What every young astronomer needs to know about spherical astronomy: Jābir ibn Aflaḥ's "Preliminaries" to his *Improvement of the Almagest*

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Abū Muḥammad Jābir b. Aflaḥ is believed to have worked in Seville in the first half of the 12th century during the reign of the Almoravids. His best-known work is his astronomical treatise, the "Improvement of the *Almagest*" (*Iṣlaḥ al-majisṭī*). It was the first serious, technical improvement of Ptolemy's work to be written in the Islamic west and it reveals Jābir as an accomplished theoretical astronomer, one devoted to logical exposition of the topic.

This paper focuses on the First Discourse of Jābir's work, which establishes the mathematical preliminaries needed for the study of astronomy based on the geometric models of heavenly bodies (Sun, Moon, planets and stars) revolving around a spherical Earth.

Early in his *Improvement*, Jābir states that one of his goals in writing the book is to write an astronomical work that would—apart from reliance on Euclid's *Elements*—stand on its own as regards the necessary mathematics. He specifically mentions that he has obviated the need for the works of the ancient writers, Theodosius and Menelaus. In the case of Theodosius, Jābir meant that the reader would not have to refer to the Greek author's *Spherics* since Jābir has selected from Books I and II of that work statements and proofs of theorems necessary for an astronomer. However, he did assume that his reader would have some basic knowledge of spherics[2] since such basic results as *Sph.* I, 12. ("Great circles on a sphere bisect each other.") were taken for granted. He also ignored propositions in *Spherics* that deal with geometrical constructions (e.g. to construct the diameter of a given sphere or to construct a circle through two given points on a sphere), which would not be useful to an astronomer.

In the case of Menelaus, the result in his *Spherics* that Arabic writers called the Sector Theorem (also known as the Transversal Theorem) was another result known to all who studied astronomy; Jābir, however, does not mention the work and replaced it with the spherical trigonometry developed in Iraq and Iran in the last half of the 10th century.[3]

One possible source for Jābir's knowledge of the results which had been discovered in Irāq and Irān in the latter half of the tenth century and which included the Sine Theorem for spherical triangles was the work of an older contemporary of Jābir, the *qāḍī* (religious judge) Ibn ʿAbdullah Muḥammad ibn Muʿādh of the Spanish province of Jaen. In fact the latter's *Book of Unknowns of*

1 The author thanks Professors José Bellver for sending him scans of the First Discourse of the *Improvement* as found in Escorial 910. He also thanks Professors Bellver, J. Hogendijk, R. Lorch and J. Samsó for aid in understanding a number of passages in the Arabic text. The author especially acknowledges his debt to the valuable study of the work in R. Lorch, "Jābir b. Aflāḥ and the Establishment of Trigonometry in the West." Items VI–VIII in Lorch 1995.

2 This was not an unreasonable assumption. Jābir specifically mentions Euclid as an author he assumes his readers know and Book 11 of the Elements is devoted to solid geometry. Indeed, since anyone in Jābir's time who was reading a treatise dealing with the *Almagest* would almost certainly have gone through a collection of books known as the Middle Books which included works on spherics by Euclid and Autolycus and possibly Menelaus.

3 For an account of this see Van Brummelen 2009, pp. 179–192.

Instruments – Observations – Theories: Studies in the History of Astronomy in Honor of James Evans, ed. Alexander Jones and Christián Carman, 2020, DOI: 10.5281/zenodo.3928498, pp. 239–259. Chapter DOI: 10.5281/zenodo.3975747.

Arcs of the Sphere was the first work to treat spherical astronomy in a purely mathematical fashion with no mention of astronomy, rather as Theodosius had done for spherical geometry. Lorch 1995 (Item VIII), however, points out that Jābir's treatise has "surprisingly little in common" with that of the *qāḍī* (and, in any case, Jābir never mentions Ibn Mu'aādh) .

By the 13th century Jābir's *Improvement* was circulating in both the western and eastern parts of the Islamic world. It was translated into both Hebrew and Latin and had a considerable impact on the development of spherical trigonometry and astronomy in Western Europe.[4]

In this paper we shall survey the contents of the First Discourse of the *Improvement* and provide translations of some of its parts, a number of which have not, to our knowledge, been translated elsewhere.[5] Our translation, which is indented, is from folios 4a – 16a of the Arabic text of the treatise contained in Escorial MS 910, and references of the form /X,Y/ refer to line Y on folio X of that work and, when the context is clear, /Y/, alone, refers to line Y on the last folio previously mentioned. Brackets < > enclose our explanatory material. References to the "Arabic version" refer to the Arabic version of Theodosius's *Spherics* (Kunitzsch and Lorch 2010). That important work also points out the major differences between the Arabic version and the Greek versions as published in Cziczenheim 2000.

