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Studies in the Astronomy of the Roman Period

I. The Standard Lunar Scheme

by

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Although modern scholarship tends to emphasize the earlier stages of Greek astronomy, say from Eudoxus to Hipparchus, our most generous and detailed direct evidence pertains to the Roman period, or roughly the first five centuries of our era. By way of the medieval manuscript traditions we possess the writings of Ptolemy, several major commentaries on them by Pappus, Theon, and Proclus, and a handful of less technical elementary works. Since the middle of the nineteenth century, excavations and accidental finds in Egypt have yielded fragments of astronomical texts and tables, written in Greek or Egyptian on papyrus, ostraca, and wooden boards, the great part dating from the first five centuries of Roman rule. These contemporary documents illustrate the astronomical practices prevalent in Roman Egypt, and they are now sufficiently numerous to allow us to trace the parallel, indeed symbiotic, existence of kinematic predictive astronomy (mostly represented by versions of Ptolemy's *Handy Tables*) and an arithmetical astronomy

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that was in large part evolved from the predictive methods known to us from Babylonian tablets of the last four centuries B.C. The most copiously documented component of this arithmetical tradition is a method of calculating lunar longitudes and arguments of latitude that, while preserving a characteristically Babylonian arithmetical model for the day-to-day variation in the moon's progress, otherwise appears to be an original invention of unknown Greek astronomers.

In a previous article surveying various ancient and medieval procedures for computing the moon's position that incorporated the approximate equation of nine anomalistic months with 248 days, I discussed three Greek papyrus fragments from Roman Egypt: *P. Ryl.* I.27, *P. Lund* inv. 35a, and *P.S.I.* XV.1493.[1] Neugebauer and van der Waerden had already established the following:[2]

(1) *P. Lund* inv. 35a is a tabular list of dates (regnal year of current emperor, Egyptian calendar month and day) approximately coinciding with the moon's reaching its apogee or day of least progress in longitude, during at least the years 69–109.[3] Beside each date is given the moon's longitude on that day, expressed as zodiacal sign and degrees (to the third fractional sexagesimal place). These 'epoch' dates are grouped in series of twelve, all separated by exactly 248 days, whereas the interval between the twelfth epoch date of a cycle and the first date of the next cycle is 303 days; hence the overall cycle is 3031 days; i.e. approximately 110 anomalistic months. The tabulated longitudes increase by constant steps of 27;43,24,56° every 248 days and of 337;31,19,7° every 3031 days. Dividing these increments by the corresponding time intervals does not lead to a consistent mean daily progress in longitude; van der Waerden suggested that the discrepancy reflects an awareness that the 248-day and 3031-day periods are only approximations of a more precise period of anomaly, and that the underlying mean motion was 13;10,34,52° per day, which is essentially correct for sidereal longitudes.

(2) *P. Ryl.* 27, written about A.D. 250, is a 'procedure text', the main part of which contains instructions for calculating epoch dates and longitudes of the kind listed in *P. Lund* inv. 35a. (For a translation of the entire text, see Appendix 2.) Two versions of the

instructions are given, the second set generating dates and longitudes exactly matching those of *P. Lund* inv. 35a, whereas the first set seems to generate cycles such that the first eleven epochs match the second through twelfth epochs in the equivalent cycle of the second set of instructions.[4] As well as the longitude, the rules generate a second number associated with each epoch: the moon's argument of latitude, measured from the northern limit in units called 'steps' (*βαθμοί*) such that 1 step equals 15°. The epoch-to-epoch increments in argument of latitude can be derived from those in longitude exactly, using a nodal motion of – 0;3,10,49,15° per day.

(3) *P.S.I.* 1493 came to light more recently than the other two papyri, and still awaits formal publication.[5] Neugebauer recognized that it is a fragment of a 'template' or table of the moon's progress since epoch through a cycle of consecutive days starting with any day of least motion. The preserved or partially preserved lines extend from day 80 to day 135. In addition to an index column giving the number of days since epoch (tabulated only every five lines), there is a column for the progress in longitude in degrees and minutes, and also for a second two-place sexagesimal quantity that cycles through the range 0 to 24. The line-to-line differences of either column behave as truncations of a linear zigzag function with the anomalistic month as period.

In my 1983 article I identified the second column in *P.S.I.* 1493 as progress in argument of anomaly since epoch, measured in steps; and I determined approximate parameters for the two zigzag functions from which it was possible to generate a reconstructed template that agreed with all preserved values in the papyrus to within 1 minute (except for isolated scribal errors). I showed that the reconstructed template could be used together with an epoch table such as *P. Lund* inv. 35a to find the moon's position on any date: one simply had to find the preceding epoch, and add together the tabulated epoch position and the progress in the template corresponding to the number of days elapsed since epoch. Passages in the *Anthologiae* of the second-century Antiochene astrologer Vettius Valens prove that he possessed just such a set of tables and used them in this way.[6]

In the course of editing the astronomical fragments among the vast collection of papyri excavated between 1896 and 1906 by Grenfell and Hunt at the site of Oxyrhynchus, several have come to light that are concerned with the same tables.[7] In fact, as a means of computing lunar positions they are only rivalled among the papyri by Ptolemy's *Handy Tables.* A predictive scheme – that is, tables and the methods of generating and using them – of such prolonged and widespread popularity deserves to have a name, and so I will here refer to it as the 'Standard Lunar Scheme'.

Through the new texts we can recover details of the Standard Scheme that were obscure before; and we may take advantage of the opportunity to correct some long-standing mistakes in the interpretation of the documents already known. The outcome will be the restoration of the Standard Scheme to its full working order, so that we can recompute lunar positions for arbitrary dates using this scheme, just as we are already able to do with Ptolemy's tables – providing us with a valuable resource for the analysis and dating of other astronomical papyri.[8] As well, we obtain a glimpse of the methodology behind the arithmetically-based astronomy that, as the papyri reveal, was prevalent in Roman Egypt. The present article undertakes the following: (1) deduction of the exact parameters of the two zigzag functions on which the Standard Scheme template is based; (2) explication of a new procedure text that demonstrates a profounder interest in and understanding of the mathematical workings of the zigzag functions than one would have expected in a Greek text; (3) explanation of the epoch increments; (4) investigation of the epoch alignments of the scheme; and (5) survey of the range of dates of the documents testifying to the use of the Standard Scheme.

1. The Template. Before turning to the Standard Scheme template I will review, in all brevity, the general definition of a linear zigzag function and the notations used here for its parameters. A zigzag function is sequence of numbers z_i generated according to simple iterative arithmetical rules involving three determining parameters: a 'minimum' m, a 'maximum' M, and an 'increment/decrement' d. Every number z_i generated by the function is considered to belong to either an increasing or decreasing 'branch' of the function. Depending on which branch z_i is in, the next value z_{i+1} is obtained as follows:

Increasing: (1)
If $z_i+d \leq M$ then $z_{i+1}=z_i+d$, on increasing branch
Otherwise $z_{i+1}=(2M-d)-z_i$, on decreasing branch
Decreasing:
If $z_i-d \geq m$ then $z_{i+1}=z_i-d$, on decreasing branch
Otherwise $z_{i+1}=(2m+d)-z_i$, on increasing branch

Consequently the sequence of z_i will be equally spaced values of a real periodic function composed of alternating linear stretches of equal but opposite slope, inflecting repeatedly at m and M, with a period

$$P=\frac{2(M-m)}{d} \tag{2}$$

and a mean value

$$\mu=\frac{M+m}{2} \tag{3}$$

Unless the double amplitude $2(M-m)$ is exactly divisible by d, the sequence of values generated by (1) will not repeat in a single period. However, since in Babylonian and Greek astronomy m, M, and d are always chosen as terminating sexagesimal numbers, there will always be a 'whole number period' Π comprising a whole number Z of real periods P such that $z_{\Pi+i}=z_i$.

