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DYNAMIC ANALYSIS OF AN ANNULAR PLATE RESTING ON THE SURFACE OF AN ELASTIC HALF-SPACE WITH DISTRIBUTIVE PROPERTIES

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ABSTRACT

This work gives a semi-analytical approach for the dynamic analysis of a plate of annular shape resting on the surface of an elastic half-space with distributive properties. Such calculations have been associated with significant mathematical challenges, often leading to unrealizable computing processes. Therefore, the dynamic analysis of beams and plates interacting with the surfaces of elastic foundations has to date not been completely solved. To advance this work, the deflections of the plate are determined by the Ritz method, and the displacements of the surface of elastic half-space are determined by studying Green's function. The coupling of these two studies is achieved by a mixed method, which allows determination of reactive forces in the contact zone and, hence, the determination of other physical magnitudes. Natural frequencies, natural shapes, and the dynamic response of a plate due to external harmonic excitation are determined. Validation with a Winkler problem illustrates the distributive property effects on the results of the dynamic analysis.

Keywords: *Green's function; distributive properties; plate; Eigen-frequencies; Eigen-shapes; dynamic response*

1. INTRODUCTION

The dynamic behavior of beams and plates resting on an elastic foundation is a topic of high interest throughout the design and construction sectors. However, the dynamic analysis of such structures is associated with significant mathematical difficulties thereby possibly leading to impractical computational processes. For these reasons, the problem of dynamic analysis of plates resting on elastic foundations is not completely solved. Thus, there is a need to develop precise methods of analysis for these structures and to provide more effective methods than those that currently exist.

The work [1] is devoted to the study of the contact problems of plates of different forms resting on various elastic foundations in polar coordinates. A solution capable of dealing with static deflections and free vibrations of an annular or circular plate is provided in [2]. The formulation of this method is based on the modified theory of constraint torque where differential partial differential equations are governed by the Hamilton principle. The calculation results given concern the case of an annular and circular plate subjected to thermal loading. The work of reference [3] deals with the vibratory analysis of an annular plate using the modified, quadrature differential method. The results are greatly improved with respect to those obtained with the quadrature differential method. In the work [4], a method allowing the treatment of the Eigen modes and the distribution of the stresses in an annular plate of conditions with the specific limits. The results were favorably compared with finite element results. In the work [5] the Hart and Holland theorem was used to prove that axisymmetric equilibrium exists and is unique in different problems of elasticity theory and to generalize it for the case of an annular plate with different boundary conditions. The reliability of a solution to the contact problems used in engineering design is given in reference [6]. This solution is based on an improved finite element method using element selection that optimizes performance. A semi-analytical method for studying contact problems, without taking into account friction forces is presented in reference [7]. A sample application of the dynamic analysis of a beam resting on foundation of Lamb's model is given.

Three-dimensional contact problems without friction have been studied using the boundary face method presented in [8]. This involves an iterative procedure to correctly determine the reaction forces in the contact zone taking into account different boundary conditions. In the work [9], the variational method is applied to study the axially symmetrical contact of a thick circular plate resting on a smooth surface of an elastic half-space subjected to a uniformly distributed load over only a portion of the plate. The results illustrate how plate deflections, contact stresses, and bending moments can be affected by plate thickness variability.

Much part of the reference [10] is devoted to the application of the principle of Ritz's method to beams and plates of different shapes with different boundary conditions. In related work, the authors of reference [11] used the Ritz method to create a semi-analytical approach to solve the contact problem of a structure on a quarter-infinite elastic surface. This problem required the coupling of the Green function with structural deformations. The results given by this approach are very reliable compared to those of other methods. The reference [12] is the basic of the theory of beams, plates and shells of different shapes and different boundary conditions, including contact problems, and external stresses. This reference is of great importance since it always represents the basic reference relative to the calculations of the structures and the destination of all the comparisons of the calculations in connection with the classical cases.

The dynamic response of a plate resting on an elastic foundation subjected to static and moving loads was investigated in [13]. In the reference [14] is derived a solution based on Kirchhoff for a plate resting on an elastic foundation; reducing the solution of the standard Eigen-value problem and the mixed element based upon a consistent mass matrix formulation. Subsequently, in [15] is applied the method of fundamental solution (MFS) to a shear deformable plate resting on an elastic foundation under either static and dynamic loads. The mixed method for the dynamic calculation of structures (using finite elements method) on a half-space (using analytical computing) was proposed much earlier in [16] and also studied in the same work the influence of plate stiffness. An analytical formulation using the principle of minimum potential energy to study the dynamic behavior of a rectangular plate resting on an elastic half-space and subjected to a uniformly distributed load is given in [17]. In that work, the compatibility at the interface of the plate and soil medium was satisfied by integrating the Boussinesq's formula that relates the contact stress and the soil surface deformation. The problems of slabs and tanks resting on a semi-infinite elastic continuum (Boussinesq's type) or on individual springs (Winkler's type) were solved using the finite element method in [18]. A new version of the differential quadrature method was proposed in [19], in order to obtain the vibration characteristics of rectangular plates resting on elastic foundations. The results of this work demonstrated, according to the author, the efficiency of the method in treating the vibration problem of rectangular plates resting on elastic foundations. In [20] giving an exact solution for functionally graded thick plates resting on a Winkler–Pasternak elastic foundation based on the three-dimensional theory of elasticity and established that the elastic foundation affects significantly the mechanical behavior of functionally graded thick plates.

