Optimal Auction Design under Non-Commitment *

Vasiliki Skreta†

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Abstract

First (or second) price auctions with optimally chosen reserve prices maximize revenue among all possible selling procedures when buyers are risk-neutral, ex-ante identical, and the seller commits to throw away the object for sale if no one bids above the reserve price. However, sellers seldom remove unsold items from the market: Real estate, used cars and art reappear in later auctions. This paper derives the profit-maximizing selling procedure when the seller, after each unsuccessful attempt to sell the item, updates her information about the buyers' willingness to pay and proposes an optimal selling procedure given the updated information. We show that first- (or second-) price auctions with optimally chosen reserve prices are revenue-maximizing when buyers are ex-ante identical. When buyers' valuations are drawn from different distributions, the seller maximizes revenue by assigning the good to the buyer with the highest virtual valuation if it is above a buyer-specific reserve price. Reserve prices drop over time. How much the optimal reserve prices drop depends on how the seller discounts the future. Inability to commit is costly for the seller. The revenue loss is highest for intermediate values of the discount factor and when the number of buyers is small.

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†*Leonard Stern School of Business, Kaufman Management Center, 44 West 4th Street, KMC 7-64, New York, NY 10012, USA, vskreta@gmail.com.*

The classic works on auctions (Myerson (1981); Riley and Samuelson (1981)) characterize the revenue-maximizing allocation mechanism for a risk-neutral seller who owns one object and faces a fixed number of buyers whose valuations are private information. An important assumption in these papers is that the seller commits to withdraw the item from the market in the event that it is not sold. This commitment assumption is far-fetched and often not met in reality. Christie's in Chicago auctions bottles of wine that failed to sell in earlier auctions. The U.S. government re-auctions properties that fail to sell: Lumber tracts, oil tracts, and real estate are put up for a new auction if no bidder bids above the reserve price.[1] As Porter (1995) reports, 46.8 percent of the oil and gas tracts with rejected high bids were put up for a new auction. In March 2010 the FCC announced that in 2011, it would re-auction part of the 700 MHz wireless spectrum that failed to sell in 2009. The key issue is not only that commitment is often unrealistic, but, more importantly, that an auction that is desirable with commitment may lead to poor outcomes if there is limited commitment.

The durable-good monopolist literature was first to study the effects of a seller's inability to commit to a given institution if it fails to realize all gains of trade (Bulow (1982); Gul et al. (1986); Stokey (1981). McAfee and Vincent (1997) study an auction setup where the seller behaves sequentially rationaly. These papers restrict the procedures the seller can employ (the seller chooses prices in the durable-goods papers and reservation prices in McAfee and Vincent (1997))[2] and show that the seller's inability to commit erodes monopoly profits. Here, we maintain the assumption that the seller behaves sequentially rationally, but we allow the seller to choose any selling procedure (mechanism). Our goal is to determine which procedure maximizes revenue and to investigate the extent to which allowing for general mechanisms enables the seller to mitigate revenue loss due to the lack of commitment.

We consider the following scenario: There is a risk-neutral seller who owns a single object and faces I risk-neutral buyers. Valuations are private, independently distributed across buyers, and constant over time. The buyers and the seller interact for two periods and discount the future with the same discount factor. At the beginning of each period, the seller proposes a mechanism to sell the object. If the object is sold, the game ends; otherwise, the seller returns in the next period and offers a new mechanism. The game ends after two periods even if the object remains unsold.[3] We show that first- (or second-) price auctions with optimally chosen reserve prices are revenue-maximizing when buyers are ex-ante identical. When buyers' valuations are drawn from different distributions, the seller maximizes revenue by assigning the good to the buyer with the highest virtual valuation if it is above a buyer-specific reserve price. Reserve prices drop over time. How much the optimal reserve prices drop depends on the discount factor. Inability to commit is costly for the seller. The revenue loss is highest for intermediate values of the discount factor and when the

[1] These examples are also mentioned in McAfee and Vincent (1997).

[2] Other papers that study reserve price dynamics without commitment are Burguet and Sákovics (1996), which examines cases of costly bidding, and Caillaud and Mezzetti (2004), which looks at sequential auctions of many identical units.

[3] The analysis of the case of $T = 2$ contains the most essential insights and can be carried out with less-burdensome notation. Section 7 presents an overview of the analysis of the case where $2 < T < \infty$.

number of buyers is small.

In the U.S. the FDIC runs a large number of auctions of distressed assets (real estate, in particular). Properties are auctioned off with reserve prices. In a number of cases, the initial reserve price is too high and the property is sold later with a lower reserve.[4] In both the U.S. and in Europe, fiscal crises have lead to a surge in distressed assets for sale.[5] The amounts that financial institutions recover from these sales is very important for their future solvency and the health of the financial sector. Given that unsold assets are placed back on the market, there is a lack of commitment to the initial reserve price. Our analysis establishes that auctions with reserve prices are revenue-maximizing and that revenue loss due to the lack of commitment is small when the seller is either very patient or very impatient and when the number of buyers is relatively large. This last finding says that when demand exceeds supply significantly, the impact of lack of commitment is very modest, but it also stresses that the auction design becomes more important in times when supply surges relative to demand, as happens during a crisis.

Methodologically, this is the first paper that solves for the optimal mechanism under limited commitment in a multi-agent environment setup allowing for a continuum of valuations, and for the possibility that the seller controls what agents observe–the *transparency* of mechanisms.[6] Mechanism design under non-commitment is notoriously difficult even in single-agent environments because, as the literature on the ratchet effect (Freixas et al., 1985; Laffont and Tirole, 1988) first observed, one cannot use the revelation principle,[7] which asserts that the choice of mechanism(s) is final. This implies that the designer can never change the rules in the future, even though it might become obvious that better ones exist. When this commitment assumption fails, there is no generally applicable canonical class of mechanisms.

Kumar (1985), who formulated what he refers to as the "noisy revelation principle," provides a first step towards providing a canonical class of mechanisms when the principal behaves sequentially rationally.[8] Despite this result, finding the optimal mechanism is not simple, as showing what posteriors are optimal is challenging. Bester and Strausz (2001) establish that for single-agent and finite-type models, it is without loss of generality to restrict attention to mechanisms with message spaces that have the same cardinality as the type space, and in which the agent reports his true type with strictly positive probability. However, Bester and Strausz (2000) show that this result fails in a two-agent example in which only one agent has

[4]See http://www.fdic.gov/buying/historical/index.html and McAfee et al. (2002), who thoroughly document this phenomenon.

[5]See Stovall and Tor (2011) or "Troubled European Assets Come to Market," in *The Wall Street Journal*, Feb. 5 2013.

[6]Bester and Strausz (2007) allow a mediator to execute the principal's mechanism and then release a noisy signal of the agent's choice. In other words, the mediator in Bester and Strausz (2007) controls what the principal observes. In our paper, the transparency of mechanisms controls what agents observe. This dimension of transparency is not relevant in Bester and Strausz (2007)'s model because there is only one agent.

[7]See Laffont and Tirole (1988), Salanie (1997), or, for a more recent treatment, Skreta (2006b).

[8]Kumar's principle states that without any loss, we can restrict attention to mechanisms where the agents report a probability distribution about their valuations. Kumar (1985)'s result captures the essential difference between mechanism design with and without commitment-namely, that without commitment, in all but the first period, the principal's posteriors beliefs are endogenous and depend on her previous mechanism choice.

private information.[9] Again, finding the optimal set of types that each type of the agent will be randomizing over (and with what probabilities) is not trivial.

This paper employs the approach developed in Skreta (2006b), which relies on characterizing equilibrium outcomes. Section 2 describes necessary conditions for an outcome to be $PBE-implementable$, and using these conditions as constraints, in Section 3 we formulate the seller's search for the revenue-maximizing sequentially rational auctions as a constraint maximization problem–Program NC. We solve Program NC in Section 4 and show that we can find an assessment that is a PBE and it implements its solution in Subsection 4.3. These are the two main steps of the proof of the main result Theorem 1. The value of commitment is discussed in Section 5, which also illustrates our characterization in a simple example.

To solve Program NC we analyze how the second-period optimal mechanism, which is a vector of functions, varies as a function of arbitrarily complex posteriors.[10] Lemma 1 formalizes the result that in the absence of commitment, eliciting information sequentially is costly, because the buyers at $t = 1$ anticipate that the seller will be exploiting this information at $t = 2$ and, hence, require that they be rewarded for it in advance. Based on Lemma 1, Proposition 2 shows that the best action for the seller is to pool all valuations below a cutoff until the second period of the game, when she has commitment power, since trying to separate them at $t = 1$ is too costly. Lemma 2 establishes that, at an optimum, valuations below the cutoff do not get the good and do not pay anything at $t = 1$. Lemma 3 shows that this cutoff is higher than the revenue-maximizing reserve price with commitment (the valuation where a buyer's *virtual* valuation is equal to the seller's value). Then, a solution of Program NC separates valuations in two groups: the-no-trade-at-$t = 1$-region, where all valuations below a cutoff are pooled together with the lowest possible valuation and never get the good, nor pay anything at $t = 1$, and the-trade-at $t = 1$-region, where the seller assigns the object to a buyer with the highest virtual valuation at $t = 1$.

Apart from the challenge arising from the lack of a "revelation principle" result that is common to single- and multi-agent environments, there are two conceptual issues specific to multi-agent environments: The first is related to the fact that what buyers (or agents, more generally) observe at each stage–the transparency of mechanisms–critically affects their beliefs about each other, which may, in turn, affect their future behavior. For example, competing in a sealed bid, versus an open outcry, auction has different implications about what buyers learn about each other. The second difference is that with multiple agents, the mechanism designer may endogenously become privately informed over time.[11] This is possible because the designer (the seller) observes more than the agents (the buyers) observe about the behavior of their competitors; think, for instance, of sealed-bid auctions. These two issues do not arise when there is a single buyer and

[9] Evans and Reiche (2008) show that the revelation principle does extend in the Bester and Strausz (2000) example if one allows ex-ante payments, but it fails even with ex-ante payments if more than one agent has private information.

[10] This problem is much more challenging compared to the one in Skreta (2006b). In that paper, the optimal $t = 2$-mechanism is a posted price, and the analysis of how the $t = 2$ optimal price varies with posteriors is much simpler.

[11] For a brief account of the literature on informed-principal problems, see Skreta (2011).

compound the difficulties arising from the lack of an appropriate "revelation principle."

In Section 6 we address the issue of transparency by introducing an alternative definition of mechanisms that not only determines who gets the object and the payments as a function of buyers' behavior, but also determines what buyers know after the mechanism is played. Formally, this is done by modeling first-period mechanisms as game forms endowed with an information-disclosure policy. Then, by applying the information-disclosure irrelevance result from Skreta (2011), we show that the set of sequentially optimal mechanisms at $t = 2$ is independent from the disclosure policy used at $t = 1$. We conclude that without any loss, we can model period-one mechanisms as game forms and assume that all buyers observe all actions chosen, which, in turn, implies that the seller does not become privately informed.

The ideas and techniques developed in the present paper have a large set of potential applications. One area in which the designer (in this case, the buyer) chooses mechanisms sequentially is that of government procurement and, in particular, defense procurement:[12] There are typically multiple stages until the final winner is determined; in each of these stages, sellers submit bids, and based on the bids, a subset of them advances to the next stage. If a bidder signals too much about his private information early on, his rents may be reduced at a later stage. Also, how bidders compete at each stage may depend on the information they obtain about their competitors in earlier stages, so the issue of transparency arises here, too.

Our techniques could be applicable in situations where other issues relating to transparency–privacy,[13] for example, are important.[14] Nowadays, keeping track of buyers has become easy and inexpensive, and the old theories that had firms or sellers treating buyers as anonymous are enhanced by ones in which the sellers not only keep track of the buyers with whom they are dealing, but also design their pricing schemes (coupons, loyalty programs) based on the information that they have obtained thus far.[15] When sellers track their interactions with various buyers, the issues of ratcheting and transparency are central.

Other related literature: "No sale" is not the only form of inefficiency of the classical optimal auction. Sometimes, it allocates the object to a buyer other than the one with the highest valuation, thus leaving resale opportunities open for the new owner. Zheng (2002) studies optimal auctions allowing for resale. With an impressive construction, that paper derives conditions under which the optimal allocation of Myerson (1981) can also be attained by a seller who cannot prevent resale. In Zheng (2002), there is no discounting. Here, we look at the complementary problem of characterizing mechanisms for a seller who cannot prevent herself from re-auctioning the good, and we allow for discounting.

[12]Bower and Dertouzos (1993) outline some of the commitment issues that arise in defense procurement.

[13]The pioneering papers on the economics of privacy are Hirshleifer (1980); Posner (1981); Stigler (1980). For a recent survey, see Hui and Png (2006). For a paper contributing to the policy debate on privacy issues, see Varian (1996).

[14]Some recent work on mechanism design that addresses these issues, marries mechanism design theory with cryptography. Two recent contributions are Izmalkov et al. (2005, 2008).

[15]See, for instance, Acquisti and Varian (2005) and Fudenberg and Villas-Boas (2007) for an excellent and comprehensive survey of the work in this area.

Other works assume sellers that have even less commitment than the seller in this paper; in particular, the seller cannot even commit to carry out the rules of the current auction. In McAdams and Schwarz (2007), the seller, after observing the bids, cannot commit not to ask for another rounds of bids. The current paper, in contrast, assumes that the seller chooses revenue-maximizing procedures and commits at each stage to carry out the mechanism for that stage. Vartiainen (2011), too, examines a symmetric model in which buyers' types are finite and there is no discounting.[16] He allows both forms of no commitment (no commitment to current or to future mechanisms), but he assumes that all actions are publicly observable, and asks what mechanism leads to a sustainable outcome given complete lack of commitment. He shows that, essentially, only the English auction achieves sustainability. Again, this paper is different from Vartiainen (2011) in that our interest is in designing optimal mechanisms and there is discounting. We also allow for a continuum of valuations and for non-transparent mechanisms.

1 The model

A risk-neutral seller, indexed by S, owns a unit of an indivisible object, and faces I risk-neutral buyers. We use the female pronoun for the seller and male pronoun for the buyers. The set of all players–the buyers and the seller–is denoted by $\bar{I} = \{S, 1, .., I\}$. The seller's valuation is denoted by $v_S \equiv 0$ and is common knowledge, whereas that of buyer i, is denoted by v_i, is private information and is distributed on $V_i = [0, b_i]$ with $0 \leq b_i < \infty$ according to F_i, which has a continuous and strictly positive density.[17] Buyers' valuations are distributed independently of one another and remain constant over time. We use $F(v) = \times_{i \in I} F_i(v_i)$, where $v \in V = \times_{i \in I} V_i$ and $F_{-i}(v_{-i}) = \times_{\substack{j \in I \\ j \neq i}} F_j(v_j)$. Time is discrete and finite, and the game lasts $T \geq 2$ periods. Everyone discounts the future with the same discount factor $\delta \in [0, 1]$. All elements of the game, apart from the realization of the buyers' valuations, are common knowledge. The seller's goal is to maximize expected discounted revenue, whereas buyers aim to maximize expected surplus.

Players' Choices: In each period, buyer i chooses an action from a measurable set $\mathcal{A}_i$. We assume without any loss that $\mathcal{A}_i$ contains all valuations $V_i \subset \mathcal{A}_i$, but it could certainly differ from V_i. Let $\mathcal{A} = \times_{i \in I} \mathcal{A}_i$. The seller chooses a mapping $g : \mathcal{A} \to [0, 1]^{\bar{I}} \times \mathbb{R}^{\bar{I}}$, which specifies for each vector of actions a, the probability that each buyer i obtains the good at t, $q_i^t(a)$, and his expected payment at t, $z_i^t(a) \in \mathbb{R}$. Because the seller collects buyers' payments and keeps the good if no buyer gets it, we have $z_S^t(a) = -\Sigma_{i \in I} z_i^t(a)$ and $q_S^t(a) = 1 - \Sigma_{i \in I} q_i^t(a)$. A *mechanism* $\mathcal{M} = (\mathcal{A}, g)$ consists of the set of actions and the mapping g.[18]

[16] In contrast to the problem in Vartiainen (2011), without discounting, our problem is trivial: In that case, the seller would wait until the last period of the game and offer the mechanism described in Myerson (1981), obtaining the highest possible revenue given the presence of asymmetric information.

[17] The fact that we take each buyer's lowest valuation to be zero is without loss. Nothing in the analysis hinges on that.

[18] Each mechanism has the same set of actions for buyer i $\mathcal{A}_i$. This is without loss of generality. To see this, consider a case where the seller offers mechanisms where at $t = 1$ the action set is $\mathcal{A}_i^1$ and at $t = 2$ it is $\mathcal{A}_i^2$. This can be equivelantly represented by the seller offering a mechanism in both periods with action set $\mathcal{A}_i = \mathcal{A}_i^1 \cup \mathcal{A}_i^2$, and by having the mapping g^1 specify for all vectors of actions where $a_i \in \mathcal{A}_i^2 \backslash \mathcal{A}_i^1$ the outcome $q_i^1(a_i, a_{-i}) = q_i^1(\hat{a}_i, a_{-i})$ and $z_i^1(a_i, a_{-i}) = z_i^1(\hat{a}_i, a_{-i})$, for

Timing: The timing of the game is as follows: At the beginning of period $t = 1$, nature determines the buyers' valuations. Subsequently, the seller chooses a mapping g, and buyers observe g and choose their actions. If there is trade, the game ends; otherwise, we move to $t = 2$. At $t = 2$, the seller chooses a mapping g, buyers observe g and choose their actions, and so forth until we reach $t = T$. The game ends at $t = T$, irrespective of whether or not trade takes place.

Histories and Strategies: What players observe determines their beliefs about the parameters that are not commonly known. A *public history* at t consists of $\{g^{(t-1)}, \chi^{(t-1)}, a^{(t-1)}\}$, where $a^{(t-1)} = a^1, ..., a^{t-1}$ consists of all vectors of actions buyers choose up to t; $g^{(t-1)}$ consists of all mappings the seller chooses up to t, and $\chi^{(t-1)}$ keeps track of whether trade has taken place up to t.[19] The seller's set of information sets, I_S, coincides with the set of public histories. Her strategy σ_S, specifies for each element in I_S a mapping g. Now, in addition to the public history, each buyer knows the realization of his own valuation, so an information set of buyer i at the beginning of period t, after the seller chooses g^t, is $\iota_i^t = \{v_i, g^{(t)}, \chi^{(t-1)}, a^{(t-1)}\}$. A behavioral strategy of buyer i, σ_i, consists of a mapping from his information sets, denoted by I_i, to a probability distribution over actions. We require strategies and beliefs to be a perfect Bayesian equilibrium (PBE).