The fragment of a lunar template in *P.S.I.* 1493 preserves a full column of 25 consecutive values of the progress in longitude, and two columns of the argument of latitude. However, all values in this copy are given only to one fractional sexagesimal place, so that it was not possible to get exact values for the differences and hence also for the parameters of the zigzag functions. In order to reconstruct the longitude function in my former paper I assumed that its whole number period was 248 days and that the mean value was the one implied by the epoch increment associated with 3031-day intervals in *P. Ryl.* 27; I then varied the increment/decrement of the zigzag function in search of a value that would reproduce the attested values in the papyrus. In this way I arrived at the following parameters:

$$\begin{aligned} d &= 0;12,50^\circ \\ M &= 14;38,59,18,37^\circ \\ m &= 11;42,10,25,17^\circ \\ \mu &= 13;10,34,51,57^\circ \\ P &= 27;33,20 \text{ days} \\ \Pi &= 248 \text{ days} \\ Z &= 9 \end{aligned} \qquad (4)$$

The initial value, giving the progress between days 0 and 1 of the template, was the minimum, m. From these parameters I derived a second zigzag function for daily motion in argument of latitude in steps by subtracting the nodal daily motion of $-0;3,10,49,15^\circ$ from the mean, minimum, and maximum and dividing by 15. The accumulated sums of both functions, truncated to minutes, were in agreement with the values in *P.S.I.* 1493 except for some discrepancies of 1 minute. Some small tolerances in the parameters were admissible without appreciable effect in the minutes place; in particular m and M of the longitudinal function could be raised by $0;0,0,0,3^\circ$ in order to make the mean exactly $13;10,34,52^\circ$ per day. However, I was unable to find satisfactory parameters that would keep the longitudinal function to just three fractional places and the latitudinal function to just four, to accord with the numbers of places used in the epoch positions.

Now among the new papyri from Oxyrhynchus is a fragment with tables on both sides (hence probably coming from a codex), the 'front' (*P. Oxy.* LXI.4151) being part of a Standard Scheme epoch table covering dates between 210 and 252, so that the papyrus must date from the early second half of the third century. On the 'back', *P. Oxy.* LXI.4164, are parts of two sets of columns of a template, with values given to three fractional places. The following numerals can be read or immediately restored in the second set of columns (the line numbers are those of the text edition):

	[13]	26	[	
	[1]3	13	[	
(10)	13	0	3[x	
	12	47	4[x	
	12	34	54	[
	12	22	4	[
	12	9	14	[
(15)	[11]	56	24	[
(16)	[11]	43	34	37
(18)	[1]1	53	36	37
	[1]2	6	26	37
(20)	12	19	16	37
	12	32	6	37
	12	44	56	37

This is obviously a stretch of the zigzag function for daily progress in longitude. The line-to-line difference, subtractive in lines 8–16, additive in 18–22, is 0;12,50,0°, i.e. just the value of d of my reconstructed function. But alas! this proves to be the only parameter that I got exactly right. Where the function passes a minimum, we can recover the exact value of m by adding the adjacent tabulated values, subtracting 0;12,50°, and dividing by 2. We thus find that m = 11;42,10,37°. There is not a well-preserved passage of a maximum in the papyrus from which we can extract M so directly. But if we assume that the mean value is 13;10,34,52°, then the rest of the parameters of the function follow

$$
\begin{aligned}
d &= 0;12,50^\circ \\
M &= 14;38,59,7^\circ \\
m &= 11;42,10,37^\circ \\
\mu &= 13;10,34,52^\circ \\
P &= \tfrac{3031}{110} \text{ days} = 27;33,16,21,49,\ldots \text{ days} \\
\Pi &= 3031 \text{ days} \\
Z &= 110
\end{aligned}
$$

Again we take m as the initial value of the function. If we now generate the first nineteen values of the zigzag function and its summation, we obtain values for the accumulated progress in longitude since epoch that agree perfectly in every place to the legible numerals from the first set of columns in *P. Oxy.* 4164:

	[11	42	10	37		11	42]	10	37
(5)	[11	55	0	37		23	37]	11	14
	[12	7	50	37		35	45]	1	51
	[12	20	40	37		48	5	42]	28
	[12	33	30	37	5	60	39	13]	5
	[12	46	20	37		73	25	33]	42
(10)	[12	59	10	37		86	24	4]4	19
	[13	12	0	37		99	36]	44	56
	[13	24	50	37		113]	1	35	33
	[13	37	40	37	10	126]	39	16	10
	[13	50	30	37		140]	29	46	47
(15)	[14	3	20	37		154	3]3	7	24
	[14	16	10	37		168	4]9	18	1
(17)	[14	29	0	37		183	1]8	18	38
(19)	[14	36	7	37	15	1]97	54	26	1[5]
(20)	[14	23	17	37	]	212	17	43	5[2]
	[14	10	27	37		2]26	28	11	[29]
	[13	57	37	37	]	240	25	4[9	6]

This confirms that the parameters given above are exact. Although this version of the template does not incorporate columns for the argument of latitude, we can use the constant nodal motion from the epochs to derive the parameters of a zigzag function:

$$\begin{aligned} d &= 0;0,51,20 \text{ steps} \\ M &= 0;58,48,39,45 \text{ steps} \\ m &= 0;47,1,25,45 \text{ steps} \\ \mu &= 0;52,55,2,45 \text{ steps} \\ P &= \tfrac{3031}{110} \text{ days} = 27;33,16,21,49,\ldots \text{ days} \\ \Pi &= 3031 \text{ days} \end{aligned} \qquad (6)$$

If we now return with these functions to *P.S.I.* 1493, we find that the values they generate, truncated to minutes, exactly reproduce all legible values in that copy of the template except for the few scribal errors.

A zigzag function is completely determined by an initial value and three independent parameters, for example the mean value, the increment/decrement, and the period. The difficulty I had in finding zigzag functions fitting the truncated values in *P.S.I.* 1493 and yet not requiring more than three and four fractional places arose merely from the assumption that a template meant to bridge intervals between epochs of 248 days should be based on zigzag functions with 248 as their whole number periods. Moreover the obvious historical precedent for using a zigzag function to represent lunar daily motion was the Babylonian function called F* by Neugebauer, which has a period of 248 days. Instead, the period of the Standard Scheme template has turned out to be 3031 days, that is, not the short interval between successive epochs but the interval between epoch *cycles*, which implies a much more accurate estimate of the anomalistic month. Did the complete template give all 3031 values of the zigzag functions? On the face of it this appears unlikely, firstly because the resulting table would have been of cumbersome length, and secondly because the spacing of the epochs is such that one would never need to use the template beyond the 248th line, or the 303rd line for the longer gaps between epoch cycles. We may hypothesize, therefore, that the template ended at the 303rd day, and that the inventors of the scheme were not bothered by the small discontinuities implicit in repeating the sequence of values of the functions prematurely, before they 'closed'.

Although the idea of using a zigzag function to represent variable lunar daily motion clearly harks back to the Babylonian F* function (the parameters of which were known to Geminus), none of the specific parameters of the Standard Scheme template have turned up so far in pre-Roman sources, whether Babylonian or Greek. The 3031-day anomalistic period is a good approximation, and can be derived by continued fractions from two Babylonian System B parameters known to Greek astronomers from Hipparchus's time:

$$1 \text{ anom. mo.} = \frac{251}{269} \text{ syn. mo.} \times 29;31,50,8,20 \text{ days/syn. mo.} \qquad (7)$$
$$= 27;33,16,26,57,28,\ldots \text{ days}$$
$$= 27 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{12 + \cfrac{1}{\ddots}}}}}$$
$$\approx \frac{3031}{110} \text{ days}$$

The mean daily motion in sidereal longitude is in excellent agreement with the value implied by the Greek 54-year eclipse period relation called the *exeligmos*:

$$19756 \text{ days} = 723 \times 360° + 32° \qquad (8)$$
$$\mu = 13;10,34,51,55,..° \text{ per day} \qquad (9)$$

(whereas the more accurate parameters of the System B lunar theory lead to a value closer to 13;10,34,51° per day). I have not succeeded in deriving the daily motion of the node, −0;3,10,49,15°, from attested parameters; from the System B parameters one finds:

$$1 \text{ drac. mo.} = \frac{5458}{5923} \text{ syn. mo.} \times 29;31,50,8,20 \text{ days/syn. mo.} \qquad (10)$$

which, combined with a mean motion in longitude of 13;10,34,52° per day, leads to a nodal motion of −0;3,10,47,40,...° per day.

