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The Roofed Spherical Sundial and the Greek Geometry of Curves

Summary

Greco-Roman sundials existed in a great variety of forms, but in most of the common types the curves traced through the day by the Sun's projection at the various stages of the year were circles, straight lines, or conic sections, that is, the kinds of line most commonly investigated in Greek geometry. The variety known as roofed spherical sundials has day curves of a more complicated character; nevertheless, the mathematicians of the time could have investigated their properties by means of trigonometrical and projective resources attested in texts such as Ptolemy's *Almagest* and Pappus's *Collection*.

Keywords: Sundials; geometry; Cetius Faventinus; Vitruvius; Pappus of Alexandria.

Griechisch-römische Sonnenuhren existierten in großer Formenvielfalt, aber bei den gängigsten Typen sind die Kurven, denen die Sonnenprojektion über den Tag und in den verschiedenen Jahresabschnitten folgt, Kreise, gerade Linien oder Kegelschnitte – also die Art von Linien, die am häufigsten in der griechischen Geometrie untersucht wurden. Bei Sonnenuhren mit Lochgnomon und halbkugelförmiger Schattenfläche (*roofed spherical sundials*) treten jedoch kompliziertere Kurven als Tageslinien auf. Nichtsdestotrotz hätten die damaligen Mathematiker deren Eigenschaften mit trigonometrischen und projektiven Mitteln untersuchen können, die in Texten wie dem *Almagest* von Ptolemaios und den *Mathematischen Sammlungen* von Pappos belegt sind.

Keywords: Sonnenuhren; Geometrie; Cetius Faventinus; Vitruvius; Pappos von Alexandria.

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Greco-Roman sundials were products of astronomy, mathematics, and craft. The underlying astronomical theory is that, from a terrestrial perspective, the Sun's movement during the course of a day and night can be idealized, with negligible inaccuracy, as uniform motion along a declination circle of the celestial sphere, i.e. a circle parallel to the celestial equator that is partly above and partly below the observer's horizon. The 'seasonal hours' of the day, which were the seasonally varying time units used in daily life, were defined astronomically as the intervals during which the Sun traverses equal twelfths of the arc of the declination circle above the horizon. The sundial displays the linear projection of the Sun's instantaneous position on the celestial sphere, through a fixed vertex point, upon an immobile surface.[1] This surface is inscribed with a grid formed by two sets of lines: projections of a subset of the declination circles corresponding to key stages of the solar year, called 'day curves', and loci of the projections of the points on all the declination circles corresponding to the endpoints of the seasonal hour arcs, called 'hour curves'. Hence the position of the Sun's projection relative to the grid lines shows not only the current seasonal hour of the day but also the current stage of the year.

The surfaces chosen for sundials were those of simple geometrical forms: planes, spheres, cones, and cylinders; and normally the vertex was the tip of a gnomon so that the projection of the Sun's position was displayed as the tip of the gnomon's shadow. The quintessential Greco-Roman sundial type from a cosmological point of view had a concave spherical surface and a gnomon whose tip was at the center of the sphere, so that the surface is an inverted but geometrically undistorted image of part of the celestial sphere, and the day curves are parallel circular arcs (Fig. 1). Another common type that preserved the day curves as parallel circular arcs had a concave right conical surface whose axis was polar, that is, perpendicular to the plane of the equator, and whose gnomon tip was on the axis. A comparatively rare limiting case of this type flattened the cone into a planar surface parallel to the equator; such equatorial sundials had to consist of a slab with two inscribed faces and two gnomons since the Sun shines on each face of the slab for only half the year.[2] In another, likewise rare, limiting case, the conical

1 In this paper I am not concerned with portable sundials, which for the most part worked on different principles from fixed-position sundials.

2 Six examples are currently known (Herrmann, Sipsi, and Schaldach 2015). Notable among them are the fragments of an exceptionally early – second half of the fourth century BC? – equatorial sundial excavated at Oropos (Archaeological Museum of Oropos, East Attika, inv. A 392, formerly Piraeus, Archeological Museum inv. 235, see Schaldach 2004 and Schaldach 2006, 116–121) and a well preserved one of unknown provenance and date (British Museum 1884,0615.1 = Gibbs 5022G, intended latitude estimated by Gibbs as 32° and by me as 33°, incorrectly identified by Winter 2013, 597 as a vertical sundial).