In our formulation, the seller observes the vector of actions buyers choose. What the seller observes determines, in some sense, her commitment power. If the seller does not observe anything, the "commitment solution" is sequentially rational. Similarly, if the seller could hire an intermediary to run the mechanism without releasing any information to her, we are again back to the commitment case studied by Myerson (1981). We assume that the seller does not have access to such intermediaries because we are interested in finding the revenue-maximizing mechanisms when the seller lacks commitment, in the sense that she cannot prevent herself from observing what happens in the current auction–the buyers actions–and from using this information to design future auctions. We also assume that all buyers observe the entire vector of actions chosen at period t. In Section 6, we establish that this is without loss, in the sense that our characterization remains unchanged even if the seller can choose what each buyer observes–that is, the seller can control the *transparency* of mechanisms.

Outcomes: The outcome of a strategy profile σ (not necessarily an equilibrium) is an *allocation rule* and a *payment rule*. The allocation rule $p_i(\sigma)(v)$, $i \in I$ specifies for all $i \in I$ the expected, discounted

some $\hat{a}_i \in \mathcal{A}_i^1$. The mapping g^2 can be amended analogously. Doing so does not add any new outcomes at period 1. Moreover, it does not alter the sequentially rational allocations at $t = 2$, since (4.3) implies that, in equilibrium, actions associated with identical $t = 1$-menus must be followed by identical $t = 2$ allocations and the proof of Proposition 2 shows that this $t = 2$ allocation is identical to the one that would be sequentially rational if we would merge all actions with identical $t = 1$-menus into one. Also, the formulation allows the action set to be larger than the type space. If the seller wants to use a mechanism with less actions then, for all "superfluous" actions $\tilde{a}_i$ and all a_{-i}, he can assign the outcome $q_i^t(a_i, a_{-i}) = q_i^t(\tilde{a}_i, a_{-i})$ and $z_i^t(a_i, a_{-i}) = z_i^t(\tilde{a}_i, a_{-i})$, corresponding to some action a_i in the restricted set of actions that the seller wants to employ.

[19] $\chi^t = \begin{cases} 1 \text{ if trade takes place at } t \\ 0 \text{ otherwise} \end{cases}.$

probability that player i ends up with the object, and $x_i(\sigma)(v)$, is the expected, discounted payment that player i will incur given σ, when the realized vector of buyers' valuations is v. These expectations are taken from the *ex-ante* point of view. Analogously, we can define the outcomes of continuation games: Specifically, fix a strategy profile σ, and suppose that the seller at $t=1$ employs a mechanism that induces the menu $\{q_i(a), z_i(a); i \in \bar{I}\}_{a\in\mathcal{A}}$.[20] Let $p_i^2(\sigma)(v,a)$ and $x_i^2(\sigma)(v,a)$ denote the allocation and payment rule arising at the continuation game that starts at $t=2$, after the vector of actions a was chosen at $t=1$, conditional on σ where the realized vector of buyers' valuations is v. To simplify the notation, we often omit σ, so, for example, we write $p_i^2(v,a)$ instead of $p_i^2(\sigma)(v,a)$. Then, let $p_i(v,a) \equiv q_i(a) + q_S(a)\delta p_i^2(v,a)$ and $x_i(v,a) \equiv z_i(a) + q_S(a)\delta x_i^2(v,a)$ denote, respectively, the probability and payment of buyer i when the vector of actions chosen at $t=1$ is a, and the realized vector of valuations is v.

Fix a strategy profile σ and let $m_i(a_i|.) : [0,b_i] \to [0,1]$ be a measurable mapping of v_i denoting the probability that buyer i is choosing action a_i at $t=1$ when his type is v_i at the strategy profile σ, and let

$$m(a|v) = \times_{i\in I} m_i(a_i|v_i) \tag{1.1}$$

denote the probability that vector a is chosen at $t=1$ when the buyers' valuations are v. Then, the allocation and payment rules implemented by a strategy profile σ are given by:

$$p_i(v) = \int_{a\in\mathcal{A}} m(a|v)\left[q_i(a) + q_S(a)\delta p_i^2(v,a)\right] da \text{ and } x_i(v) = \int_{a\in\mathcal{A}} m(a|v)\left[z_i(a) + q_S(a)\delta x_i^2(v,a)\right](v) da, \tag{1.2}$$

where $m(a|v)$ is defined in (1.1). We also let $P_i(v_i) \equiv E_{v_{-i}}[p_i(v)], X_i(v_i) \equiv E_{v_{-i}}[x_i(v)]$ (respectively, $P_i^2(v_i,a_i) \equiv E_{v_{-i},a_{-i}}\left[p_i^2(v,a)|v_i,a_i\right]$ and $X_i^2(v_i,a_i) \equiv E_{v_{-i},a_{-i}}\left[x_i^2(v,a)|v_i,a_i\right]$) denote the expectations of p_i and x_i (respectively, of p_i^2 and x_i^2) from i's perspective.

Beliefs: Let $V_i(a_i)$ denote the set that contains all valuations v_i's for which $m_i(a_i|.) > 0$; its convex hull is denoted by $\bar{V}_i(a_i) \equiv [\underline{v}_i(a_i), \bar{v}_i(a_i)]$. If $\int_{V_i(a_i)} m_i(a_i|t_i)\, dF_i(t_i) > 0$, the seller's posterior about i's valuation conditional on a_i is

$$f_i(v_i|a_i) = \begin{cases} \frac{m_i(a_i|v_i))f_i(v_i)}{\int_{V_i(a_i)} m_i(a_i|t_i)dF_i(t_i)}, \text{ for } v_i \in V_i(a_i) \\ 0 \text{ otherwise} \end{cases}. \tag{1.3}$$

Let, also, $\bar{V}_{-i}(a_{-i}) = \times_{\substack{j\in I \\ j\neq i}} \bar{V}_i(a_i)$, $\bar{V}(a) = \times_{i\in I}\bar{V}_i(a_i)$. Because buyers behave non-cooperatively, they choose their actions at $t=1$ independently from one another. Then, upon observing the vector of actions a, with $\int_{\bar{V}(a)} m(a|t) f(t)dt > 0$, the seller's posterior about the buyers' valuations is

$$f(v|a) \equiv \begin{cases} \times_{i\in I} f_i(v_i|a_i), \text{ for } v \in V(a) \\ 0 \text{ otherwise} \end{cases}. \tag{1.4}$$

These beliefs matter when the seller keeps the object at $t=1$–that is, when $q_S(a) > 0$. This is because at a PBE, the seller's choice must be a best response at each information set conditional on beliefs, which

[20] To simplify on notation, we omit the superscript for time by writing $\{q_i(a), z_i(a); i \in \bar{I}\}_{a\in\mathcal{A}}$ instead of $\{q_i^t(a), z_i^t(a); i \in \bar{I}\}_{a\in\mathcal{A}}$.

must be consistent with Bayes' rule. At the same time, the $t = 1$ and the $t = 2$ choices of the seller together determine the feasible $m(a|v) = \times_{i \in I} m_i\,(a_i|v_i)$. This is because for buyer i to be choosing action a_i with probability less than one–that is, $m_i\,(a_i|v_i) < 1$–it must be the case that i at valuation v_i is indifferent between action a_i and some other action $\hat{a}_i$. These interdependencies are at the heart of the intricacies of the problem of characterizing revenue-maximizing auctions without commitment.

Goal and solution approach: The goal is to identify a PBE where the seller's expected revenue is maximal. As discussed in the introduction, the standard revelation principle is inapplicable given that the seller behaves sequentially rationally. We sidestep the difficulties that arise from the lack of an applicable "revelation principle" by examining equilibrium outcomes rather than strategies: The seller seeks an allocation rule and a payment rule that maximize expected discounted revenue among all allocation and payment rules implemented by a PBE of the game. In other words, the seller seeks:

$$\max_{p,x} \int_V \Sigma_{i \in I} x_i(v) dF(v) \tag{1.5}$$

subject to p, x being PBE implementable.

Our first goal is to translate the requirement that p and x be PBE-implementable into properties of p and x. We call the PBE-implementable allocation and payment rules *feasible.*

In the remainder of the paper, we analyze the solution of the problem when $T = 2$, and in Section 7, we discuss how the characterization extends, by induction, to longer-horizon games.

2 PBE-implementable allocation and payment rules

In this section, we derive necessary conditions for allocation and payment rules to be PBE-implementable.

Resource Constraints: The allocation rule p has to satisfy *resource constraints,* (**RES**): $0 \leq p_i(v)$ and $\Sigma_{i \in I} p_i(v) \leq 1$ and for all $v \in \times_{i \in I} V$ and $i \in I$.

Participation Constraints: Following Myerson (1981), we assume that the seller employs mechanisms that guarantee that buyers' expected discounted payoff is higher than their outside options, which we normalize to zero. We call these *participation constraints:*

$$\mathbf{PC}_i : U_i(p, x, v_i) \equiv P_i(v_i) v_i - X_i(v_i) \geq 0, \text{ for all } i \in I, v_i \in V_i.$$

Best-Response Constraints: At a PBE, buyer i's and the seller's strategy must be best responses at each information set. When buyer i with valuation v_i chooses some action a_i at $t = 1$, and play proceeds according to σ, his expected payoff is given by $U_i^{a_i}(v_i) \equiv P_i^{a_i}(v_i) v_i - X_i^{a_i}(v_i)$, where

$$P_i^{a_i}(v_i) \equiv \int_{a_{-i} \in \mathcal{A}_{-i}} \int_{\bar{V}_{-i}(a_{-i})} \left[q_i(a) + q_S(a) \delta p_i^2(v, a) \right] m(a_{-i}|v_{-i}) f_{-i}(v_{-i}) dv_{-i} da_{-i} \tag{2.1}$$

and $X_i^{a_i}(v_i) \equiv \int_{a_{-i}\in\mathcal{A}_{-i}} \int_{\bar{V}_{-i}(a_{-i})} \left[z_i(a) + q_S(a)\delta x_i^2(v,a)\right] m(a_{-i}|v_{-i}) f_{-i}(v_{-i}) dv_{-i} da_{-i}$.[21] For valuations that are choosing both a_i and $\hat{a}_i$ with positive probability–in other words, that are mixing between these two actions, best-response constraints imply that they must be indifferent: That is, $U_i^{a_i}(v_i) = U_i^{\hat{a}_i}(v_i)$, which implies that a.e. it must be the case that $P_i^{a_i}(v_i) = P_i^{\hat{a}_i}(v_i)$ and $X_i^{a_i}(v_i) = X_i^{\hat{a}_i}(v_i)$. Also, best-response constraints imply that there is no type of buyer i that can strictly benefit by mimicking another type of i. We call these the *anti-mimic (or incentive) constraints)* for i :

$$\mathbf{IC}_i\text{: } P_i(v_i)v_i - X_i(v_i) \geq P_i(v_i')v_i - X_i(v_i'), \text{ for all } i \in I, v_i, v_i' \in V_i.$$

Sequentially-Rationality Constraints: At a PBE, the seller's strategy must be a best response at each information set. In other words, whenever trade does not take place at $t = 1$–that is, for all vectors $a \in \mathcal{A}$, such that $q_S(a) > 0$–the continuation allocation and payment rule p^2, x^2 must be a best response given (1.4). Because all players observe the vector of actions chosen at $t = 1$, posterior beliefs are common knowledge. Then, at each continuation game at the beginning of the last period of the game $T = 2$, the seller's problem is equivalent to finding the revenue-maximizing auction in a static setup, with the only difference that beliefs are endogenous and depend on the vector of actions that was chosen at $t = 1$. Hence, we can appeal to the revelation principle as usual. Myerson (1981) solves this problem assuming that all buyers' distributions of valuations have strictly positive densities. In general, distributions of valuations may fail to have densities or fail to have strictly positive densities. In such cases, we cannot straightforwardly express the buyers' virtual valuations. We can, however, use the method of Skreta (2007) or of Monteiro and Svaiter (2010), or we can appropriately approximate the problem as we explain in Appendix E, so as to get meaningful expressions of virtual valuations.

Let $J_i(v_i\,|a_i\,)$ denote buyer i's posterior virtual valuation at $t = 2$ when he chooses a_i at $t = 1$.[22] Also, let $U_i^{2(a)}(p^2, x^2, \underline{v}_i\,(a_i))$ denote buyer i's expected payoff at $t = 2$ after the vector of actions a is chosen at $t = 1$, the seller employs mechanism p^2, x^2, and i's valuation is at its lowest possible level given a_i, namely $\underline{v}_i\,(a_i)$. From Myerson (1981)'s analysis, it follows that the seller's problem at the continuation game that starts at $t = 2$ after a vector of actions a with $q_S\,(a) > 0$ was chosen at $t = 1$, is

$$\max_{p^{2(a)},x^{2(a)}} \int_{\bar{V}(a)} \Sigma_{i\in I} p_i^2(v,a) J_i(v_i\,|a_i\,) f(v|a) dv - \Sigma_{i\in I} U_i^{2(a)}(p^2, x^2, \underline{v}_i\,(a_i)), \tag{2.2}$$

subject to: (i) $P_i^2(v_i, a)$ increasing in v_i on $\bar{V}_i\,(a_i)$; (ii) $0 \leq p_i^2(v,a) \leq 1$, (iii) $U_i^{2(a)}(p^2, x^2, \underline{v}_i\,(a_i)) \geq 0$, and (iv) $\Sigma_{i\in I} p_i^2(v,a) \leq 1$ for all $v \in \bar{V}\,(a)$.

Myerson (1981) shows that a revenue-maximizing mechanism assigns the object with probability one to

[21] We can, at the cost of additional notation, define $P_i^{a_i}$ and $X_i^{a_i}$ along sequences of actions off-the-path, (for example, when either a buyer deviated at $t = 2$ by misreporting in a mechanism at $t = 2$, or the seller proposed a mechanism not specified by his equilibrium strategy). We decide not to do so because, for our solution approach, it suffices to investigate properties of $P_i^{a_i}$ and $X_i^{a_i}$ along the equilibrium path.

[22] If the posterior is well-behaved, then it is written as usual: $J_i\,(v_i|a_i) = v_i - \frac{(1-F_i(v_i|a_i))}{f_i(v_i|a_i)}$.

the buyer (or in the case of ties, to one of the buyers) with maximal posterior virtual valuation,[23] provided that their reported valuation is above a buyer-specific reserve price. The reserve price at $t = 2$, $r_i^2(a_i)$, depends on i's action at $t = 1$. Skreta (2006a) establishes that the optimal reserve price is given by[24]

$$r_i^2(a_i) \equiv \inf\left\{ v_i \in \bar{V}_i(a_i) \text{ s.t. } \int_v^{\tilde{v}} \left[sf_i(s|a_i) - \int_s^{b_i} f_i(t|a_i)dt \right] ds \geq 0, \text{ for all } \tilde{v} \in [v, \bar{v}_i(a_i)] \right\}. \tag{2.3}$$

Note that (2.3) implies that $r_i^2(a_i) \geq \underline{v}_i(a_i)$–the smallest valuation on the support of $F_i(.|a_i))$.

Let $v_j = v_j(v_i, a)$, denote j's valuation that satisfies

$$Q(v_i, v_j|a) \equiv J_i(v_i|a_i) - J_j(v_j|a_j) = 0. \tag{2.4}$$

If $v_j < v_j(v_i, a)$, i's posterior virtual valuation is higher than j's, and the reverse if $v_j > v_j(v_i, a)$. Let $v_{-i}(v_i, a_i, a_{-i})$, be the vector consisting of all $v_j(v_i, a)$ with j different from i. An optimal allocation at $t = 2$ $p_i^{2(a)}$ is described fully by boundaries $r_i^2(a_i)$ and $v_{-i}(v_i, a_i, a_{-i})$.

$$\begin{aligned} p_i^2(v, a) &= 1 \text{ if } v_i \geq r_i^2(a_i) \text{ and } v_{-i} \leq v_{-i}(v_i, a_i, a_{-i}), \\ p_i^2(v, a) &= 0 \text{ otherwise, and} \\ x_i^2(v, a) &= p_i^2(v, a)v_i - \int_0^{v_i} p_i^2(t_i, v_{-i}, a)dt_i. \end{aligned} \tag{2.5}$$

In what follows, we call the requirement that p^2 and x^2 satisfy (2.5) sequential-rationality constraints, and we denote them by $\textbf{SRC}(a)$, since the vector of actions a summarizes the seller's relevant information at $t = 2$.

3 Formulating the seller's problem

We start by expressing the seller's revenue as a function of the allocation rule. Lemma 2 in Myerson (1981), establishes that the constraints IC_i, PC_i and RES are equivalent to: $P_i(v_i)$ is increasing in v_i; $U_i(p, x, v_i) = \int_0^{v_i} P_i(t_i)dt_i + U_i(p, x, 0)$; $U_i(p, x, 0) \geq 0$; and, finally, $\Sigma_{i\in I} p_i(v) \leq 1$, $p_i(v) \geq 0$ for all i and $v \in V$. Following Myerson (1981), the seller's expected revenue can be expressed as $\int_V \Sigma_{i\in I} x_i(v)dF(v) = \int_V \Sigma_{i\in I} p_i(v) J_i(v_i)f(v)dv - \Sigma_{i\in I} U_i(p, x, 0)$, where $J_i(v_i) \equiv v_i - \frac{(1-F_i(v_i))}{f_i(v_i)}$ denotes buyer i's (prior) virtual valuation. As is standard, we assume that $J_i(v_i)$ is strictly increasing in v_i for all i.[25]

We now formulate a constrained maximization problem, which we call **Program NC**:

$$\max_{\{m_i, q_i(a), i\in \bar{I}\}_{a\in\mathcal{A}}} \int_V \Sigma_{i\in I} p_i(v) J_i(v_i)f(v)dv - \Sigma_{i\in I} U_i(p, x, 0),$$

[23] If $J_i(v_i|a_i) = v_i - \frac{(1-F_i(v_i|a_i))}{f_i(v_i|a_i)}$ is increasing for all $v_i \in \bar{V}_i(a_i)$ and $i \in I$, the problem is regular, meaning that the point-wise optimum is incentive-compatible. If not, we replace them with their "ironed" versions according to a procedure described in Myerson (1981), Skreta (2007) or Monteiro and Svaiter (2010). In what follows, in order to avoid extra notation, when we write J_i, we will mean its *ironed* version.