The current work provides a semi-analytical approach to determine the natural frequencies and natural shapes of an annular plate resting on an elastic foundation with distributives properties (Boussinesq's type) and its response to an external vertical harmonic excitation (Fig. 1). The following as aspects are neglected: damping, inertia of the elastic foundation, and friction in the contact zone between the plate and the surface of the elastic foundation. The approach is based on the mixed method known as Zhemochkin's method [21], wherein the annular plate is divided into a finite number of identical elements, and in the center of each element is placed a rigid link, through which the contact between the plate and the surface of elastic foundation is achieved. The approach assumes that contact between the surfaces of the structure and the elastic foundation is replaced by a finite contact through rigid links and that the mass of each element is concentrated in its center.

2. Problem Definition

2. 1. Description of the method of analysis

In brief, the procedure to study free vibrations of an annular plate resting on a surface of elastic foundations of a Boussinesq type is as follows. An annular plate with a mass, M, and cylindrical rigidity, D, rests on the surface of an elastic foundation with distributive properties with a modulus of elasticity, E, and a Poisson's ratio, ν.

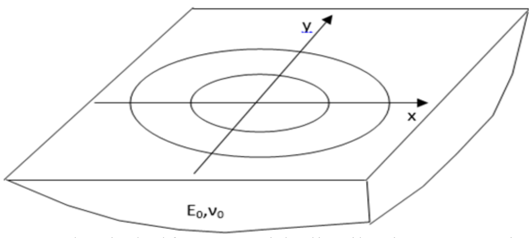

Figure 1. Plate resting on elastic half space with distributive properties

The essential parameters of Zhemochkin's method for the study of the dynamic of a plate resting on an elastic foundation are shown in Fig. 2.

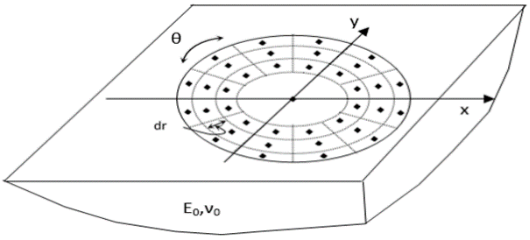

Figure 2. Discretization of the studied system

The inertial forces are applied only on the plate, since the mass of the elastic foundation is excluded, while the efforts of connection are applied on the plate and on the surface of the elastic foundation Fig. 3.

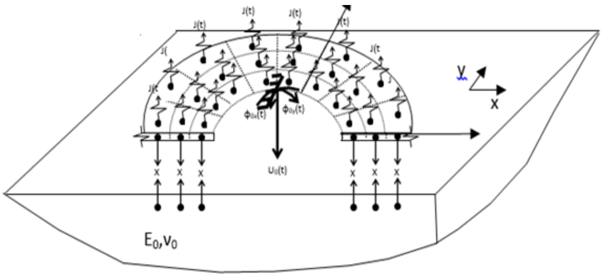

Figure 3. Illustration of the method of analysis

The canonical system of equations for the dynamic study of an annular plate resting on the surface of an elastic foundation with distributive properties is expressed in Eq. (1) as per [21]:

$$\begin{cases} \vdots \\ \sum_{j=1}^{n}\left(v_{ij}+W_{ij}\right)X_j(t)-\sum_{j=1}^{n}W_{ij}J_j(t)+\ell_{ix}\varphi_{0x}(t)+\ell_{iy}\varphi_{0y}(t)+u_0(t)+\Delta_{iP}=0; \quad i=1,\dots,n \\ \vdots \\ \sum_{j=1}^{n}\left[X_j(t)-J_j(t)\right]\ell_{jx}=\mathrm{I}_{0x}\,\ddot{\varphi}_{0x}(t); \\ \sum_{j=1}^{n}\left[X_j(t)-J_j(t)\right]\ell_{jy}=\mathrm{I}_{0y}\,\ddot{\varphi}_{0y}(t); \\ \sum_{j=1}^{n}\left[X_j(t)-J_j(t)\right]=\mathrm{M}\,\ddot{u}_0(t) \end{cases} \tag{1}$$

