g(\theta))}{dg(\theta')}\right|_{\theta'=\theta} = \frac{u_1'(\theta)}{g'(\theta)}\left(u_1(\theta)-g(\theta)-1\right)+\frac{u_2'(\theta)}{g'(\theta)}\left(u_2(\theta)-g(\theta)-1\right)-(T-2) \tag{85}$$

Subtracting the 2nd expression from the 1st, we get

$$\left(u_1'(\theta)+u_2'(\theta)\right)\left(g(\theta)-\theta\right) = (T-2)(1-g'(\theta))$$

If we define $a(\theta) \equiv \ln\frac{g(\theta)-\theta}{g(\theta_1)-\theta_1}$, so that $a'(\theta) = \frac{g'(\theta)-1}{g(\theta)-\theta}$, this becomes

$$u_1'(\theta)+u_2'(\theta) = -(T-2)a'(\theta) \Rightarrow u_1(\theta)+u_2(\theta) = k_u-(T-2)a(\theta),\ k_u \text{ a constant} \tag{86}$$

Now: the disutility from telling the truth is

$$\begin{aligned} D_u(\theta|\theta) &\equiv D_u(\theta) = \left(u_1(\theta)-\theta-1\right)^2+\left(u_2(\theta)-\theta-1\right)^2+(T-2) \\ &\Rightarrow \frac{D_u'(\theta)}{2} = \left(u_1'(\theta)-1\right)\left(u_1(\theta)-\theta-1\right)+\left(u_2'(\theta)-1\right)\left(u_2(\theta)-\theta-1\right) \end{aligned} \tag{87}$$

Substituting (84) into this expression, we get

$$\begin{aligned} \frac{D_u'(\theta)}{2} &= T-2-\left(u_1(\theta)+u_2(\theta)\right)+2(\theta+1) \\ &= T+2\theta-k_u+(T-2)a(\theta)\ \text{(by (86))} \end{aligned}$$

Integrating w.r.t. θ, we get

$$D_u(\theta) = D_u(0)+2\theta\left(T+\theta-k_u\right)-2(T-2)\int_0^\theta (a(\theta'))d\theta'$$

Setting $u_1(0) \equiv u_0$ and using expression (86) to obtain $u_2(0) = k_u-(T-2)a(0)-u_1(0) = k_u-u_0$, this becomes

$$\begin{aligned} D_u(\theta) &= \underbrace{(u_0-1)^2+(k_u-u_0-1)^2+(T-2)}_{D_u(0)}+2\theta(T+\theta-k_u)+2(T-2)\int_0^\theta a(\theta')d\theta' \\ &= 2\left(\frac{k_u}{2}-u_0\right)^2+2\left(\frac{k_u}{2}-\theta-1\right)^2+2\theta\left(T-2\right)+2(T-2)\int_0^\theta a(\theta')d\theta'+(T-2) \end{aligned}$$

It will be convenient to change variables: rather than describing g as a function from $[0,\theta_1] \to [\theta_2,\theta_3]$, and using $a(\theta) \equiv \ln\frac{g(\theta)-\theta}{g(0)-0}$, we "flip" variables, describing each type $\theta \in [0,\theta_1]$ as a parametric function $x(a)$ of the variable a, and each type in $[\theta_2,\theta_3]$ as a parametric function $g(a)$ of the variable a, where a takes on all values between 0 and $a_1 = \ln\frac{g(\theta_1)-\theta_1}{g(0)-0}$, and $g(a), x(a)$ hold the relationship $(g(a)-x(a)) = (g(0)-x(0))\,e^a$. With this, rewriting $\int_0^\theta a(\theta')d\theta'$ as $\int_0^a sx'(s)ds$, and noting that $\theta = \int_0^a x'(s)ds$, our above disutility expression for type $\theta = x(a)$ becomes

$$D_u(x(a)) = 2\left(\frac{k_u}{2} - x(a) - 1\right)^2 + 2(T-2)\int_0^a (s+1)x'(s)ds + (T-2) + 2(T-2)C_u \qquad (88)$$

where $C_u \equiv \frac{\left(\frac{k_u}{2}-u_0\right)^2}{T-2}$ may be any non-negative constant. Setting this equal to type $x(a)'s$ truth-telling disutility (evaluate (87) at $\theta = x(a)$), using $u_2(a) = k - (T-2)a - u_1(a)$ (from (86)), and solving for $u_1(a), u_2(a)$, we obtain

$$\begin{aligned} u_1(a) &= \frac{k_u}{2} - \frac{T-2}{2}a - \sqrt{T-2}\sqrt{C_u + \frac{k_u}{2}a - a(x(a)+1) - \frac{T-2}{4}a^2 + \int_0^a (s+1)x'(s)ds} \qquad (89) \\ &= \frac{k_u}{2} - \frac{T-2}{2}a - \sqrt{T-2}\sqrt{C_u + \frac{k_u}{2}a - a(g(a)+1) - \frac{T-2}{4}a^2 + \int_0^a (s+1)g'(s)ds} \end{aligned}$$

with $u_2(a) = k_u - (T-2)a - u_1(a)$. Evaluating this at $x(a) = \theta_3 + a - \theta_3 e^a$ and $\frac{k_u}{2} = K + \theta_3$ gives precisely our expression $u_1(a, \alpha_a)$ in (16) evaluated at $\alpha_a = \frac{1}{2}$; the expressions in (16), (17) were "rescaled" (via the coefficients on the square roots) such that both disutility and average actions are independent of α_a.