> [4a, 12] First Discourse – Preliminaries
>
> Let us begin with an explanation of the terms used. First: The **pole** of the circle drawn on the surface (*ẓahr*[6]) of the sphere is the point of the surface (*basīṭ*) of the sphere such that all lines[7] produced from it to the circumference of the circle are equal to one other.[8] And the **great circle** /15/drawn on the sphere is the circle whose center is the center of the sphere, and it divides the sphere into two halves. And, among the angles bounded by the arcs of great circles,[9] a **right angle** is one that, if we make its vertex a pole and make a circle with any distance [about that vertex] then the arc of the circle that is cut off between the two sides [of the angle] is a quarter of that circle and this arc is called the arc of the angle if it is part of a great circle.[10] And if that arc is greater than a quarter circle then that angle is obtuse, and if it is less then it is acute /20/ and this arc is called the arc of the angle if it is from a great circle.[11]

4 The Latin translation was done by Gerhard of Cremona, who referred to the author as 'Geber.'

5 Prof. Lorch, in his study mentioned above, gives a valuable, brief account of some of the important contents of this part of the *Islaḥ*. And see (Berggren 2006, pp. 539–544) for a survey of the mathematics and translations of some of its key results.

6 The Arabic edition of the *Spherics* uses *saṭḥ* for the surface of a sphere.

7 The Arabic here is *khuṭūṭ*, which could refer to straight lines or arcs of circles. In the *Spherics* "straight lines" are specified.

8 Prior to this point in his text Theodosius has defined "sphere," "center of the sphere" and "axis" of rotation of the sphere. Jābir assume his reader does not need to have these defined.

9 Theodosius gives a careful explanation of the general notion of angles between great circles on a sphere based on the idea of inclination of planes, which is also found in El. XI, Def. 6 and 7. (Throughout this paper the abbreviation "*El.*" refers to Euclid's *Elements*.)

10 A marginal note adds "And the perpendiculars produced from [a point in] their common section in the planes of each of them contain a right angle." (This is, in fact, a consequence of *El.* XI, Definitions 3 and 4.)

11 Jābir made the same remark for the special case of a right angle a few lines earlier.

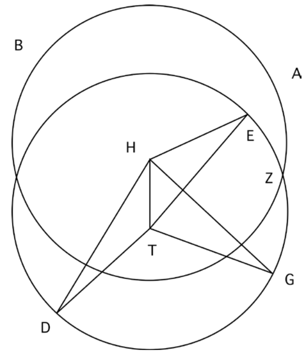

Figure 1. Proposition 1.

And the **Sine** of an arc[12] is half the chord of its double, and it is also the perpendicular falling from an extremity [or the arc] onto the diameter produced from the second extremity.[13]

And the **complement** of an arc is the difference between it and a quarter circle,[14] and similarly the complement of an angle is the difference between it and a right angle, whether the angle is less than a right angle or greater. And of two angles, which, together, are equal to two right angles, each is called **supplementary**.[15] And they are those the sum of whose arcs is equal to a semicircle, and the Sines of their two arcs are one and the same line.[16] Similarly, of two arcs that, together, are a semicircle, each of them is called supplementary. [4b, 1]

Jābir now states and proves eleven theorems from Books I and II of Theodosius's *Spherics*. Since it is clear from the introduction of his *Almagest* that Ptolemy assumes that his reader knows the basics of this subject Jābir's inclusion of theorems from Theodosius's work is very much in order. The fact that he gives not only the theorems he needs but their proofs seems to indicate that the astronomer should know not only what is true but why it is.

We have translated the theorems but the proofs only in those cases where Jābir's proof is significantly different from that in the Arabic version. Footnotes note minor differences.

Theorems from Spherics, Book I

[4b.1][Proposition 1. (= *Sph.* I, 1)] If a plane [*saṭḥ*] cuts a sphere the section common to that plane and the surface [*basīṭ*] of the sphere is the circumference of a circle.[17]

Jābir's proof (Fig. 1) differs from the Arabic version in a number of respects: The diagram in the Arabic has only one circle, representing the plane section of the sphere, but Jābir adds a second

12 The material from here to the beginning of the theorems from Theodosius is not in the *Spherics*.

13 We have capitalized "sine" to remind the reader that here, as in all medieval literature, the sine of an arc is a geometrical object, a line segment, not a ratio. For this reason it is usually capitalized in modern translations.

14 A marginal note adds, "If the arc is smaller or larger than a quarter of a circle."

15 The Arabic word here means 'joined.'

16 The medieval Sine of an angle was a line segment (half the chord of double the angle in a reference circle). The supplementary arcs play an important role in Jābir's discussion of spherical triangles.