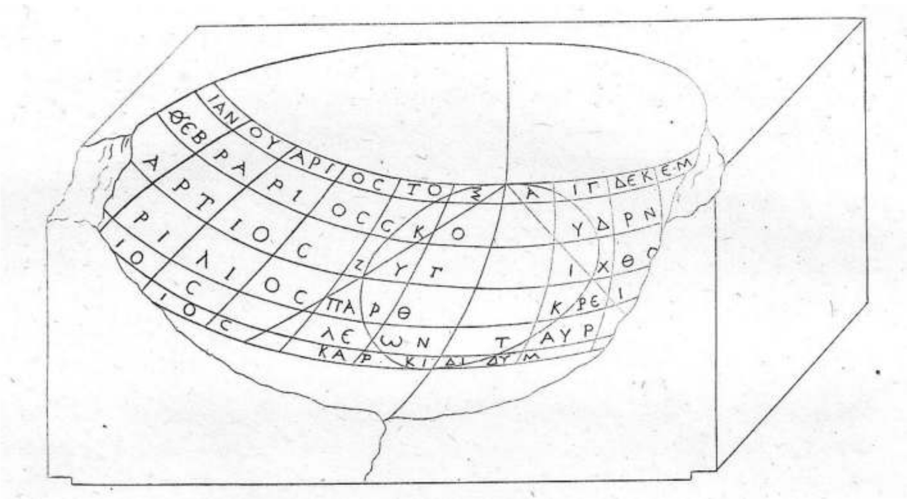

Fig. 1 Spherical sundial, Vatican, Musei Vaticani inv. 2439 = Gibbs 1068G, found before 1820 on the Esquiline, Rome, likely first century AD and certainly after 8 BC; drawing from Guattani 1811, 102. Gibbs (1976, 184) estimates the latitude for which the sundial was made as 42°, appropriate for Rome (actual latitude 41° 54′). The sundial bowl would have faced south, and the marble block out of which it was sculpted would have been rectangular except for the south face, which would probably have sloped forward from bottom to top so that the upper rim of the bowl could accommodate the projections of the entire arcs of the eastern and western horizons over which sunrises and sunsets take place through the year. The lost gnomon would have been mounted vertically from the middle of the bottom front edge, roughly where the present broken edge shows a notch. The grid is exceptionally elaborate and carefully executed, with labeling inscriptions in Greek. The arcs running from left to right are the day curves correspond to the dates of the Sun's entry into the zodiacal signs, including the winter solstice (top), equinoxes (middle), and summer solstice (bottom). Eleven hour curves separating the twelve hours of day cross the day curves. The circle is an image of the ecliptic divided into twelve equal sectors, used to locate the day curves for the zodiacal sign entries between the solstices and equinoxes, while the two oblique lines indicate the lengthening of days through the course of the year relative to the winter equinox.

surface was stretched out into a concave cylindrical surface with a polar axis.[3] All the foregoing types can be grouped in a general category of polar-axial sundials.

Since the surface generated by the straight lines passing through a fixed vertex and through all points of a declination circle is a right cone, a Greek geometer would immediately have recognized that the projections of declination circles on planar surfaces are

3 A remarkable example of this type, consisting of a cylindrical hole perforating a slab in the plane of the equator, was excavated at Aï Khanoum (No. 2 in Veuve 1982; see also Savoie 2007); it was constructed for latitude 37°, which is approximately correct for Aï Khanoum, except that the hour curves would best fit a latitude of about 25°. The other two polar cylindrical sundials known to me, resembling conventional spherical or conical sundials, are Gibbs 6002G (found at Cumpăna, Rumania, Constanta Archeological Museum inv. 1657) and Gibbs 1053G (uncertain provenance and date, in archeological storage at Thessaloniki, classified by Gibbs as spherical but see Schaldach 2006, 140).

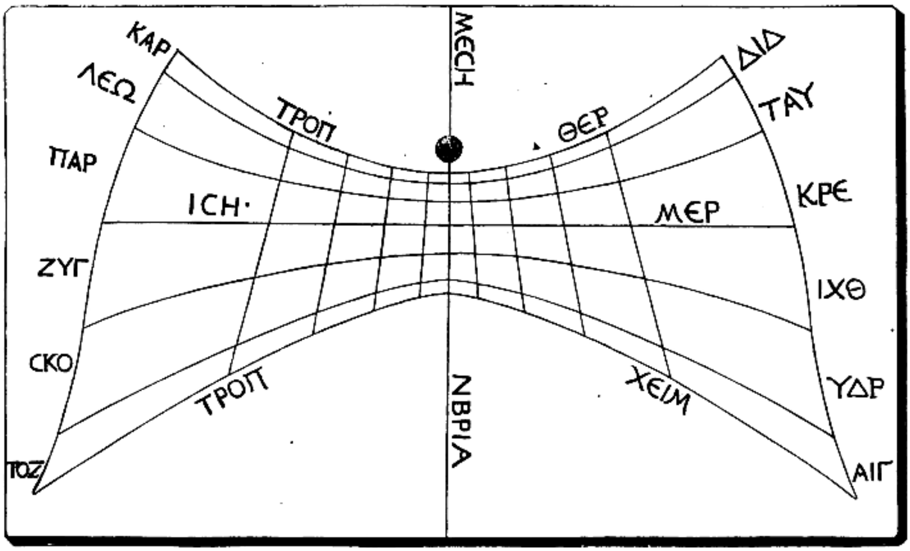

Fig. 2 Horizontal sundial, Naples, Museo Archeologico Nazionale inv. 3075 = Gibbs 4007, found in 1865 at Pompeii, probably first century AD and certainly not later than AD 79; drawing from Museo archeologico nazionale di Napoli 1867, 16. Only the hole for the vertical gnomon survives; the slab would have been oriented with the top edge (as shown) facing south. The hyperbolic day curves correspond to the dates of the Sun's entry in the zodiacal signs, including the summer solstice (top), equinoxes (straight line at middle), and winter solstice (bottom). labeled with abbreviated names in Greek. The hour curves have been drawn pointwise, and exhibit unexplained sinuosities; in most horizontal sundials the hour curves are drawn as straight lines. Basing the estimate on measurements along the meridian hour curve from the photograph, the sundial was constructed for approximate latitude 41°, appropriate for Pompeii (actual latitude 40° 45′).

conic sections.[4] Aside from the equatorial type that we have just described, the surfaces of ancient planar sundials were either parallel to the horizon or perpendicular to it and facing any horizontal direction.[5] Assuming a terrestrial location between the Tropic of Cancer and the Arctic Circle, the day curves of a horizontal or vertical sundial will always be hyperbolas except for the equinoctial curve, which must be a straight line since the tip of the gnomon lies in the plane of the celestial equator.[6] Although no ancient discussion of the day curves of planar sundials as conic sections survives, there is no doubt that their properties were well within the grasp of Hellenistic geometers; and in

4 Neugebauer (1948) went so far as to suggest that the Greek study of conic sections originated in sundial theory, though he conceded that this hypothesis was difficult to reconcile with the specific orientations of cone and plane by which the curves were generated in the period before Apollonius's *Conics*.

5 In practice one finds vertical dials built to face the four cardinal directions as well as the four directions at 45° from them; the octagonal Tower of the Winds at Athens has sundials facing all eight directions.

6 The conventional axiom of Greek cosmology that 'the Earth has the ratio of a point to the cosmos' implies that the tip of a gnomon is, for all observational purposes, at the center of the celestial sphere.