The amplitude of the Standard Scheme longitudinal function, 2;56,48,30°, is only about $^5/_7$ of its counterparts in the Babylonian lunar schemes, and this surely reflects a new estimate of the size of the lunar anomaly obtained by analysis of eclipse observations.[9] We can recover only a rough approximation of the anomaly assumed by the inventors of the Standard Scheme, because the choice of amplitude of a zigzag function is subject to arithmetical constraints that limit usable numbers to a few fairly widely spaced values, none of which might have been very close to the theoretically preferable amplitude. Since in one whole number period the zigzag function must traverse the amplitude $2Z$ times by steps of d, we have

$$2Z\times(M-m) = d\times\Pi \tag{11}$$

with the consequence that $2(M-m)$ times a small power of 60 must be exactly divisible by Π, and d times the same small power of 60 must be divisible by Z. The choice of d is also influenced by the fact that the more sexagesimal places it has, the more laborious the computation of the table becomes; hence in Babylonian astronomy d usually has only one place. If, in constructing a zigzag function for the Standard Scheme, one had wanted to keep d to just *two* places and all other parameters to just four, then d would have had to be selected from multiples of 0;1,50°, including consecutively 0;11,0°, 0;12,50°, and 0;14,40°. In other words, one would have ended up with exactly the same function whether one started with the assumption that the maximum equation of the moon was 5°, or the assumption that the moon's eccentricity was 1/12 the size of its orbit, or the assumption that the variation in the moon's daily motion in longitude is 3°.

2. *The procedure text P. Oxy. LXI.4136.* That the astronomers who invented the Babylonian planetary and lunar schemes possessed mastery in handling periodic arithmetical functions is not in dispute. As yet we have less detailed evidence for Greek adaptations of the Babylonian schemes, and it might reasonably have been expected that such changes as were made were superficial; for the basic computational methods would surely have been much more susceptible of transmission than the mathematical methodology by which the schemes were originally crafted. We have seen now that the inventors of the Standard Scheme were able to construct a new zigzag function with a desired periodicity and with an economy of sexagesimal arithmetic that would not look out of place in a cuneiform tablet. How much deeper did their understanding of the mathematical properties of their creation go?

A new fragment of a procedure text, *P. Oxy.* LXI.4136, casts unexpected light on this question. The front of this papyrus substantially preserves the top 22 lines of a column of text with traces of the 23rd line and of the left edge of a subsequent column. On the back are horoscopes (published as *P. Oxy.* LXI.4241) cast for dates in 162, 191, and 201, so that the date of writing of the front is probably the late second or early third century. Several scribal errors prove that this is a copy.

The text bears no title, and nowhere mentions the moon. Only the numerical details of the argument reveal to us that it is concerned with a Standard Scheme template giving the progress in longitude since epoch (i.e. the running totals of the longitudinal zigzag function) and perhaps also the zigzag function for daily motion itself. (There is no allusion in the preserved lines to arguments of latitude.) The template is supposed to be so laid out that a new column is begun after each passage of a minimum in the zigzag function; hence each column will have had either 28 or 27 lines. A complete period of 3031 days, supposing one had the patience to compute it, would require 110 columns, and the 248th and 303rd days would have ended the ninth and eleventh columns respectively.

The problem that the text addresses is strictly mathematical: how to find the total progress in longitude over each column of the template without having to add each value separately. This is a special case of the general problem, to find the sum of *n* successive values of a zigzag function, which has been treated in modern times by Neugebauer because of its pertinence to the editor's task of establishing continuity between Babylonian astronomical tablets.[10] For the ancient computer of the tables it might have been useful to have an independent check of the totals at the end of each column. Let us proceed through the instructions of the text step by step, while attempting to explain what the instructions mean.

> (lines 1–7) How one should determine the total for 27-line and 28-line columns. Multiply the mean [course(?)] by the 28 lines; there results 368;56,16 [16 degrees(?)]. And multiply the amount added for 28-line columns, 0;5,4[3, by] 7, which is $^1/_4$ of the 28 lines; there results 0;40,1. (Subtracting) this from the foregoing, there remains 368;16,15,16, [which] is the total for all 28-line columns.

We may begin by justifying the last point made, that one total applies to all 28-line columns of the template. In any zigzag function, if an even number of values happen to fall between two consecutive minima, an equal number of them must be on the increasing and on the decreasing branch. Consider any pair of such 'even-numbered' intervals between minima. If the first value in the increasing branch of one of these intervals is greater than the first value in the increasing branch of the other interval by some amount δ, then obviously all the remaining values in the increasing branch of the first interval must be greater than their counterparts in the other interval by δ. But it follows that the first value in the *decreasing* branch of the first interval must be *less* than its counterpart in the other interval by δ, since each is derived from the last value in the increasing branch by the formula

$$z_{i+1} = (2M-d)-z_i; \qquad (12)$$

and so the same difference will subsist between all corresponding values in the descending branches. Thus the differences all cancel out, and the sums of values of the zigzag function over the two even-numbered intervals are equal.

In determining what this constant sum is, the text uses a quantity 0;5,43, which is called 'the amount added for 28-line columns'. This is in fact the constant difference between any pair of values of the zigzag function separated by 28 lines, so long as both are on the same kind of branch, either increasing or decreasing. In general, we can determine the difference between two values on increasing branches separated by n lines and by k complete branches as:

$$\varepsilon_n = n\times d-k\times(M-m) \qquad (13)$$

so that in the present instance

$$\varepsilon_{28} = 28\times 0;12,50^\circ-2\times 2;56,48,30^\circ = 0;5,43^\circ \qquad (14)$$

If z_i lies on an increasing branch, then $z_{i+28}=z_i+\varepsilon_{28}$ unless this exceeds M; whereas if z_i lies on a decreasing branch, $z_{i+28}=z_i-\varepsilon_{28}$ unless this is less than m.

Consider now a sequence of 28 values generated by the zigzag function's rules in such a way that the middle pair, numbers 14 and 15, are on an increasing branch and exactly centred on the mean value, i.e. $z_{14}=\mu-{}^{d}/_{2}$, and $z_{15}=\mu+{}^{d}/_{2}$. From symmetry it is obvious that the 28 values must be distributed with the first seven in a decreasing branch, the middle fourteen in an increasing branch, and the last seven in a decreasing branch; and their total will be exactly $28\times\mu$. (This symmetrical arrangement is possible only because 28 is divisible by 4.) Now if we subtract ε_{28} from each of the first seven values z_1 through z_7 and move them from the beginning to the end of the sequence, we will obtain a sequence of 28 numbers that either forms a complete 28-line pair of branches of the zigzag function or, if the new last number turns out to be less than m, at least has the same total as a 28-line cycle. Thus the text finds for 28-line cycles:

$$s = 28\times 13;10,34,52° - 7\times 0;5,43° = 368;16,15,16° \qquad (15)$$

(lines 8–14) For the 27-line columns, subtract from the [368];16,15,16 the minimum course, 11;42,10,37 degrees, [and there results] 356;34,4,39 degrees. And add to these the excess of the amount subtracted for [27-line columns] over the amount added for 28-line columns, 0;1,24; there results 35[6;35,28],39. This is the size of the total for the third column. And since this falls short of 360 degrees by 3;24,31,21,...

The rule that any interval of a zigzag function between two consecutive minima containing an even number of values has a constant total is a special case of a general theorem that the sum of any even number of values centred on a minimum or maximum is constant. It follows that the total of values in an interval between minima containing an *odd* number of values is not constant; for it will be composed of an even number of values centred on the maximum plus one extra value at the beginning or end, and a particular number can appear at most twice in the whole number period of the zigzag function. Our text therefore first deduces the total for the first 27-line column as a correction to the total for 28-line columns, and then goes on to find the total for each subsequent 27-line column as a correction to its predecessor.