[24] This expression shows how to obtain the optimal reserve price when i's distribution of valuations does not necessarily have a positive density, nor satisfies the monotone hazard rate property.

[25] This assumption allows us to avoid the complications that result from not having well-defined virtual valuations or from having to iron them and to focus on the ones that arise from the sequential-rationality constraints.

where $p_i(v) = \int_{a\in\mathcal{A}} m(a|v)\left[q_i(a) + q_S(a)\delta p_i^2(v,a)\right] da$ and subject to:

For all $a_i \in \mathcal{A}_i$, $m_i(a_i|.) : [0, b_i] \to [0,1]$ measurable and $m(a|v) = \Pi_{i\in I} m_i(a_i|v_i)$

$IC_i : P_i(v_i)$ increasing in v_i, for all $v_i \in V_i$ and $i \in I$;

$IC_i(a_i) : P_i^{a_i}(v_i) = P_i^{\hat{a}_i}(v_i)$ for $v_i \in V_i(a_i) \cap V_i(\hat{a}_i)$ a.e.;

$PC_i : U_i(p, x, 0) \geq 0$, for all i;

RES: for all $v \in V$, $0 \leq p_i(v) \leq 1$, $\Sigma_{i\in I} p_i(v) \leq 1$, and $i \in I$

and $\sum_{i=S}^{\bar{I}} q_i(a) = 1$, $q_i(a) \geq 0$, for all $i \in \bar{I}$;

$SRC(a)$: for all a s.t. $q_S(a) > 0$ and $\int_V m(a|v) f(v) dv > 0$, p^2, x^2 are given by (2.5);

$Beliefs$: posterior beliefs are given by (1.4) .

To solve the problem, we have to optimally specify the available $t = 1$ menus $\{q_i(a), z_i(a); i \in \bar{I}\}_{a\in\mathcal{A}}$; and the probabilities that buyers choose actions; that is, the m_i's, which, in turn, determine the $t = 2$ sequentially rational menus through (2.5).

The following Proposition relates the value of Program NC with what the seller can achieve at a PBE:

Proposition 1. *The value of Program NC is an upper bound for what the seller can achieve at a PBE.*

Proof. First, recall that the revelation principle applies at $t = 2$ since it is the last period of the game. Hence, the conditions we impose on each p^2, x^2 are *necessary* and *sufficient*, so they characterize the entire set of feasible continuation allocation and payment rules. However, the remaining conditions are only *necessary*, implying that the feasible set of Program 1 is a superset of the set of $PBE-$implementable allocation and payment rules. □

In what follows, we obtain a solution of Program NC, and construct an assessment that is a PBE and that implements it. Hence, the upper bound is, indeed, achieved.

4 Revenue-maximizing sequentially rational auctions

The main result of this paper is the characterization of the revenue-maximizing sequentially rational auctions. We first state the result and its implications, and then establish it.

Theorem 1. *In revenue-maximizing sequentially rational auctions, the seller allocates at $t = 1$ the good to the buyer with the highest virtual valuation if it is above a buyer-specific reserve price. If no trade takes place at $t = 1$, at $t = 2$, the seller allocates the object to the buyer with the highest posterior virtual valuation if it is above the seller's value.*

An interesting and practically relevant implication of Theorem 1 is the following Corollary:

Corollary 1. *When buyers are ex-ante symmetric, the symmetric equilibrium of the game in which the seller runs a second-price (SPA) or a first-price (FPA) auction in each period with optimally chosen reserve prices, generates maximal revenue for the seller.*

Proof. At a sequential SPA with reserve prices, conditional on submitting a bid above the reserve price, it is a weakly dominant strategy for a buyer to submit a bid equal to his true valuation. Then, at a symmetric equilibrium of an SPA with a reserve price, the object is assigned to the buyer with the highest valuation–who, due to symmetry, is the buyer with the highest virtual valuation–among all buyers that submit a bid above the reserve price that the seller has posted at $t = 1$. If no one bids above the reservation price at $t = 1$, we go to $t = 2$. Given ex-ante symmetric buyers, at a symmetric equilibrium, the buyers are symmetric at $t = 2$, as well. At $t = 2$, an SPA assigns the object to the buyer with the highest valuation–who, due to symmetry, is also the buyer with the highest posterior virtual valuation–if his valuation is above the reservation price posted at $t = 2$.[26] Similar arguments hold for first-price auctions, FPA. □

We establish Theorem 1 in Subsections 4.2-4.4 as follows: We solve Program NC and then find an assessment that is a PBE of our game and implements this solution. Then, Proposition 1 allows us to conclude that this is the highest revenue that the seller can expect at a PBE.

4.1 Benchmark: The solution ignoring sequential-rationality constraints

To better understand the solution of Program NC, it is helpful to consider the solution of a relaxed program–called Program C–which ignores the sequential-rationality constraints. Myerson (1981) solves this relaxed program and shows that, at the optimum, the object goes to the buyer with the highest virtual valuation, provided that it is above the seller's value: This "commitment solution,"as Figure 4.1 illustrates, is characterized by boundaries that determine the area where a buyer or the seller is awarded the good with probability one. The boundary between buyers i and j is determined by the equality of their virtual valuations–that is, the locus where $Q(v_i, v_j) = J_i(v_i) - J_j(v_j) = 0$; the boundary between buyer i and the seller is the reserve price r_i^* that satisfies (2.3) for the prior. With some abuse of notation, let 0 denote the action chosen by valuation 0 of a buyer. Then, using this paper's formulation, the solution can be written as: $m_i(0|v_i) = 1$ for all $v_i \in [0, r_i^*]$; $m_i(v_i|v_i) = 1$ for all $v_i > r_i^*$.[27] The optimal $t = 1$-menu is: $q_S(a_i, a_{-i}) = 1$ if for all i $a_i = 0$; otherwise, $q_S = 0$; $q_i(0, a_{-i}) = 0$ for all a_{-i}, and $q_i(v_i, v_{-i}) = 1$ if i has the maximum virtual valuation, zero otherwise, while the optimal $t = 2$-menu is: $p_i^2(v, a) = q_i(v)$ for all v and all i.

This solution trivially satisfies the sequential-rationality constraints if $\times_{i\in I}[0, r_i^*]$ is empty since, then, $q_S(a) = 0$ for all vectors of actions chosen with positive probability. If $\times_{i\in I}[0, r_i^*]$ is non-empty, however,

[26] If ties occur–a probability zero event given increasing virtual valuations–the lowest-index buyer among the ones who tie is assigned the good with probability one.
[27] Roughly, the m_i specify the optimal pooling and separating regions.

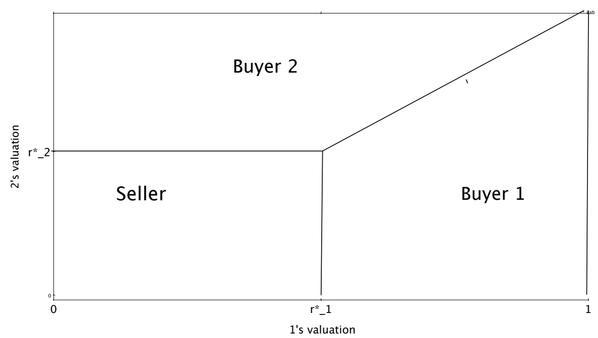

Figure 4.1: The revenue-maximizing allocation ignoring sequential-rationality constraints when $I = 2$

the solution with commitment imposes that $p_i^2(v, a) = 0$ for all i when v is in $\times_{i \in I}[0, r_i^*]$, which is not sequentially rational because when the posterior beliefs have support $[0, r_i^*]$, the seller would at least want to give the good to i when his valuation is r_i^*.[28]

4.2 The solution of Program NC

The fact that at the optimum of Program C, the solution pools the lower end of each buyer's valuations, implies that the cost of separating them (in terms of rents that they should be provided) outweighs the benefit, so the net benefit of separation is negative. When we add the sequential-rationality constraints, the cost of separation increases because, now, the rewards of separating a valuation must include the anticipated reduction in the rents that occurs at $t = 2$ because the seller exploits the information he obtained at $t = 1$.

As a first step, we show that at a solution of Program NC, the seller pools the lower end of valuations at $t = 1$; formally $m_i(0|v_i) = 1$ for all $v_i \in [0, \bar{v}_i]$ and $i \in I$, for some $\bar{v}_i \in V_i$. When $m_i(0|v_i) = 1$ for all $v_i \in [0, \bar{v}_i]$, the seller's posterior about i's valuation when i chooses action 0 at period 1 is:

$$f_i(v_i|\bar{v}_i) = \begin{cases} \frac{f_i(v_i)}{F_i(\bar{v}_i)}, & \text{for } v_i \in [0, \bar{v}_i] \\ 0, & \text{otherwise} \end{cases} . \tag{4.1}$$

[28] It is immediate to see that, then, the posterior virtual valuation at r_i^* is equal to $r_i^* > 0$.

This first result is based on the following observation:

Lemma 1. *For $v_i \in \bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i)$ a.e., it holds that $U_i^{a_i}(v_i) = U_i^{\hat{a}_i}(v_i)$, $P_i^{a_i}(v_i) = P_i^{\hat{a}_i}(v_i)$ and $X_i^{a_i}(v_i) = X_i^{\hat{a}_i}(v_i)$.*

Consider a valuation in $V_i(a_i)$ that, in addition to a_i, chooses another action $\hat{a}_i$–that is $m_i(\hat{a}_i|v_i) > 0$. Then, best-response constraints imply that $U_i^{a_i}(v_i) = U_i^{\hat{a}_i}(v_i)$. Lemma 1 establishes that $U_i^{\hat{a}_i}(v_i) = U_i^{a_i}(v_i)$ for essentially all valuations in $\bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i)$, even for those that do not choose $\hat{a}_i$. This, in turn, implies that $\frac{dU_i^{a_i}(v_i)}{dv_i} = \frac{dU_i^{\hat{a}_i}(v_i)}{dv_i}$ a.e. on $\bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i)$, which is equivalent to

$$P_i^{a_i}(v_i) = P_i^{\hat{a}_i}(v_i), \text{ a.e. on } \bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i). \tag{4.2}$$

Recalling (2.1) and (2.5), we know that i, when choosing an action a_i at $t = 1$, gets the good at t_2 when $v_{-i} \leq v_{-i}(v_i, a_i, a_{-i})$; then, (4.2) can be more explicitly rewritten as:[29], [30]

$$\begin{aligned}
&\Sigma_{a_{-i}\in\mathcal{A}_{-i}} \int_{\bar{V}_{-i}} [q_i(a) - q_i(\hat{a}_i, a_{-i})]\, m_i(a_{-i}|v_{-i}) f_{-i}(v_{-i}) dv_{-i} \\
&= \delta\Sigma_{a_{-i}\in\mathcal{A}_{-i}} \int_{\bar{V}_{-i}} \left[q_S(\hat{a}_i, a_{-i}) p_i^{2(\hat{a}_i, a_{-i})}(v) - q_S(a) p_i^{2(a)}(v)\right] m_i(a_{-i}|v_{-i}) f_{-i}(v_{-i}) dv \qquad (4.3) \\
&= \delta\Sigma_{a_{-i}\in\mathcal{A}_{-i}} \left[\int_{\underline{v}_{-i}(a_{-i})}^{v_{-i}(v_i,a_i,a_{-i})} q_S(a) m_i(a_{-i}|v_{-i}) f_{-i}(v_{-i}) dv_{-i} - \int_{\underline{v}_{-i}(a_{-i})}^{v_{-i}(v_i,\hat{a}_i,a_{-i})} q_S(\hat{a}_i, a_{-i}) m_i(a_{-i}|v_{-i}) f_{-i}(v_{-i}) dv_{-i}\right].
\end{aligned}$$

Equality (4.3) tell us (even) if mixing can reduce the probability that i gets the good at $t = 2$ (which is desirable from the ex-ante perspective), this reduction must be balanced exactly with the expected probability of trade at $t = 1$, to maintain the indifference. Hence, sustaining the mixings requires distorting the $t = 1$ probabilities at an amount equal to the expected change in the $t = 2$ probability of trade. Equation (4.3) is the basis for the proof of the following Proposition:

Proposition 2. *At a solution of Program NC, the seller pools the lower end of valuations at $t = 1$: that is $m_i(0|v_i) = 1$ for all $v_i \in [0, \bar{v}_i]$ and $i \in I$, for some $\bar{v}_i \in V_i$.*

Proof. See Appendix B. □

Lemma 1 tells us that the seller must reward separating types at $t = 1$ for the rent loss they expect at $t = 2$ due to the sequential-rationality constraints. For this reason sequential-rationality constraints increase the cost of "separations" or complicated mixing, which is the reason behind Proposition 2: The more information the seller can condition on at $t = 2$, the bigger the cost in terms of ex-ante revenue for the seller. In addition, having more than one vector of actions at which the seller keeps the good with positive probability at $t = 1$

[29] We are integrating over $\bar{V}_{-i}$ instead of over $\bar{V}_{-i}(a_{-i})$, since this makes no difference, as $m_i(a_{-i}|v_{-i}) = 0$ for all $v_{-i} \in \bar{V}_{-i} \backslash \bar{V}_{-i}(a_{-i})$.

[30] At $v_i = r_i^2(a_i)$, i's posterior virtual valuation is zero (equal to the seller's value) and $v_{-i}(v_i, a_i, a_{-i}) = r_{-i}^2(a_{-i})$, whereas at $v_i < r_i^2(a_i)$, i's posterior virtual valuation is below zero and $v_{-i}(v_i, a_i, a_{-i}) = 0_{-i}$ (i does not obtain the good at $t = 2$). Note, also, that $v_{-i}(v_i, a_i, a_{-i}) = 0$, when given a_i, trade takes place with probability one at $t = 1$ for all a_{-i}, or when i's posterior valuation is below all his competitors' posterior virtual valuations.

adds additional sequential-rationality constraints, which increase further the cost of separation. Hence, at a solution of Program NC, the seller pools the lower end of valuations at $t = 1$–that is, the seller chooses $m_i(0|v_i) = 1$ for all $v_i \in [0, \bar{v}_i]$ and $i \in I$, for some $\bar{v}_i \in V_i$.

Given this result, we now proceed to specify the optimal $t = 1$-menu corresponding to the vector of actions where all buyers choose action 0, which we denote by $\mathbf{0}$.

Lemma 2. *At an optimal first-period menu, either $q_S(\mathbf{0}) = 1$ or $q_S(\mathbf{0}) = 0$. Moreover, if $q_S(\mathbf{0}) = 1$, then $q_i(0, a_{-i}) = 0$, for all a_{-i}–that is, buyer i never obtains the good at $t = 1$ when he chooses action 0, regardless of the other buyers' actions.*

Proof. Proposition 2 establishes that at an optimum $m_i(0|v_i) = 1$ for all $v_i \in [0, \bar{v}_i]$. Then, the seller's expected revenue can be written as:

$$R = \int_{\times_{i\in I}[0,\bar{v}_i]} \Sigma_{i\in I}\left[q_i(\mathbf{0}) + \delta q_S(\mathbf{0}) p_i^2(v, \mathbf{0})\right] J_i(v_i) f(v) dv + \int_{V|\times_{i\in I}[0,\bar{v}_i]} \Sigma_{i\in I} p_i(v) J_i(v_i) f(v) dv, \tag{4.4}$$

which is linear in the menu $q_i(\mathbf{0})$, for $i \in \bar{I}$. Linearity implies that if we choose $q_S(\mathbf{0})$ so as to maximize (4.4), ignoring all but the resource constraint of Program NC, at such a relaxed solution, either $q_S(\mathbf{0}) = 1$ or $q_S(\mathbf{0}) = 0$.

When $q_S(\mathbf{0}) = 1$, then $q_i(\mathbf{0}) = 0$ for all i, which says that no buyer obtains the good at $t = 1$ for vectors of valuations in $\times_{i\in I}[0, \bar{v}_i]$. Because at a vector of actions $0, a_{-i}$,with a_{-i}, different from $\mathbf{0_{-i}}$, $v_i \in [0, \bar{v}_i]$, whereas $v_j \geq \bar{v}_j$ for all $j \in I, j \neq i$, then at an optimum $q_i(0, a_{-i}) = 0$, for all a_{-i}: If i does not get the good when $v_j \leq \bar{v}_j$ for all $j \in I, j \neq i$, he should not be getting it when competitors have higher realized valuations. □

At a revenue-maximizing allocation, the seller keeps the good when all buyers' valuations lie below some cutoff. At the "commitment solution,"the cutoff is r_i^*, the valuation where i's virtual valuation is zero (the seller's valuation). We now show that with sequential-rationality constraints, the cutoff $\bar{v}_i$ must be (weakly) larger than this "static" cutoff:

Lemma 3. *At a solution of Program NC, $\bar{v}_i \geq r_i^*$: that is, $\times_{i\in I}[0, \bar{v}_i]$ contains the region where all virtual valuations are below the seller's value.*

The intuition behind Lemma 3 is that the seller anticipates at $t = 1$ her temptation at $t = 2$ to over-assign the object compared to what is revenue-maximizing from the $t = 1$ perspective. The reason for this over-assignment is that at $t = 2$, the seller's posterior is (4.1), and i's posterior virtual valuation, which is equivalent to $v_i - \frac{[F_i(\bar{v}_i) - F_i(v_i)]}{f_i(v_i)}$, is overestimated by the term $\frac{[1-F_i(\bar{v}_i)]}{f_i(v_i)}$ compared to the prior virtual valuation $v_i - \frac{[1-F_i(v_i)]}{f_i(v_i)}$.[31]

[31] This intuition oversimplifies matters. The cutoffs (the $\bar{v}_i$'s) also determine the ranking of the buyers' posterior virtual

The optimal level of $\bar{v}_i$ depends on the discount factor δ, which, as we illustrate in examples in Section 5, determines the revenue loss due to sequential rationality.

Lemma 3 tells us that at a solution of Program NC, the seller keeps the good at $t = 1$ for all vectors of valuations where virtual valuations are below the seller's value (and maybe a larger set of valuations). This implies that in the remaining area of vectors of valuations, that is in $V \backslash \times_{i \in I} [0, \bar{v}_i]$, at least one buyer has a virtual valuation that is higher than the seller's value. For $V \backslash \times_{i \in I} [0, \bar{v}_i]$ we choose the first-period menu that maximizes (4.4) pointwise (for each v), subject to the resource constraint. Clearly, this point-wise optimum of a relaxed program is the best that the seller can achieve.