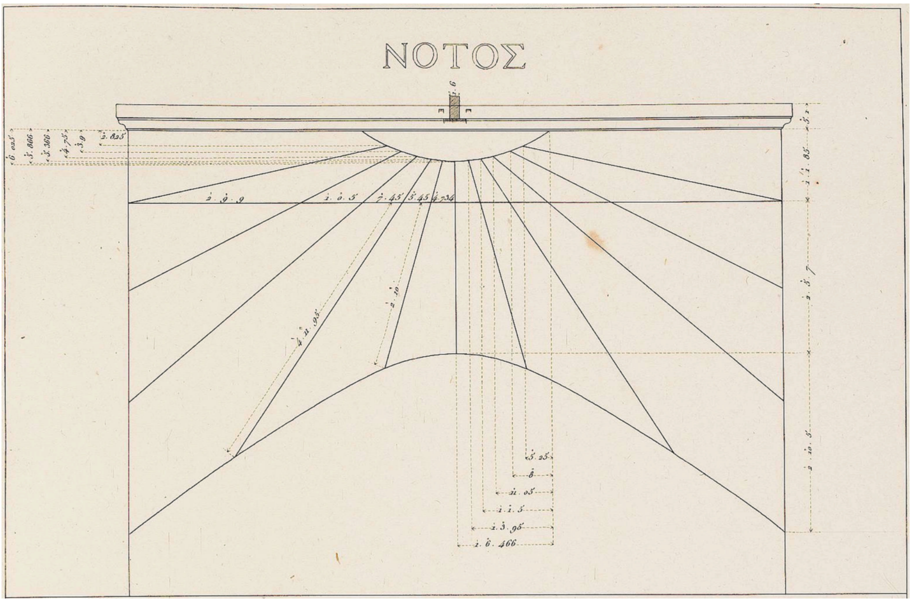

Fig. 3 Vertical sundial on the south face of the Tower of the Winds = Gibbs 5001, Athens, c. 100 BC; drawing from Stuart and Revett 1825, pl. xix. The hyperbolic day curves correspond to the winter solstice (top), equinoxes (horizontal straight line at middle), and summer solstice (at bottom); they are executed with astonishing accuracy, and were probably based on measurements made on the walls of the Tower after it had been erected (Schaldach 2006, 68–81 and 169–181). The hour curves are inscribed as straight lines. The gnomons on the present-day monument are inaccurate modern restorations.

fact some of the day curves on the best executed planar sundials have the appearance of being the products of theoretical construction calibrated by empirical data (examples Figs. 2–3).

The hour curves, by contrast, would have been beyond the resources of Greek mathematics to handle except in an approximative pointwise manner. If the time units employed had been equinoctial hours (equal twenty-fourths of a day and night, counted from noon or midnight), the hour curves in any polar-axial sundial would have divided all the day curves in similar arcs of 15°, and hence they would have been easy geometrical objects to handle: great circle arcs on spherical sundials, and straight lines on the other types, all lying in planes passing through the polar axis. These same planes that correspond to the equinoctial hours would project on a planar sundial as straight lines, albeit no longer equally spaced. Use of seasonal hours, however, results, for all the sundial types, in hour curves that have rather messy analytical representations that do not lend themselves to geometrical construction in the Greek manner.

Of course *any* shape of surface could be used for a sundial if one does not require the day curves to be circles, straight lines, or conic sections. In practice the designers of Greco-Roman sundials exercised this freedom only in limited ways. One recurring, though not very common, type employed a concave or convex cylindrical surface with a vertical axis.[7] In such sundials, the day curve for the equinoxes is an arc of an ellipse, but the other day curves would not have been tractable by ancient mathematical methods.

The type of Greco-Roman sundial usually designated in English as 'roofed spherical sundial' (the French name *cadran à œilleton* is better) stands out as by far the most popular of the designs whose day curves are not straight lines, circles, or conic sections. According to the latest published information, about thirty-two examples of this type are known, either as existing at present or having existed since the 16th century.[8] This amounts to something between one-fifteenth and one-twentieth of the currently known Greco-Roman sundials, a fraction comparable to that accounted for by horizontal sundials.[9] The great majority are from Italy and fully a third from Aquileia, which thus appears to have been a center of their production in the imperial period.[10] The earliest, however, appears to be the south face of an elaborate late Hellenistic multiple sundial excavated in 1905 in the sanctuary of Posidon and Amphitrite on Tinos, which bears inscriptions associating it with Andronikos Kyrrhestes (c. 100 BC?), the architect of the Tower of the Winds in Athens.[11]

Vitruvius writes tersely of a type of sundial named "hemispherical" (*hemicyclium*),[12] "hollowed out of a rectangular block and undercut in accordance with the latitude".[13]

7 Several examples of concave vertical cylindrical sundial surfaces are elements of rather baroque Roman-period multiple sundials (Gibbs 7004–7007), probably all from Italy. An apparently self-standing one is Gibbs 6001, from Volubilis. The cistern-annex on the south side of the Tower of the Winds bears a convex vertical cylindrical sundial, a type otherwise known only from miniature portable sundials.

8 Gibbs (1976, 195–218) lists twenty-three while Winter (2013, 59) lists twenty-eight, among which 'Durostorum 1' (Silistra, Archeological Museum inv. 517) and 'Serdica' (Sofia) are misidentified, whereas one should add Winter's 'Leptis Magna 3' (a photo of which appears on the book's cover), 'Salona' (Split, Archeological Museum, incorrectly classified by Winter 2013, 539 as a conventional spherical sundial, but see Gibbs 1976, 210, No. 2016G), as well as the following four sundials that are entirely missing from Winter's book: Gibbs 2014G (Trieste, Civic Museum of History and Art), 2021 (Vatican Museum inv. 53875 = PN 5), 2023G (Berlin, Antikensammlung SK1049), and a sundial from Villa B at Oplontis published (without inventory number) in Catamo et al. 2000, 217–218. Bonnin (2012, 22) speaks of thirty-three known roofed spherical sundials without providing a list. Hannah and Magli (2011) have proposed that the Pantheon was a kind of monumental roofed spherical sundial.