17 Unlike its modern definition as "a plane curve" the circle in the ancient and medieval world was, as Euclid put it, "a plane figure contained by one line...."

circle intersecting that and representing the sphere itself; The Arabic proof begins with the easy case, when the section of the sphere contains the center of the sphere, a case Jābir ignores; Although Jābir and the Arabic version then both drop a perpendicular from the center of the sphere onto the plane section, the Arabic then requires that one join the center of the sphere to two points chosen at random on the periphery of the section. Jābir requires that for three points. Both versions then require that one join the foot of the perpendicular to the points chosen on the periphery.

These differences in the diagram continue in the body of the proof itself. The Arabic version uses *El.* I, 47 (not cited) to conclude that the squares on the lines joining the foot of the perpendicular to the two points chosen on the circumference are equal. Hence the two lines are equal. Jābir neither mentions squares nor that his three right triangles have not only equal hypotenuses but a common side as well. Presumably he thought it was obvious from this that the triangles are congruent and hence the third sides must be equal as well.

A consequence of the proof of this proposition is that the perpendicular from the center of the sphere to the plane of a small circle passes through the center of that circle.

The following, Proposition 2, asserts the equivalence of three conditions relative to a small circle in a sphere and a diameter of the sphere: (1) A line joining the center of a circle in a sphere to the center of the sphere is perpendicular to the circle, (2), Extended in both directions that line passes through the poles of the circle and conversely, and (3) a line passing through the poles of a circle in the sphere passes through its center and that of the sphere.

These statements are parts of the contents of *Sph.* I, 7 – 11 Jābir proves them with reference to Fig. 2, where E is the center of the circle, Z the center of the sphere, and points T and H are the poles of circle ABGD.

> [4b, 13] [Proposition 2. (= *Sph.* I, 7 & 8)] <i> If there is on a sphere a circle that is not a great circle[18] and we join its center with the center of the sphere by a line then it <the line> is perpendicular to the plane of the circle. <ii> And if it is extended in both directions it will pass through the two poles.
>
> And conversely, <iii> if a line is produced from the center of the sphere perpendicular to the plane of a circle then it will pass through its center and, if it is produced in both directions, it passes through its poles.
>
> And <iv> if a line passes through its two poles then it passes through its center and the center of the sphere.

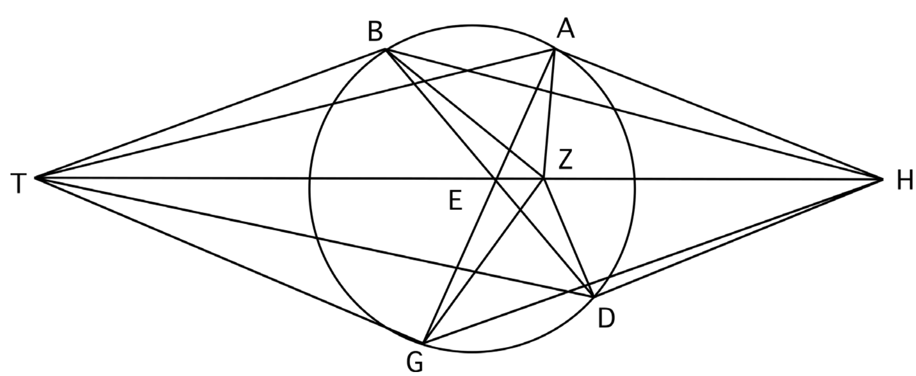

Figure 2. Proposition 2.

18 The condition “that is not a great circle” is missing in the Greek and Arabic versions. Its inclusion here is testimony to Jābir’s pedagogical intentions.

Jābir's proof of i: With reference to Figure 2 Jābir marks two arbitrary points A, B on the circumference of the given circle and joins them to its center, E. He then extends these radii to meet the circle again at G and D. And, since E is the center, the radii EA, EB, etc are all equal. He than joins A, B, G and D to Z, the center of the sphere. Since Z is the center of the sphere the lines AZ, BZ, etc are also equal. And since the line joining the centers, EZ, is common <to all four triangles> angles AEZ and GEZ, are equal (*El.* I, 8). Hence (*El.* 1, Def. 10) they are right angles right angles. Similarly angles BEZ and DEZ are right. Jābir then concludes that EZ is perpendicular to the plane of circle ABGD (*El.* XI, 4).In the Arabic version lines AZ, BZ, GZ and DZ neither appear nor are needed.

For whatever reason, Jābir took A and B as arbitrary points and then showed that all four angles formed with EZ by the two diameters going through A and B are right. (Perhaps he felt that the student might not immediately grasp the importance of the fact that BD was an arbitrary diameter of the circle.)

Jābir's proof of ii: He extends line EZ in both directions tot meet the sphere at H and T and join both of these with points A, B, G and D with lines HA, HB,... and TA, TB.... Because E is the center of circle ABGD the four lines AE, BE,... are equal. Since EZ is perpendicular to the plane of the circle ABGD the four angles AEH, ... DEH are right. And AH is common to all four triangles AEH, ... , BEH so the lines AH, ... , DH are equal, and therefore H is a pole of circle ABGD. A similar argument shows T is a pole.