A proposed solution is: $m_i(0|v_i) = 1$ for all $v_i \in [0, \bar{v}_i]$; $m_i(v_i|v_i) = 1$ for all v_i. The optimal $t = 1$ menu is: $q_S(a_i, a_{-i}) = 1$ if for all i $a_i = 0$, otherwise $q_S = 0$; $q_i(0, a_{-i}) = 0$ for all a_{-i}, $q_i(v_i, v_{-i}) = 1$ if i has the maximum virtual valuation; zero otherwise, while p^2 is given by (2.5), for beliefs given by (4.1). The resulting allocation and payment rules are:

$$\begin{aligned} \text{for } v &\in V \backslash \times_{i \in I} [0, \bar{v}_i] : \; p_i^*(v_i, v_{-i}) = \begin{cases} 1 \text{ if } i \in \arg\max_{i \in I^1(v)} J_i(v_i) \\ 0 \text{ otherwise} \end{cases}, \\ \text{for } v &\in \times_{i \in I}[0, \bar{v}_i] : \; p_i^*(v_i, v_{-i}) = \delta p_i^2(v), \end{aligned} \tag{4.5}$$

where $I^1(v) = \{i \in I : v_i \geq \bar{v}_i\}$

$$x_i^*(v) = p_i^*(v) v_i - \int_0^{v_i} p_i^*(t_i, v_{-i}) dt_i.^{32} \tag{4.6}$$

We now verify that indeed this proposed solution is feasible for Program NC.

Lemma 4. *The allocation rule in (4.5) is feasible for Program NC.*

Proof. First, observe that p^* in (4.5) satisfies resource constraints. Moreover, it satisfies the sequential-rationality constraints since p^2 in (4.5) is given by (2.5) for beliefs given by (4.1), and $\mathbf{0}$ is the only vector of actions that leads to no trade at $t = 1$ (hence, it is the only vector of actions relevant for sequential-rationality constraints). We now show that P_i is increasing in v_i. We actually establish a stronger result-namely, that $p_i^*(v)$ is increasing in v_i for each v_{-i} : From standard arguments, it is easy to see that $p_i^*(v)$ is increasing in v_i for $v_i \in [0, b_i] \backslash \{\bar{v}_i\}$ and for all v_{-i}. Hence, it remains to show that it does not drop at $\bar{v}_i$: When at least one $v_j > \bar{v}_j$, we have that $p_i^*(\bar{v}_i - \varepsilon, v_{-i}) = 0$ (where $\varepsilon > 0$), whereas $p_i^*(\bar{v}_i + \varepsilon, v_{-i})$ is either 0 or 1. In both cases, it is increasing. Now, when for all $j \neq i$, $v_j < \bar{v}_j$, we have that $p_i^*(\bar{v}_i - \varepsilon, v_{-i})$ is equal to either 0 or δ, whereas $p_i^*(\bar{v}_i + \varepsilon, v_{-i}) = 1$ for that region of v_{-i}; hence, again, p_i^* is increasing. Finally, it is routine to verify that given the payments specified in (4.6), the participation constraints are satisfied. □

valuations. It is then conceivable that the ranking that is supported by some vector of cutoffs with $\bar{v}_i < r_i$ for some i is optimal because it reduces the costs of sequential-rationality constraints. We show (in Appendix C) that this cannot happen because, roughly, the effect of $\bar{v}_i$'s on the ranking of posterior virtual valuations is secondary.

[32]Because virtual valuations are increasing, so are posterior virtual valuations; see Lemma 6 in Appendix C. Ties, then, occur with probability zero and can be broken arbitrarily.

Remark 1. *From Lemma 4, we can conclude that (4.5) is dominant-strategy incentive-compatible since p_i^* is increasing for each realization of v_{-i}.*[33]

After substituting (4.5) in the objective function of Program 1, it reduces to the problem of finding $\bar{\mathbf{v}} = (\bar{v}_1, ..., \bar{v}_I)$, with $\bar{v}_i \in [0, b_i]$; that solves

$$\max_{\bar{\mathbf{v}} \in \times_{i\in I}[0,b_i]} R(\bar{\mathbf{v}}) \equiv \delta \int_{\times_{i\in I}[0,\bar{v}_i]} \Sigma_{i\in I} p_i^2(v_i)\, J_i(v_i) f(v) dv + \int_{V \setminus \times_{i\in I}[0,\bar{v}_i]} J^{\max}(v)\, f(v) dv, \tag{4.7}$$

where $J^{\max}(v) = \max_{i\in I}\{J_1(v_1), ..., J_I(v_I)\}$ and p^2 satisfies (2.5) for beliefs given by (4.1). This program is tremendously simpler than the one we set out to solve: Instead of maximizing over an infinite dimensional space, we are choosing the vector $\bar{v}$, which is a finite-dimensional object, out of a compact set $\times_{i\in I}[0, b_i]$. We illustrate how to obtain this solution in an example in Section 5.

4.3 Implementation

We have, thus far, obtained a solution of Program NC described in (4.5) and (4.6). Since this solution satisfies only necessary conditions of being PBE-implementable, to complete our characterization, we construct a strategy profile that is a PBE and that implements it:

Buyers' strategies: When the seller proposes the mechanism described by (4.8) and (4.9) below, at period $t = 1$, $m_i(0|v_i) = 1$ for all $v_i \in [0, \bar{v}_i]$; $m_i(v_i|v_i) = 1$ for all $v_i > \bar{v}_i$. At $t = 2$, buyers report their valuations truthfully. When the seller deviates, buyer i chooses actions described at any continuation equilibrium.

Beliefs: Given the buyers' behavior, when trade does not occur at $t = 1$, the seller's posterior beliefs along the path are given by (4.1).

Seller's Strategy: The first-period mechanism has an allocation mapping

$$q_i(v_i, v_{-i}) = \begin{cases} 1 \text{ if } i \in \arg\max_{i\in I^1(v)} J_i(v_i) \text{ and } v_i \in [\bar{v}_i, b_i] \text{ for all } i \in I \\ \quad 0 \text{ otherwise} \end{cases}, \tag{4.8}$$

where $I^1(v) = \{i \in I : v_i \in V_i \text{ and } v_i \geq \bar{v}_i\}$. At $t = 2$ the seller proposes a direct-revelation mechanism with an allocation rule described in (2.5).[34]

Buyer i pays only when he wins the object: If i wins while facing some competition at $t = 1$–that is, when $I^1(v) \neq i$–he pays the lowest possible valuation that would still allow him to win given v_{-i}–that is, $r_i^1(v_{-i}) = \inf\{v_i \text{ such that } q_i(v) = 1\}$.[35] If i does not face any competition at $t = 1$–that is, if $I^1(v) = \{i\}$–i

[33] This observation is employed later, in Section 6, to establish that it is without any loss to assume that the seller employs fully transparent mechanisms.

[34] Note that according to this mechanism, buyer i is not allocated the good if he chooses an out-of-equilibrium action.

[35] The payment rule in this region is analogous to the one described by the optimal mechanism in Myerson (1981). The difference is that, here, not all buyers report a valuation above the cutoff at $t = 1$.

pays a buyer-specific reserve price $\bar{r}_i^1$, so the payment mapping is:

$$\begin{aligned} z_i(0, a_{-i}) &= 0, \text{ for all } a_{-i} \in \mathcal{A}_{-i} \\ z_i(v_i, v_{-i}) &= \begin{cases} r_i^1(v_{-i}) \text{ if } i \in \arg\max_{i \in I_1(v)} J_i(v_i) \text{ and } I^1(v) \neq i \\ \bar{r}_i^1 \text{ if } I^1(v) = i \\ 0 \text{ otherwise} \end{cases}. \end{aligned} \tag{4.9}$$

The payment rule (4.9) is constructed from (4.6), as follows. We rewrite (4.6)

$$\begin{aligned} x_i(v) &= q_i(a(v))v_i - \int_0^{v_i} q_i(a(t_i, v_{-i}))dt_i + \\ &\quad \delta \left[q_S(a(v))p_i^2(v)v_i - \int_0^{v_i} q_S(a(t_i, v_{-i}))p_i^2(t_i, v_{-i})dt_i \right]. \end{aligned} \tag{4.10}$$

Given the proposed strategies for the buyers and the seller, for $v \in \times_{i \in I}[0, \bar{v}_i]$, (4.10) reduces to $x_i(v) = \delta \left[p_i^2(v)v_i - \int_0^{v_i} p_i^2(t_i, v_{-i})dt_i \right] = \delta x_i^2(v)$, where p^2 and x^2 are given by (2.5). This is because for $v \in \times_{i \in I}[0, \bar{v}_i]$, $q_S = 1$, and the first two terms of (4.10) are zero. Now, for $v \in V \backslash \times_{i \in I} [0, \bar{v}_i]$, (4.10) reduces to:

$$x_i(v_i, v_{-i}) = q_i(v)v_i - \int_{\bar{v}_i}^{v_i} q_i(t_i, v_{-i})dt_i - \delta \int_{\max\left\{r_i^2(\bar{v}_i), r_j^2(v_{-j})\right\}}^{\bar{v}_i} p_i^2(t_i, v_{-i})dt_i, \tag{4.11}$$

where $r_i^2(\bar{v}_i)$ is given by (2.3) given beliefs (4.1) and $r_j^2(v_{-j}) = \inf\{v_i \in [0, \bar{v}_i] \text{ such that } p_i^2(v) = 1\}$.

There are two cases to consider, depending on whether or not buyer i faces some competition at $t = 1$. When buyer i faces some competition at $t = 1$–that is, when $I^1(v) \neq \{i\}$–i pays only when he wins, and his payment is equal to $r_i^1(v_{-i})$. When $v_{-i} \in \times_{j \neq i}[0, \bar{v}_j]$, i does not face competition at $t = 1$, and using q_i from (4.8) and p^2 from (2.5), (4.11) can be further simplified to

$$x_i(v) = (1 - \delta)\bar{v}_i + \delta \max\left\{r_i^2(\bar{v}_i), r_j^2(v_{-j})\right\}. \tag{4.12}$$

However, it is not possible to implement (4.12) as is, because it varies with v_{-i}, whereas all buyers $-i$ with valuations in $[0, \bar{v}_j]$ pool at $t = 1$ and choose action 0. But given that in the case under consideration, buyer i faces no competition at $t = 1$, we can use (4.12) to determine a personalized reserve price, $\bar{r}_i^1$, that satisfies

$$\bar{r}_i^1 = \frac{1}{F_{-i}(\bar{v}_{-i})} E_{v_{-i} \in \times_{j \neq i}[0, \bar{v}_j]} \left[(1 - \delta)\bar{v}_i + \delta \max\left\{r_i^2(\bar{v}_i), r_j^2(v_{-j})\right\} \right]. \tag{4.13}$$

Since buyer i does not know v_{-i}, from his perspective, he is indifferent between paying $\bar{r}_i^1$ whenever all other buyers choose 0 (which occurs with probability $F_{-i}(\bar{v}_{-i})$), or incurring payments according to (4.12). Moreover, these two different payment methods are equivalent from the seller's perspective, because they are associated with the same allocation rule.

It is immediate to see that this strategy profile implements (4.5) and (4.6). Now, we establish that it is a PBE. Given the buyers' strategies, the seller's strategy is a sequential best response since, at $t = 2$, she proposes a direct revelation mechanism with an allocation rule described in (2.5), which is revenue-maximizing given her posterior beliefs. Since $\bar{\mathbf{v}}$ is optimally chosen, the seller cannot do any better by changing the regions where trade takes place at $t = 1$, versus $t = 2$. The fact that buyers' strategies are best responses at $t = 2$ follows immediately from the incentive compatibility of the direct mechanism that the seller employs at $t = 2$. Hence, the only requirement that we still need to verify is that buyers' strategies are best responses at $t = 1$. This follows from the incentive compatibility of (4.5) established in Proposition 4 and from the fact that payments are constructed from (4.6). Putting all the pieces together, we have shown that:

Proposition 3. *The allocation rule described in (4.5) is PBE-implementable.*

4.4 Proof of theorem 1

In Proposition 1, we argued that the value of Program NC is an upper bound for how much the seller can achieve at a PBE. We then showed that the allocation rule in (4.5) with optimally chosen $\bar{v}$ solves Program NC. Finally, Proposition 3 states that this solution can be implemented by an assessment that is a PBE. Hence, this assessment is a revenue-maximizing PBE.

5 Illustration and the value of commitment

Here, we provide an illustration of the result of Theorem 1 in a simple example: Suppose that there are I buyers whose valuations are distributed uniformly on $[0, 1]$, that the seller's valuation is zero, and that $T = 2$. For this example, the commitment benchmark–that is, a revenue-maximizing auction without the sequential-rationality constraints–is a second-price auction with a reserve price of $\bar{v}_i^C = 0.5$ for all $i \in I$. When $I = 2$, the seller's expected revenue is 0.4166.

With sequential-rationality constraints, at $t = 1$, buyer i gets the object if $v_i \geq v_j$, for all $i \neq j$, and $v_i \geq \bar{v}_i$. Given a vector of first-period cutoffs $\bar{\mathbf{v}} = (\bar{v}_1, ..., \bar{v}_I)$, the posterior is $f_i(v_i|\bar{v}_i) = \frac{1}{\bar{v}_i}$ and i's posterior virtual valuation is $J_i(v_i\,|\bar{v}_i) = 2v_i - \bar{v}_i$. In is easy to see that the optimal $t = 2$ reserve price $r_i^2\,(\bar{v}_i) = \frac{\bar{v}_i}{2}$. Then, at a revenue-maximizing mechanism at $t = 2$, buyer i obtains the object if $v_i \geq v_j - \frac{(\bar{v}_j - \bar{v}_i)}{2}$ and $v_i \geq \frac{\bar{v}_i}{2}$ for all $j \neq i$.

Imposing symmetry on the solution,[36] (4.7) for this example reduces to:

$$I\delta \int_{\frac{\bar{v}}{2}}^{\bar{v}} (2v - 1)v^{I-1} dv + I \int_{\bar{v}}^{1} (2v - 1)v^{I-1} dv. \tag{5.1}$$

It is straightforward to verify that the revenue-maximizing first-period cutoff is:

$$\bar{v} = \frac{\delta - 2^I\delta + 2^I}{\delta - 2^{I+1}\delta + 2^{I+1}}. \tag{5.2}$$

[36] Imposing symmetry on the solution ($\bar{v}_i = \bar{v}$ for all $i \in I$) allows us to obtain it analytically in a clean way. If we allow $\bar{v}_i \neq \bar{v}_j$ the expression of expected revenue becomes more complex and the solution more tedious, but of course the solution turns out to be symmetric. Details are available from the author upon request.

Comparative statics: We first observe that the first-period cutoff, described in (5.2), is increasing in δ. Differentiating (5.2) with respect to δ, we get $\frac{\partial \bar{v}}{\partial \delta} = \frac{2^I + 2^{I+1}}{(\delta - 2^{I+1}\delta + 2^{I+1})^2} > 0$. Differentiating (5.2) with respect to I, we get that $\frac{\partial \bar{v}}{\partial I} = \frac{\frac{1}{2} 2^I I \delta(\delta-1)}{(\delta - 2^{I+1}\delta + 2^{I+1})^2} \leq 0$. Hence, the first-period cutoff decreases, and it converges to $\frac{1}{2}$ as I gets large. The left panel of the following figure depicts how the first-period cutoff varies with the discount factor: and where each curve corresponds to a different number of buyers; the dotted line to $I = 2$, the

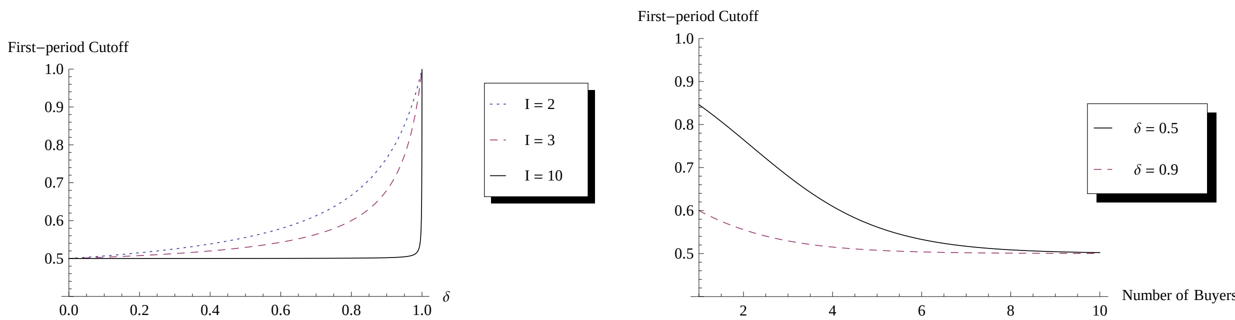

Figure 5.1: First-Period Cutoff

dashed line to $I = 3$ and the solid line to $I = 10$. The right panel depicts how the first-period cutoff varies with the number of buyers and where the dotted curve corresponds to a discount factor of $\delta = 0.5$, and the solid line corresponds to a discount factor of $\delta = 0.9$.

Expected revenue as a function of the discount factor when $I = 2$ and $T = 2$ is given by $\frac{8}{3(7\delta-8)^3}\left(53\delta^3 - 183\delta^2 + 210\delta - 80\right) - \frac{1}{12}\delta\frac{(3\delta-4)^2}{(7\delta-8)^3}(21\delta - 16)$ and is depicted by the solid line in the following Figure 5.2 below. The dotted line on the same graph depicts the expected revenue in the commitment benchmark. It is clear that the revenue loss is zero for extreme values of the discount factor and highest for δ close to 0.8. Instead of the first-period cutoff, we can also look at at the first-period reserve price. For the example under consideration and when $I = 2$, the first-period reserve price derived in (4.13) becomes $\bar{r}_i^1 = \bar{v}(1 - 0.375\delta)$, with $\bar{v} = \frac{3\delta-4}{7\delta-8}$. Interestingly, the first-period reserve price varies non-monotonically with the discount factor.