9 Gibbs inventories 276 sundials, Winter roughly 400, while the forthcoming catalog by Jérôme Bonnin will list at least 563 (http://bsa.biblio.univ-lille3.fr/blog/2012/09/horologia-romana, visited on 17/7/2017).

10 Schaldach 1997, 35–36; Winter 2013, 60.

11 Tinos, Archeological Museum inv. A 139 = Gibbs 7001G. Whether the Tinos sundial was constructed by Andronikos or merely honors his memory is disputable. On the vexed problem of dating Andronikos and the Tower of the Winds see Schaldach 2006, 61–63.

12 The Greek word ἡμικύκλιον can mean a hemisphere as well as (more commonly) a semicircle.

13 Vitruvius, *De Architectura* Book 9, 8, translated from Rose 1899, 233.

A fuller description provided by the third-century architectural writer Cetius Faventinus removes any doubt that this type is our roofed spherical sundial:[14]

> Let the clock [*horologium*] that is called *hemicyclion* be formed in a similar manner from a stone or a marble block having its four sides broader at the top and narrower at the bottom, so that it has its sides wider behind and on the sides, but let the front lean forward somewhat and make a greater shadow. On the underside of this front let a circumference [*rotunditas*] be drawn with a compass, and let this be hollowed out inwards and make the shape of a hemisphere. In this cavity let there be three circles [*circuli*], one close to the top of the clock, the second through the middle of the cavity, and let the third be marked close to the edge. Next from the smaller circle to the greater seasonal[?] circle[15] [*circulum horalem*] let eleven straight [!] lines be drawn at equal spacing, which are to indicate the hours. Through the middle of the hemisphere, above the smaller circle, let there be a smooth plate of more delicate thickness, so that with a circular finger-size hole having been opened up [*aperta rotunditate digitali*] the ray of the sun, passing within more easily, may indicate the hours through the numbers of the lines. Then at the season of winter it will provide the numbers of the hours through the smaller circle, and in the season of summer it will step through the intervals of the greater circle.[16]

As Cetius writes, the operating surface of a roofed spherical sundial (examples Figs. 4, 5, 6, and 7) is a concave hemisphere that is oriented facing southwards and slightly downwards so that the body of the sundial overhangs the surface, hence the modern designation 'roofed'.[17] At the highest point of the hemisphere is an orifice covered by

14 Cetius Faventinus 310.13–311.2, translated from Rose 1899, 302–303.

15 It is not clear what Cetius intends by *horalis*, a very rare word that one would expect to mean 'pertaining to hours'. The "circle" in question is a day-curve, not an hour-curve.

16 Following the passage translated here, Cetius speaks of two vertical sundial faces oriented eastwards and westwards, but (contrary to the interpretation in Schaldach 1997, 37–38) this must refer to the other type of sundial that he earlier described, the *pelicinum*, which comprised a pair of vertical sundials facing southeast and southwest and joined at the meridian hour line; the sentences in question are likely displaced. For the correct identification of the *pelicinum* see Traversari 1989 and Bonnin 2015, 30–32; incorrect identifications abound. Bonnin (2015, 29–30) doubts whether Cetius is correct in applying the name *hemicyclium* to the roofed vertical type, and demonstrates the existence of a rare roofed conical type, which will not be discussed in the present article.

17 Discussing the Berlin sundial, Staatliche Museen zu Berlin, Antikensammlung inv. SK1049 = Gibbs 2023G, Woepcke (1848 [1842], 38–39) proposed that the sundial would have been mounted lying on the face that we would call its back, and with the face that we would call its top facing south, with disastrous results for his analysis of it. The lion's feet should have made the correct orientation obvious. The mistake, and Woepcke's consequent identification of the type with Vitruvius's *antiboreum* (*De Architectura* Book 9, 8), persist even in fairly recent works on ancient sundials, e.g. Rohr 1970, 14; this despite the fact that other publications starting with Kenner 1880 had shown roofed spherical sundials

Fig. 4 Roofed spherical sundial, Museo Arqueológico Nacional, Madrid, inv. 33185 = Gibbs 2020, excavated at Baelo Claudia, 1st century AD. This is the general design that Cetius Faventinus knew, a rectangular block with a forwards-sloping south face, and most of the extant roofed sundials follow it. The hemispherical sundial surface is approximately tangent to the top face of the block. The original eyehole would have been a perforation in a metal plate mounted covering the large hole at the top; the plate now occupying this position is a modern restoration. The loop-shaped curves are the day curves for the winter solstice (smallest), equinoxes, and summer solstice (largest, close to the rim of the bowl). Measurement of the inclination of the equinoctial day curve from the digital model shows that the sundial was constructed for a latitude of approximately 48° 30′, whereas the latitude of Baelo Claudia is near 36°.

a plate perforated in an eyelet, through which sunlight penetrates from above. A small spot of light thus falls on the surface at the point that is the projection of the Sun's position on the celestial sphere through the eyelet, which thus functions as a gnomon in reverse.[18]

Situating the vertex of projection on the spherical surface instead of at its center results in a complete change in the geometry of the Sun's projected paths compared to a conventional spherical sundial. At both sunrise and sunset the Sun's projection coincides with the eyelet (treating the eyelet as a geometrical point), so that each day curve is a closed loop. Since the eyelet lies in the plane of the celestial equator and the intersection of any plane with a sphere is a circle, the equinoctial day curve is a complete circle; but, notwithstanding what Cetius writes, this is not true of the day curves for the solstices or for any of the other day curves; in fact, unlike the days curves of polar and planar sundials, those of the roofed spherical sundial do not even lie in a single plane. For the

in their proper orientation, the correctness of which was decisively established by the mathematical analysis in Drecker 1925, 25–34. See Schaldach 2016 for further references.