Apart from Z being labeled D, H labeled Z, T labeled H and D being labeled T the proof in the Arabic version is the same.

<iii> and <iv) The last part of <iii> is, of course, immediate from the first part, whose proof, Jābir says, may be obtained from the last part of <ii>. As for <iv> Jābir does not prove it, although <iv> (together with the statement that the line joining the poles is perpendicular to the circle) is carefully proved as Theorem 11 of the Arabic version.

> [5a, 7] [Proposition 3 (= *Sph.* I, 15/Arabic I,16[19])] If a great circle passes through the pole of a circle on a sphere[20] then it divides it into two halves and it stands on the surface [of the circle] at right angles.[21] (Fig. 3)

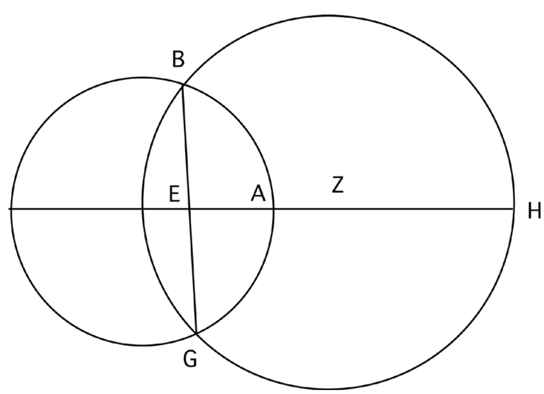

Figure 3. Proposition 3.

19 Theorem I, 9 of the Arabic text is not found in the Greek, so from I, 10 onward the Arabic numbers are one greater than the Greek.

20 The Arabic version changes this, unnecessarily to 'If a great circle in a sphere cuts one of the circles in the sphere and passes through its poles...." Jābir agrees with the Greek version.

21 The Arabic version has 'and cuts it at right angles,' although it often uses Jābir's phrase as well.

The main difference between Jābir's proof (which identifies the circle as ABG and that in the Arabic version, which identifies it as ABGD) is that whereas the latter begins by joining the poles with a straight line, Jābir's proof produces that line by joining the center of the sphere to the center of the circle, which line – by previous results – when extended in both directions passes through the poles. Both make implicit reference to *El.* XI, 18 when they argue that the great circle is perpendicular to the given circle because it contains a perpendicular to that circle. But, the Arabic version then obtains the result that the great circle bisects the given circle by Arabic I, 14, which says that when a great circle on the sphere is perpendicular to a given circle it bisects it.[22] Jābir, by introducing the line connecting the poles as a line going through the centers of the sphere and the circle knows immediately that the great circle passes through the center of the circle and, so, being a diameter, must bisect it.[23]

> [5a, 20] [Proposition 4 (= *Sph.* I, 13/Arabic I,14)] And, analogously, we will show the converse, that the surface of any great circle perpendicular to the surface of circle ABG passes through its poles.

In the Arabic version the conclusion reads "divides it into halves and passes through its poles."

> The proof is that if we produce from the center of the sphere a perpendicular to the common section of the great circle and the circle ABG, i.e. to the line BG, and it is the line ZE, it will be perpendicular to the plane of circle ABG,[24] and so it will pass through its center. And if it is extended in both directions, so that it meets the surface of the sphere, then it passes through its two poles, as we proved. And this line, EZ, is in the plane of the great circle.

Propositions from Spherics, II

In the Arabic version of the *Spherics* the following proposition is stated without reference to the converse or the statement dealing with the case when it is known only that a great circle passes through the poles of one of the circles.

> [5b, 2][Proposition 5 (=*Sph.* II, 9)] If two circles intersect each other on a sphere and a great circle passes through their poles then it divides the intersecting arcs of those two circles into halves. And the converse is also true. And, like that, if it [the great circle] passes through the pole of one of the two and it cuts the second into two halves then it also passes through its pole.

Jābir's proof uses the same basic idea as that in the Arabic version but has some variations. It was certainly not copied from the Arabic, so we reproduce it here although it is not so clear as the Arabic.

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22 As proof of this latter statement Proposition 14 offers no more than the fact that the figure in question is a semicircle.

23 That a diameter of a circle bisects it is stated in the definition of "diameter" in *El.* I, Def. 17, although as Heath 1956, Vol. I, p. 186 points out (citing Simson) it may be proved from propositions in *El.* III.

24 The text in the Arabic version, for justification, argues that since the great circle and the circle are at right angles to each other and from a point in the great circle a line is constructed in its plane perpendicular to the common section it follows that that line is perpendicular to the circle. This is no doubt true, but the editors in the mathematical notes refer to *El.* XI, 4, which deals with a different situation.