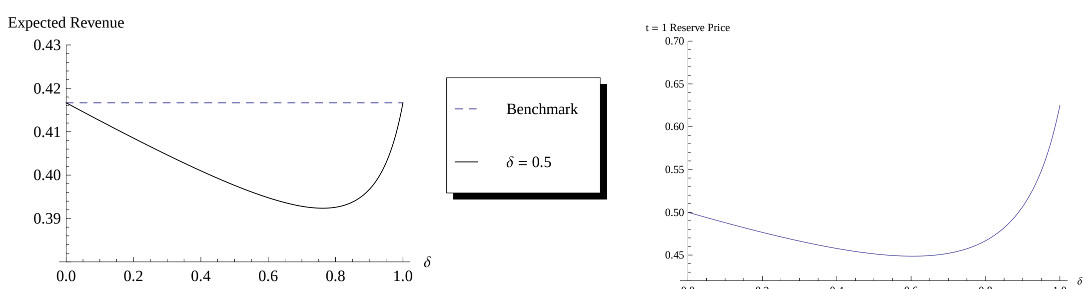

Figure 5.2: Left Panel: Revenue; Right Panel: First-period reserve price

Following an analogous procedure, we can obtain the optimal cutoffs for the case where $I = 2$ as a function of the discount factor for longer games.[37] In the following graphs, the number of buyers is held at two, and we depict the first-period cutoff as a function of the discount factor for games of various lengths.

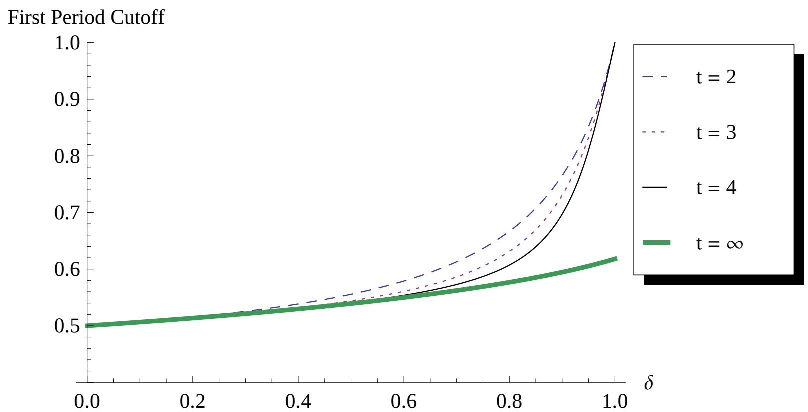

Figure 5.3: First-Period Cutoff and Length of T

The dashed line corresponds to $t = 2$, the dotted line to $t = 3$ and the solid line to $t = 4$. The last thick line is the first-period cutoff of a stationary equilibrium of the $t = \infty$ game of McAfee and Vincent (1997). This cutoff is implicitly given by a solution of $2\overline{v} - 1 = \delta\overline{v}^3$.[38]

McAfee and Vincent (1997) have studied reserve price dynamics in an infinite-horizon version of our model with symmetric buyers and the "gap case." [39] In that case, there is always some period in which trade takes place with probability one. Our results show that in the finite version of their model, the revenue a seller achieves in their equilibrium is actually the highest possible.

5.1 How much is commitment worth?

For the cases of extreme discount factors–namely, for $\delta = 0$ and for $\delta = 1$–the revenue loss due to lack of commitment is zero: When $\delta = 0$, the future does not matter at all, so the sequential-rationality constraints disappear, and the optimal vector of cutoffs is given by the vectors of valuations where all buyers' virtual valuations are equal to the seller's valuation; that is, $\overline{\mathbf{v}}(\delta = 0) = (r_1, ..., r_I)$. When $\delta = 1$, waiting is costless,

[37] For $T = 3$, we get that $v_1 = \frac{640\delta - 500\delta^2 + 88\delta^3 + 27\delta^4 - 256}{1344\delta - 1144\delta^2 + 284\delta^3 + 27\delta^4 - 512}$; $v_2 = \left(\frac{3\delta - 4}{7\delta - 8}\right) v_1$; and $v_3 = \frac{1}{2}\left(\frac{3\delta - 4}{7\delta - 8}\right) v_1$, whereas for $T = 4$, we get that

$$v_1 = \frac{4\delta + 4A^2\delta^2 - 4A^2\delta - 4A^2B^2\delta^2 + 3A^2B^2\delta^3 - 4}{8\delta + 8A^3\delta^2 - 8A^3\delta - 8A^3B^3\delta^2 + 7A^3B^3\delta^3 - 8}$$

where $A = \frac{(640\delta - 500\delta^2 + 88\delta^3 + 27\delta^4 - 256)}{1344\delta - 1144\delta^2 + 284\delta^3 + 27\delta^4 - 512}$ and $B = \left(\frac{3\delta - 4}{7\delta - 8}\right)$; $v_2 = Av_1$; $v_3 = ABv_1$ and $v_4 = \frac{1}{2}ABv_1$.

[38] More specifically, McAfee and Vincent (1997) consider a uniform $[0, 1]$, infinite-horizon, no-gap example, and establish that it has a symmetric linear stationary equilibrium characterized by two numbers: γ and r.

[39] The optimal mechanism for every discount factor in an infinite horizon version of the game is an open and quite challenging question.

so the seller can wait until the last period of the game and offer an optimal mechanism without sequential-rationality constraints. This can be achieved by selecting $\bar{\mathbf{v}}$ to be equal to the vector of the highest possible valuations of all buyers: $\bar{\mathbf{v}}(\delta = 1) = (b_1, ..., b_I)$. This can also be seen in Figure 5.1 above. For intermediate discount factors, an optimal vector of cutoffs is somewhere between $\bar{\mathbf{v}}(0)$ and $\bar{\mathbf{v}}(1)$.

For a fixed discount factor, the value of commitment increases with the length of the horizon, or, put differently, the costs of sequential rationality increase with the number of periods. McAfee and Vincent (1997) establish a generalized version of the Coase conjecture which states that in their environment (symmetric, IPV, infinite-horizon, gap), as δ approaches one, the seller's expected revenue is the same as in a game with no reserve price (that game has, by the revenue equivalence theorem, the expected revenue of the efficient allocation). Then, an upper bound of the value of commitment in terms of the seller's expected revenue is equal to the difference in revenue between the optimal Myerson benchmark and the revenue generated by an efficient auction.

Theorem 1 implies that when the horizon is finite the seller's maximal revenue is in between the Myerson and the efficient benchmark. The difference between these two benchmarks depends on the number of buyers, the discount factor, and the distribution of valuations. If the seller faces many buyers, then the probability that trade occurs at period one is very high, and her lack of commitment becomes less important: It is when she faces a small number of buyers that the design matters.

6 Mechanisms with Variable Transparency

We have, thus far, assumed that all players observe the entire vector of actions chosen at period t; that is, we have assumed that the seller employs fully transparent mechanisms. This can be restrictive, since in dynamic settings where we require players to behave sequentially rationally, what buyers observe determines their beliefs about their competitors. Their observations may affect their future behavior and, hence, the set of continuation equilibrium outcomes. Here, we allow for the seller to control how much buyers observe in each period. We model this with information-disclosure policies.

An *information-disclosure policy* is a mapping from the vector of actions chosen by the buyers, to a vector of messages, one for each buyer: $c^t : \mathcal{A} \to \Delta(\Lambda)$, where $\Lambda := \times_{i \in I} \Lambda_i$, and Λ_i is the set of messages that the seller can send to buyer i. It is easy to see that this formulation encompasses full disclosure, no disclosure, as well as intermediate cases.

We assume that *all* players observe the mechanism (which includes the information-disclosure policy), and whether or not trade takes place. In this modified environment, an information set of the seller at the beginning of period t is $\iota_S^t = \{M^{(t-1)}, \chi^{(t-1)}, a^{(t-1)}, \lambda^{(t-1)}\}$, and that of buyer i is $\iota_i^t = \{v, M^{(t)}, \chi^{(t-1)}, a_i^{(t-1)}, \lambda_i^{(t-1)}\}$.

Let p^{FT}, x^{FT} denote a mechanism chosen by the seller at $t = 2$ under full transparency. Here, we establish that, no matter what disclosure policy the seller employs at $t = 1$, the revenue-maximizing mechanism at $t = 2$ coincides with p^{FT}, x^{FT}. Note, first, that the seller can guarantee the same revenue as under full transparency regardless of the disclosure policy that she employs at $t = 1$. This is because p^{FT}, x^{FT} is dominant-strategy incentive-compatible (which follows from the fact that it is strictly increasing in v_i for all v_{-i}). However, can she do better?[40] Or, more generally, would her choice differ if she employed some other information-disclosure policy at $t = 1$? The following result establishes that the answer to both these questions is no.

Proposition 4. *No matter what disclosure policy the seller employs at* $t = 1$, *the set of revenue-maximizing mechanisms at* $t = 2$ *is identical to the one derived under full transparency.*

Proposition 4 states that the set of sequentially-rational mechanisms at $t = 2$, and, hence, the revenue generated at $t = 2$ are independent of the disclosure policy employed at $t = 1$. Since disclosure policies can affect revenue only through changing the revenue-maximizing choices at $t = 2$, we can conclude the following:

Corollary 2. *Without loss of generality, we can assume that all buyers observe the entire vector of actions chosen in the first period.*

This result establishes that the earlier analysis, which imposed that the seller employ fully transparent mechanisms, is without loss of generality.

Discussion: Proposition 4 allows us to sidestep a conceptual difficulty that arises because of the lack of a canonical class of mechanisms for mechanism-design problems of limited commitment: It is without loss to model a mechanism as a game form and to assume that all agents observe each other's actions.

Some may argue that the transparency issue of the first-period mechanism can be addressed using ideas from Myerson's (1986) work on multi-stage games with communication. In that paper, the revelation principle asserts that the set of communication equilibria is largest when, at each stage, the mediator privately recommends to each player which action to choose. In our setup, this would translate to saying that the seller simply recommends to the buyers only which actions to choose, without disclosing any additional information. However, it is easy to see, following the exact same logic of Myerson (1986)'s revelation principle, that when the seller discloses no information to the buyers at $t = 2$, this makes the set of incentive-compatible mechanisms at $t = 2$ as large as possible[41] and enlarges the seller's choices. This is a good thing for her if we are in a static setup, but it may not be so if there is limited commitment because, now, the seller has a larger set of possible deviations at $t = 2$, which may make the sequential-rationality constraints costlier. The key difference between Myerson's and our setup is that in this paper, the information disclosed to the

[40] Disclosure policies affect buyers' beliefs about each other, possibly creating correlation in their belief part of their private information, which could allow the seller to generate higher revenue at $t = 2$. Affecting buyers' beliefs is irrelevant, given that the revenue-maximizing mechanism assuming the seller's information is common knowledge is dominant-strategy incentive-compatible.

[41] See Skreta (2011) for further details.

buyers at some stage, affects the set of feasible choices for the seller at a subsequent stage. This is not the case in Myerson's (1986) setup, in which the set of choices (actions) available to the players is exogenous to the communication device.

7 Analysis of the Problem when $2 < T < \infty$: An Overview

We have shown that if $T = 2$, the seller maximizes expected revenue by employing a "Myerson" auction in each period. Here, we sketch out how we can extend by induction this result for any T finite.

We describe the induction step for $T = 3$: First, we establish the analog of Lemma 1, whose proof remains essentially unchanged, with the only modification that the allocations at $t = 2$ are now continuation allocations of a two-period game that starts at $t = 2$.[42] Given Lemma 1, we can establish Proposition 2 using arguments parallel to the ones used in the $T = 2$ case: When two different actions of a buyer are followed by identical $t = 2$ continuation allocations, we can merge the actions, and that does not change the sequentially-rational continuation allocation at $t = 2$. Actions followed by different continuation allocations increase the cost of sequential-rationality constraints. Hence, pooling the lower end of valuations at $t = 1$ is optimal. Given Proposition 2, the analog of Lemma 2 for the $T = 3$ game follows, using arguments identical to those in the $T = 2$ case.

From Proposition 2, it follows that the relevant posteriors after all buyers choose the vector $\mathbf{0}$ are truncations of the priors–given by (4.1). Lemma 6 (in the Appendix) allows us, then, to conclude that that posterior virtual valuations are strictly increasing in v_i. Then, we can apply our $T = 2$ result to obtain the revenue-maximizing sequentially-rational allocation rule at the continuation game that starts at $t = 2$ after the vector $\mathbf{0}$ is chosen at $t = 1$, which is dominant-strategy incentive-compatible (recall Remark 1).

To establish Lemma 3, we need to show that the $t = 1$ boundary $\bar{v}_i$ is greater or equal to r_i^* (the commitment reserve price) for all $i \in I$ in the case where the problem lasts three periods. We already know from Lemma 2 of Skreta (2006b) that the $T = 3$-boundaries $r_i^3(\bar{v}_i^2(\bar{v}_i))$ are increasing in $\bar{v}_i^2(\bar{v}_i)$ (or, more generally, $r_i^T(\bar{v}_i^{T-1}(\bar{v}_i))$ are increasing in $\bar{v}_i^{T-1}(\bar{v}_i)$.) The proof of the analog of Lemma 3 mimics the one for the two-period version of the game, after we establish that the boundaries $\bar{v}_i^2(\bar{v}_i)$ are increasing in $\bar{v}_i$. The $t = 2$-boundaries $\bar{v}_i^2(\bar{v}_i)$ give the valuation of i below which the seller keeps the object at $t = 2$ when the posteriors are given by equation (4.1). When $T = 3$, posterior virtual valuations at the continuation game that starts at $t = 2$ are given by $J_i(v_i|\bar{v}_i) = v_i - \frac{F_i(\bar{v}_i) - F_i(v_i)}{f_i(v_i)}$. Notice that $J_i(v_i|\bar{v}_i)$ is decreasing in $\bar{v}_i$, since F_i is increasing in $\bar{v}_i$. Then, because posterior virtual valuations fall when $\bar{v}_i$ increases, the region where the seller keeps the good at $t = 2$ increases; that is, $\bar{v}_i^2(\bar{v}_i)$ is (weakly) increasing in $\bar{v}_i$. With this simple observation in hand, we can straightforwardly establish Lemma 3 for the case in which the game lasts for three periods. Lemma 4 and Proposition 3 are, then, routine to generalize. Finally, given that the $T = 2$

[42] A moment's thought reveals that the logic of footnote 45 in the Appendix applies here as well.

allocation is dominant-strategy incentive-compatible (recall Remark 1), we can easily establish the analog of Proposition 4 for $T = 3$ and for longer finite games, and we conclude that we can, without loss, assume that at $t = 1$, all buyers observe the entire vector of actions chosen.

Given the result for $T = 3$, we can continue to get the result for $T = 4$ and so forth, since all steps followed in the case of $T = 3$ immediately generalize to the case of $T = 4$ and so forth.

8 Robustness and Concluding Remarks

We have shown that when sellers cannot resist the temptation to place back on the market items that remain unsold, simple selling procedures are optimal: When buyers are ex-ante symmetric, first- or second-price auctions with optimally chosen reservation prices are revenue-maximizing. Lack of commitment is costly for the seller, especially when demand is thin and when the seller is moderately patient. Governments sell important assets, such oil tracts, timber tracts, spectrum and treasury bills, through auctions. Optimal design is especially important for revenue generation when the number of buyers who participate in the auction is very small and there is little competition. This is usually the case for auctions of very valuable assets. This observation, together with the fact that a large fraction of items that remain unsold are placed back on the market, makes the characterization obtained in this paper a relevant extension of the optimal-auction literature.

We now offer a few remarks on the generality of the solution. It depends (i) on the set of mechanisms that the seller employs; (ii) on the generality of the buyers' strategies; (iii) on what the seller observes during play; and (iv) on the length of the time horizon. With respect to the definition of "mechanisms," we have been very general: We have assumed that a mechanism consists of some abstract game form endowed by an information-disclosure policy as a way of capturing different scenarios of what buyers observe during the play of an auction. Regarding the generality of buyers' strategies, we have not imposed any restrictions: We allow for mixed strategies, and for a non-convex set of types that may be choosing the same actions. Finally, regarding what the seller observes during play, we have assumed that she observes the vector of actions that buyers choose at each stage. This assumption makes the non-commitment constraints quite strong, and intentionally so: The point of our analysis is to find what is best for the seller given that she cannot commit. If we had assumed that the seller observes nothing over time, then, trivially, the commitment solution is sequentially-rational.

A limitation of this work is that we analyze a finite-horizon problem. This goes somewhat against the spirit of our analysis since the seller does have commitment power in the last period. However, in many situations in practice, financial or political constraints impose a hard deadline by which an asset must be sold. For example, financial institutions have a certain period of time to sell distressed confiscated property. Technically, an infinite-horizon mechanism-design problem in which the designer behaves sequentially rationally is very complex and beyond the scope of the present paper. With such a problem, continuation allocation rules need not be revenue-maximizing, so Lemma 1 may not hold. In addition, it seems impossible to express the problem in a recursive way, thus precluding the use of dynamic programming techniques. It is very likely that a characterization would be based on an argument that relies on the properties of the revenue-maximizing mechanisms in the finite horizon established in this paper, making the current analysis a key stepping stone towards the solution of an infinite-horizon problem.

In recent years, motivated by the large number and the importance of applications,[43] there has been substantial work on dynamic mechanism design.[44] A key assumption in that literature is that the principal can fully commit ex-ante to the mechanism for the entire relationship. The assumption of commitment implies that the principals may behave in a time-inconsistent manner that could be inappropriate for certain applications. Designing multi-period incentives schemes under various assumptions of commitment is an important area since contracting parties often renegotiate or change a contract if it becomes clear that there exist others that dominate it. A step in this direction is the work of Hörner and Samuelson (2011), who study a revenue-management problem in the absence of commitment when the seller is posting prices. However, general dynamic mechanism design in the absence of commitment is largely an understudied area. We hope and expect that the ideas and tools developed in this paper will be useful for further work.