18 If the eyelet is circular and the edge around it is thin, the projected spot of light will be circular no matter where it falls on the spherical shell. In principle this is a better way of marking the Sun's position than a gnomon shadow because the center of the spot can be easily judged by eye.

Fig. 5 View from the west side of a three-dimensional digital model of the bowl and front and top faces of the Baelo Claudia roofed sundial, with other faces cut away. A recessed area of the top face accommodates a metal plate perforated with the eyehole.

Fig. 6 Roofed spherical sundial, Louvre inv. ME1178, acquired 1999, reportedly found at a Roman Villa in Carthage, likely first century AD and certainly after 8 BC. The day curves correspond to the entries of the Sun into the zodiacal signs, and are labeled in Greek. The hour curves are executed with greater theoretical accuracy than those of the Berlin sundial. A plate perforated with the eyehole would have been mounted over the present hole at the top. Savoie and Lehoucq (2001) determined the latitude for which the sundial was constructed to be approximately 41°, much too far north for Carthage (latitude 36° 51′), more nearly appropriate for Rome.

Fig. 7 View from the west side of a digital model of the bowl of the Louvre sundial. (Model reconstructed by photogrammetry.)

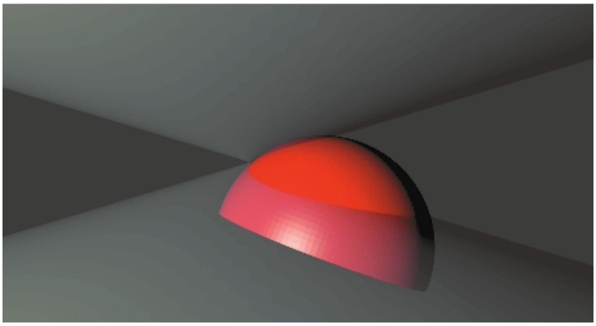

Fig. 8 Double-napped cone corresponding to solar declinations $\pm\delta$ intersecting sundial bowl, with its vertex at the eyehole point. The figure is oriented so that horizontal lines are parallel to the equator.

general case of a declination circle for declination δ, the day curve is part of the intersection of the spherical surface with a double-napped right cone of aperture $(180^\circ - 2\delta)$ whose vertex lies on the surface of the sphere and whose axis is perpendicular to the plane of the equator (Fig. 8). The portion of this intersection lying on the northern nap is the day curve for declination $+\delta$, and the portion on the southern nap is the curve for $-\delta$ (Fig. 9). Since Greek geometry only dealt with single-napped cones, an ancient geometer would have regarded the day curve as a complete line of intersection of a cone and a sphere, though for purposes of mathematical analysis it would have been useful to work with the curves for equal positive and negative declinations simultaneously.

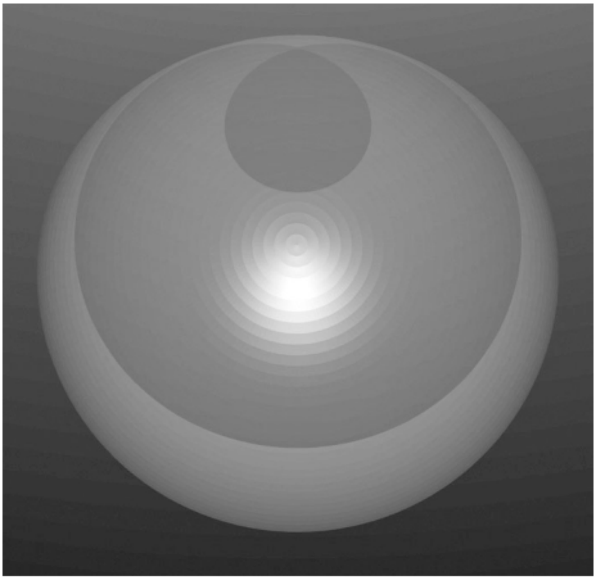

Fig. 9 Curves of intersection of the cone of Fig. 8 with the sundial bowl, viewed from directly in front of the bowl. The outline of the paler gray region is the day curve for $+\delta$, and that of the darker region is the day curve for $-\delta$.

According to Pappus, Greek mathematicians recognized three classes of lines as tractable mathematical objects: straight lines and circles, which could by hypothesis be invoked in a given plane without having to justify their generation; conic sections, which could be generated by the intersection of the plane with simple three-dimensional surfaces (cones and cylinders); and miscellaneous other curves, which could be generated either by imagined 'mechanical' contrivances or by geometrical constructions involving the intersections of three-dimensional surfaces.[19] He delineates a hierarchy of geometrical problems, according to which a problem that can be solved using just straight lines and circles is called 'planar' (ἐπίπεδον) and *should* only be solved using these 'planar' objects, while a problem that is not planar but that can be solved by introducing one or more conic sections is 'solid' (στερεόν), and one that can be solved using another variety of curved line is 'curvilinear' (γραμμικόν). Pappus attributes to the geometers a strict view that it was "no small fault" when a problem was solved by curves that are not proper to its classification, which would mean that special curves should only be invoked when a problem cannot be solved using just straight lines, circles, and conics, in practice special curves were sometimes applied to 'solid' problems, perhaps because it was easier to devise an apparatus for drawing them.