Now, for our S.O.C.'s: differentiating (83) w.r.t. θ' gives

$$\begin{aligned} \frac{1}{2}\frac{dD_u(\theta'|\theta)}{d\theta'} &= \underbrace{u_1'(\theta')\left(u_1(\theta') - \theta' - 1\right) + u_2'(\theta')\left(u_2(\theta') - \theta' - 1\right) - (T-2)}_{=0 \text{ by } (84)} \\ &\quad + (\theta' - \theta)\underbrace{\left(u_1'(\theta') + u_2'(\theta') + T - 2\right)}_{=\frac{d}{d\theta'}(u_1(\theta') + u_2(\theta') + (T-2)\theta')} \end{aligned}$$

This implies that a sufficient condition for truth-telling to indeed yield a *minimum* on disutility is that the average action induced by each type θ, $u_1(\theta) + u_2(\theta) + (T-2)\theta$, be increasing: in this case, $\frac{dD_u(\theta'|\theta)}{d\theta'}$ is positive for any $\theta' > \theta$ (as type θ contemplates mimicking types θ' further above him, disutility increases, making him worse off), and negative for $\theta' < \theta$ (as he moves further below the truth, disutility increases, also making him worse off), but zero at $\theta' = \theta$: thus, telling the truth is better than mimicking any other type in the interval.

To sum up, this has shown that given arbitrary interval endpoints $\theta_1, \theta_2, \theta_3$, functions $x : [a_1, 0] \to [0, \theta_1]$ and $g : [a_1, 0] \to [\theta_2, \theta_3]$, and with $a_1 \equiv \ln \frac{g(a_1) - x(a_1)}{g(0) - x(0)} = \ln \frac{\theta_2 - \theta_1}{\theta_3 - 0}$ and $g(a) - x(a) = \theta_3 e^a$, if we want an equilibrium in which types $x(a), g(a)$ recommend $u_1(a)$ for one period, then $u_2(a)$ for one period, then separate and reveal the truth, then truth-telling satisfies the F.O.C. for disutility minimization iff u_1, u_2 are as specified by (89) and $u_2(a) = k_u - (T-2)a - u_1(a)$. If we additionally impose the requirement that average action be increasing in type, then we satisfy also the S.O.C.'s: this requires that each of $x'(a), g'(a)$ is either negative or ≥ 1. Analogously, for arbitrary functions $z : [a_\gamma, 0] \to [\theta_1, \theta_2]$, $h : [a_\gamma, 0] \to [\theta_3, \frac{1}{b}]$, with $a_\gamma = \ln \frac{h(a_\gamma) - z(a_\gamma)}{h(0) - z(0)} = \frac{\theta_3 - \theta_2}{\frac{1}{b} - \theta_1}$ and $h(a) - z(a) = (\frac{1}{b} - \theta_1)e^a$, if we want an equilibrium in which types $z(a), h(a)$ recommend $v_1(a)$ for $\frac{T-\tau}{2}$ periods, then $v_2(a)$ for $\frac{T-\tau}{2}$ periods, then separate and reveal the truth, the F.O.C.'s for truth-telling to minimize disutility yield the following equations:

$$v_1(a) = \frac{k_v}{2} - \frac{\tau}{T-\tau}a - \sqrt{\frac{2\tau}{T-\tau}}\sqrt{C_v + \frac{k_v}{2}a - a(h(a)+1) - \frac{\tau}{2(T-\tau)}a^2 + \int_0^a (s+1)h'(s)ds} \qquad (90)$$

with $v_2(a) = k_v - \frac{2\tau}{T-\tau}a - v_1(a)$, k_v and C_v constants. And the S.O.C.'s, guaranteeing that truth-telling indeed yields a disutility *minimum* over the interval, reduce to the requirement that each of $z'(a), h'(a)$ is either negative or ≥ 1. The proof that no expert type wishes to deviate after the initial recommendation follows almost trivially from the prescribed strategies.