The number of values in the k'th branch of a zigzag function is

$$l_k = [k\times{}^{P}/_{2}]-[(k-1)\times{}^{P}/_{2}]$$

where $[x]$ denotes the integer part of x. With $P=\frac{3031}{110}$ we obtain the following pattern of lengths of branches and columns:

column	*increasing branch*	*decreasing branch*	*total*
1	13	14	27
2	14	14	28
3	13	14	27
4	14	14	28
5	13	14	27
6	14	14	28
7	14	13	27
8	14	14	28
9	14	13	27
10	14	14	28
11	14	14	28
12	13	14	27
(etc.)			

The status of the first column of the function is actually ambiguous, because the first tabulated value, for the daily motion between days 0 and 1, is right on the minimum. Counting this value as part of the first column, we obtain a 28-line column, the total of which is already known to be 368;16,15,16°. But if we start with the first value that is *within* the increasing branch, we have a 27-line column, and its total is obviously

$$s_1 = 368;16,15,16° - m = 356;34,4,39° \qquad (16)$$

Now the 'odd' value in the first column is the last one, which is on the decreasing branch; and the same is true of the 'odd' value in the next 27-line column, the third. Since the total of the other 26 values in both columns must be the same, we need only find the difference between these odd values, which are separated by 28 days plus 27 days. We already have found that

$$\varepsilon_{28} = 28 \times 0;12,50° - 2 \times 2;56,48,30° = 0;5,43° \qquad (17)$$

By the same reasoning,

$$\varepsilon_{27} = 27 \times 0;12,50° - 2 \times 2;56,48,30° = -0;7,7° \qquad (18)$$

The text calls this quantity 'the amount subtracted for 27-line columns'. So the difference between values on the same kind of branch separated by 55 days is

$$\varepsilon_{55} = 0;5,43° - 0;7,7° = -0;1,24° \qquad (19)$$

Since the 'odd' values in the first and third columns are on the decreasing branch, we must *add* 0;1,24° to the total for column 1 to obtain the total for column 3:

$$s_3 = s_1 + 0;1,24° = 356;35,28,39° = 360 - 3;24,31,21° \qquad (20)$$

(lines 15–23) ... subtract from (3;24,31,21°) the 1,24,0; there remains for the fifth column 3;23,7,21. Again, from the 0;1,24 subtract 1/110 of the increment, namely 0;0,7; there remains 0;1,17. Add this to the amount by which the fifth column falls short of a circle, namely 3;23,7,21, and there will be for the seventh column 3;24,24,21. Add to this the entire excess, 0;1,24; there results for the ninth column 3;25,48,21. In the same way let there be added [the 0;1,24; there results for the eleventh column 3;27,12,21.]

The relationship between the fifth and third columns is the same as that between the third and first; hence

$$s_5 = s_3 + 0;1,24° = 360 - 3;23,7,21° \qquad (21)$$

With the seventh column, however, the 'odd' value is now the first one of the increasing branch instead of the last of the decreasing branch, so that the constant difference does not apply between these columns. How does one find the difference in this situation? Neugebauer has shown that the 'odd' values of all 'odd number' intervals in a zigzag function themselves form a derivative zigzag function, which in the present instance has these parameters:[11]

$$\begin{aligned} d &= 0;1,24° \\ M &= 11;52,9,7° \\ m &= 11;49,17,37° \\ P &= \tfrac{49}{12} = 4;5 \\ \Pi &= 49 \end{aligned} \qquad (22)$$

Since the total for each 27-line column is simply the constant sum of 26 values centred on the maximum plus the 'odd' value, it is generated by a synchronous zigzag function with the same amplitude and d, but these bounds:

$$\begin{aligned} M &= 356;36,56,9^\circ = 360^\circ - 3;23,3,51^\circ \\ m &= 356;34,4,39^\circ = 360^\circ - 3;25,55,21^\circ \end{aligned} \qquad (23)$$

If we consider only those stages of a zigzag function where either a maximum or a minimum is passed, so that the difference between consecutive values is not $\pm d$, and reverse the signs of the differences at the minima, we will obtain a 'saw function', i.e. a sequence of values increasing (or decreasing) cyclically by a constant increment d between two limiting values M and m. For the zigzag functions defined in (22) and (23), the parameters of the saw function representing differences at maxima and minima are:

$$\begin{aligned} d &= -0;0,7^\circ \\ M &= +0;1,24^\circ \\ m &= -0;1,24^\circ \end{aligned} \qquad (24)$$

and hence the actual differences form the sequence +0;1,24°, −0;1,17°, +0;1,10°,... −0;1,10°, +0;1,17°. These will be the differences between consecutive 27-line columns of the Standard Scheme template whenever the 'odd' value changes from being on the increasing to being on the decreasing branch and *vice versa*.

Our text only gets as far as the deduction of the second difference in the sequence, which is the difference between the totals for the fifth and seventh columns of the template. It is clear, however, that the author knows the pattern of successive subtractions of 0;0,7°. He is also aware that 0;0,7° here is, in terms of the parameters of the longitudinal zigzag function (5), $^d/_Z$, a quantity that we have seen must have been a determining factor in choosing the amplitude and increment/decrement of the zigzag function (cf. (11) above).

Thus we have

$$s_7 = s_5 - 0;1,17^\circ = 360 - 3;24,24,21^\circ \qquad (25)$$

and, by the same means as before,

$$s_9 = s_7 - 0;1,24° = 360 - 3;25,48,21° \qquad (26)$$

In the complete 3031-line period of the longitudinal function, the next 27-line column is the twelfth, in which the 'odd' value has moved back into the increasing branch. To find the total for this branch, therefore, the author would have subtracted 0;0,7° from the difference 0;1,17 found between the totals for columns 5 and 7, and *subtracted* this from the shortfall 3;25,48,21° associated with column 9. Only slight traces of line 23 of the papyrus survive, but from these and the legible end of line 22 it is clear that, on the contrary, the text prescribed simply *adding* some amount to 3;25,48,21° to get the total for column y, such that $y>10$.

This proves that the template *as actually written out* ended with the 303rd day. Thus curtailed, column 11, which should properly have 28 lines, becomes artificially a 27-line column, obviously still with its 'odd' value in the increasing branch, and so

$$s_{11} = s_9 - 0;1,24° = 360 - 3;27,12,21° \qquad (27)$$

At this point the text has accomplished what it set out to do. What was contained in subsequent lines, and in the next column of text vestigially preserved along the right edge of the papyrus fragment, can only be guessed.

The manner of exposition of *P. Oxy.* 4136 is curious. The author is not content simply to give a list of totals for each column of the template, which would have sufficed for checking the sums if one wanted to recompute the template from its parameters. On the other hand, while he gives arithmetical derivations for each total that have a manifest origin in a mathematical analysis of the longitudinal zigzag function, he does not discuss this analysis or attempt to explain why the derivations work. He obviously possessed a level of theoretical understanding that he did not see fit to impart to the user of the text.

3. *The epoch increments.* The mean μ of the Standard Scheme's longitudinal zigzag function (5) is exactly 13;10,34,52°, and the latitudinal function (6) is presumed to have had $\mu = 0;52,55,2,45$ steps. Yet if we divide the increments in longitude and argument

of latitude for the epoch dates prescribed by *P. Ryl.* 27 (and confirmed in *P. Lund* inv. 35a) by the corresponding numbers of days, we find:[12]

days	*total longitude*	*daily*	*total arg. of latitude in steps*	*daily*
248	9×360°+ 27;43,24,56°	13;10,34,41,...°	9×24+ 2;43,28,34,0	0;52,55,2,4,...
3031	110×360°+ 337;31,19,7°	13;10,34,51,57,...°	111×24+ 9;12,43,48,15	0;52,55,2,44,...

Writing before the existence of the Standard Scheme template was known, van der Waerden hypothesized that the increments were supposed to represent the true progress of the moon over intervals of whole numbers of days that only approximately equalled a whole number of anomalistic months.[13] Assuming a reasonably accurate value for the length of the anomalistic month, one would find that 248 days is approximately 0;0,33 days longer than 9 anomalistic months, so that in the 248 days between epoch dates there would be a brief interval of motion at close to minimum speed in addition to the whole anomalistic months, during which true progress should equal mean progress. Analogously one might explain the smaller discrepancy in the 3031-day increments as due to the assumption of an anomalistic month very slightly shorter than $\frac{3031}{110}$ days.