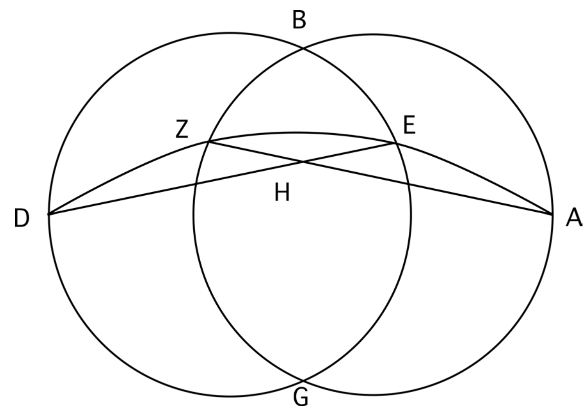

Figure 4. Proposition 5.

So, let the two circles AGB and /5/ GDB intersect each other on a sphere, and let the great circle AEZD pass through their poles. I say it divides arcs BAG, BZG, BEG[25] and BDG into two halves.

Its Proof (Fig. 4): We make line BG the common section between the two circles ABG, BDG and line AZ the common section between the great circle and circle ABG and the common section between it and circle BDG and line ED.

And so, because the great circle AEZD passes through the two poles of the two circles it will pass through their centers and the center of the sphere. And it will divide /10/ the common section between the two circles, the line BG, at a point H in the surface of the great circle,

And because the point H is on the line BG, which is the common section to the two circles ABG, GBD, it is in each of their two surfaces. And so the point H is on the intersection of the two common sections between the two of them and the great circle.

And because of that the intersection of the two lines AZ, ED is necessarily at point H of the line BG.

Also, because the great circle passes through the poles of the two circles, each of the two of them will stand on it at a right angle, and it cuts /15/ each of them into halves [*Sph.* I, 6].

And [so] their common section, line BG, is perpendicular to the great circle, [*El.* XI, 19]. And so it is perpendicular to each of the two lines AZ, ED, which are diameters of the two circles.

And so, for that reason, arc AB is equal to arc AG and, similarly, arc GD is like arc BA and arc GE is like arc BE and arc GZ is like arc ZB,[26] which is what we wanted /20/ to prove.

And the converse of that is proved by reversing the argument and by an easy demonstration (*bi-bay*īnat *qarīb*). And like that it will also be clear that if the great circle passes through the pole of one of the two [circles] and divides the arc of the other circle into halves then it passes through its two poles also. The proof is finished.[27]

The following theorem is the first reference to parallelism of circles. Its first statement appears as Theorem II, 2 in the Arabic, and its second as Theorem II, 1. However, Jābir begins with the converse and states both parts before beginning the proof of either.

25 Text has "H"

26 One notes Jābir's use of two quite different words in this argument for the same concept: *musāwī* (equal) and *mithl* (like).

27 This statement of the converse is missing in the Arabic version.

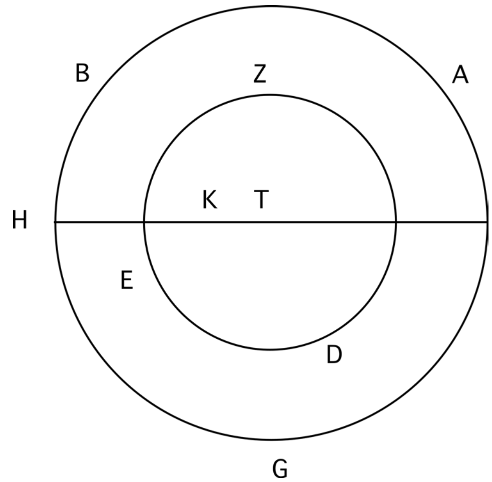

Figure 5. Proposition 6.

[5b, 24] [Proposition 6 (=*Sph.* II, 2)] Circles drawn around a single pole are parallel to each other. And if circles are parallel they are drawn around a single pole.

Proof (Fig. 5): We join the center of the sphere, point T, with the pole of the two circles, point H, with the line HT, and it will pass through the centers of the two circles and be perpendicular to their surfaces. And if a single line is perpendicular to two [plane] surfaces they are mutually parallel [*El.* XI, 14]. And so the surfaces of the two circles are parallel. End of proof.

And [*Sph.* II, 1 and 2] if the two circles are parallel then their poles are one and the same.

Proof: We join the center of the sphere, point T, with point K, the center of circle ABG,[28] and we extend it to the surface of the second circle and to the surface [*basīṭ*] of the sphere,[29] point H. And line TK is perpendicular to both surfaces.[30] And so it is perpendicular to the surface of circle DEZ, so it will pass through its center.