It remains to show that no expert type wishes to mimic the initial recommendation of any type from any *other* interval. This reduces to the additional requirements that at each endpoint $\theta_i \in \{\theta_1, \theta_2, \theta_3\}$, the average action is non-decreasing at θ_i (if discontinuous), and type θ_i is indifferent between the two sequences that he can induce. Our construction chooses the specific parametrizations $g(a) = \theta_3 + a$ and $h(a) = \frac{1}{b} + a$, with $x(a) = g(a) - \theta_3 e^a$, $z(a) = h(a) - (\frac{1}{b} - \theta_1)e^a$. Then we have $x'(a) = 1 - \theta_3 e^a \leq 1 - \theta_3 e^{-2}$, $z'(a) = 1 - 2e^{a-a_\gamma} \leq -1$, and $g'(a) = h'(a) = 1$, which clearly satisfy the S.O.C.'s (provided that $\theta_3 e^{-2} \geq 2$; we in fact will restrict to $\theta_3 e^{-2} \geq 8$). With this, the expressions in (89), (90) become (with $K \equiv \frac{k_u}{2} - \theta_3$)

$$u_1(a) = K + \theta_3 - \frac{T-2}{2}a - \sqrt{T-2}\sqrt{C_u + Ka - \frac{T}{4}a^2} \tag{91}$$

$$v_1(a) = \frac{k_v}{2} - \frac{\tau}{T-\tau}a - \sqrt{\frac{2\tau}{T-\tau}}\sqrt{C_v + \left(\frac{k_v}{2} - \frac{1}{b}\right)a - \frac{T}{2(T-\tau)}a^2} \tag{92}$$

We chose $a_1 = -2$ and $g'(a) = 1$ because this is in fact the *only* way that the indifference constraint at θ_2 can hold simultaneously with both the indifference constraint at θ_3, and the increasing-average-action requirement at θ_2. We chose $h'(a) = 1$ just for simplicity. With it, the remaining increasing-average-action constraints do not bind, and the indifference conditions reduce to the following requirements on the relationships between k_v, C_v, τ (parameters from the v_t-functions) and k_u, C_u, T (parameters from the u_t-functions):

$$\begin{array}{l} \frac{k_v}{2} - \theta_3 = \frac{2K + \tau a_\gamma}{T-\tau} \\ C_v = \frac{(T-2)C_u}{\tau} + \frac{\frac{T-\tau-2}{\tau}K^2 - 2Ka_\gamma - \frac{T}{2}a_\gamma^2}{(T-\tau)} \\ \frac{\tau}{T-2} = \beta = \frac{(\theta_2-\theta_1)(\theta_2-\theta_1-2)}{(\frac{1}{b}-\theta_1)(\frac{1}{b}-\theta_1-2)} \end{array} \tag{93}$$

With this, the expressions in (91), (92) simplify exactly to the expressions for $u_1(a, \alpha_a)$ in (16) and $v_1(a, \alpha_0)$ in (18), at $\alpha_a = \frac{1}{2}$, $\alpha_0 = \frac{T-\tau}{4}$; in Proposition D3, we then rescaled (16), (18) for other values of α_a, α_0 in such a way that incentives were not affected.

D.3.2 For the decision-maker:

Suppose the decision-maker receives the recommendation $u_1(a, \alpha_a)$ in period 1. If he assigns probabilities $(p_a, 1 - p_a)$ to types $x(a), g(a)$, then his disutility from following all recommendations is

$$\begin{aligned} &p_a \left(2\alpha_a \left(u_1(a, \alpha_a) - x(a)\right)^2 + 2(1-\alpha_a)\left(u_2(a, \alpha_a) - x(a)\right)^2\right) \\ &+(1-p_a)\left(2\alpha_a \left(u_1(a, \alpha_a) - g(a)\right)^2 + 2(1-\alpha_a)\left(u_2(a, \alpha_a) - g(a)\right)^2\right) \end{aligned}$$

Substituting in the expression for $u_1(a, \alpha_a)$,$u_2(a, \alpha_a)$ from (16), (17), and expanding, this becomes

$$2p_a\left(K + \theta_3 - \frac{T-2}{2}a - x(a)\right)^2 + 2(1-p_a)\left(K + \theta_3 - \frac{T-2}{2}a - g(a)\right)^2 + 2(T-2)C_u + 2(T-2)a\left(K - \frac{T}{4}a\right)$$

The best possible deviation is to choose the myopically optimal action $p_a x(a) + (1 - p_a) g(a)$ in all T periods, resulting in disutility

$$\begin{aligned} & Tp_a \left(p_a x(a) + (1-p_a)g(a) - x(a)\right)^2 + T(1-p_a)\left(p_a x(a) + 1 - p_a)g(a) - g(a)\right)^2 \\ = \; & Tp_a(1-p_a)\left(g(a) - x(a)\right)^2 \end{aligned}$$