Now that we know that the template was based on zigzag functions with the longer period of 3031 days, we can ask what accumulated progress in longitude and argument of latitude it assigned to day 248. To obtain the total progress in longitude we need merely add together the totals for the first nine 'columns' of the template found in *P. Oxy.* 4136:

$$\Sigma_{248} = 5\times 8;16,15,16° + s_3 + s_5 + s_7 + s_9 = 27;43,24,56° \qquad (28)$$

and this is just the prescribed increment in longitude. Adding the uniform nodal motion for 248 days and converting to 'steps', we also obtain exactly the increment in argument of latitude. This confirms, in more concrete terms, van der Waerden's explanation of the 248-day increments: they are a consequence of the fact that the anomalistic month *implied by the 3031-day cycle* falls slightly short of $\frac{248}{9}$ days.

One might expect that the overall sum of Π consecutive values of a zigzag function must equal $\mu\Pi$; but this is only true on one of two conditions: either Π must be an even number, or, if Π is odd, one of the values generated by the function must be μ.[14] In the case of the Standard Scheme longitudinal function, the period, 3031 days, is odd; and we know, because the initial value chosen is m, that all values generated by the function can be expressed as

$$z = m + 0;0,7^\circ \times j \tag{29}$$

for some whole number j. Thus the closest value to μ generated by the function is 13;10,36,37°, which is 0;0,1,45° too large; and like every other generated value other than m itself, 13;10,36,37° appears twice in each whole number period: once in an increasing and once in a decreasing branch. If we were to locate this number where it occurs in an increasing branch and reduce its value by 0;0,1,45°, adjusting the other 3030 values accordingly, we would of course obtain a sequence totalling $\mu\Pi$, i.e. 110×360°+ 337;31,20,52°. In so doing, we would have raised each of the 1515 values on increasing branches of the function by 0;0,1,45°, and lowered each of the 1515 values on decreasing branches by the same amount, with no overall effect on the total. But the unique occurrence of m in the sequence, which is on neither an increasing nor a decreasing branch, would also be raised by 0;0,1,45° with no other value to compensate. And so the total over an entire 3031-day period of the template function turns out to be

$$\Sigma_{3031} = 337;31,20,52^\circ - 0;0,1,45^\circ = 337;31,19,7^\circ, \tag{30}$$

and this is precisely what is prescribed for the epoch advance in longitude in 3031 days; again the advance in argument of latitude is derived from this value and the constant nodal motion.

Hence the discrepancy between the mean value of the zigzag function and the average over a 3031-day interval in the epoch table is not on account of an assumed anomalistic month different from $\frac{3031}{110}$ days, but rather a consequence of choosing m as the first value of the zigzag functions. The inventors of the Standard Scheme obviously knew that the initial value influenced the periodic total, because the correct total was actually used as the increment between epochs. If they did not take care to use an initial value that would bring μ into the sequence, it may have been because they believed μ to be only a rounding to three fractional places of a slightly smaller mean motion in longitude, for example the 13;10,34,51,55,...° per day implicit in the 'exeligmos' period relation (9). Incidentally, I think it is unlikely that the total was found by actually computing and adding all 3031 values of the zigzag function; the kind of analysis that underlay *P. Oxy.* 4136 was probably used here too.

To summarize: the parameters of the longitudinal zigzag function and the constant nodal motion are the foundations upon which the entire Standard Scheme was constructed. The only further thing needful was an empirical determination of a single epoch date and corresponding lunar position in longitude and argument of latitude.

4. The alignment of the scheme's epochs. In addition to the two sets of rules for finding epoch dates and positions in *P. Ryl.* 27, which we may call Rule A (lines 2–31 of the papyrus) and Rule B (lines 32–50), we now have four fragments of Standard Scheme epoch tables on papyrus:

	Range of years	*Calendar*
P. Lund inv. 35a	59–108	Egyptian
P. Oxy. LXI.4149	96–166	Alexandrian
P. Oxy. LXI.4150	187–198	Egyptian
P. Oxy. LXI.4151	210–252	Egyptian

Our first concern in investigating the alignment of the dates and corresponding positions of the scheme's epochs must be to check how closely these sources agree in the dates and positions.

Neugebauer found two discrepancies between the series of epochs generated by the Rule A and Rule B.[15] The first is in the alignment of the 3031-day epoch cycles: apparently the first eleven epochs in each cycle according to Rule A have exactly the same assigned positions in longitude and argument of latitude as the last eleven epochs in the corresponding cycle according to Rule B. But also Neugebauer found a shift of 1 day in the epoch dates generated by the two rules, so that for any an epoch with the same assigned positions generated by both rules, the date according to

Rule A is one day later than the date according to Rule B. This alleged second discrepancy, which has been repeated in all subsequent discussions of the Standard Scheme papyri, is entirely illusory; and it is also probable, as we shall see, that the rules, correctly understood, generated the very same series of epochs without exception.

In either rule, we begin by choosing an Egyptian calendar year within which we want to find an epoch. Rule B assumes that we know the year as counted from the first regnal year of Commodus, which is equivalent to Augustus 190 or A.D. 160/161. If we call this year number x, then we take $y = x + 92$ – which is (though the text does not say so) the year reckoned from the first regnal year of Vespasian – and then find a and b such that

$$y = 25a + b \qquad (31)$$

We then calculate an interval of days

$$c = 32a + 365b, \qquad (32)$$

which is further broken down into intervals of 3031 and 248 days:

$$3031e + 248f + g = c \qquad (33)$$

Lastly, taking

$$d = 293 - g, \qquad (34)$$

we 'count off' (ἀπόλυε) g days from the first day of the year, Thoth 1, to obtain the epoch date. In analysing this procedure Neugebauer tacitly assumes that the 'counting off' is inclusive, i.e. that $d = 1$ signifies Thoth 1 and so on; and this is certainly correct. For example, if we apply Rule B to the year Nero 14, which is effectively Vespasian 0 or Commodus −92, we obtain $y = 0$, and hence $c = 0$ and $d = 293$, which signifies the 23rd day of the tenth month, i.e. Payni 23. This very date is given as the first of an epoch cycle in *P. Lund* inv. 35a. In fact except when two epoch dates fall within a single Egyptian year (in which case Rule B only generates a single date), absolutely every epoch date in each of the five known Standard Scheme epoch tables is generated by Rule B.[16]

The meaning of Rule B is this: firstly, to break down the interval from a base epoch on Nero 14, Payni 23 to Payni 23 of the given year x (which we can call the 'anniversary' of the base epoch) into $(3a+e)$ complete cycles of 3031 days, f 248-day intervals, and g excess days; secondly, to count back g days from Payni 23 (the 293rd day of the year) to obtain an epoch date. The computation is abbreviated in steps (31) and (32) by taking account of the fact that 25 Egyptian years equal three epoch cycles plus 32 days. In step (33) the text fails to warn that f should not be allowed to exceed 11, that is, that the epoch cycle comprises eleven 248-day intervals plus one 303-day interval, not twelve 248-day intervals plus one 55-day interval. Because of this limitation, g can in some instances be greater than 293, in which case the epoch date is pushed back into the end of the preceding year.

The interpretation of Rule A involves complications not present in Rule B. Here the year is to be specified by two more than the number of *completed* years (τὰ πλήρη ἔτη) counting from the first regnal year of Augustus (−29/28); i.e. for year x of Augustus we take $y = x+1$. Again we find a and b to satisfy (31), and then calculate

$$c = 32a+365b+61 \qquad (35)$$

We determine g from c according to (33), again with the unstated proviso that f should not exceed 11. Then we are given two alternative formulas for d:

$$d = \begin{cases} 303-g \text{ in the case of `nodes'} \\ 248-g \text{ in the case of no `nodes'} \end{cases} \qquad (36)$$

We are told to 'count off' (διέκβαλε) d from the first day (νεομηνία) of Thoth.[17] In this instance Neugebauer understood the counting of days as exclusive, i.e. $d = 1$ would signify Thoth 2, but there is really no reason to interpret the operation differently here from in Rule B. The alleged one-day discrepancy between equivalent epoch dates found by the two rules is merely the consequence of Neugebauer's inconsistency in 'counting off'.