X_j : connection's force applied on the plate and on the surface of the elastic foundation; J_j : inertia's force applied only on the plate; v_{ik} : Green's function defining the displacement of the surface of the elastic foundation at the point i due to a force X_j applied in the point j of the same surface; $\varphi_{0x}, \varphi_{0y}$: angle of rotation of the plate relative to the axes Ox and Oy at the embedding point; u_0 : initial displacement of the plate at the embedding point; W_{ik} : deflection of the plate at a point i due to a force X_j applied at the point j of the plate; Δ_{iP} : function characterizing the deflection of the plate at a point i due to an external force P_j applied at the point j of the plate (for free vibrations $\Delta_{iP}=0$); ℓ_{ix}, ℓ_{iy} : arms of the elements of the plate relative to the axes of coordinate; M : total mass of the plate; $\mathrm{I}_{0x}, \mathrm{I}_{0y}$: inertia's moments of the plate relative to the axes of the coordinates.

The relationship for the free vibrations is shown in equation (2) as per [7]:

$$X_k(t)=X_k e^{i\omega t};\varphi_{0x}(t)=\varphi_{0x}e^{i\omega t};\varphi_{0y}(t)=\varphi_{0y}e^{i\omega t};u_0(t)=u_0 e^{i\omega t};J_k(t)=J_k e^{i\omega t} \tag{2}$$

Taking into account (2), the system from (1) takes the following form:

$$\begin{cases} \vdots \\ \sum_{j=1}^{n} (v_{ij} + W_{ij}) X_j - \sum_{j=1}^{n} W_{ij} J_j + \ell_{ix} \varphi_{0x} + \ell_{iy} \varphi_{0y} + u_0 = 0; \quad i = 1, \ldots, n \\ \vdots \\ \sum_{j=1}^{n} [X_j - J_j] \ell_{jx} = \mathrm{I}_{0x} \ddot{\varphi}_{0x}; \\ \sum_{j=1}^{n} [X_j - J_j] \ell_{jy} = \mathrm{I}_{0y} \ddot{\varphi}_{0y}; \\ \sum_{j=1}^{n} [X_j - J_j] = \mathrm{M} \ddot{u}_0 \end{cases} \tag{3}$$

2. 2. Green function defining the vertical displacement of the surface of elastic half space

Since the elastic foundation is the elastic half-space with distributives properties (principle of Boussinesq), then the expression of takes the form of Eq. (4) as per [22]:

$$v_{ij} = \frac{1 - \nu_0^2}{\pi E_0} \frac{P}{\Omega} \int_{\varphi=\varphi 1}^{\varphi=\varphi 2} \int_{a=r1}^{a=r2} \frac{a da d\varphi}{\sqrt{a^2 + r^2 - 2ar\cos(\varphi - \theta)}} \tag{4}$$

E_0 and ν_0 : Modulus of elasticity and Poisson's ratio of the elastic half-space; P : External force applied on the surface of the elastic half-space, Ω : Surface of the loaded element, a and φ : Polar coordinates of the center of the loaded element; r and θ: Polar coordinates of the center of the element where the displacement is determined; $r1$ and $r2$: Rays defining the loaded element; $\varphi 1$ and $\varphi 2$: Angles made by the rays of the loaded element with the x-axis; $\sqrt{a^2 + r^2 - 2ar\cos(\varphi - \theta)}$: Distance between the center of the loaded element and the element where the displacement is determined, Fig. 4.

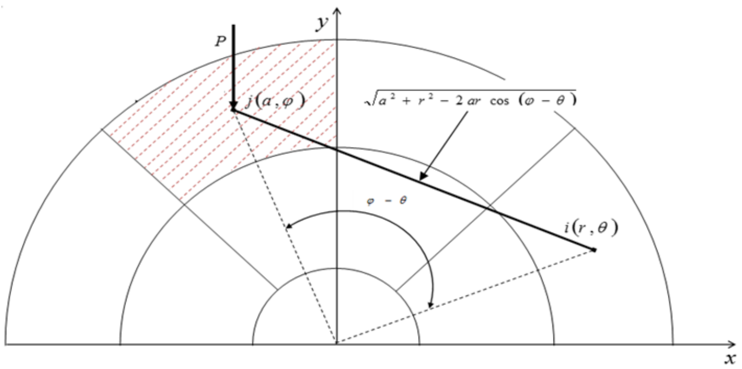

Figure 4. Geometry of the loaded element

After integration, expression (4) becomes:

$$v_{ij} = \frac{1 - \nu_0^2}{\pi E_0} \frac{1}{\Omega} F_{ij} \tag{5}$$

$$
\begin{aligned}
P_{ij} = &-\frac{\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi1]}\;\mathrm{EllipticE}\left[\frac{\theta-\phi1}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]}{\sqrt{\frac{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi1]}{(r-r1)^2}}}+\frac{\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi1]}\;\mathrm{EllipticE}\left[\frac{\theta-\phi1}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{\sqrt{\frac{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi1]}{(r-r2)^2}}}+\frac{\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi2]}\;\mathrm{EllipticE}\left[\frac{\theta-\phi2}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]}{\sqrt{\frac{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi2]}{(r-r1)^2}}}-\\
&\frac{\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi2]}\;\mathrm{EllipticE}\left[\frac{\theta-\phi2}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{\sqrt{\frac{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi2]}{(r-r2)^2}}}+\frac{2\,r^2\sqrt{\frac{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi1]}{(r-r1)^2}}\;\mathrm{EllipticF}\left[\frac{\theta-\phi1}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]}{\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi1]}}-\frac{\left(r^2+r1^2\right)\sqrt{\frac{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi1]}{(r-r1)^2}}\;\mathrm{EllipticF}\left[\frac{\theta-\phi1}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]}{\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi1]}}-\\
&\frac{2\,r^2\sqrt{\frac{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi1]}{(r-r2)^2}}\;\mathrm{EllipticF}\left[\frac{\theta-\phi1}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi1]}}+\frac{\left(r^2+r2^2\right)\sqrt{\frac{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi1]}{(r-r2)^2}}\;\mathrm{EllipticF}\left[\frac{\theta-\phi1}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi1]}}-\frac{2\,r^2\sqrt{\frac{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi2]}{(r-r1)^2}}\;\mathrm{EllipticF}\left[\frac{\theta-\phi2}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]}{\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi2]}}+\\
&\frac{\left(r^2+r1^2\right)\sqrt{\frac{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi2]}{(r-r1)^2}}\;\mathrm{EllipticF}\left[\frac{\theta-\phi2}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]}{\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi2]}}+\frac{2\,r^2\sqrt{\frac{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi2]}{(r-r2)^2}}\;\mathrm{EllipticF}\left[\frac{\theta-\phi2}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi2]}}-\frac{\left(r^2+r2^2\right)\sqrt{\frac{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi2]}{(r-r2)^2}}\;\mathrm{EllipticF}\left[\frac{\theta-\phi2}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi2]}}-\\
&r\,\mathrm{Log}\left[r1-r\,\mathrm{Cos}[\theta-\phi1]+\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi1]}\right]\mathrm{Sin}[\theta-\phi1]+r\,\mathrm{Log}\left[r2-r\,\mathrm{Cos}[\theta-\phi1]+\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi1]}\right]\mathrm{Sin}[\theta-\phi1]+\\
&r\,\mathrm{Log}\left[r1-r\,\mathrm{Cos}[\theta-\phi2]+\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta-\phi2]}\right]\mathrm{Sin}[\theta-\phi2]-r\,\mathrm{Log}\left[r2-r\,\mathrm{Cos}[\theta-\phi2]+\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta-\phi2]}\right]\mathrm{Sin}[\theta-\phi2]
\end{aligned}
$$

Fig. 5 illustrates the displacements of the surface of half space with distributive properties given by (5) in the contact zone due to a concentrated force applied on the plate.

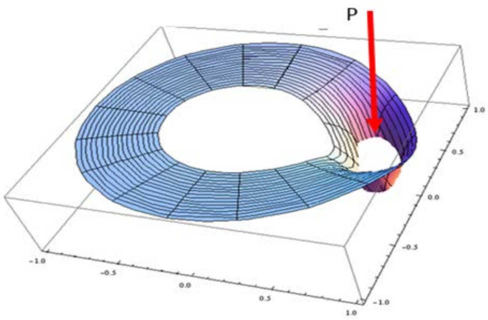

Figure 5. Surface's deformation of half space due to concentrate force applied on the plate

2. 3. Function of the deflections of plate

In Eq. (6) W_{ij} is the function defining the deflections of the annular plate in a point i due to the force X_j applied in a point j. Based on the Clebsch's solution [12], W_{ij} takes the following form:

$$W_{ij}(x,y)=W_0(x,y)+\sum_{n=1}^{\infty}A_nW_n(x,y)+\sum_{n=1}^{\infty}B_nW_n(x,y) \tag{6}$$