Therefore, incentive compatibility of our strategies for the decision-maker refers that the following expression (the gain to deviating at $\{x(a), g(a)\}$) be weakly negative for all $a \in [-2, 0]$:

$$\begin{aligned} & 2p_a \left(K + \theta_3 - \frac{T-2}{2}a - x(a)\right)^2 + 2(1-p_a)\left(K + \theta_3 - \frac{T-2}{2}a - g(a)\right)^2 \\ & + 2(T-2)C_u + 2(T-2)a\left(K - \frac{T}{4}a\right) - Tp_a(1-p_a)\left(g(a) - x(a)\right)^2 \end{aligned} \tag{94}$$

Substituting in $g(a) = \theta_3 + a, x(a) = \theta_3 + a - \theta_3 e^a$ and solving for K, we obtain that the decision-maker's gain to deviating at information set $\{x(a), g(a)\}$ is negative if and only if

$$\begin{aligned} K \;\; & \in \;\; \left[a - p_a\theta_3 e^a - \sqrt{T-2}\Delta(a),\; a - p_a\theta_3 e^a + \sqrt{T-2}\Delta(a)\right] \\ \text{where } \Delta(a) \;\; & \equiv \;\; \sqrt{\frac{1}{2}p_a(1-p_a)\left(\theta_3 e^a\right)^2 + p_a\left(\theta_3 a e^a\right) - \frac{a^2}{2} - C_u} \end{aligned} \tag{95}$$

For there to exist a value of K which satisfies this expression, we need $\Delta(a)$ to be real-valued, i.e. the term in square roots must be positive; at $a = -2$, this holds iff

$$p_a \in \left[\frac{\theta_3 e^{-2} - 4}{2(\theta_3 e^{-2})} - \frac{1}{2}\sqrt{\frac{\theta_3 e^{-2} - 8}{\theta_3 e^{-2}}}, \frac{\theta_3 e^{-2} - 4}{2(\theta_3 e^{-2})} + \frac{1}{2}\sqrt{\frac{\theta_3 e^{-2} - 8}{\theta_3 e^{-2}}}\right]$$

which in turn is *possible* (for some belief system) to satisfy only if $\theta_3 \geq 8e^2$. For our construction, with $g' = h' = 1$, this corresponds to

$$a_\gamma = \ln\frac{\theta_3 - \theta_2}{\frac{1}{b} - \theta_1} = \ln\frac{2}{-a_\gamma + 2 + \theta_3 e^{-2}} \leq -1.7726 \Leftrightarrow b \leq \frac{1}{60.885}$$

That is, it is possible to satisfy the decision-maker IC constraints in a straightforward manner only if the bias satisfies $b \leq \frac{1}{60.885}$, which is why our construction specifies $b < \frac{1}{61}$.

Similarly, if the decision-maker receives the recommendation $v_1(a)$ in period 1 and assigns probabilities $q_a, 1 - q_a$ to types $z(a), h(a)$, his maximum gain to deviating is

$$\begin{aligned} & q_a\left(2\alpha_0\left(v_1(a,\alpha_0) - z(a)\right)^2 + (T - \tau - 2\alpha_0)\left(v_2(a,\alpha_0) - z(a)\right)^2\right) \\ & + (1-q_a)\left(2\alpha_0\left(v_1(a,\alpha_0) - h(a)\right)^2 + (T - \tau - 2\alpha_0)\left(v_2(a,\alpha_0) - h(a)\right)^2\right) - Tq_a(1-q_a)\left(h(a) - z(a)\right)^2 \end{aligned}$$

Recalling that the expressions in $(18), (19)$ were scaled to make the above expression independent of α_0, we can without loss of generality set $\alpha_0 = 1$, in which case v_1 is given by (92), and $v_2 = k_v - \frac{2\tau}{T-\tau}a - v_1$; substituting into the above expression for the decision-maker's gain to deviating, we obtain

$$(T-\tau)\left(q_a\left(\frac{k_v}{2}-\frac{\tau}{T-\tau}a-z\right)^2+(1-q_a)\left(\frac{k_v}{2}-\frac{\tau}{T-\tau}a-h\right)^2\right)+2\tau C_v$$
$$+2\tau a\left(\frac{k_v}{2}-\frac{1}{b}-\frac{T}{2(T-\tau)}a\right)-Tq_a(1-q_a)(h-z)^2 \tag{96}$$

Setting $h(a)=\frac{1}{b}+a$, $z(a)=\frac{1}{b}+a-(\frac{1}{b}-\theta_1)e^a=\frac{1}{b}+a-2e^{a-a_\gamma}$ and solving for $\frac{k_v}{2}$, we obtain that the decision-maker's gain to deviating at $\{z(a),h(a)\}$ is negative if and only if

$$\frac{k_v}{2} \in \left[\frac{1}{b}+a-q_a\left(2e^{a-a_\gamma}\right)-\sqrt{\frac{2\tau}{T-\tau}\widetilde{\Delta}(a)},\ \frac{1}{b}+a-q_a\left(2e^{a-a_\gamma}\right)+\sqrt{\frac{2\tau}{T-\tau}\widetilde{\Delta}(a)}\right]$$
$$\text{where } \widetilde{\Delta}(a) \equiv \sqrt{\frac{1}{2}q_a(1-q_a)(2e^{a-a_\gamma})^2+q_a\left(2ae^{a-a_\gamma}\right)-\frac{a^2}{2}-C_v}$$