'Nodes' (σύνδεσμοι) in this context certainly refers, as Neugebauer recognized, to the 303-day gaps between epoch cycles. There is some uncertainty about how the user of Rule A was supposed to recognize when a 'node' occurs, that is, whether to use the first or second alternative in (36) to determine the epoch date. In lines 13–14 of *P. Ryl.* 27 is a statement beginning, 'nodes occur...'; but the remainder of the sentence is either peculiarly or sloppily written in the papyrus, and neither the original editors nor Neugebauer were able to interpret it.[18] My reading of the line is essentially the same as Neugebauer's, and can be translated: 'nodes occur at 6 and 14 (and) 23.' What this means becomes clearer if we compare the computation of epoch dates according to Rules A and B.

Let us take the epoch dates computed according to Rule B for a sequence of years close to the time when *P. Ryl.* 27 was written, and set beside them the dates generated by Rule A, assuming either 'nodes' or 'no nodes', and the values of *b* and *f* obtained in carrying out Rule A:

Date (Rule B) Era Augustus	*Rule A, 'nodes'*	*Rule A, 'no nodes'*	*b (Rule A)*	*f (Rule A)*
274 III 23	**II 123**	V 18	0	1
275 VIII 4	**VIII 4**	IX 29	1	3
276 IV 7	**IV 7**	VI 2	2	4
277 VIII 18	**VIII 18**	X 13	3	6
278 IV 21	**IV 21**	VI 16	4	7
279 IX 2	**IX 2**	X 27	5	9
280 VI 30	V 5	**VI 30**	6	10
281 III 3	I 8	**III 3**	7	11
282 VII 14	**VII 14**	IX 9	8	1
283 III 17	**III 7**	V 12	9	2
284 VII 28	**VII 28**	IX 23	10	4

The epoch positions according to Rule A are obtained by adding appropriate increments to a base epoch on the date Cleopatra 20 (= Augustus − 2 = −32/31) Epeiph 4.[19] Neugebauer quite reasonably assumed that an epoch cycle was supposed to begin with this epoch, and consequently that 'nodes' corresponded to years for which $f = 11$ only.[20] Hence he concluded that Rule A generates the same epoch dates as Rule B only for the last eleven epochs of Rule A cycle, so that in Augustus 280 for example the two rules would yield epochs 55 days apart.

It is here, however, that the rubric that 'nodes occur at 6 and 14 (and) 23' comes into play. The number b found in the process of carrying out Rule A marks the place of the year in question in a cycle of 25 Egyptian years, which is only a little longer than three epoch cycles ($25 \times 365 = 9125$, whereas $3 \times 3031 = 9093$). Because of this, the three years in a 25-year period when the 'nodes' or ends of epoch cycles occur will usually be the same as in the preceding 25-year period, though occasionally they slip one year back in the cycle. Now supposing that Rule A was really supposed to generate epochs compatible with Rule B. Then for the long span of years Augustus 222–338 (A.D. 192/193–308/309) the first epoch dates following 'nodes' would always fall in years for which b is 6, 14, or 23.[21] The rule of thumb for finding the epoch date is that one must subtract g from 303 for epochs falling between a 'node' and the end of an epoch cycle, that is, in years such that b has reached or surpassed 6, 14, or 23, and f has not yet dropped back to 1.

Thus we have an interpretation of the obscure sentence in the papyrus that not only supplies the information needed by the user of Rule A, but makes Rule A's epochs consistent with those of Rule B.[22] This is what we might have expected in view of the unanimity of the extant epoch tables in the alignment of the cycles, and the fact that the text only distinguishes Rule B from Rule A as being 'more concise' (*συντομώτερον*) – as indeed it is.

However, I believe that Neugebauer was nevertheless fundamentally right about the *original* form of Rule A. It has escaped notice that if the base epoch on Epeiph 4 of Augustus −2, from which Rule A builds up its epoch positions, was the first of an epoch cycle as Neugebauer presumed, then the last epoch date of the preceding cycle fell precisely on the first day of the same year, Thoth 1. It was probably just this pleasing coincidence that led to the choice of Augustus −2 as the base year.

We have seen that Rule B was designed to find the epoch date preceding the anniversary of the base epoch in a given year. If an analogous rule existed using the base epoch of Epeiph 4, Augustus −2, it would have worked exactly like Rule B's steps (31) through (34) except that we would begin with $y = x + 2$, and in step (34) we would subtract g from 304 to obtain d. Here the fact that the preceding epoch cycle ended on the first day of the year becomes not only pretty but useful: it guarantees that in counting back *g* days from Epeiph 4 of the given year we will never have to cross over into the preceding year, as occasionally happens with Rule B. Rule A in *P. Ryl.* 27 has been modified in three ways from this simpler original form. First, in the form we have it the rule finds as *g* the number of days by which the *last day* of the year *before* the given year comes after an epoch; this is why we start with the completed years of Augustus instead of the regnal year, and why we add the 61 days that separate the base epoch from the end of the year. Secondly, instead of using *g* to count back to the preceding epoch, we count forward to the *next* epoch. It is this change that forces the introduction of cases of 'nodes' and 'no nodes', because the 303-day gap between epochs sometimes comes before the next epoch; but as a compensation we are assured of finding the first epoch date of every given year. Thirdly, the *b* values associated with 'nodes' in the papyrus were subsequently made to agree with the base epoch of Rule B instead of the original base epoch of Rule A.

Whether or not Rules A and B generate the same epoch dates for every given year, it is true that whenever they do so they also generate identical epoch positions in longitude and argument of latitude, except that Rule A's latitudinal epochs are exactly 0;2 steps smaller than Rule B's on account of a scribal or arithmetical error in the text of one of the rules.[23] The longitudes also exactly match those given (to the full three fractional places) in *P. Lund* inv. 35a for the years Nero 9 through Domitian 3 (A.D. 62–84). Similarly the longitudes in *P. Oxy.* 4149 for the years Nerva 1 through Trajan 6 (96–103) and Trajan 15 through 17 (111–114), which are preserved to the degree and in some instances to the minute, agree exactly so far as they go with the rules. In *P. Oxy.* 4150, where the longitudes (for 187–198) apparently have been truncated to minutes, they are less than the longitudes generated by the rules by a quantity that, if consistent, was between 0;3,27° and 0;3,37°. On the other hand, *P. Oxy.* 4151 (on the other side of the same fragment as *P. Oxy.* 4164) gives epoch longitudes, truncated to minutes, for the years Elegabalus 1 through Severus Alexander 8 (217–229) that systematically exceed those generated by the rules by between 0;1,35° and 0;1,37°.[24] In relation to the moon's swift daily progress these deviations are quite small. In Vettius Valens there are two lunar longitudes expressed in whole degrees, computed for dates in the years Hadrian 3 and 4 (118/119 and 119/120) by the Standard Scheme, and these confirm that his epoch longitudes were within half a degree of those generated by the rules.[25]

Aside from two dubious traces at the left edge of *P. Oxy.* 4150, the only preserved arguments of latitude in a lunar epoch table are those given, truncated to minutes, for seven of the epochs in *P. Oxy.* 4151. These are all less than the values generated by Rule A by an amount increasing progressively from 0;30 steps to 0;33 steps. (The cause of the increase in the deviation is that the computer of the table used a truncated value, 2;43 steps, for the increment over 248 days.) The discrepancy is so large, amounting to about half a day's motion of the moon in argument of latitude, that one suspects a systematic error on the part of the computer of this table. A single citation by Vettius Valens of the epoch position in argument of latitude for Hadrian 3, Mesore 30, gives 12;18 steps, whereas Rule A gives 12;19,57,43,15 steps, and Rule B 12;21,57,43,15 steps.[26] Until an epoch table comes to light with arguments of latitude exactly matching either Rule A or B, I see no way to decide which one has the correct base epoch value.

The epochs are always expressed as a calendar date without mention of hours, but must be meant to apply to a specific time of day serving as the beginning of the day's lunar motion. Since there is no direct textual evidence in the papyri indicating what time of day is intended, we have to determine which of the likely choices brings the epochs into best agreement with the moon's actual situation on those dates. The historically plausible conventions are to treat the day as beginning at noon, sunset (effectively 6 hours past noon), midnight, or sunrise (effectively 18 hours past noon).