And since a line passes through the center of the sphere and the center of a circle drawn on the sphere it will also pass through its poles. And so line TK passes through the pole of the circle DEZ. But it also passes through the pole of circle ABG, and so its pole is one point, and that is point H.

End of proof.

Jābir, having now shown that circles being parallel is equivalent to their having the same poles, now has what he needs to prove the next theorem.

[6a, 12] [Proposition 7 (=*Sph.* II, 10[31])] Great circles passing through the two poles of parallel circles cut off similar arcs from them in the surface between them.

28 The proof in the *Spherics* begins with a line passing through the poles of circle ABG, and the diagram there does not indicate the center of the sphere.

29 This extension to the sphere is clearly indicated in the Greek and Arabic diagrams, but in Jābir the extension stops at the larger circle. And, indeed, Jābir calls H, the point of intersection of the line joining the center of the sphere and the pole with the larger circle, "the pole."

30 Any line from the center of the sphere to the center of a circle on that sphere is perpendicular to the circle. So TK is perpendicular to circle ABG. And because the two circles are parallel TK is also perpendicular to DEZ And so it passes through its center,

31 II, 10, in both the Greek and Arabic, also adds that the arcs of the great circles between the two parallels are equal.

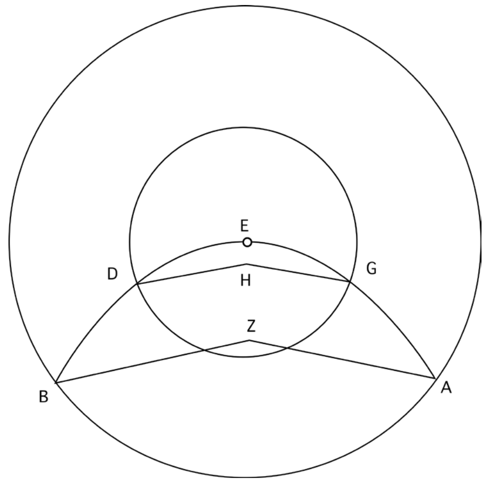

Figure 6. Proposition 7.

Jābir's proof (Fig. 6) has some gaps, which the longer version in the *Spherics* supplies with a step-by-step construction of the centers of the parallel circles.

> Let there be on a sphere two parallel circles, AB and GD, and through their two poles, the point E, let there pass two great circles, AGE and BDE. Then I say that the arcs AB, GD of the two parallel circles are similar.
>
> Proof: We make the center of [circle] AB the point Z and the center of circle GD the point H, and we join point Z with the two points A, B by the two lines AZ, BZ. And we also join point H with the two points G, D with the lines GH, DH. And because the two circles AB, GD are parallel, and the two circles AGE and BDE cut them, the common sections are parallel [*El.* XI,16[32]] and so line AZ is parallel to line GH and likewise, BZ is parallel to DH. So [*El.* XI, 10] angle AZB is equal to angle GHD, and, so, arc AB is similar to arc GD, which is what we wanted to prove.

> [6a, 24] [Proposition 8 (= part of *Sph.* II, 19, Arabic II, 18)[33]] Let there be on a sphere two parallel and equal circles ABGD, EZHT and let a great circle AQSE that does not pass through their poles cut them. (Fig. 7)
>
> Let the two common sections with the two circles ABGD and EZHT be the two lines AG, EH. /27/ Then I say that it [the great circle] cuts each one of them [the parallel circles] into unequal sections and that their alternate segments /6b, 1/ are equal. I mean that segment ABG is equal to segment ETH and the segment ADG is equal /2/ to segment EZH. And the great circle parallel to the two circles ABGD, EZHT in what is between them cuts arcs AE and GH from circle AQSE into halves.

Proposition 8 is proved very differently in *Sph.* II, 19 by use of *Sph.* II, 18, which says that if a great circle cuts equal and parallel circles, then the arcs of that great circle between the great circle parallel to the two parallel circles and the two parallel circles are equal.

32 Because El. XI, 16 requires that the lines GH and EZ lie in the same plane, but for that Jābir needs to show – as the *Spherics* does – that H lies in the plane of the great circle EA. Similarly for DH and BZ.

33 The Greek and Arabic version, however, state the theorem for an arbitrary number of parallel circles and state that, of their segments cut off in one of the hemispheres formed by the greatest of the parallels, those between the greatest of the parallels and the visible pole are larger than semicircles and the remaining [between the greatest of the parallels and the hidden pole] are smaller.