This constraint by itself is problematic. To understand the difficulty, note that at $a=a_\gamma$, there exists a value of k_v satisfying the above expression only if $\widetilde{\Delta}(a_\gamma)$ is real-valued, requiring

$$2q_a(1-q_a+a_\gamma)-\frac{a_\gamma^2}{2}-C_v\geq 0$$

We showed in the previous paragraph that the IC constraints at information sets of the form $\{x(a),g(a)\}$ can only hold if $a_\gamma\leq -1.7726$, in this case, the first term in the above inequality is negative (since we need $q_a\geq 0$ and $1+a_\gamma<0$), the second term is clearly negative, and the third must be negative (i.e. we need $C_v\geq 0$) in order for the functions $v_1,v_2$to be real-valued at $a=0$. Therefore, if the decision-maker finds it optimal to follow all recommendations sent by pairs $\{x(a),g(a)\}$, then he necessarily will have an incentive to deviate if his information set contains only types $\{z(a_\gamma),h(a_\gamma)\}=\{\theta_2,\theta_3\}$. To solve this problem, we will "bunch" pairs - scaling our action functions such that whenever the decision-maker would have an incentive to deviate after a recommendation v_1 sent by a pair $\{z(a),h(a)\}$, he believes that the recommendation is also sent (for the same length of time) by a pair $\{x(a'),g(a')\}$, and such that the expected benefit to following the recommendation (likelihood that it was sent by the pair $(x(a'),g(a'))$, times the gain in this case) exceeds the cost (which is the likelihood that it was sent by pair $(z(a),h(a))$, times the cost in this case).

D.3.3 Decision-maker Beliefs

Our incentive constraints for the DM were specified in terms of arbitrary probabilities p_a,q_a, which in turn depend both on his prior F, and on the precise details of our construction. As explained in Section 4, we assume that the DM is Bayesian. For our construction, (5) becomes:

- after a message (or message sequence) sent by types $\{x(a),g(a)\}$, $a\in[-2,0]$:

$$\frac{p_a}{1-p_a}\equiv\frac{\Pr(x(a))}{\Pr(g(a))}=\frac{f\left(x(a)\right)}{f\left(g(a)\right)}\cdot\left|\frac{x'(a)}{g'(a)}\right|=\frac{f\left(x(a)\right)}{f\left(g(a)\right)}(\theta_3e^a-1)$$

- after a message (or message sequence) sent by types $\{z(a),h(a)\}$, $a\in[a_\gamma,0]$:

$$\frac{q_a}{1-q_a}\equiv\frac{\Pr(z(a)}{\Pr(h(a)}=\frac{f\left(z(a)\right)}{f\left(h(a)\right)}\cdot\left|\frac{z'(a)}{h'(a)}\right|=\frac{f\left(z(a)\right)}{f\left(h(a)\right)}(2e^{a-a_\gamma}-1)$$

- and, after a message sent by types $\{z(a), h(a), x(\widehat{a}(a)), g(\widehat{a}(a))\}$, with $u_1(\widehat{a}(a)\alpha_{\widehat{a}}) = v_1(a, \alpha_0)$:

$$\frac{\Pr(x(\widehat{a}(a)))}{\Pr(z(a))} = \frac{f(x(\widehat{a}))}{f(z(a))} \cdot \left| \frac{x'(\widehat{a}))}{z'(a)} \right| \cdot \left| \widehat{a}'(a) \right|$$

so, denoting $\widehat{I}(\widehat{a}) = \{x(\widehat{a}), g(\widehat{a})\}$ and $I(a) = \{z(a), h(a)\}$, the decision-maker's beliefs at the pooled (4-type) information set $\widehat{I}(\widehat{a}) \cup I(a)$ satisfy

$$\begin{aligned} \frac{P^*}{1-P^*} &\equiv \frac{\Pr(\widehat{I}(\widehat{a}))}{\Pr(I(a))} = \frac{\Pr(x(\widehat{a})) \cdot \left(1 + \frac{\Pr(g(\widehat{a}))}{\Pr(x(\widehat{a}))}\right)}{\Pr(z(a)) \cdot \left(1 + \frac{\Pr(h(a))}{\Pr(z(a))}\right)} \\ &= \frac{f(x(\widehat{a}))}{f(z(a))} \cdot \left(\frac{\theta_3 e^{\widehat{a}(a)} - 1}{2e^{a-a_\gamma} - 1} \right) \left| \widehat{a}'(a) \right| \cdot \left(\frac{q_a}{p_{\widehat{a}}} \right) \end{aligned}$$

(with p_a, q_a as defined in the first two bullet points).