Because of the accuracy of the underlying parameters (the 3031-day period, the mean motion of 13;10,34,52° per day in longitude, and the node's motion of −0;3,10,49,15° per day), any single epoch generated by the rules will suffice for investigating the epoch time. Thus we may choose the epoch Nero 14, Payni 23, for which we have:

Nero 14 Payni 23
epoch position in longitude 132;34,40,38° (37)
epoch position in arg. of latitude (Rule A) 23;36,37,45,45 steps
longitudinal progress between Payni 23 and 24 is minimum

For comparison we can use Ptolemy's lunar theory as expressed in the *Handy Tables*, with two provisos: first, that the moon's true longitude must be computed without taking its second anomaly into account, since the Standard Scheme assumes the existence of only a single anomaly of nearly the same magnitude as Ptolemy's first anomaly; and secondly, that Ptolemy's tropical longitudes (λ_T) must be converted to an appropriate sidereal frame of reference (λ_S). To accomplish this last, we will use the 'trepidation' formula prescribed by Theon of Alexandria, according to which for a year y counted from the first regnal year of Augustus,

$$\lambda_S = \lambda_T + 8° - \frac{y}{80} \qquad (38)$$

On these terms we compute the moon's sidereal longitude at 6 hour intervals for several days before and after Payni 23:

	0 hours noon	diff.	6 hours sunset	diff.	12 hours midnight	diff.	18 hours sunrise	diff.
X 20	93;14°	12;28°	96;22°	12;26°	99;29°	12;23°	102;36°	12;21°
X 21	105;42°	12;19°	108;48°	12;17°	111;52°	12;16°	114;57°	12;14°
X 22	118;1°	12;12°	121;5°	12;11°	124;8°	12;10°	127;11°	12;9°
X 23	130;13°	12;8°	133;16°	12;8°	136;18°	12;7°	139;20°	12;7°
X 24	142;21°	12;8°	145;24°	12;8°	148;25°	12;8°	151;27°	12;9°
X 25	154;29°	12;13°	157;32°	12;14°	160;33°	12;16°	163;36°	12;18°
X 26	178;51°		181;55°		184;59°		188;5°	

Assumption of evening epoch for the Standard Scheme clearly brings about the closest agreement in the longitudes. The daily progresses in longitude are a little more symmetrically disposed around the progress from Payni 23 to 24 for midnight or morning epoch, but the actual longitudes for these times are uncomfortably far from the 132;34,40,38° of the epoch table. I would agree with van der Waerden that a precise alignment in anomaly is less to be expected than accurate longitudes.[27]

If we use the *Handy Tables* to compute the moon's argument of latitude, we find:

	0 hours	6 hours	12 hours	18 hours
X 23	23;23,44 steps	23;35,56 steps	23;49,20 steps	0;3,20 steps

Here again evening epoch produces very close agreement.

We have also seen that Vettius Valens used a version of the Standard Scheme according to which the epochs in longitude and argument of latitude were within a small fraction of a day's motion of the values generated by the rules. But one of the passages involving computations of lunar longitude by the Standard Scheme expressly assumes that the scheme yields the moon's longitude at sunset, so that longitudes at other times must be found by interpolation.[28] The conclusion seems inescapable that the epochs of the Standard Scheme were always to be understood as pertaining to sunset, i.e. effectively 6 equinoctial hours after noon on the specified day.

5. *Dates of use of the Standard Scheme.* With the exception of the template *P.S.I.* 1493, all the known documents exemplifying the use of the Standard Scheme are in some sense dated. The four epoch tables listed at the beginning of the preceding section all use regnal years of emperors, from which it of course follows that each was written later than the last date covered. This is as we might expect, since the chief use of the scheme was probably to obtain lunar positions for *past* dates for astrological purposes. Another series of papyri in which we can sometimes find lunar longitudes computed by the Standard Scheme are the 'ephemerides', that is, tables organized according to calendar months with day-to-day calendrical equivalences, lunar longitudes, and often the positions of the other heavenly bodies and other data tabulated line by line. Ephemerides were in all probability compiled either immediately before or very soon after the dates that they cover.

The table below lists the pertinent documents known at present according to the earliest possible year of writing. As it shows, the Standard Scheme was already in existence by the beginning of the second century, and continued to be used as late as the middle of the fourth, almost two centuries later than the publication of the

Handy Tables. One might guess from the attested dates that the scheme was put together about the end of the first century; but it must be kept in mind that we still have comparatively few astronomical papyri from the first century and earlier.[29]

Year	*Text*	
108	*P. Lund* inv. 35a	Epochs, A.D. 59–108
111	*P. Oxy.* 4176	Ephemeris, A.D. 111
127	*P. Oxy.* 4177	Ephemeris, A.D. 121–127
166	*P. Oxy.* 4149	Epochs, A.D. 96–166
173	Vettius Valens	*Anthologiae*, completed after A.D. 173
198	*P. Oxy.* 4150	Epochs, A.D. 187–198
201	*P. Oxy* 4136/4241	Procedure text, with horoscopes A.D. 162, 191, 201
251	*P. Ryl.* 27	Procedure text, with regnal canon, A.D. 140–251
252	*P. Oxy.* 4151/4164	Epochs, A.D. 210–252, with template
261	*P. Oxy.* 4178	Ephemeris, A.D. 261
348	*P. Oxy.* 4179	Ephemeris, A.D. 348

Appendix 1: Summary of parameters of the Standard Scheme

Base epochs of epoch cycles according to Rule B.:

Augustus −2 Phaophi 26 (Egyptian) = −32 October 25, 6 P.M.

	longitude	*argument of latitude*
	42;18,51,14°	9;3,52,6,45 steps

Nero 14 (=Augustus 97) Payni 23 (Egyptian)
= Pachon 30 (Alexandrian) = 68 May 25, 6 P.M.

	132;34,40,38°	23;36,37,45,45 steps

Base epoch according to Rule A:
Augustus −2 Epeiph 4 = −32 June 30, 6 P.M.

	70;2,16,10°	11;47,20,40,45 steps

Epoch increments:

time	*longitude*	*argument of latitude*
248 days	27;43,24,56°	2;43,28,34,0 steps
3031 days	337;31,19,7°	9;12,43,48,15 steps

Zigzag functions for daily progress:

	longitude	*argument of latitude*
d	0;12,50°	0;0,51,20 steps
M	14;38,59,7°	0;58,48,39,45 steps
m	11;42,10,37°	0;47,1,25,45 steps
μ	13;10,34,52°	0;52,55,2,45 steps
P	$\frac{3031}{110}$ days	$\frac{3031}{110}$ days
Π	3031 days	3031 days
Z	110	110

Initial value (progress from day 0 to day 1) $= m$

Total progress on day 0: 0° and 0 steps.

Appendix 2: P. Ryl. 1.27

A. Corrections to the transcription of Hunt and Smyly (Hunt [1911] 48–56).

Corrections by Neugebauer [1949] are identified as (N), those by van der Waerden [1958] as (vdW). I have checked all readings against a colour photograph kindly put at my disposal by the John Rylands Library.

14 *ṭως̅ κ(αὶ) ι̅δ̅ κ̅γ̅* (N). The symbol transcribed as 'ς' (i.e. the numeral 6) looks rather as here printed, and resembles no letter elsewhere in the papyrus. Elsewhere the symbol for 6 is written like 'c' with an elongated top but no tail. Hunt took it for a cursive form of ξ.

18 *γ̅ λ̅η̅ ι̅α̅* (vdW).