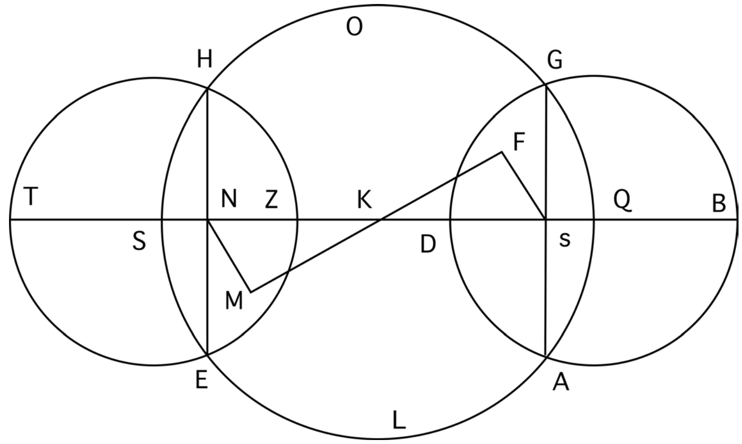

Figure 7. Proposition 8.

The following is Jābir's proof of the result.

Proof: /4/ We make the common section between the great circle AQSE and the great circle that passes over its /5/ two poles and the two poles of the parallel circles the line QsN and the common section between it [AQSE] and the greatest of the parallel circles the line LKO. Let K be the center of the sphere and the centers of /7/ the two parallel circles the points F and M. And let us join them with the center of the sphere by the two lines FK, MK. And we join the two lines, Fs and MN.

Because the centers of the two parallel circles ABG, EZH are joined with the center of the sphere by the two lines KF and KM, each of them is perpendicular to the surface of the two circles. And because the two circles are parallel, the two lines FK and KM form a continuous straight line. And because the two circles are equal to each other the two lines FK and MK that join their centers and the center of the sphere are equal. And each of the two angles, F and M, is a right angle. And the two [vertical] angles at K are equal, so [*El.* VI, 4] the two triangles FKs[34] and KNM are similar. And, because their two sides FK, KM /14/ are equal the two sides Ks and KN are equal. But they are perpendicular to /15/ the two lines AG, EH. And for that reason the [two] chord[s], AG and EH, are equal [*El.* III, 14]. And they are in two equal circles, so [*El.* 28 & 24] segment ABG is equal to the alternate segment ETH. And, likewise, segment ADG is equal to the alternate segment EZH. And because the two lines AG, EH are equal and are parallel to the diameter LO, the arcs AL, LE, GO, [O]H are equal to each other. And this is what we wanted.

[6b, 24] [Proposition 9 (=*Sph.* II, 11 & 12)[35]] (Fig. 8) If a segment of a circle is set up at right angles on a diameter of a circle ABG, say segment ADG, and on its circumference an arbitrary[36] point, D, is marked, and from the circumference of circle ABG in both directions from the point G, two equal arcs are cut off, BG, GE, and two lines, DE and DB, are joined then I say that

34 "F" is missing in the text but is supplied in the margin.

35 *Spherics* II, 11 & 12 state Proposition 9 and its converse for the case of segments erected on the diameters of two equal circles. Proposition 9 is, of course, an immediate consequence of II, 11 & 12.

36 Instead of "arbitrary" (*kayfa mā waqaʿa*) the Arabic version requires that the points marked on the two equal segments divide the arcs so that the part nearer the end of the diameter are less than the other part. Jābir makes the same requirement in his statement of the converse.

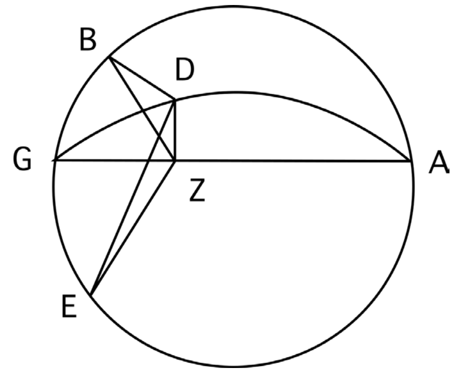

Figure 8. Proposition 9.

they are equal, and (conversely) if they [the lines] are equal and the point D divides the arc ADB into two unequal parts then the arcs GB, GE are equal.

And, conversely, the proof will be evident that if BD and DE are equal and point D divides arc AGD into unequal parts, then the two arcs BG and GE will be equal. And for the very same reason it will be necessary when there are set up on the diameters of equal circles segments of equal circles. And that is what we sought to prove.

And, now that that has been established, let us establish a necessary premise, namely:

[The Rule of Four Quantities][37]

[7a, 14] If there are two great circles on a sphere, [neither passing through the pole of the other] and if two points are marked on the circumference of one of them, or one point on the circumference of each of them, however it may fall, and from each of the two points is produced an arc of a great circle which contains with the arc of the second circle a right angle then the ratio of the Sine of the arc that is between one of the two points and one of the two points of intersection[38] to the Sine of the arc produced from that point to the second circle is as the ratio of the Sine of the arc that is between the second and between the other of the two points of the intersection to the Sine of the arc produced from that point to the second circle.