E Proof of Proposition 4

To prove that the decision-maker's incentive constraints are relaxed as he becomes more patient, thus completing the proof of Proposition 4, it suffices to prove that the "time ratio" terms in (7), (8), and $(9)$are increasing in r^{DM} (so that a decrease in r^{DM} below r^E causes a decrease in the expressions, thus making deviations even less attractive). For future reference, recall that our parameter outline in Section D.2.1 specified $T - \tau \leq 4$, $2\alpha_0 \leq 2\alpha_a$, and $T \geq 6$: by (6), this implies

$$t_1(\alpha_0) \leq t_2(\alpha_a) \leq t_3 = \frac{1}{r^E} \ln\left(\frac{1}{1-2\phi r^E}\right), \; t_4 \leq \frac{1}{r^E} \ln\left(\frac{1}{1-4\phi r^E}\right), \text{ and } \widehat{T} \geq \frac{1}{r^E} \ln\left(\frac{1}{1-6\phi r^E}\right) \tag{97}$$

Now: to prove that $\left(\frac{\int_0^{t_1(\alpha_0)} e^{-r^{DM}} dt}{\int_0^{\widehat{T}} e^{-r^{DM}} dt}\right)$ and $\left(\frac{\int_0^{t_2(\alpha_a)} e^{-r^{DM}} dt}{\int_0^{\widehat{T}} e^{-r^{DM}} dt}\right)$ are increasing in r, we will show that $\frac{d}{dr}\left(\frac{\int_0^t e^{-rt} dt}{\int_0^T e^{-rt} dt}\right) > 0$ for any $t < T$ (so, by (97), this in particular holds at $t \in \{t_1(\alpha_0), t_2(\alpha_a)\}$ and $T = \widehat{T}$). We have,

$$\frac{d}{dr}\left(\frac{1-e^{-rt}}{1-e^{-rT}}\right) = \frac{\left(1-e^{-rT}\right) te^{-rt} - Te^{-rT}\left(1-e^{-rt}\right)}{\left(1-e^{-rT}\right)^2} = (T-t)e^{-r(T+t)} \frac{1 - \left(\frac{Te^{rt} - te^{rT}}{T-t}\right)}{\left(1-e^{-rT}\right)^2}.$$

This is positive whenever $\left(\frac{Te^{rt} - te^{rT}}{T-t}\right) < 1$; and since $\frac{d}{dr}\left(\frac{Te^{rt} - te^{rT}}{T-t}\right) = \frac{Tt}{(T-t)}\left(e^{rt} - e^{rT}\right) < 0$ for $t < T$, it follows that the term $\left(\frac{Te^{rt} - te^{rT}}{T-t}\right)$ is decreasing in r, hence maximized at $r = 0$, where it exactly equals 1; for any $r > 0$, the term $\left(\frac{Te^{rt} - te^{rT}}{T-t}\right)$ is strictly below 1, implying that $\frac{d}{dr}\left(\frac{1-e^{-rt}}{1-e^{-rT}}\right) > 0$, as desired.

Next, we prove that $\left(\frac{\int_{t_1(\alpha_0)}^{t_4} e^{-r^{DM}} dt}{\int_{t_1(\alpha_0)}^{\widehat{T}} e^{-r^{DM}} dt}\right)$ is increasing in $e^{-r^{DM}}$. We have,

$$\frac{d}{dr}\left(\frac{e^{-rt_1} - e^{-rt_4}}{e^{-rt_1} - e^{-r\widehat{T}}}\right) = \frac{(T-t_4)e^{-r(T+t_4)} + (t_4 - t_1)e^{-r(t_4+t_1)} - (T-t_1)e^{-r(T+t_1)}}{\left(e^{-rt_1} - e^{-r\widehat{T}}\right)^2}$$

We want to show that this expression is positive $\forall r > 0$. Suppose, by contradiction, that it is negative: then,

$$\frac{(T-t_4)}{(T-t_1)}e^{-r(T+t_4)} + \frac{(t_4-t_1)}{(T-t_1)}e^{-r(t_4+t_1)} - e^{-r(T+t_1)} < 0 \tag{98}$$

The derivative of the LHS of this expression w.r.t. r, divided by $(T+t_1)$, is

$$-\left(\frac{T-t_4}{T-t_1}\right)\left(\frac{T+t_4}{T+t_1}\right)e^{-r(T+t_4)} - \left(\frac{t_4-t_1}{T-t_1}\right)\left(\frac{t_4+t_1}{T+t_1}\right)e^{-r(t_4+t_1)} + e^{-r(T+t)}$$