According to [1], for the case presented herein, the terms of the expression (6) that satisfy the boundary conditions of the studied plate are as follows:

$$W_{ij}(x,y)=W_0(x,y)+A_{21}W_1(x,y)+B_{21}W_2(x,y) \tag{7}$$

where

$$W_0(r,\theta)=\frac{Pb^2}{16\pi D}\left\{\left(\frac{a^2-2ar\cos[\varphi-\theta]+r^2}{b^2}\right)\ln\left[\left(\frac{a^2-2ar\cos[\varphi-\theta]+r^2}{b^2}\right)\right]+\right.$$
$$\left.+\frac{4ar\cos[\varphi-\theta]}{b^2}\left(\ln\left[\frac{ar}{b^2}\right]+1\right)-\left(\frac{a^2}{b^2}\right)\ln\left[\frac{a^2}{b^2}\right]-\left(\frac{r^2}{b^2}\right)\ln\left[\frac{r^2}{b^2}\right]\right\}$$
$$W_1(r,\theta)=A_{21}\frac{r^3}{b^3}\cos(\theta);W_2(r,\theta)=B_{21}\frac{r^3}{b^3}\sin(\theta)$$

b : some dimension introduced to check the units, it is equal to 1 and in this case considered the ray of the external contour of the plate; z : ray of the internal contour of the plate; $D=\frac{Eh^3}{12(1-\nu^2)}$: plate's cylindrical rigid; E,ν : modulus of elasticity and Poisson's ratio of the material of the plate; h : plate's thickness; $\beta=2(1-\nu)$; r and θ : coordinates of the point i on the plate where the displacement is determined; a and φ : coordinates of the point j where the force X_j is applied, Fig. 6.

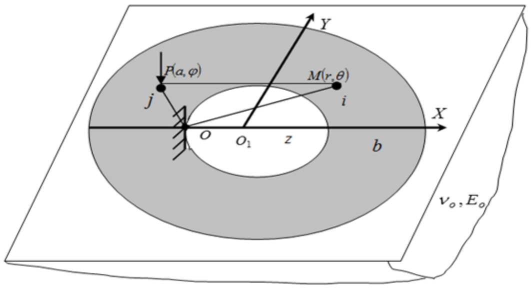

Figure 6. Annular plate embedded at a point located on the internal contour and free on all of its external contours

The coefficients A_{21} and B_{21}, are determined by the Ritz's method as presented in [11] by considering the deformation's energy of the plate. After simplification, the resolution system take the following matrix form:

$$\begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix}\begin{bmatrix} A_{21} \\ B_{21} \end{bmatrix}=\begin{bmatrix} I_1 \\ I_2 \end{bmatrix};$$

$$H_{11}=\frac{8D\pi(8-\beta)}{b^6}\left[-4z^4\cos^4(\theta)+\frac{1}{4}\left(z\cos(\theta)+\sqrt{b^2-z^2\sin^2(\theta)}\right)^4\right];$$

$$H_{22}=\frac{8D\pi(8-\beta)}{b^6}\left[-4z^4\cos^4(\theta)+\frac{1}{4}\left(z\cos(\theta)+\sqrt{b^2-z^2\sin^2(\theta)}\right)^4\right];$$

$$I_1=\frac{1}{16ab^3}P(\beta-8)\cos(\varphi)\left[8a^2z^2\cos^2(\theta)-16z^4\cos^4(\theta)-2a^2\left(z\cos(\theta)+\sqrt{b^2-z^2\sin^2(\theta)}\right)^2+\right.$$
$$\left.+\left(z\cos(\theta)+\sqrt{b^2-z^2\sin^2(\theta)}\right)^4\right]+\frac{a^3}{b^3}P\cos(\varphi);$$

$$I_2=\frac{1}{16ab^3}P(\beta-8)\sin(\varphi)\left[8a^2z^2\cos^2(\theta)-16z^4\cos^4(\theta)-2a^2\left(z\cos(\theta)+\sqrt{b^2-z^2\sin^2(\theta)}\right)^2+\right.$$
$$\left.+\left(z\cos(\theta)+\sqrt{b^2-z^2\sin^2(\theta)}\right)^4\right]+\frac{a^3}{b^3}P\sin(\varphi)$$

Finally, these coefficients take the following expressions:

$$A_{21} = \left\{ b^3 P\cos(\varphi)\left[16a^4 + (\beta-8)\left(8a^2z^2\cos^2(\theta) - 16z^4\cos^4(\theta) - \right.\right.\right.$$
$$\left.\left.\left. -2a^2\left(z\cos(\theta)+\sqrt{b^2-z^2\sin^2(\theta)}\right)^2 + \left(z\cos(\theta)+\sqrt{b^2-z^2\sin^2(\theta)}\right)^4\right)\right]\right\} / ;$$
$$\left\{128aD\pi(8-\beta)\left[-4z^4\cos^4(\theta)+\frac{1}{4}\left(z\cos(\theta)+\sqrt{b^2-z^2\sin^2(\theta)}\right)^4\right]\right\}$$
$$B_{21} = \left\{ b^3 P\sin(\varphi)\left[16a^4 + (\beta-8)\left(8a^2z^2\cos^2(\theta) - 16z^4\cos^4(\theta) - \right.\right.\right.$$
$$\left.\left.\left. -2a^2\left(z\cos(\theta)+\sqrt{b^2-z^2\sin^2(\theta)}\right)^2 + \left(z\cos(\theta)+\sqrt{b^2-z^2\sin^2(\theta)}\right)^4\right)\right]\right\} / .$$
$$\left\{128aD\pi(8-\beta)\left[-4z^4\cos^4(\theta)+\frac{1}{4}\left(z\cos(\theta)+\sqrt{b^2-z^2\sin^2(\theta)}\right)^4\right]\right\}$$

Figure 7 illustrates the displacements of the plate given by (7) due to a vertical concentrate load applied at a point near to its embedment point.