A Proof of Lemma 1

Proof. Our goal is to establish that $U_i^{\hat{a}_i}(v_i) = U_i^{a_i}(v_i)$ for all, but measure zero of valuations in $\bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i)$. First, consider a valuation in $V_i(a_i)$ that, in addition to a_i, chooses another action $\hat{a}_i$, that is, $m_i(\hat{a}_i|v_i) > 0$. Then, best-response constraints imply that at valuation v_i, buyer i is indifferent between a_i and $\hat{a}_i$, that is $U_i^{\hat{a}_i}(v_i) = U_i^{a_i}(v_i)$. Valuations in $\bar{V}_i(a_i) \backslash V_i(a_i)$ do not choose action a_i at $t=1$, $(m_i(a_i|v_i) = 0)$; still, we now show that they must be indifferent between choosing a_i and the action $\hat{a}_i$ they are actually choosing. We argue by contradiction: Consider a $\tilde{v}_i \in \bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i)$ that strictly prefers $\hat{a}_i$ to a_i, implying that:

$$P_i^{\hat{a}_i}(\tilde{v}_i)\tilde{v}_i - X_i^{\hat{a}_i}(\tilde{v}_i) > P_i^{a_i}(v_i)\tilde{v}_i - X_i^{a_i}(v_i), \tag{A.1}$$

for all options on the menu $\{P_i^{a_i}(v_i), X_i^{a_i}(v_i)\}_{v_i \in V_i(a_i)}$. Given the linearity and single crossing property of payoffs, $(A.1)$ implies that there is an open set of valuations neighboring $\tilde{v}_i$ that strictly prefer $\hat{a}_i$ to a_i at $t=1$. Let $(\tilde{v}_i^L, \tilde{v}_i^H)$ be the largest such neighborhood, implying that $\tilde{v}_i^L$ and $\tilde{v}_i^H$ (at least weakly) prefer a_i. Then $\tilde{v}_i^L$ and $\tilde{v}_i^H$ either choose a_i with strictly positive probability, or they are on the boundary of types that choose a_i with strictly positive probability. Given that there is a hole of $(\tilde{v}_i^L, \tilde{v}_i^H)$ in the support of the seller's posterior at $t=2$ after he observes a_i at $t=1$, and because the seller employs an optimal mechanism at $t=2$, we have[45]

$$P_i^{a_i}(\tilde{v}_i^H)\tilde{v}_i^H - X_i^{a_i}(\tilde{v}_i^H) = P_i^{a_i}(\tilde{v}_i^L)\tilde{v}_i^H - X_i^{a_i}(\tilde{v}_i^L). \tag{A.2}$$

[43]For example, the classical airline revenue-management problem, the allocation of advertising inventory, the design of incentive schemes that take into account inter-temporal considerations of managers, etc.

[44]See, for example, the survey of Bergemann and Said (2011), and the references therein.

[45] This is easy to see: At an optimal mechanism at $t=2$, after a vector of actions a_i, a_{-i} was chosen at $t=1$, the incentive constraint for v_i^H not to mimic v_i^L is tight for all a_{-i}, because otherwise the seller could generate strictly more revenue at $t=2$, by increasing the payments for all valuations greater or equal to v_i^H. This result is analogous to the one in the optimal-pricing problem with two types, where the price is chosen to be just big enough to make the high type indifferent about mimicking the low one.

Note that $(A.1)$ trivially implies that

$$P_i^{\hat{a}_i}(\tilde{v}_i)\tilde{v}_i - X_i^{\hat{a}_i}(\tilde{v}_i) > P_i^{a_i}(\tilde{v}_i^L)\tilde{v}_i - X_i^{a_i}(\tilde{v}_i^L). \tag{A.3}$$

Now observe that it is not possible that $P_i^{\hat{a}_i}(\tilde{v}_i) = P_i^{a_i}(\tilde{v}_i^L)$ because, then, (A.3) implies $X_i^{\hat{a}_i}(\tilde{v}_i) < X_i^{a_i}(\tilde{v}_i^L)$ and, in that case, both $\tilde{v}_i$ and $\tilde{v}_i^L$ strictly prefer to choose $\hat{a}_i$ at $t = 1$, which contradicts the definition $\tilde{v}_i^L$. If $P_i^{\hat{a}_i}(\tilde{v}_i) > P_i^{a_i}(\tilde{v}_i^L)$, then because $\tilde{v}_i^H > \tilde{v}_i$, (A.3) implies that

$$P_i^{\hat{a}_i}(\tilde{v}_i)\tilde{v}_i^H - X_i^{\hat{a}_i}(\tilde{v}_i) > P_i^{a_i}(\tilde{v}_i^L)\tilde{v}_i^H - X_i^{a_i}(\tilde{v}_i^L),$$

which, together with (A.2), implies that type $\tilde{v}_i^H$ strictly prefers $\hat{a}_i$. Contradiction.

If $P_i^{\hat{a}_i}(\tilde{v}_i) < P_i^{a_i}(\tilde{v}_i^L)$, then, because $\tilde{v}_i^L < \tilde{v}_i$, (A.3) implies that

$$P_i^{\hat{a}_i}(\tilde{v}_i)\tilde{v}_i^L - X_i^{\hat{a}_i}(\tilde{v}_i) > P_i^{a_i}(\tilde{v}_i^L)\tilde{v}_i^L - X_i^{a_i}(\tilde{v}_i^L),$$

which implies that type v_i^L strictly prefers $\hat{a}_i$. Contradiction. □

B Appendix B: Proof of Proposition 2

The essence of the proof can be summarized as follows: When two different actions of a buyer are followed by identical $t = 2$ continuation allocations, we can merge the actions (Case 1 below), and that does not change the sequentially-rational continuation allocation at $t = 2$. Actions followed by different continuation allocations increase the cost of sequential-rationality constraints (Case 2). Hence, pooling the lower end of valuations at $t = 1$ is optimal.

Case 2 employs an auxiliary result Lemma 5 (proved below), that establishes the following: Let $[0, v_i]$ denote the convex hull of valuations that choose action 0 at a PBE, and let[46] $r_i^2(0)$ denote the optimal reserve price for buyer i at $t = 2$ after he chose 0 at $t = 1$. Suppose that valuations in $[0, v_i]$ choose another action a_i and $r_i^2(a_i) \in [0, v_i]$, then, if the reserve prices are relevant in the sense that i sometimes pays the reserve,[47] then it holds that $r_i^2(a_i) = r_i^2(0)$. This tells us that in equilibrium the mixing (the $m_i(.|a_i)$ and $m_i(.|0)$) *must* be such that the reserve prices at $t = 2$ are identical.

Proof. The proof of this result is based on equation (4.3). First, note that if $v_{-i}(v_i, a_i, a_{-i}) < v_{-i}(v_i, \hat{a}_i, a_{-i})$ for some a_{-i}, this means that the posterior virtual valuation at v_i given $\hat{a}_i$ is higher than the one given a_i, so it holds that $v_{-i}(v_i, a_i, \tilde{a}_{-i}) < v_{-i}(v_i, \hat{a}_i, \tilde{a}_{-i})$ for all $\tilde{a}_{-i}$. This implies, then, that either $v_{-i}(v_i, a_i, a_{-i}) \leq v_{-i}(v_i, \hat{a}_i, a_{-i})$ for all a_{-i}, or $v_{-i}(v_i, a_i, a_{-i}) \leq v_{-i}(v_i, \hat{a}_i, a_{-i})$ for all a_{-i}, or $v_{-i}(v_i, a_i, a_{-i}) = v_{-i}(v_i, \hat{a}_i, a_{-i})$ for all a_{-i}.

Case 1–Same t=2 boundaries: If $r_i^2(\hat{a}_i) = r_i^2(a_i)$ and $v_{-i}(v_i, a_i, a_{-i}) = v_{-i}(v_i, \hat{a}_i, a_{-i})$ for all a_{-i} and v_i, the second-period allocation is identical for all a_{-i}, regardless of whether buyer i chose a_i or $\hat{a}_i$ at $t = 1$. We now show that if we merge these actions (buyer i pools and chooses one of the 2), the $t = 2$ mechanism will remain optimal and is identical to the one that is sequentially-rational if all valuations in $\bar{V}_i(a_i)$ pool and choose action a_i. Given that the seller's response does not differ at $t = 2$, regardless of whether or not she can condition on having observed that buyer i chose a_i or $\hat{a}_i$, nothing will change in the seller's response when she conditions on the information that i chose *either* a_i *or* $\hat{a}_i$. Hence, consolidating these two actions into one, say a_i, does not change anything in terms of the seller's choice at $t = 2$. We formalize this intuition below:

For transparency of the arguments below, suppose that, other than a_i, $\hat{a}_i$ is the only action that is chosen with positive probability by valuations in $V_i(a_i)$; then, we have that $m_i(a_i|v_i) + m_i(\hat{a}_i|v_i) = 1$ for all $V_i(a_i)$.[48] Because $r_i^2(\hat{a}_i) = r_i^2(a_i)$ and $v_{-i}(v_i, a_i, a_{-i}) = v_{-i}(v_i, \hat{a}_i, a_{-i})$ for all a_{-i} and v_i, we have that the allocation rule $p_i^2(v, \hat{a}_i, a_{-i}) = p_i^2(v, a_i, a_{-i})$ for all $i \in I$, v and all vectors of actions $a_{-i} \in \mathcal{A}_{-i}$ that are chosen with

[46] Recall the definition of the optimal reserve prices $r_i^2(a_i)$ in (2.3).

[47] This is always true since i will pay the reserve when all other buyers' realized valuations are zero.

[48] As we explain below, the arguments generalize in a straightforward way if valuations in $V_i(a_i)$ choose other actions associated with identical $t = 2$-menus.

strictly positive probability. From now on, we call this commonly optimal allocation rule p^2 (it also depends on a_{-i}, which is hold fixed below). Given that p^2, x^2 is revenue-maximizing at $t = 2$ given a_i, for all a_{-i}, it satisfies that[49]

$$\begin{aligned} & \int_V \Sigma_{i\in I} p_i^2(v_i, v_{-i}) \left[v_i f_i(v_i|a_i) - (1 - F_i(v_i|a_i))\right] f_{-i}(v_{-i}|a_{-i}) dv \\ \geq & \int_V \Sigma_{i\in I} \tilde{p}_i^2(v_i, v_{-i}) \left[v_i f_i(v_i|a_i) - (1 - F_i(v_i|a_i))\right] f_{-i}(v_{-i}|a_{-i}) dv \end{aligned} \tag{B.1}$$

for all $\tilde{p}^2$ feasible. By substituting the exact expressions of $f_i(v_i\,|a_i)$ and of $F_i(v_i|a_i)$, and by multiplying through with the constant $\int_{\underline{v}_i}^{\bar{v}_i} m_i(a_i|t_i) f_i(t_i) dt_i$ one can easily see that (B.1) is equivalent to

$$\begin{aligned} & \int_V p_i^2(v_i, v_{-i}) \left[v_i m_i(a_i|t_i) f_i(t_i) - \left(\int_{\underline{v}_i}^{\bar{v}_i} m_i(a_i|t_i) f_i(t_i) dt_i - \int_{\underline{v}_i}^{v_i} m_i(a_i|t_i) f_i(t_i)\right)\right] f_{-i}(v_{-i}|a_{-i}) dv \\ & + \int_V \Sigma_{\substack{j\in I \\ j\neq i}} p_j^2(v_j, v_{-j}) \left[v_j f_j(v_j|a_j) - (1 - F_j(v_j|a_j))\right] f_{-i-j}(v_{-i-j}|a_{-i-j}) m_i(a_i|v_i) f_i(v_i) dv \\ \geq & \int_V \tilde{p}_i^2(v_i, v_{-i}) \left[v_i m_i(a_i|t_i) f_i(t_i) - \left(\int_{\underline{v}_i}^{\bar{v}_i} m_i(a_i|t_i) f_i(t_i) dt_i - \int_{\underline{v}_i}^{v_i} m_i(a_i|t_i) f_i(t_i)\right)\right] f_{-i}(v_{-i}|a_{-i}) dv \\ & + \int_V \Sigma_{\substack{j\in I \\ j\neq i}} \tilde{p}_j^2(v_j, v_{-j}) \left[v_j f_j(v_j|a_j) - (1 - F_j(v_j|a_j))\right] f_{-i-j}(v_{-i-j}|a_{-i-j}) m_i(a_i|v_i) f_i(v_i) dv, \end{aligned} \tag{B.2}$$

for all $\tilde{p}^2$.

Similarly, given that p^2, x^2 is revenue-maximizing at $t = 2$ given $\hat{a}_i, a_{-i}$, it satisfies that

$$\begin{aligned} & \int_V p_i^2(v_i, v_{-i}) \left[v_i m_i(\hat{a}_i|t_i) f_i(t_i) - \left(\int_{\underline{v}_i}^{\bar{v}_i} m_i(\hat{a}_i|t_i) f_i(t_i) dt_i - \int_{\underline{v}_i}^{v_i} m_i(\hat{a}_i|t_i) f_i(t_i)\right)\right] f_{-i}(v_{-i}|a_{-i}) dv \\ & + \int_V \Sigma_{\substack{j\in I \\ j\neq i}} p_j^2(v_j, v_{-j}) \left[v_j f_j(v_j|a_j) - (1 - F_j(v_j|a_j))\right] f_{-i-j}(v_{-i-j}|a_{-i-j}) m_i(\hat{a}_i|v_i) f_i(v_i) dv \\ \geq & \int_V \tilde{p}_i^2(v_i, v_{-i}) \left[v_i m_i(\hat{a}_i|t_i) f_i(t_i) - \left(\int_{\underline{v}_i}^{\bar{v}_i} m_i(\hat{a}_i|t_i) f_i(t_i) dt_i - \int_{\underline{v}_i}^{v_i} m_i(\hat{a}_i|t_i) f_i(t_i)\right)\right] f_{-i}(v_{-i}|a_{-i}) dv \\ & + \int_V \Sigma_{\substack{j\in I \\ j\neq i}} \tilde{p}_j^2(v_j, v_{-j}) \left[v_j f_j(v_j|a_j) - (1 - F_j(v_j|a_j))\right] f_{-i-j}(v_{-i-j}|a_{-i-j}) m_i(\hat{a}_i|v_i) f_i(v_i) dv, \end{aligned} \tag{B.3}$$

for all $\tilde{p}^2$.

Recall that for transparency of the arguments, we have supposed that $\hat{a}_i$ is the only action other than a_i that is chosen with positive probability by valuations in $V_i(a_i)$ (otherwise, we would have to write more inequalities like the ones above, and then add up); then, we have that $m_i(a_i|v_i) + m_i(\hat{a}_i|v_i) = 1$ for all $v_i \in \bar{V}_i(a_i)$, and by adding the two inequalities $(B.2)$ and (B.3), we get that

$$\begin{aligned} & \int_V p_i^2(v_i, v_{-i}) \left[v_i f_i(t_i) - \left(\int_{\underline{v}_i}^{\bar{v}_i} f_i(t_i) dt_i - \int_{\underline{v}_i}^{v_i} f_i(t_i)\right)\right] f_{-i}(v_{-i}|a_{-i}) dv \\ & + \int_V \Sigma_{\substack{j\in I \\ j\neq i}} p_j^2(v_j, v_{-j}) \left[v_j f_j(v_j|a_j) - (1 - F_j(v_j|a_j))\right] f_{-i}(v_{-i}|a_{-i}) f_i(v_i) dv \\ \geq & \int_V \tilde{p}_i^2(v_i, v_{-i}) \left[v_i f_i(t_i) - \left(\int_{\underline{v}_i}^{\bar{v}_i} f_i(t_i) dt_i - \int_{\underline{v}_i}^{v_i} f_i(t_i)\right)\right] f_{-i}(v_{-i}|a_{-i}) dv \\ & + \int_V \Sigma_{\substack{j\in I \\ j\neq i}} \tilde{p}_j^2(v_j, v_{-j}) \left[v_j f_j(v_j|a_j) - (1 - F_j(v_j|a_j))\right] f_{-i}(v_{-i}|a_{-i}) f_i(v_i) dv, \end{aligned}$$

[49] Note, that it is without loss to take $p_i^2(., \hat{a}_i, a_{-i})$ (resp. $p_i^2(., a_i, a_{-i})$) to be defined on V_i since for $v_i \in V_i \backslash V_i(a_i)$, $m_i(a_i|v_i) = 0$.

for all $\tilde{p}^2$, which is equivalent to saying that the allocation rule p^2 is revenue-maximizing at $t = 2$ when all valuations in $\bar{V}_i(a_i)$ choose a_i with probability one, in which case i is using a partitional strategy. This implies that if we are in this case, we can without any loss assume that all valuations in $\bar{V}_i(0) = [0, \bar{v}_i]$ choose action 0 with probability 1 and the result follows.

Case 2–Different t=2 boundaries:[50] We start by showing that pooling all valuations below $v_i(0)$ with valuation 0 minimizes the distortions with respect to reserve prices–the boundary that determines when the seller should keep the good rather than giving it to buyer i: Let $r_i^2(\bar{v}_i)$ denote the optimal reserve price given posterior beliefs (4.1). Lemma 2 in Skreta (2006b) establishes that $r_i^2(\bar{v}_i)$ is increasing in $\bar{v}_i$, which immediately implies that $r_i^2(\bar{v}_i) \leq r_i^2(b_i) = r_i^*$. We now establish that at any PBE, truncations of the prior lead to the highest sequentially-rational reserve prices at $t = 2$, compared to any other equilibrium-feasible posterior with the same highest possible valuation in the support $\bar{v}_i$:

Let $r_i^2(0)$ denote the optimal reserve price for buyer i at $t = 2$ after i chooses action 0 at $t = 1$. Lemma 5 (below) implies that if $r_i^2(\hat{a}_i) < \bar{v}_i(0)$, then it must be the case that $r_i^2(\hat{a}_i) = r_i^2(0)$. Suppose that we merge all actions associated with the same reserve prices into one of them and call the merged action, 0^m, as follows: The probability that valuation v_i chooses action 0^m is given by $m_i(0^m|v_i) = m_i(0|v_i) + m_i(\hat{a}_i|v_i) + \ldots$, where we add over all actions for which the reserve price is equal to $r_i^2(0)$. We denote the optimal reserve price given the resulting posterior as $r_i^2(0^m)$. From the analysis of the previous case, it follows that that $r_i^2(0^m) = r_i^2(\hat{a}_i) = r_i^2(0)$. Now, we move on to compare $r_i^2(0^m)$ and $r_i^2(\bar{v}_i)$, which is the optimal reserve price given posterior beliefs (4.1).

Claim: At any PBE, it holds that $r_i^2(0^m) \leq r_i^2(\bar{v}_i)$.