Substituting in $e^{-r(T+t_1)} > \frac{(T-t_4)}{(T-t_1)}e^{-r(T+t_4)} + \frac{(t_4-t_1)}{(T-t_1)}e^{-r(t_4+t_1)}$ from (98) , and factoring, we obtain that this is greater than

$$\left(\frac{T-t_4}{T-t_1}\right)\frac{(t_4-t_1)}{(T-t_1)}\left(e^{-r(t_4+t_1)} - e^{-r(t_4+T)}\right)$$

which is strictly positive $\forall r > 0$ by $t_1 < t_4 < T$. That is, the LHS of (98) is increasing in r whenever it is negative; therefore, if it is negative at some $\widehat{r} > 0$, it is also negative for all $r < \widehat{r}$. So, in particular, (98) can only hold for some $\widehat{r} > 0$ if it also holds strictly at $r = 0$; but since this is in fact *not* the case – the LHS of (98) is exactly zero at $r = 0$ – we conclude that (98) cannot hold for any $r > 0$. Therefore, as desired, we have that $\frac{d}{dr}\left(\frac{e^{-rt_1}-e^{-rt_4}}{e^{-rt_1}-e^{-r\widehat{T}}}\right) > 0\ \forall r > 0$.

And finally, to prove that $\frac{\int_{t_1(\alpha_0)}^{t_4} e^{-r^{DM}}dt}{\int_0^{\widehat{T}} e^{-r^{DM}}dt}$ and $\frac{\int_{t_2(\alpha_a)}^{t_3} e^{-r^{DM}}dt}{\int_0^{\widehat{T}} e^{-r^{DM}}dt}$ are increasing in r^{DM}, we will show that the condition $\frac{d}{dr}\left(\frac{\int_t^{t+\Delta} e^{-r\tau}d\tau}{\int_0^{\widehat{T}} e^{-r\tau}d\tau}\right) > 0$ becomes more difficult to satisfy as $r, t, t+\Delta$ increase, and easier to satisfy as $\widehat{T}$ increases: so, it is sufficient to prove that the inequality holds if we replace $r, t, t+\Delta$ with upper bounds, and $\widehat{T}$ with a lower bound. We specifically need $\frac{d}{dr}\left(\frac{\int_t^{t+\Delta} e^{-r\tau}d\tau}{\int_0^{\widehat{T}} e^{-r\tau}d\tau}\right) > 0$ to hold for $(t, t+\Delta) \in \{(t_1(\alpha_0), t_4), (t_2(\alpha_a), t_3)\}$, $T = \widehat{T}$, and $r \leq r^E$, which gives the upper bounds (by (97)) $r = r^E$, $t = \frac{1}{r^E}\ln\left(\frac{1}{1-2\phi r^E}\right)$, and $t+\Delta = \frac{1}{r^E}\ln\left(\frac{1}{1-4\phi r^E}\right)$, and the lower bound $\widehat{T} = \frac{1}{r^E}\ln\left(\frac{1}{1-T\phi r^E}\right) \geq \frac{1}{r^E}\ln\left(\frac{1}{1-6\phi r^E}\right)$.

To this end, we differentiate $\frac{\int_t^{t+\Delta} e^{-r\tau}d\tau}{\int_0^{\widehat{T}} e^{-r\tau}d\tau} = \left(\frac{e^{-rt}-e^{-r(t+\Delta)}}{1-e^{-r\widehat{T}}}\right)$, obtaining

$$\begin{aligned}\frac{d}{dr}\left(\frac{e^{-rt}-e^{-r(t+\Delta)}}{1-e^{-r\widehat{T}}}\right) &= (1-e^{-r\widehat{T}})\left((t+\Delta)e^{-r(t+\Delta)} - te^{-rt}\right) - \widehat{T}e^{-rT}\left(e^{-rt} - e^{-r(t+\Delta)}\right) \\ &> 0 \text{ whenever } \frac{\widehat{T}}{(e^{r\widehat{T}}-1)} < \frac{(t+\Delta)e^{rt} - te^{r(t+\Delta)}}{e^{r(t+\Delta)} - e^{rt}} \end{aligned} \tag{99}$$

We first show that this becomes harder to satisfy as r increases. We have,

$$\begin{aligned}\frac{d}{dr}\left(\frac{t+\Delta-te^{r\Delta}}{e^{r\Delta}-1} - \frac{\widehat{T}}{(e^{rT}-1)}\right) &= -\Delta^2\frac{e^{r\Delta}}{(e^{r\Delta}-1)^2} + \widehat{T}^2\frac{e^{\widehat{T}r}}{\left(e^{\widehat{T}r}-1\right)^2} \\ &< 0 \text{ whenever } e^{r\widehat{T}} - \frac{\widehat{T}}{\Delta}e^{\left(\frac{T+\Delta}{2}\right)r} + \frac{\widehat{T}}{\Delta}e^{\left(\frac{\widehat{T}-\Delta}{2}\right)r} - 1 > 0 \end{aligned} \tag{100}$$