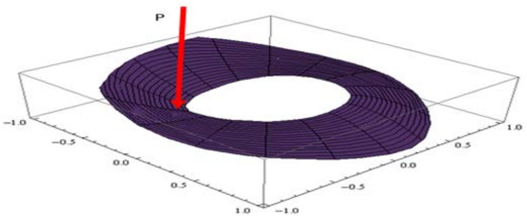

Figure 7. Plate's deformation due to the concentrate force applied near to the embedded point

2. 4. Remarks

Since the displacements of the surface of a half-space are considered equal to the plate deflection (i.e. $W_i = v_i$), so the inertia force J_i can be given by the following expression as per [7]:

$$J_i = -M_i \frac{d^2W_i(t)}{dt^2} = -M_i \frac{d^2v_i(t)}{dt^2} = M_i\omega^2 v_i = M_i\omega^2\left(\frac{1-\nu_0{}^2}{\pi E_0}\right)\sum_{j=1}^{n} X_j F_{ij}, \qquad (8)$$

$$v_i = \frac{1-\nu_0{}^2}{\pi E_0}\sum_{j=1}^{n} X_j F_{ij}$$

F_{ij} given by (5).

3. Sample Application

A plate with the following properties is considered:

$b = 1\,m;\ z = 0.5\,m;\ \nu = 1/3;\ E = 2\times10^{11}\,N/m^2;\ h = 0.01m;\ \nu_0 = 1/3.\ E_0 = 10^8\,N/m^2$

The plate is divided into 24 identical elements where the embedment point is as shown on figure 8.

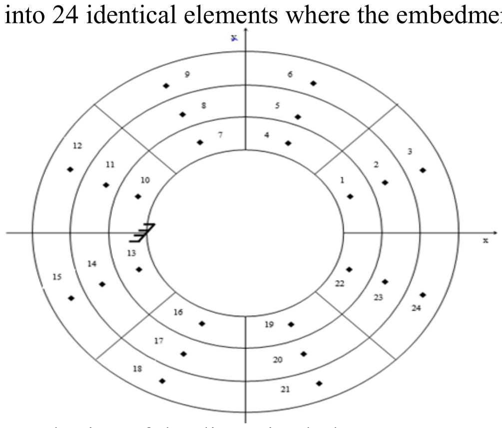

Figure 8. Elements' numbering of the discretized plate

Using the system described in (3) and taking into account the expressions of each of its parameters, as well as mathematical simplifications, the matrix system (9) is obtained:

$$[A]\begin{Bmatrix} X_1 \\ \vdots \\ X_{24} \\ \varphi_{0x} \\ \varphi_{0y} \\ u_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{Bmatrix} \qquad (9)$$

As the terms of the matrix $[A]$ are very long expressions resulting from mathematical transformations, they are not included herein.

3. 1. Determination of the plate's Eigen frequencies

The determination of the plate's Eigen frequencies resting on the surface of a half-space with distributive properties is done by the resolution of the equation of the determinant of the matrix $[A]$ of the system (9). Within this long and complicated equation, the roots represent the Eigen frequencies values of the plate. The calculation of the roots of the equation is executed numerically using "Mathematica". Fig. 8 represents a close up of one of roots of the determinant representing the plate's Eigen frequencies.

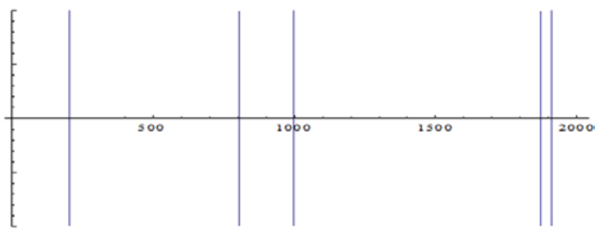

Figure 9. Graphical close up of the roots of determinant representing an Eigen frequency

The spectrum of Eigen frequencies in (Hz) is:
$\omega_1 = 200.732814$, $\omega_2 = 811.125487$, $\omega_3 = 992.481452$,...