An ancient mathematician would easily have seen that the day curves of any sundial are the intersections of cones defined as above with the sundial surfaces. What is less obvious is whether a mathematician would have been capable of discovering and

19 Pappus, *Collection* Book 4, cited after Hultsch 1876–1878, Vol. 1, 270–272.

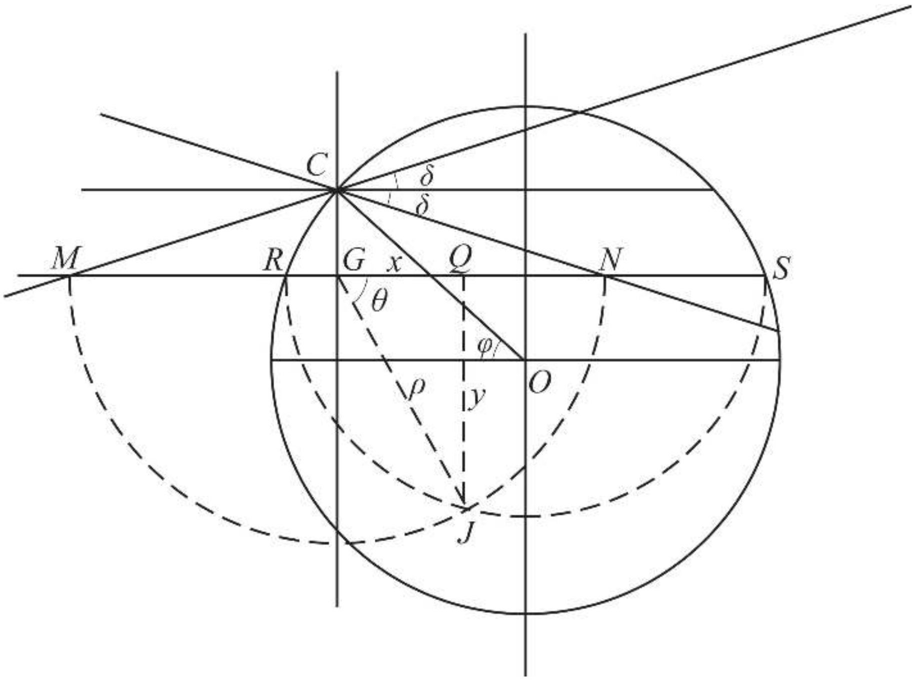

Fig. 10 Conditions determining the coordinates x, y (i.e. GQ, QJ as defined in the text) of a point J lying on the day curve for declination $+\delta$ (adapted from Drecker 1925, Plate 4, Fig. 43). Solid lines are in the meridian plane through the center O of the sphere of the sundial bowl, broken lines in an arbitrary plane of section parallel to the equator and passing through RS. The diagram is oriented so that lines parallel to the equator are horizontal. C is the eyehole point; the semicircle on diameter MN is the section of the declination cone cut by the arbitrary plane, and the semicircle on diameter RS is the section of the sphere cut by the plane.

demonstrating properties of the day curves in a roofed spherical sundial. Curves in three dimensions were certainly objects of study; examples include the helix, the hippopede of Eudoxus, and the intersection of a torus and a cylinder employed by Archytas in his construction of the two mean proportionals. Pappus, *Collection* 4 contains a discussion of a technique of generating surfaces ('cylindroids') as the loci of straight lines perpendicular to a given plane and passing through points of a given curve;[20] by taking the intersection of such a cylindroid with a planar or curved surface, one could obtain new and possibly more mathematically tractable curves as a form of projection of the original curves. The fact that certain 'mechanically' generated curves such as the quadratrix could also be related to intersections of surfaces was a matter of interest.

Expressed in suitable orthogonal coordinates x, y, and z, the intersection of a sphere with a cone would be the solution of a pair of quadratic equations. Drecker demon-

20 Hultsch 1876–1878, Vol. 1, 258–264.

strated that if we set the origin at the eyehole point, the x-axis oriented south-north in the plane of the equator, and the y-axis oriented east-west in the plane of the equator, then (Fig. 10):[21]

$$\left(x^2 + y^2 - 2r\cos(\varphi)\cos^2(\delta)\,x\right)^2 = r^2\sin^2(\varphi)\sin^2(2\delta)\left(x^2 + y^2\right) \qquad (1)$$

where r is the diameter of the sphere and φ is the terrestrial latitude for which the sundial is constructed. Disregarding z, this quartic equation describes the day curve projected orthogonally into the plane of the equator (Fig. 11).[22] As Drecker remarks, it is the equation of a limaçon of (Étienne) Pascal. A characteristic property of the limaçon becomes apparent if we express the equation in polar form:

$$\rho = 2r\cos(\varphi)\cos^2(\delta)\cos(\theta) \pm r\sin(\varphi)\sin(2\delta) \qquad (2)$$

Since

$$\rho = 2r\cos(\varphi)\cos^2(\delta)\cos(\theta) \qquad (3)$$

is the equation of a circle passing through the origin, the limaçon is the locus of points at a constant distance from the circle as measured along any straight line through the origin. Hence the limaçon is also known as the conchoid of a circle, an analogue of the conchoid of Nicomedes which is the locus of points at a constant distance from a straight line as measured along any straight line that passes through an origin not lying on the given straight line.

The conchoid of Nicomedes was a 'mechanical' curve (in principle drawable by means of a special compass) introduced in the Hellenistic period as a way of allowing certain so-called *neusis* constructions, which are constructions that can be reduced to the postulates of *Elements* Book 1 only in special conditions. A *neusis* is the construction of a straight line passing through a given polar point, such that the part of the line cut off between two intersections with given straight lines or circular arcs has a given length. When at least one of the given lines is a straight line, the *neusis* can be performed by drawing the conchoid of Nicomedes generated from the given polar point and the given straight line, and then finding the intersections of the conchoid with the other given line. Certain geometrical problems (for example in Archimedes *On Spirals*) could be reduced to *neuses* in which one of the bounding lines was a circle and the polar point

21 Drecker 1925, 26–27.

22 Drecker also shows that the orthogonal projection of the day curves for ±δ in the meridian plane is a parabola. I doubt that this would have been realized in antiquity; the body of the sundial obstructs this perspective from view.