28–29a λ̅β̅ [λ̅γ̅] | [μ̅]δ̣̅ ν̣̅[α̅], ἐ̣π̣[ὶ δὲ πλάτου]ϲ̣ γ̅ ι̅δ̅ [κ̅θ̅ λ̅δ̅] | [ι̅ϛ̣̅ ϲυνάξαϲ πάντα τὸν ἀριθμόν, ἆρον ἀπ'] (numerals N. vdW)
30 αὐτ(ῶν) μήκουϲ μ̅[θ̅] ν̣̅ζ̅ μ̣̅γ̣̅ ν̅ (N)
32 {κε̣} In the scribe's exemplar this was probably a marginal correction of the erroneous κ of line 34.
34 κ̅⟨ε̅⟩ (N)
45a Between lines 45 and 46 the scribe skipped over the following text:
λ̅γ̅ ν̅ζ̅ κ̅α̅, πλάτουϲ ἐπὶ γ̅ λ̅η̅ (N. vdW)
53 ι̅θ̅ ι̅ϛ̅ (not ι̅θ̅ κ(αὶ)).
55 ἀριθ[μὸν πλήρ]ωϲον
56–56a τ̣̅ϛ̅ [κ(αὶ) τοὺϲ] κύκλου[ϲ τοὺϲ ἀνὰ ι̅]η̣̅ ποίηϲο(ν) | [ἐπὶ ι̅θ̅ ...]
75 κ̣̅ (not ι̣̅[ϛ̅] (N)

B. Translation of the papyrus.
Restorations of missing text are enclosed in square brackets, explanatory glosses in parentheses. I have corrected scribal errors, indicating the affected words or numerals by italics. The translation follows the lines of the text. Lines 29a and 56a are restorations of lines lost at the bottom of the columns (one or two lines of the table in column iii may also have disappeared). Line 45a was accidentally skipped by the scribe.

Col. i Moon.
(Take) the completed years (of Augustus). Add 2. Divide by 25.
(Multiply) the remainder by 365, and the cycles
of 25 by 32. Then add 61.
After adding to make one number, divide
if you can by 3031, and (divide) the remainder
by 248, and subtract the remainder
in the case of nodes
from 303, in the case of no nodes
from 248. And count off the remainder
from the first day of Thoth
and the result is the day of the epoch
according to the Egyptian calendar. The nodes

are in the 6th and 14th and 23rd (year). The degree is found as follows. Multiply the cycles of 25 in the case of longitude by 292;33,57,21 degrees, in the case of latitude by 3;38,11, 24,45 'degrees' (i.e. steps). (Multiply) the (cycles) of 3031 in the case of longitude by 337;31,19, 7 degrees, in the case of latitude by 9;12,43, 48,15. And (multiply) the (cycles) of *248* in the case of longitude by 27;43,24,56 degrees, in the case of latitude by 2;43,28,34,0. And if you subtracted from 248, add on in the case of longitude another 27;43, 24,56 degrees, in the case of latitude 2;43,28,34,0. If (you subtracted) from 303, in the case of longitude (add on) 32;[33,] [4]4,5[1], in the case [of latitude] 3;14,[29,34,] [15. After adding to make one number, subtract from]
Col. ii them in longitude 4[9];57,43,50, in latitude 12;12,39,19,15. Then count off (the longitude) from Leo (0°).

Another shorter way, from the beginning.

(Take) all the years from Commodus (year 1). Add 92. Divide by *25*. Then (multiply) the remainder by *365*, and the cycles of 25 by 32. After adding to make one number, divide if you can by 3031, and the remainder by *248*, and in this way the remainder will be the number (of days) falling short of 298 (days). Count these (i.e. 298 minus the remainder) from Thoth 1. And there results the day of the epoch according to the Egyptian calendar. The degrees are obtained as follows. Multiply the cycles of *3031* in the case of longitude by 337; 31,19,7 degrees, in the case of latitude by 9;12,43, 48,15. (Multiply) the (cycles) of 25 in the case of longitude by 292; [33,57,21 degrees, in the case of latitude by 3;38,] 11,24,45. (Multiply) the (cycles) of 248 in the case of longitude by 27;43,24,56, in the case of latitude by 2;43, 28,34. *Then add on* in longitude 12;34, 40,38 degrees, in latitude subtract 0;21,22,14,15. Then count off from Leo (0°).

On the node.

On the node. Divide the completed years by

18. (Multiply) the remainder by 19;16, the Egyptian months by 1;35, the days by 0;3,10. After adding to make one number, take its complement into 360, [and] multiply [the] cycles [of] 18 [by 19. (Remainder of instructions lost.)]

Col. iii Solstices [and equinoxes that] Ptolemy observed. Year 46[3 from the] death of Alexander. Summer solstice Mesore 11/12, hour 7 of night. Add 92;30 days. *This is* the era (?) of the observations. Autumnal equinox Hathyr [9], approximately 1 hour after sunrise. Add 88;7,30 days. Winter solstice Mecheir 7, hour 4 of day. Add *90;7,*30 days. Spring equinox Pachon 7, approximately 1 hour after noon. Add 94;30 days. The year is Aelius Antoninus 3. Hence take the years from year 4 (of Antoninus) to the current year, and form 1/4 of this, and subtract from the total of days 0;0,12 for each year. And add the remainder to (the date of) each of the observations.

Remaining years of Aelius Antoninus: 20.

221	Commodus	32	Year 1 is (?)	190	294
246	Severus	25	Year 1	222	
250	Impious (i.e. Elegabalus)	4	Year 1	247	
263	Alexander	13	Year 1	251	
266	Maximinus	3	Year 1	264	
272	Gordianus	6	Year 1	267	
278	Philips	6	Year 1	273	
2[80]	Decius	2	Year 1	279	
[282]	Gallus	[	Year 1]	28[1]	

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NOTES

1. Jones (1983) 14-23.
2. Knudtzon & Neugebauer (1947); Neugebauer (1949); van der Waerden (1958).
3. When complete, *P. Lund* inv. 35a covered the years Nero 6 through Trajan 12. Traces of numerals in columns i and x are the remains of a column, found also in some of the new epoch tables from Oxyrhynchus, giving the deviation in days between equivalent Egyptian and Alexandrian calendar dates. Hence there is no evidence that the table extended back as far as Tiberius 21 (as suggested by Neugebauer (1975) 813).
4. On the supposed discrepancies between dates computed according to the two sets of rules, see section 4 below.
5. Partial transliteration and discussion in Neugebauer (1975) 822–823.
6. Jones (1983) 27–30.
7. To appear in Jones, *APO*, as *P. Oxy.* LXI.4133–4300.
8. Complete tables for the Standard Scheme, covering the first five centuries of the Roman period, will appear in Jones, *APO*, Appendix H.
9. Roughly speaking, the amplitudes of the Babylonian daily motion functions approximate the *maximum* variation in the moon's actual progress, whereas the Standard Scheme's amplitude approximates the variation predicted by a single-anomaly model deduced from observations at syzygies.
10. Neugebauer (1948).
11. Neugebauer (1948) 270–275.
12. Increments are also prescribed for the 303 days ending each 3031-day cycle, but these are simply the differences between eleven times the 248-day increments and the 3031-day increments, and thus of no independent significance.
13. Van der Waerden (1958) 182.
14. Neugebauer (1948) 268–269.

15. Neugebauer (1949) 17–18.
16. In *P. Oxy.* 4149 the dates have been converted to the Alexandrian calendar.
17. Neugebauer mistakes this for an allusion to a new moon on this date.
18. My former effort to explain this line (Jones (1983) 15) is incorrect.
19. Neugebauer has Epeiph 5 because of his different convention for counting the days off.
20. Neugebauer (1949) 7.
21. After the cycle beginning with Augustus 338 the first epochs recede from year 22 into the latter part of year 22, but this does not affect Rule A until Augustus 396 because it is designed to give only the *first* epoch date in years containing two epochs.
22. A minor difference is that in years with two epoch dates, Rule A will always give the first date, while Rule B sometimes gives the second.
23. Neugebauer (1949) 19
24. For the last five preserved epochs in this papyrus, it seems that the copyist meant to give the longitudes to at least one further fractional place; but the numbers are arithmetically inconsistent and must be garbled in some way.
25. Jones (1983) 28–29.
26. Jones (1983) 30. (The epoch position of 12;43 steps cited there is incorrect.)
27. Van der Waerden (1958) 187.
28. Jones (1983) 29.
29. There is only one known ephemeris (*P. Oxy.* 4175) containing legible lunar longitudes and antedating *P. Oxy.* 4176; the lunar longitudes – given to degrees only – in this fragment from an ephemeris covering parts of Augustus 6 and 7 (= −23) were not computed by the Standard Scheme.