The derivative of the LHS of (100) w.r.t. r, setting $\lambda \equiv \frac{\widehat{T}}{\Delta}$, is

$$\frac{1}{2}\Delta\lambda e^{r\Delta\left(\frac{\lambda-1}{2}\right)}\left(2e^{r\Delta\left(\frac{\lambda+1}{2}\right)}+(\lambda-1)-(\lambda+1)\,e^{r\Delta}\right).$$

This is positive for any $r > 0$, since the bracketed term equals zero at $r = 0$, and is increasing in r (the derivative is $\Delta(\lambda+1)\,e^{r\Delta}\left(e^{r\Delta\left(\frac{\lambda-1}{2}\right)}-1\right)$, which is positive by $\lambda = \frac{\widehat{T}}{\Delta} > 1$). That is, the LHS of the expression in (100) is increasing in r; and since it is exactly equal to zero at $r = 0$, we conclude that (100) holds: that is, our desired inequality (99) becomes harder to satisfy as r increases. Also, $\frac{t+\Delta-te^{r\Delta}}{e^{r\Delta}-1}$ is decreasing in both t and Δ (since $\frac{d}{dt}\left(\frac{t+\Delta-te^{r\Delta}}{e^{r\Delta}-1}\right) = -1$, and $\frac{d}{d\Delta}\left(\frac{t+\Delta-te^{r\Delta}}{e^{r\Delta}-1}\right) = \frac{\left((1-r\Delta)e^{r\Delta}-1\right)}{\left(e^{r\Delta}-1\right)^2}$, which is negative: the denominator is positive, and the numerator is decreasing in r (derivative w.r.t. r is $-r\Delta^2 e^{r\Delta}$) with a maximum value, at $r = 0$, of zero).

So, it is sufficient to prove that (99) holds at $r = r^E$, $t = \frac{1}{r^E}\ln\left(\frac{1}{1-2\phi r^E}\right)$, $t+\Delta = \frac{1}{r^E}\ln\left(\frac{1}{1-4\phi r^E}\right)$, $\widehat{T} = \frac{1}{r^E}\ln\left(\frac{1}{1-\phi T r^E}\right)$ (where T is the horizon from the original construction; recall that Section D.2.1 specifies $T \geq 6$) : here, (99) becomes

$$\begin{aligned} & \frac{\frac{1}{r^E}\ln\left(\frac{1}{1-T\phi r^E}\right)}{e^{\ln\left(\frac{1}{1-T\phi r^E}\right)}-1} < \frac{\frac{1}{r^E}\ln\left(\frac{1}{1-4\phi r^E}\right)e^{\ln\left(\frac{1}{1-2\phi r^E}\right)}-\frac{1}{r^E}\ln\left(\frac{1}{1-2\phi r^E}\right)e^{\ln\left(\frac{1}{1-4\phi r^E}\right)}}{e^{\ln\left(\frac{1}{1-4\phi r^E}\right)}-e^{\ln\left(\frac{1}{1-2\phi r^E}\right)}} \qquad (101) \\ \Leftrightarrow\ & \frac{2}{T}\left(1-T\phi r^E\right)\ln\left(1-T\phi r^E\right)-\left(1-4\phi r^E\right)\ln\left(1-4\phi r^E\right)+\left(1-2\phi r^E\right)\ln\left(1-2\phi r^E\right) > 0 \end{aligned}$$

Note that the LHS of (101) exactly equals zero at $\phi r^E = 0$; so, to show that (101) holds $\forall r^E > 0$, we just need to show that the LHS expression is increasing in ϕr^E. To this end, note that

$$\begin{aligned} & \frac{d^2}{d(\phi r^E)^2}\left(\frac{2}{T}\left(1-T\phi r^E\right)\ln\left(1-T\phi r^E\right)-\left(1-4\phi r^E\right)\ln\left(1-4\phi r^E\right)+\left(1-2\phi r^E\right)\ln\left(1-2\phi r^E\right)\right) \\ = \ & \frac{2\left(T-6+8\phi r^E\right)}{\left(1-T\phi r^E\right)\left(4\phi r^E-1\right)\left(2\phi r^E-1\right)} \end{aligned}$$

This is positive (by $T \geq 6$ and $2\phi r^E < 4\phi r^E < T\phi r^E < 1$), and therefore the first derivative w.r.t. r of the LHS of (101) reaches a minimum at $r = 0$, where it equals zero. We conclude that the LHS of (101) is increasing in ϕr^E, therefore strictly positive for any $\phi r^E \in (0, \frac{1}{T})$, as desired.

This completes the proof that all incentive constraints for the decision-maker are relaxed as he becomes more patient; therefore, our modified timeline yields a fully revealing equilibrium (for some priors) for any $r^{DM} \leq r^E$. ■

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