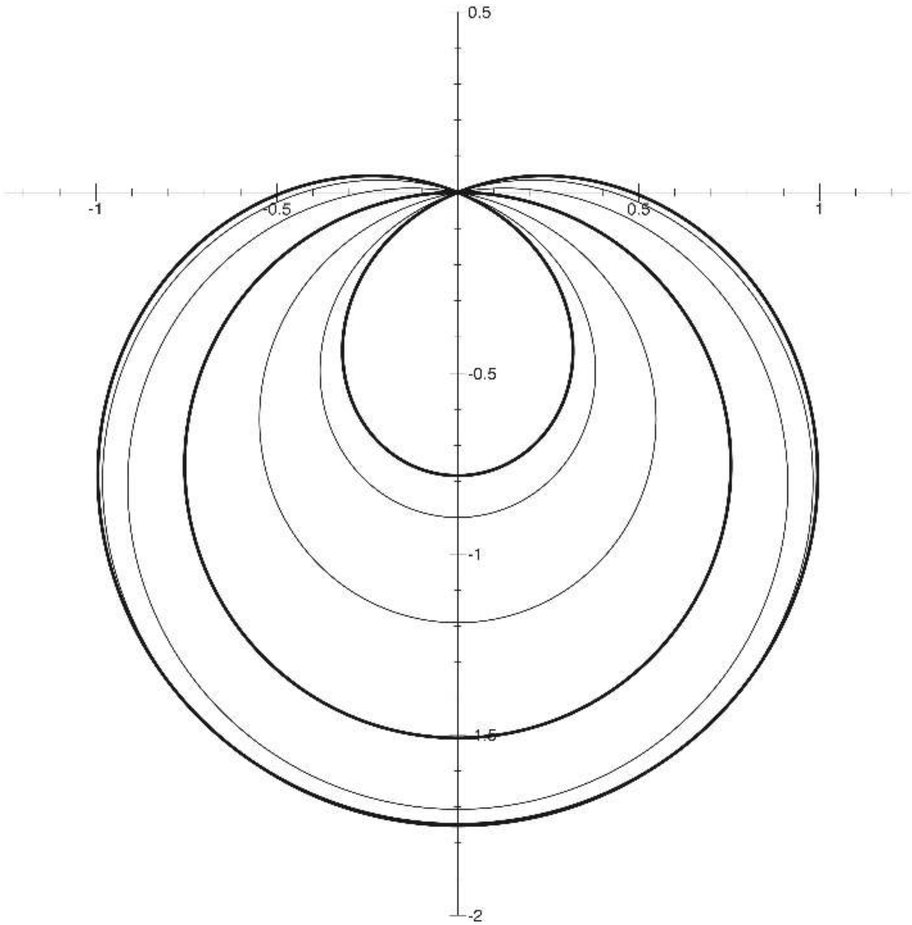

Fig. 11 Orthogonal projection into an equatorial plane of day curves for $\varphi = 41°$, declinations corresponding to the Sun's entry into the zodiacal signs. Thicker curves are for the winter solstice (innermost), equinoxes, and summer solstice (outermost). Cf. Fig. 6 right.

lies on the same circle, and there is good reason to believe that the limaçon of Pascal was known in antiquity as a resource for resolving such *neuses*.[23]

The fact that the day curves of a roofed spherical sundial are projections of limaçons on the spherical surface is mathematically appealing, and a geometer familiar with the planar curves might have suspected it simply from the look of the day curves on an empirically constructed roofed spherical sundial. But could the geometer have *proved* it? Drecker's analytical approach to the problem does not translate well into a synthetic form that one could imagine being discovered in antiquity. However, a deduction of the

23 Knorr 1986, 222 and 258.

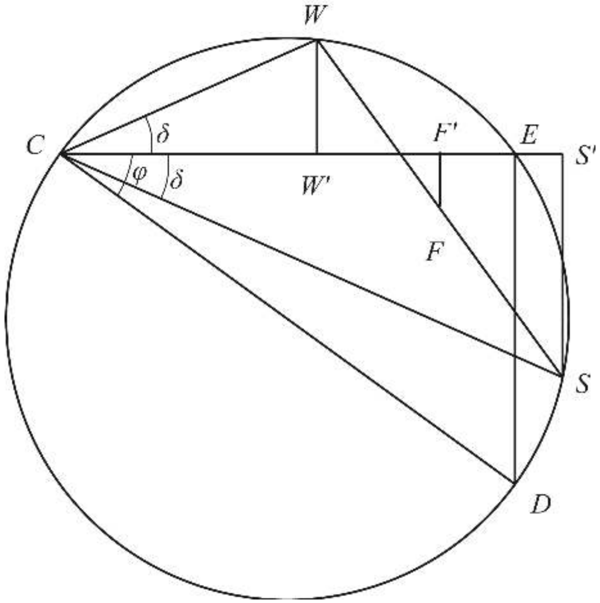

Fig. 12 Meridian section of sundial sphere.

limaçon would have been within reach of someone equipped with the basic theorems underlying the planar trigonometry of Book 1 of Ptolemy's *Almagest*. In the following conjectural reconstruction I employ for the sake of clarity modern trigonometric functions instead of Ptolemy's chord function.

Fig. 12 shows the cross-section in the plane of the meridian of the complete sphere to which the sundial's bowl belongs, oriented so that the intersection of the meridian and equatorial planes is horizontal in the diagram. *C* is the eyehole point, *E* is the projection of the Sun at noon on an equinox, and *W* and *S* are respectively the projections of the Sun at noon on dates when the Sun's declination is $-\delta$ and $+\delta$. *CD*, the diameter of the sundial sphere passing through *C*, is perpendicular to the horizon of the locality for which the sundial has been constructed. Hence

$$CE = 2r\cos(\varphi) \tag{4}$$

$$CW = 2r\cos(\varphi + \delta) = 2r\left[\cos(\varphi)\cos(\delta) - \sin(\varphi)\ \sin(\delta)\right] \tag{5}$$

$$CS = 2r\cos(\varphi - \delta) = 2r\left[\cos(\varphi)\cos(\delta) + \sin(\varphi)\ \sin(\delta)\right] \tag{6}$$