3. 2. Determination of the plate's natural shapes

Determination of the natural shapes of the square plate resting on the surface of a half-space with distributive properties requires determination of the natural shape corresponding to each Eigen frequency. To achieve this, the first equation of the matrix system (9) is excluded, and the system is solved with $n+2$ and $n+2$ unknowns. Subsequently, the natural shapes of the plate are determined by the formula (8) introduced by Zhemochkin and Sinitsyn [20]. For a given discretization: $i = 1,\ldots,n$; $n = 24$. The application of these operations allows the determination of the plate's natural shapes for the first three following forms:

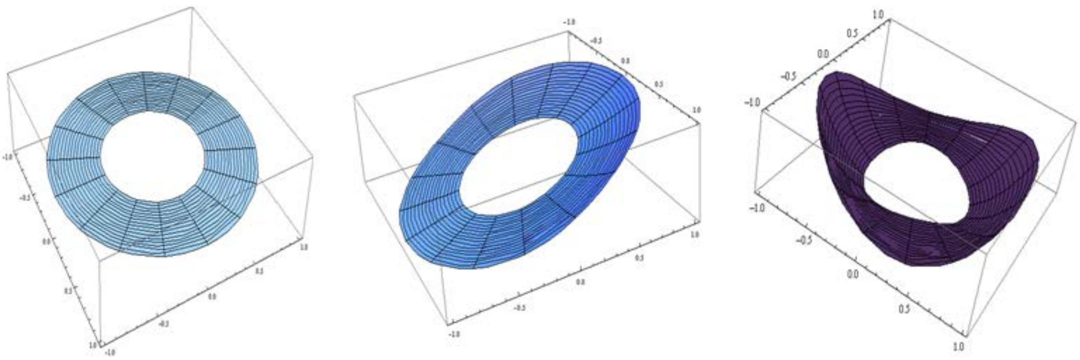

Figure 10. Natural shapes of the plate corresponding to the first three Eigen frequencies

3.3 Response of the plate resting on the surface of a half-space with distributive properties due to external vertical harmonic excitations

The forced vibrations of the plate resting on the surface of a half-space with distributive properties concerns its reaction due to the dynamic external vertical loads. The case presented is of the reaction of the plate caused by the vertical harmonic load applied at point p coinciding with the center of the element 2 (Fig. 8). The external vertical harmonic load is varied according to Eq. (10):

$$P_p = P_2 = P_0 \cos(2 \times \pi \times f \times t) \tag{10}$$

where P_0 : amplitude of excitation; f : frequency of excitation; t : time of excitation. The values: $P_0 = 100000N$, $f = 250Hz$ are considered. Obtaining the response requires the resolution of the system (1). In this case, the parameter Δ_{ip} is determined by a following formula (11) [20]:

$$\Delta_{ip} = \sum_{p=1}^{n} W_{ip} P_p = W_{i2} P_2 \tag{11}$$

W_{ip} : plate deflections in the point i due to the external load P_p applied on the plate at a point p. These deflections are due to the external loads P_p as determined by (7). The solution of the system (1) for the forced oscillations gives the unknowns X_i values varying with the time of excitation. Unknowns X_i represent reactive efforts in the contact zone. In this case the value's variability is expressed by Eq. (12):

$$X_i(t) = X_i(t_0)\cos(2 \times \pi \times f \times t), (i = 1, \ldots, 25) \tag{12}$$

Finally, the plate displacements during the time of excitation t, representing its response are determined by Eq. (13):

$$v_i = \frac{1 - \nu_0^{\,2}}{\Omega \pi E_0} \sum_{j=1}^{24} X_j F_{ij} \tag{13}$$

Fig. 11 shows the vertical displacements variability v_i of the entirety of the plate resting on the surface of a half-space with distributive properties in 3D due to the dynamic external load P_2 at each moment $t_j = t_0 + \Delta t$. Here, $t_0 = 0\,s; \Delta t = 0.001\,s$. Furthermore, the maximum displacement is obtained at the point in which the external excitation is applied.