We argue by contradiction. Suppose that $r_i^2(0^m) > r_i^2(\bar{v})$. First, observe that all actions chosen by valuations below $r_i^2(0^m)$ lead to the same reserve price, so the seller's posterior after observing the merged action 0^m is

$$f_i(v_i|0^m) = \begin{cases} \frac{f_i(v_i)}{F(r(0^m)) + \int_{r(0^m)}^{\bar{v}_i} m_i(0^m|s) f(s) ds}, & v_i \in [0, r_i^2(0^m)) \\ \frac{m_i(0^m|v_i) f_i(v_i)}{F(r(0^m)) + \int_{r(0^m)}^{\bar{v}_i} m_i(0^m|s) f(s) ds}, & v_i \in [r_i^2(0^m), \bar{v}_i] \end{cases}. \tag{B.4}$$

Define $A(s) \equiv s f_i(s) - F_i(\bar{v}_i) - F_i(s) = s f_i(s) - \int_s^{\bar{v}_i} f_i(t)\, dt$ and $B(s) \equiv s f_i(s) - \int_s^{\bar{v}_i} m_i(0^m|(t) f_i(t)\, dt$. Note, that since $m_i(0^m|(t) \leq 1$, we have that for all s $A(s) \leq B(s)$. Recalling (2.3) and the definition of $r_i^2(\bar{v})$, it follows that $\int_{r_i^2(\bar{v})}^{\tilde{v}} A(s)\, ds$ for all $\tilde{v} \geq r_i^2(\bar{v})$, which, because $A(s) \leq B(s)$, immediately implies that

$$\int_{r_i^2(\bar{v})}^{\tilde{v}} B(s)\, ds \text{ for all } \tilde{v} \geq r_i^2(\bar{v}_i),$$

contradicting the definition of $r_i^2(0^m)$. To put it very roughly, (B.4) puts less weight on the higher valuations compared to (4.1).

What about the inter-buyer boundaries? This question can be addressed with a direct argument for the case that buyers are ex-ante symmetric: In that case, at the commitment solution, the revenue-maximizing boundary dividing the regions where i and j get the object is the forty-five degree line. When, at period one, we pool valuations in $[0, \bar{v}_i]$ and $\bar{v}_i = \bar{v}_j = \bar{v}$ for all $i, j \in I$ buyers are still symmetric in the eyes of the seller at $t = 2$, so the optimal boundary dividing the regions where i and j get the object remains the forty-five degree line. The common cutoff $\bar{v}$ implies a common reservation price $r^2(\bar{v})$ at $t = 2$. Then, the distortions that arise from the sequential-rationality constraints arise only because $r^2(\bar{v})$ is below r^*, but the boundary that specifies which buyer gets the object is not distorted. Hence, truncation achieves the ex-ante optimal boundary and pooling all valuations below $\bar{v}_i$ is optimal.

When buyers are asymmetric, the shape of the ex-ante optimal boundary can be very complex and depends on the particular distributions. For this case, we employ an indirect argument: If $v_{-i}(v_i, a_i, a_{-i}) \leq v_{-i}(v_i, \hat{a}_i, a_{-i})$ for all a_{-i}, or $v_{-i}(v_i, a_i, a_{-i}) \leq v_{-i}(v_i, \hat{a}_i, a_{-i})$, then (4.3) implies that the first-period menu

[50] Note that the left-hand side of (4.3) is constant. Note, also, that (4.3) has to hold for $v_i \in \bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i)$ a.e.. Unless $v_{-i}(v_i, a_i, a_{-i}) = v_{-i}(v_i, \hat{a}_i, a_{-i})$ for all a_{-i} and v_i, condition (4.3) is very difficult to be sustained for all v_i. Indeed, as we establish in Lemma 5 in Appendix B, equilibrium considerations imply that if $r_i^2(\hat{a}_i)$ is in $\bar{V}_i(a_i)$, then it must hold that $r_i^2(\hat{a}_i) = r_i^2(a_i)$; that is, the boundaries $v_{-i}(v_i, a_i, a_{-i}) = v_{-i}(v_i, \hat{a}_i, a_{-i})$ for $v_i \leq r_i^2(a_i)$. However, establishing this equality for all $v_i \in \bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i)$ seems impossible.

must compensate buyer i for the discrepancy in the $t = 2$ probability of getting the object. If we merge these actions, two things happen: First, there is no need to distort the period-1 allocation and, second, there are fewer sequential-rationality constraints to satisfy. This increases the overall value of the seller's program. $\square$

Lemma 5. *If $r_i^2(\hat{a}_i)$ is in $\bar{V}_i(a_i)$ and i obtains the object with positive probability at $t = 2$ when his valuation is equal to the reserve price $r_i^2(\hat{a}_i)$, then $r_i^2(\hat{a}_i) = r_i^2(a_i)$.*

Proof. We argue by contradiction: Suppose that $r_i^2(\hat{a}_i) < r_i^2(a_i)$. Depending on whether $r_i^2(\hat{a}_i) > \underline{v}_i(\hat{a}_i)$ or $r_i^2(\hat{a}_i) = \underline{v}_i(\hat{a}_i)$, there are two cases to consider:

Case 1: $r_i^2(\hat{a}_i) > \underline{v}_i(\hat{a}_i)$: If $r_i^2(\hat{a}_i) > \underline{v}_i(\hat{a}_i)$, then Lemma 1 implies that for $v_i \in [\underline{v}_i(a_i), r_i^2(\hat{a}_i))$, $P_i^{\hat{a}_i}(v_i)$ is constant and equal $P_i^{(1)a_i}$, where $P_i^{(1)a_i}(v_i) \equiv \int_{a_{-i}\in\mathcal{A}_{-i}} \int_{\bar{V}_{-i}(a_{-i})} q_i(a) m(a_{-i}|v_{-i}) f_{-i}(v_{-i}) dv_{-i} da_{-i}$ respectively for $\hat{a}_i$. Moreover, since $[\underline{v}_i(a_i), r_i^2(\hat{a}_i))$ is contained in $[\underline{\mathrm{v}}_i(a_i), r_i^2(a_i))$, we get that

$$P_i(v_i) = P_i^{\hat{a}_i}(v_i) = P_i^{(1)\hat{a}_i} = P_i^{(1)a_i}, \tag{B.5}$$

for $v_i \in [\underline{v}_i(a_i), r_i^2(a_i))$. On the other hand, for all $v_i \in \left[r_i^2(\hat{a}_i), r_i^2(a_i)\right]$, we have that

$$P_i^{\hat{a}_i}(v_i) \geq P_i^{(1)\hat{a}_i} + \Sigma_{a_{-i}\in\mathcal{A}_{-i}} \delta q_S(a) \int_{\underline{v}_{-i}(a_{-i})}^{r_{-i}(a_{-i})} m_{-i}(a_{-i}|v_{-i}) f_{-i}(v_{-i}) dv_{-i} > P_i^{(1)a_i},$$

which follows because buyer i obtains the object with positive probability at $t = 2$ when his valuation is at the reserve price $v_i = r_i^2(\hat{a}_i)$, which contradicts $(B.5)$. Analogously, we can argue that the case $r_i^2(\hat{a}_i) > r_i^2(a_i)$ is impossible.

Case 2: $r_i^2(\hat{a}_i) = \underline{v}_i(\hat{a}_i)$: First, suppose that $\underline{v}_i(\hat{a}_i) > r_i^2(a_i)$, while $r_i^2(\hat{a}_i) = \underline{v}_i(\hat{a}_i)$: When $r_i^2(\hat{a}_i) = \underline{v}_i(\hat{a}_i)$, the posterior virtual valuation at $\underline{v}_i(\hat{a}_i)$ is positive, implying that i obtains the good with positive probability at $t = 2$. Also, because the seller chooses an optimal mechanism at $t = 2$, it must be the case that the surplus for valuation $\underline{v}_i(\hat{a}_i)$ at $t = 2$ must be zero; that is , $P_i^2(\underline{v}_i, \hat{a}_i)\underline{v}_i(\hat{a}_i) - X_i^2(\underline{v}_i, \hat{a}_i) = 0$ implying that

$$\begin{aligned} P_i^{\hat{a}_i}(\underline{v}_i(\hat{a}_i))\,\underline{v}_i(\hat{a}_i) - X_i^{\hat{a}_i}(\underline{v}_i(\hat{a}_i)) &= P_i^{(1)\hat{a}_i}\underline{v}_i(\hat{a}_i) - X_i^{(1)\hat{a}_i} \\ &= \Big[P_i^{(1)\hat{a}_i} + \delta P_i^2(\underline{v}_i, \hat{a}_i)\Big]\underline{v}_i(\hat{a}_i) - \Big[X_i^{(1)\hat{a}_i} + \delta X_i^2(\underline{v}_i, \hat{a}_i)\Big], \end{aligned} \tag{B.6}$$

where $X_i^{(1)\hat{a}_i}(v_i) \equiv \int_{a_{-i}\in\mathcal{A}_{-i}} \int_{\bar{V}_{-i}(a_{-i})} z_i(\hat{a}_i, a_{-i}) m(a_{-i}|v_{-i}) f_{-i}(v_{-i}) dv_{-i} da_{-i}$. This equality implies that for $v_i < \underline{v}_i(\hat{a}_i)$, it must be the case that

$$P_i^{a_i}(v_i) \leq P_i^{(1);\hat{a}_i} \tag{B.7}$$

otherwise, it is easy to see that $\underline{v}_i(\hat{a}_i)$ would have an incentive to deviate.[51]

From $(B.6)$ and $(B.7)$, it follows that for $r_i^2(a) < v_i < \underline{v}_i(\hat{a}_i)$,

$$P_i^{a_i}(v_i) = P_i^{(1)a_i} + \Sigma_{a_{-i}\in\mathcal{A}_{-i}} \delta q_S(a_i, a_{-i}) \int_{a_{-i}}^{v_{-i}(v_i, a_i, a_{-i})} m^{a_{-i}}(v_{-i}) f_{-i}(v_{-i}) dv_{-i} \leq P_i^{(1)\hat{a}_i}. \tag{B.8}$$

From the previous considerations, it follows that at $\underline{v}_i(\hat{a}_i) + \varepsilon$, it must be the case that

$$P_i^{\hat{a}_i}(\underline{v}_i(\hat{a}_i) + \varepsilon) = P_i^{(1)\hat{a}_i} + \Sigma_{a_{-i}\in\mathcal{A}_{-i}} \delta q_S(\hat{a}_i, a_{-i}) \int_{\underline{v}_i(a_{-i})}^{v_{-i}\left(\underline{v}_i(\hat{a}_i) + \varepsilon, \hat{a}_i, a_{-i}\right)} m^{a_{-i}}(v_{-i}) f_{-i}(v_{-i}) dv_{-i} = P_i^{a_i}(\underline{v}_i(\hat{a}_i) + \varepsilon). \tag{B.9}$$

[51] To see this, note that if $P_i^{a_i}(v_i) > P_i^{(1)\hat{a}_i}$, then the fact that at a PBE strategies are best responses implies that

$$P_i^{a_i}(v_i)\, v_i - X_i^{a_i}(v_i) \geq P_i^{(1)\hat{a}_i} v_i - X_i^{(1)\hat{a}_i},$$

but, then, since $v_i < \underline{v}_i(\hat{a}_i)$ and $P_i^{a_i}(v_i) > P_i^{(1)\hat{a}_i}$, we have that

$$P_i^{a_i}(v_i)\, \underline{v}_i(\hat{a}_i) - X_i^{a_i}(v_i) > P_i^{(1)\hat{a}_i} \underline{v}_i(\hat{a}_i) - X_i^{(1)\hat{a}_i},$$

which contradicts $(B.6)$.

Then, the requirement from Lemma 1 that $P_i^{a_i}(v_i) = P_i^{\hat{a}_i}(v_i)$, for v_i a.e. in $\bar{V}_i(a_i)$, together with $(B.8)$ and $(B.9)$, imply that for some $\varepsilon > 0$ small enough, we have that

$$P_i^{a_i}\left(\underline{v}_i\left(\hat{a}_i\right)+\varepsilon\right) - P_i^{a_i}\left(\underline{v}_i\left(\hat{a}_i\right)\right)$$

$$\geq \Sigma_{a_{-i}\in\mathcal{A}_{-i}}\delta q_S(\hat{a}_i, a_{-i})\int_{\underline{v}_i(a_{-i})}^{v_{-i}\left(\underline{v}_i(\hat{a}_i)+\varepsilon,\hat{a}_i,a_{-i}\right)} m^{a_{-i}}(v_{-i})f_{-i}(v_{-i})dv_{-i};$$

thus, P_i^2 must jump at $v_i+\varepsilon$, which is impossible since $J_i\left(\underline{v}_i\left(\hat{a}_i\right)|a_i\right) \geq J_i\left(\underline{v}_i\left(\hat{a}_i\right)+\varepsilon|a_i\right)$ for ε sufficiently close to zero. To see this, note that at $\underline{v}_i\left(\hat{a}_i\right)$, $J_i\left(\underline{v}_i\left(\hat{a}_i\right)|a_i\right) = \underline{v}_i\left(\hat{a}_i\right) - \frac{\int_{\underline{v}_i(\hat{a}_i)}^{\bar{v}_i(a_i)} m_i^{a_i}(t_i)f_i(t_i)dt_i}{f_i\left(\underline{v}_i(\hat{a}_i)\right)}$ whereas at $\underline{v}_i\left(\hat{a}_i\right)+\varepsilon$, it is $J_i\left(\underline{v}_i\left(\hat{a}_i\right)+\varepsilon|a_i\right) = \underline{v}_i\left(\hat{a}_i\right)+\varepsilon - \frac{\int_{\underline{v}_i(\hat{a}_i)+\varepsilon}^{\bar{v}_i(a_i)} m_i^{a_i}(t_i)f_i(t_i)dt_i}{m_i^{a_i}\left(\underline{v}_i(\hat{a}_i)+\varepsilon\right)f_i\left(\underline{v}_i(\hat{a}_i)+\varepsilon\right)}$,[52] which cannot jump at $\underline{v}_i\left(\hat{a}_i\right)+\varepsilon$, because $m_i^{a_i}\left(\underline{v}_i\left(\hat{a}_i\right)+\varepsilon\right) \leq 1$ and because f_i is continuous (which implies that the ironed virtual valuation cannot jump either).[53]

Hence, the only possibility is the other case, in which $\underline{v}_i(\hat{a}_i) < r_i^2(a_i)$, which immediately implies that $r_i^2\left(\hat{a}_i\right) < r_i^2(a_i)$. Again, observe, that since $v_i = r_i^2\left(\hat{a}_i\right)$ makes no rents at $t=2$, we have

$$P_i^{(1)a_i} r_i^2\left(\hat{a}_i\right) - X_i^{(1)a_i} = P_i^{(1)\hat{a}_i} r_i^2\left(\hat{a}_i\right) - X_i^{(1)\hat{a}_i} = P_i^{\hat{a}_i}\left(r_i^2\left(\hat{a}_i\right)\right) r_i^2\left(\hat{a}_i\right) - X_i^{\hat{a}_i}\left(r_i^2\left(\hat{a}_i\right)\right), \tag{B.11}$$

where $P_i^{\hat{a}_i}\left(r_i^2\left(\hat{a}_i\right)\right) = P_i^{(1)\hat{a}_i} + \Sigma_{a_{-i}\in\mathcal{A}_{-i}}\delta\int_{\underline{v}_{-i}(a_{-i})}^{r_{-i}(a_{-i})} q_S(a)\delta m_{-i}(a_{-i}|v_{-i})f_{-i}(v_{-i})dv_{-i} \geq P_i^{(1)a_i}$, but then $(B.11)$ implies that all valuations in $\left[r_i^2\left(\hat{a}_i\right), r_i^2(a_i)\right]$ strictly prefer action $\hat{a}_i$, implying that $m_i\left(a_i|r_i^2(a_i)\right) = 0$. This contradicts the definition of $r_i^2(a_i)$ because (2.3) implies that the reserve price cannot be equal to a valuation where the density is zero. $\square$

C Proof of Lemma 3

In order to establish Lemma 3, we prove an intermediate Lemma.

Lemma 6. *If $J_i(v_i) = v_i - \frac{[1-F_i(v_i)]}{f_i(v_i)}$ is increasing in v_i, then so is $J_i(v_i\,|\bar{v}_i\,) = v_i - \frac{F_i(\bar{v}_i)-F_i(v_i)}{f_i(v_i)}$.*

Proof. In order for $J_i(v_i\,|\bar{v}_i\,)$ to be increasing in v_i, the following inequality must hold:

$$f_i'(v_i)[F_i(\bar{v}) - F_i(v_i)] \geq -2f_i^2(v_i). \tag{C.1}$$

Now, if $J_i(v_i)$ is increasing in v_i, we have that:

$$f_i'(v_i)[1 - F_i(v_i)] \geq -2f_i^2(v_i). \tag{C.2}$$

If $f_i' \geq 0$, (C.1) is automatically satisfied. If $f_i' < 0$, then we have that

$$f_i'(v_i)\left[F_i(\bar{v}) - F_i(v_i)\right] \geq f_i'(v_i)\left[1 - F_i(v_i)\right] \geq -2f_i^2(v_i).$$

$\square$

[52] For transparency of the arguments we suppose (without loss) that, other than a_i, $\hat{a}_i$ is the only action that is chosen with positive probability by valuations in $V_i(a_i)$.

[53] Note that

$$\underline{v}_i\left(\hat{a}_i\right) - \frac{\int_{\underline{v}_i(\hat{a}_i)}^{\bar{v}_i(a_i)} m_i^{a_i}\left(t_i\right)f_i\left(t_i\right)dt_i}{f_i\left(\underline{v}_i\left(\hat{a}_i\right)\right)} - \underline{v}_i\left(\hat{a}_i\right) + \varepsilon + \frac{\int_{\underline{v}_i(\hat{a}_i)+\varepsilon}^{\bar{v}_i(a_i)} m_i^{a_i}\left(t_i\right)f_i\left(t_i\right)dt_i}{m_i^{a_i}\left(\underline{v}_i\left(\hat{a}_i\right)+\varepsilon\right)f_i\left(\underline{v}_i\left(\hat{a}_i\right)+\varepsilon\right)} \geq 0 \tag{B.10}$$

as $\varepsilon \to 0$ because $\frac{\int_{\underline{v}_i(\hat{a}_i)+\varepsilon}^{\bar{v}_i(a_i)} m_i^{a_i}(t_i)f_i(t_i)dt_i}{m_i^{a_i}\left(\underline{v}_i(\hat{a}_i)+\varepsilon\right)f_i\left(\underline{v}_i(\hat{a}_i)+\varepsilon\right)} > \frac{\int_{\underline{v}_i(\hat{a}_i)}^{\bar{v}_i(a_i)} m_i^{a_i}(t_i)f_i(t_i)dt_i}{f_i\left(\underline{v}_i(\hat{a}_i)\right)}$ as $\varepsilon \to 0$, which follows from the fact that $m_i^{a_i}\left(\underline{v}_i\left(\hat{a}_i\right)+\varepsilon\right) \leq 1$.