Let W' and S' be the orthogonal projections of W and S in the plane parallel to the equator that contains C and E, and let F' be their midpoint. Then

$$CF' = \cos^2(\delta)\, CE \tag{7}$$

which, we note, does not depend on φ. Moreover,

$$F'W' = F'S' = 2r\sin(\delta)\cos(\delta)\sin(\varphi) \tag{8}$$

We now consider (Fig. 13) a different cross-section of the sphere in an arbitrary plane containing C and perpendicular to the equator. Points E'', W'', and S'' are projections of the Sun at certain times of day (not necessarily the same times) on an equinox and on dates when the Sun's declination is respectively $-\delta$ and $+\delta$ as before. Let $D''C$ be the diameter of the circle of the cross-section, and let φ'' be angle $D''EC''$. By the same argument used to find CF, we have

$$CF''' = \cos^2(\delta) CE'' \tag{9}$$

Hence CF'''/CE'' is constant and equal to CF'/CE, so that F''' lies on the circle with diameter CF' in the equatorial plane through C. Again,

$$F'''\, W''' = F'''\, S''' = 2r''\sin(\delta)\cos(\delta)\sin(\varphi'') \tag{10}$$

But

$$\sin(\varphi'') = \frac{E''D''}{2r''} = \frac{ED}{2r''} = \left(\frac{r}{r''}\right)\sin(\varphi) \tag{11}$$

So

$$F'''\, W''' = F'''\, S''' = 2r\sin(\delta)\cos(\delta)\sin(\varphi) = F'\, W' = F'\, S' \tag{12}$$

which establishes that W''' and S''' lie on the two branches of a limaçon generated by the equatorial circle on diameter CF' with C as pole.

The day curves projected into the equatorial plane exhibit an asymmetry that has an analogue in planar sundials. As we have seen, the day curves corresponding to solar declinations of $+\delta$ and $-\delta$ are the intersections of the sundial surface with the two naps of a single double-napped cone. Hence from the modern perspective the two hyperbolic day curves for equal but opposite declinations on a planar sundial are the two branches of a single hyperbola (a Greek geometer would have called them a pair of 'opposite' hyperbolas); but the straight line that is the day curve for the equinoxes is not equidistant

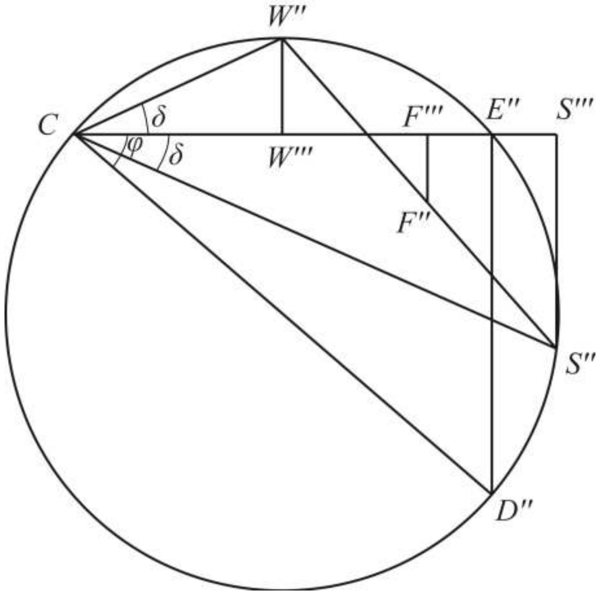

Fig. 13 Section of sundial sphere in arbitrary plane through eyehole point *C* and perpendicular to the equator.

from the branches – this is very obvious in horizontal and south-facing vertical sundials. Similarly in the roofed spherical sundial, since the diameter of the generating circle of the limaçon projected in the plane of the equator for declination $\pm\delta$ is not CE but CF', which is progressively smaller than CE as δ increases, the projections of the day curves for positive declinations are crowded closer together than those for negative declinations. However, on the spherical surface itself the arcs separating the day curves for $\pm\delta$ from the equinoctial day circle, measured along the circular section through C and perpendicular to the equator as in Fig. 13, are equal to each other and subtend the same central angles (2δ) as their counterparts in the plane of the meridian.[24] Since it is easy to draw a series of these circular sections on the spherical bowl – they are the circles through C whose centers lie on the great circle of the sphere parallel to the equator – this property would make it easy to accurately construct the day curves pointwise. Moreover, a viewer standing reasonably close to the sundial will see a clearer separation between the day curves near that of the summer solstice than one gets in the equatorial projection.

24 Gibbs (1976, 98–99, note 10) incorrectly states that the arcs separating the day curves for $\pm\delta$ from the equinoctial curve are constant as measured along great circles through C.

Vitruvius ascribes the *hemicyclium* or roofed spherical sundial to Berossus the Chaldean, the Babylonian scholar who reportedly resided in Kos in the third century BC. We may reasonably be skeptical about this attribution.[25] But it is interesting to observe the company Berossus keeps in Vitruvius's list of inventors of sundial types, among whom we find Eudoxus, Aristarchus of Samos, Apollonius, and Dionysodorus, all of whom were distinguished mathematicians or mathematical astronomers. Whatever the specific accuracy of these credits, Vitruvius leaves us in no doubt that sundial design was regarded as field appropriate for a mathematician, and that the great variety of sundial types was a manifestation of scientific creativity. Many of the known examples of roofed spherical sundials were prestige objects exhibiting a high level of ornamental as well as geometrical skill in their sculpture. The comparative popularity of the type likely resulted in part from certain practical advantages. Unlike vertical sundials, they yielded an easy reading of the hour at all seasons and all times of day; while, unlike conventional spherical or conical sundials, they were well suited to mounting at eye height or above. But beyond this, the unobvious beauty of the inscribed curves of a well-executed roofed spherical sundial would have pleased the mind as well as the eye of the connoisseur.

25 See Steele 2013 for a judicious consideration of the astronomical and astrological testimonia concerning Berossus, concluding that some of the reports may be genuine but that Berossus had little or no connection with genuine Babylonian astronomy. The alleged invention of the sundial is mentioned on pp. 118–119.

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