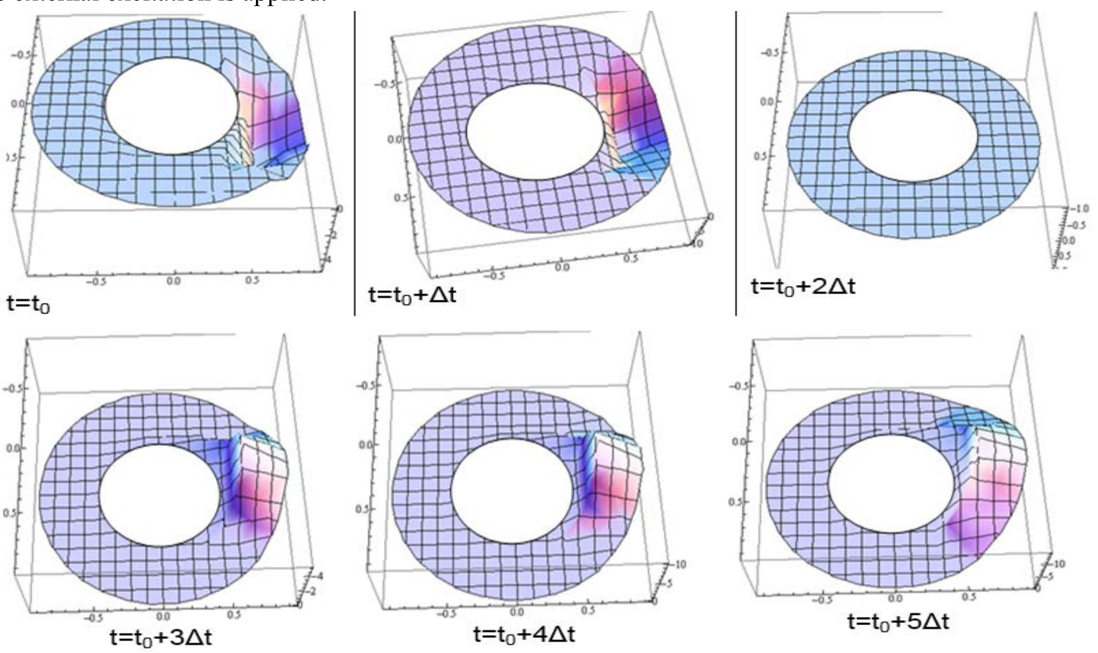

Figure 11. Response of an annular plate resting on an elastic foundation with distributive properties due to a harmonic load applied at the center of element 2

Finally, Figure 12 shows the projection of the response (iso-values) of the annular plate at a given moment, where it is clearly seen that the distribution of the response displacements is not the same. The maximum displacements are in the zone near to the point of application of the load, and the further away from this point, the displacements are minimized.

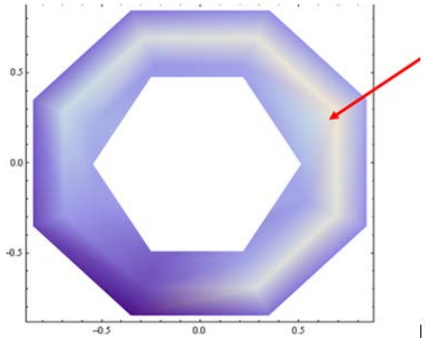

Figure 12. Distribution of the plate displacements due to the loading applied at the center of element 2 at a given moment

4. Results comparison

To ensure the reliability of the results, the same annular plate resting on two different types of elastic foundations (Boussinesq's model and Winkler's model) are considered. The plates are excited by a harmonic load, expressed by equation (10) and applied to the centre of the element 2 (fig. 8). Fig. 13 compares the vertical displacement variability of plate at the the dynamic load application point, where is the foundation with distributive properties (Boussinesq's model) and the foundation with a spring model (Winkler's model). Of note is that the vertical displacements of the plate for the half-space with distributive properties are always less than the same type of displacement, when the distributive properties of half-space are neglected, thus proving that the half-space with distributive properties fundamentally influences dynamic analysis.

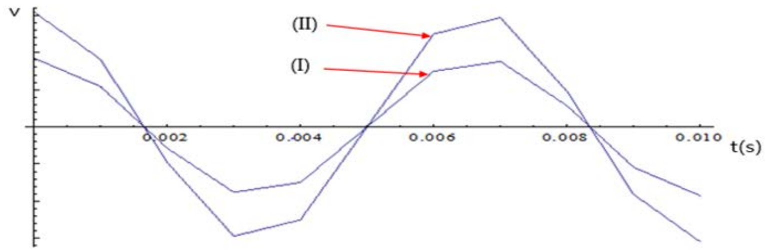

Figure 13. Comparison of the displacement variability of a point on the plate by two different models: (I) Boussinesq's model and (II) Winkler's model

5. Conclusions

Using a semi-analytic approach, a dynamic analysis of an annular plate resting on the surface of an elastic half-space with distributive properties was achieved to determine the Eigen frequencies, natural shape, plate response to external harmonic dynamic loads. Determination of the reactive forces in the contact zone representing the interaction phenomena between the plate and half-space surface is necessary to find the any physical magnitude using formulas of the elasticity's theory. For this purpose, it is imperative to study the Green's function defining the vertical displacements of the contact zone (i.e. contact problem phenomena). The calculated results were compared satisfactorily to the same plate resting on the surface of an elastic foundation using Winkler's model. Additionally, the obtained solution is semi-analytical and can, therefore, be readily computed to be more compatible with engineering applications. As such, this work represents a fundamental advance in the solving of complicated dynamic problems. The next step is to study the plates resting on an elastic foundation with inertial properties (Lamb's problem).

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