If the difference in $(B.10)$ is strictly positive, then the posterior virtual valuation drops at $\underline{v}_i\left(\hat{a}_i\right)$, so we have to consider the ironed virtual valuation, which will be flat around a neighborhood of $\underline{v}_i\left(\hat{a}_i\right)$–call it $\bar{J}$–which satisfies $J_i\left(\underline{v}_i\left(\hat{a}_i\right)+\varepsilon|a_i\right) \leq \bar{J} \leq J_i\left(\underline{v}_i\left(\hat{a}_i\right)|a_i\right)$. These arguments imply that around the neighborhood of $\underline{v}_i\left(\hat{a}_i\right)$, $J_i\left(\underline{v}_i\left(\hat{a}_i\right)|a_i\right)$ does not jump upwards.

From Lemma 2, it follows that buyer i never gets the object at $t = 2$ when his valuation is below the optimal second-period reserve price denoted by $r_i^2(\bar{v}_i)$, which satisfies (2.3) for beliefs given by (4.1). With the help of Lemma 2 and this last observation, the sellers' expected revenue can be rewritten as:

$$\Sigma_{i\in I}\left(\int_{r_i^2(\bar{v}_i)}^{\bar{v}_i}\delta P_i^2(v_i)J_i(v_i)f_i(v_i)dv_i+\int_{\bar{v}_i}^{b}P_i(v_i)J_i(v_i)f_i(v_i)dv_i\right). \tag{C.3}$$

We employ $(C.3)$ to show that at a solution of Program NC, $\bar{v}_i \geq r_i^*$.

Proof. Our goal is to establish a solution $\bar{v}_i \geq r_i^*$ for all $i \in I$, where r_i^* is given by (2.3) for the prior. In order to do so, we evaluate the impact of a marginal increase in $\bar{v}_i$ on the seller's revenue and show that it is strictly positive whenever $\bar{v}_i < r_i^*$, implying that at a revenue-maximizing assignment, it must be the case that $\bar{v}_i \geq r_i^*$.

As a preliminary step, we show that r_i^2 is a continuous function of $\bar{v}_i$, and, hence, it is differentiable almost everywhere. From Lemma 6, we know that if $J_i(v_i)$ is increasing, so is $J_i(v_i\,|\bar{v}_i\,)$. Hence, r_i^2 is unique. Moreover, since f_i is continuous, so is F_i, which ensures that r_i^2 is a continuous function of $\bar{v}_i$, and, hence, it is differentiable almost everywhere.[54]

Now, when we increase $\bar{v}_i$ for some buyer, this has a direct and an indirect effect on $(C.3)$. The direct effect is a change in the range of integration for i. The indirect effect is a change on p_j^2, for all $j \in I$, and it results because an increase in $\bar{v}_i$ changes the ranking of the posterior virtual valuations: Recall that buyer i wins the object at $t = 2$ if his posterior virtual valuation is the highest, and it is above the seller's value-that is, we must have $v_i - \frac{F_i(\bar{v}_i)-F_i(v_i)}{f_i(v_i)} \geq v_j - \frac{F_j(\bar{v}_j)-F_j(v_j)}{f_j(v_j)}$ and $v_i - \frac{F_i(\bar{v}_i)-F_i(v_i)}{f_i(v_i)} \geq 0$.

We now examine each of these effects separately. The direct effect of increasing $\bar{v}_i$ is:

$$\begin{aligned}&\delta P_i^2(\bar{v}_i)J_i(\bar{v}_i)f_i(\bar{v}_i)-\delta P_i^2(r_i^2(\bar{v}_i))J_i(r_i^2(\bar{v}_i))f_i(r_i^2(\bar{v}_i))\frac{\partial r_i^2(\bar{v}_i)}{\partial\bar{v}_i}-P_i(\bar{v}_i)J_i(\bar{v}_i)f_i(\bar{v}_i) \qquad \text{(C.4)}\\ =\;&-\left(P_i(\bar{v}_i)-\delta P_i^2(\bar{v}_i)\right)J_i(\bar{v}_i)f_i(\bar{v}_i)-\delta P_i^2(r_i^2(\bar{v}_i))J_i(r_i^2(\bar{v}_i))f_i(r_i^2(\bar{v}_i))\frac{\partial r_i^2(\bar{v}_i)}{\partial\bar{v}_i}>0.\end{aligned}$$

This inequality results from the following observations: At a solution of Program 2, P_i must be increasing, implying that $P_i(\bar{v}_i) \geq \delta P_i^2(\bar{v}_i)$. Also, from Lemma 2 in Skreta (2006b), we have that $r_i^2(\bar{v}_i)$ is increasing in $\bar{v}_i$, from which we obtain that $\frac{\partial r_i^2(\bar{v}_i)}{\partial\bar{v}_i} \geq 0$. From the last two observations, it follows that this partial effect (C.4) is strictly positive for $\bar{v}_i < r_i^*$ since if this is the case, we have $J_i(\bar{v}_i) < 0$, and, hence, $J_i(r_i^2(\bar{v}_i)) < 0$.

We now move on to establish that the indirect effect of $\bar{v}_i$ on expected revenue is zero:

$$\Sigma_{i\in I}\int_{r_i^2(\bar{v}_i)}^{\bar{v}_i}\delta\frac{\partial P_i^2(v_i)}{\partial\bar{v}_i}J_i(v_i)f_i(v_i)dv_i=0.$$

To see this, let $p^2(v)$ denote the allocation at $t = 2$ given $\bar{v}_i$, and let $\hat{p}^2(v)$ denote the allocation rule at $t = 2$ given cutoff $\bar{v}_i + \varepsilon$ when the realized vectors of valuations are v. The vector of types where these two rules differ are at points where the ranking of i's virtual valuation flips and are located at points where virtual valuations are equal to each other. In other words, $p_i^2(v) = \hat{p}_i^2(v)$ for all $v \in V_{-i} \times [0, \bar{v}_i]$, except the vectors of valuation where the ranking of virtual valuations changes. This happens along the boundaries where posterior virtual valuations are equal, which is a set of measure zero.

Therefore, the direct effect of increasing $\bar{v}_i$, as captured in (C.4), is equal to the total. We can, then, conclude that at a revenue-maximizing PBE, it must be the case that $\bar{v}_i \geq r_i^*$. □

[54] This cutoff can be alternatively obtained as the solution of the optimal price by a monopolist who is facing a downward-sloping demand $[F_i(\bar{v}_i) - F_i(r_i^2)]$. The monopolist problem is $[F_i(\bar{v}_i) - F_i(r_i^2)]r_i^2$. The first-order necessary conditions for a maximum (which are also sufficient given MHR), are $[F_i(\bar{v}_i)-F_i(r_i^2)]-f_i(r_i^2)r_i^2 = 0$ or, since $f_i(r_i^2) > 0$, $v_i - \frac{F_i(\bar{v}_i)-F_i(r_i^2)}{f_i(r_i^2)} = 0$. Then, the continuity of r_i^2 follows by the continuity of the seller's objective function and the Theorem of the Maximum.

D Proof of Proposition 4

Proof. When the seller has observed actions a and employed some disclosure policy that revealed a vector of messages λ, her revenue conditional on a and λ[55] can be expressed as (2.2), with an additional term as follows:

$$\int_{\bar{V}(a)} \Sigma_{i\in I} p_i(v,a,\lambda) J_i(v_i|a_i) f(v|a) dv - \Sigma_{i\in I} E_{v_{-i}} \left[u_i(\underline{v}_i(a_i), v_{-i}, a, \lambda) \,|a, \lambda \right]$$
$$+ \underbrace{\int_{\bar{V}(a)} \Sigma_{i\in I} \tau_i(v,a,\lambda) f(v\,|a\,) dv}_{\text{information fees}}, \tag{D.1}$$

where $u_i(v,a,\lambda) = p_i(v,a,\lambda)v_i - x_i(v,a,\lambda)$ and

$$-\tau_i(v,a,\lambda) \equiv u_i(v,a,\lambda) - \int_{\underline{v}_i(a_i)}^{v_i} p_i(t_i, v_{-i}, a, \lambda) dt_i - u_i(\underline{v}_i(a_i), v_{-i}, a, \lambda).^{56} \tag{D.2}$$

The first two terms are the same as in (2.2) because, roughly, a buyer can still "mimic" the behavior of the same set of valuations, as in the case where all the information that the seller has is public. The additional term $\int_{\bar{V}(a)} \Sigma_{i\in I} \tau_i(v,a,\lambda) f(v\,|a\,) dv$ results from the fact the seller's and the buyers' beliefs differ, and it can be thought of as the sum of transfers that the seller hopes to extract from the buyers for providing them with information about their competitors.

Step 1: Fix a strategy profile, and let $\pi\,(a)$ denote the ex-ante probability that a is the vector of actions chosen. In this step, we show that

$$\int_{a\in\mathcal{A}} \pi(a) \int_{V(a)} \Sigma_{\lambda\in\Lambda} c(\lambda\,|a\,) \tau_i(v,a,\lambda) f(v\,|a\,) dv da = 0, \tag{D.3}$$

where $c(\lambda\,|a\,)$ denotes the probability that the disclosure policy reveals λ when the vector of actions chosen is a.

We first show that for a mechanism that satisfies IC_i for all $i \in I$, it must hold that

$$E_{v_{-i},a_{-i},\lambda_{-i}} \left[\tau_i(v,a,\lambda) \,|a_i, \lambda_i \right] = 0. \tag{D.4}$$

To see this, note that, by definition, at a truth-telling equilibrium it must be the case that

$$U_i(v_i, a_i, \lambda_i) = E_{v_{-i},a_{-i},\lambda_{-i}} \left[u_i(v,a,\lambda) \,|a_i, \lambda_i \right]. \tag{D.5}$$

Also, following Myerson (1981), we can express buyer i's expected payoff at an incentive-compatible mechanism as

$$U_i(v_i, a_i, \lambda_i) = E_{v_{-i},a_{-i},\lambda_{-i}} \left[\int_{\underline{v}_i(a_i)}^{v_i} p_i(t_i, v_{-i}, a, \lambda) dt_i + u_i(\underline{v}_i(a_i), v_{-i}, a, \lambda) \,|a_i, \lambda_i \right]. \tag{D.6}$$

Combining (D.5) and (D.6), we obtain that

$$E_{v_{-i},a_{-i},\lambda_{-i}} \left[u_i(v,a,\lambda) - \int_{\underline{v}_i(a_i)}^{v_i} p_i(t_i, v_{-i}, a, \lambda) dt_i - u_i(\underline{v}_i(a_i), v_{-i}, a, \lambda) \,|a_i, \lambda_i \right] = 0. \tag{D.7}$$

With the help of (D.2), (D.7) can be rewritten as $E_{v_{-i},a_{-i},\lambda_{-i}} \left[\tau_i(v,a,\lambda) \,|a_i, \lambda_i \right] = 0$, establishing (D.4). Adding over all v_i, λ_i, a_i, we get (D.3).

Step 2: We show that given any disclosure policy, at a solution of the informed seller problem, it must hold that $\Sigma_{i\in I} \int_{V(a)} \tau_i(v,a,\lambda) f(v\,|a\,) dv = 0$ for all $i \in I$ and all a, a.e..

[55] If the seller employs partially revealing disclosure policies, she maintains some private information herself, thus endogenously becoming an informed principal. Then, a and λ, determine the seller's type.

[56] For more details, see Skreta (2011).

For some disclosure policy c, let p, x denote a revenue-maximizing mechanism given c. From (D.3), it follows that if for a positive measure of a, λ we have that $\Sigma_{i\in I}\int_{V(a)}\tau_i(v,a,\lambda)f(v\,|a)dv > 0$, then, there exists a positive measure of $\hat{a}, \hat{\lambda}$ with

$$\Sigma_{i\in I}\int_{V(\hat{a})}\tau_i(v,\hat{a},\hat{\lambda})f(v\,|\hat{a})dv < 0. \tag{D.8}$$

We know that p^{FT}, x^{FT} is feasible for the seller, given c for all realizations of a and λ because it is dominant-strategy incentive-compatible. Then, if p, x is a revenue-maximizing mechanism at $t = 2$ given a disclosure policy c, then it must be the case that for each a, λ, the seller's revenue at p, x is as least as high as at p^{FT}, x^{FT} for all a, λ. For $\hat{a}, \hat{\lambda}$, this implies that

$$\begin{aligned}
&\int_{V(\hat{a})}\Sigma_{i\in I}p_i(v,\hat{a},\hat{\lambda})J_i(v_i,\hat{a}_i)f(v\,|\hat{a})dv - \Sigma_{i\in I}E_{v_{-i}}\left[u_i(\underline{v}_i(a_i),v_{-i},\hat{a},\hat{\lambda})\,\middle|\hat{a},\hat{\lambda}\right] \\
&+\int_{V(\hat{a})}\Sigma_{i\in I}\tau_i(v,\hat{a},\hat{\lambda})f(v\,|\hat{a})dv \\
\geq\ &\int_{V(\hat{a})}\Sigma_{i\in I}p_i^{FT}(v,\hat{a},)J_i(v_i,\hat{a}_i)f(v\,|\hat{a})dv,
\end{aligned} \tag{D.9}$$

where in the LHS of the inequality, we use the fact that at p^{FT}, x^{FT} the payoffs to the lowest valuations are zero.

Because p^{FT} maximizes (2.2), it satisfies

$$\begin{aligned}
\int_{V(\hat{a})}\Sigma_{i\in I}p_i^{FT}(v,\hat{a},)J_i(v_i,\hat{a}_i)f(v\,\middle|\hat{\hat{a}})dv \quad \geq \quad &\int_{V(\hat{a})}\Sigma_{i\in I}p_i(v,\hat{a},\hat{\lambda})J_i(v_i,\hat{a}_i)f(v\,|\hat{a})dv \\
&-\Sigma_{i\in I}E_{v_{-i}}\left[u_i(\underline{v}_i(\hat{a}_i),v_{-i},\hat{a},\hat{\lambda})\,\middle|\hat{a},\hat{\lambda}\right],
\end{aligned} \tag{D.10}$$

which, together with (D.8), contradicts (D.9).

This step allows us to conclude, that, regardless of the realized vectors of actions at $t = 1$, and the disclosure policy, the best that the seller can do at $t = 2$ is to choose a mechanism that maximizes the first two terms of $(D.1)$, which coincide with (2.2). The result follows. $\square$

E Implications of Sequential Rationality

By the revelation principle, we can restrict attention to incentive-compatible direct revelation mechanisms, consisting of an assignment rule $p^2 : V(a) \longrightarrow \Delta(\bar{I})$ and a payment rule $x^2 : V(a) \longrightarrow \mathbb{R}^{\bar{I}}$. When posterior densities are zero, the sets of valuations $V_i(a_i)$ and $V(a) = \times_{i\in I}V_i(a_i)$ are not necessarily convex and two difficulties arise: First, we cannot express expected revenue only as a function of the allocation rule, and second, the formula of posterior virtual valuation is not well-defined for valuations where the posterior density is zero. To address the first difficulty, we establish that our problem of interest is equivalent to a convexified problem; to address the second, we approximate the problem of interest with one in which virtual valuations are well-defined and show that the solution of the approximate problem is arbitrarily close to the one of the problem we are interested in.

The Convexified Problem: We consider an artificial problem that has the same objective function as the problem of interest, but a different feasible set because we impose incentive and participation constraints on the convex hulls of $V_i(a_i)$ and $V(a)$. Proposition 1 in Skreta (2006) shows that these problems are equivalent in the following sense: One can obtain a solution of the program of interest by solving the convexified problem and by restricting the solution to the actual set of valuations. Conversely, any solution of the program of interest, can be extended appropriately to the convex hull of valuations, and it is a solution of the convexified problem. Hence, without loss of generality, we can consider the convexified problem. The intuition for this result is that "adding" types that occur with probability zero does not change the value of the program. For the convexified problem, we have, as usual, a "revenue equivalence theorem," and we use standard arguments to express the seller's problem as follows:

$$\max_{p^2,x^2} \int_{\bar{V}(a)} \Sigma_{i\in I} p_i^2(v,a)\left[v_i f_i(v_i|a_i) - (1-F_i(v_i|a_i))\right] f_{-i}(v_{-i}|a_{-i})dv - \Sigma_{i\in I} U_i^{2(a)}(p^2, x^2, \underline{v}_i(a_i)), \tag{E.1}$$

subject to: $P_i^2(v_i,a)$ increasing in v_i on $\bar{V}_i(a_i)$; $0 \leq p_i^2(v,a) \leq 1$ and $\Sigma_{i\in I} p_i^2(v,a) \leq 1$ for all $v \in \bar{V}(a)$.

Unfortunately, we still have to deal with the fact that a buyer's virtual valuation is not necessarily well-defined because the density f_i can be zero for some valuations on $\bar{V}_i(a)$. The inability to divide by $f_i(v_i|a_i)$ to obtain the virtual valuation creates difficulties, as one cannot compare the benefit from assigning the good to one buyer versus another.[57] We address this issue by considering another artificial program that is "close" to the convexified problem:

The Approximate Problem: We approximate the objective function with one where the density of each i is replaced by $f_i^{\varepsilon}(v_i|a_i) = \begin{cases} f_i(v_i|a_i) \text{ if } f_i(v_i|a_i) > 0 \\ \varepsilon \text{ otherwise} \end{cases}$, where $\varepsilon > 0$, arbitrarily small. The resulting problem, has the *same* constrained set as the program in $(E.1)$, because in $(E.1)$ incentive and participation constraints are imposed on $\bar{V}(a)$, which includes vectors of valuations that occur with probability zero. Moreover, the objective function is arbitrarily close to the one in $(E.1)$: It is routine to check that the objective function is continuous in ε, and the feasible set is sequentially compact in the topology of point-wise convergence (for similar arguments, see the Technical Appendix of Skreta (2006b)). Then, the Theorem of the Maximum implies that the value and the solution of the approximate problem are arbitrarily close to the ones of $(E.1)$.

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[57]In $(E.1)$, each buyer is assigned a different weight in the objective function: For example, buyer i's weight is given by $f_{-i}(v_{-i}|a_{-i})$.

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