So, let the two circles AGDB and AEZB be great circles on the sphere and first of all let there be marked on the circumference of circle ABGD two points G, D and let there be produced from them to the circumference of circle AEZ at two right angles [arcs GE and DZ]. Then I say that the ratio of the Sine of arc AG to the Sine of arc GE is as the ratio of the Sine of arc AD to the Sine of arc DZ.[39]

37 Jābir has no special name for this theorem. That Ibn Muʿādh skips it, and goes directly from the Sector Theorem (which Jābir skips) to the Sine Theorem is one more indication of Jābir's independence from Ibn Muʿādh's work.

38 This refers to point A in the diagram.

39 Kūshyār in his *Zīj al-jāmīʿ* states what is a special case of this theorem when arc AD in Figure 9 is a quadrant. He then states, as a corollary, the general case that Jābir and Abū al-Wafāʾ in his *Zīj al-majisṭī* state.

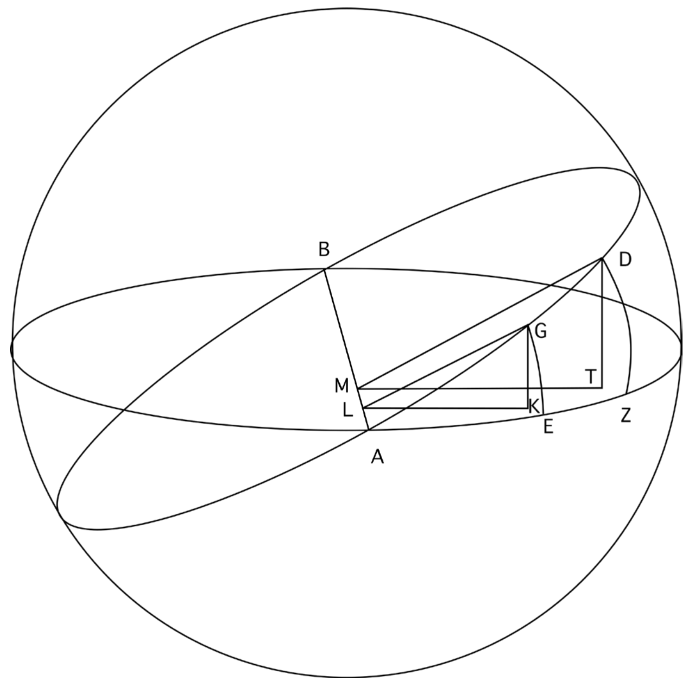

Figure 9. The Rule of Four Quantities.

The idea is[40] that the perpendiculars, DT and GK, from D and G onto the surface of circle AEZ, lie in the great circles containing arcs DZ and GE respectively.[41] Then, if we draw the perpendiculars, DM and GL, from D and G onto the diameter AB, and draw lines TM and KL we have created two right triangles, DMT and GLK, with MT parallel to LK and KG parallel to DT. Hence the two angles KGL and TDM are equal.[42] So the two triangles are similar. Thus GL:GK = DM:DT. But GL = Sin(arcGA) and GK = Sin(arc GE). Also, line DM = Sin(arc AD) and DT = Sin(arc DZ). The conclusion of the first part of the theorem follows.

Jābir now reduces the proof of the second part, when the two points are on opposite sides of A, to the first part.[43] (Fig. 10)

> [7b, 6] Let the point N be marked on the circumference of the circle AEZ and let there be produced from it an arc of a great circle that contains with the arc of circle AGB a right angle, namely arc *Ns*, and let angle *s* be right. Then I say that the ratio of Sine of arc AG to Sine of arc GE equals the ratio of the Sine of arc BN to [Sine] of arc Ns.

---

40 What follows is a summary of the author's translation in Berggren 2016, pp. 540-43, which also contains Jābir's proof of the Sine Theorem. Jābir's use of these theorems to prove two rules about spherical triangles with only one right angle may be found on pp. 543-44 . (The figure in Berggren 2016 is inexact since it gives the impression that the perpendiculars DT and GK do not lie in radii to Z and E.)

41 Both Abū al-Wafā' (Berggren 2007, 621-3) and Kūshyār (Debarnot 1985, 142-4) describe the perpendiculars DT and GK as perpendiculars from D and G onto the radii from the center of the sphere to the points Z and E respectively. Mathematically, of course, Jābir's approach comes to the same thing.

42 Kūshyār, too, argues that these two angles are equal, but Abū al-Wafā' argues that the other pair of acute angles in these two triangles are equal.

43 Kūshyār proves this, as a corollary to his 'Rule of Four Quantities' by regarding each of the two triangles AGE and AsN in Fig. 10 as being contained in a larger right triangle whose hypotenuse is a quadrant.

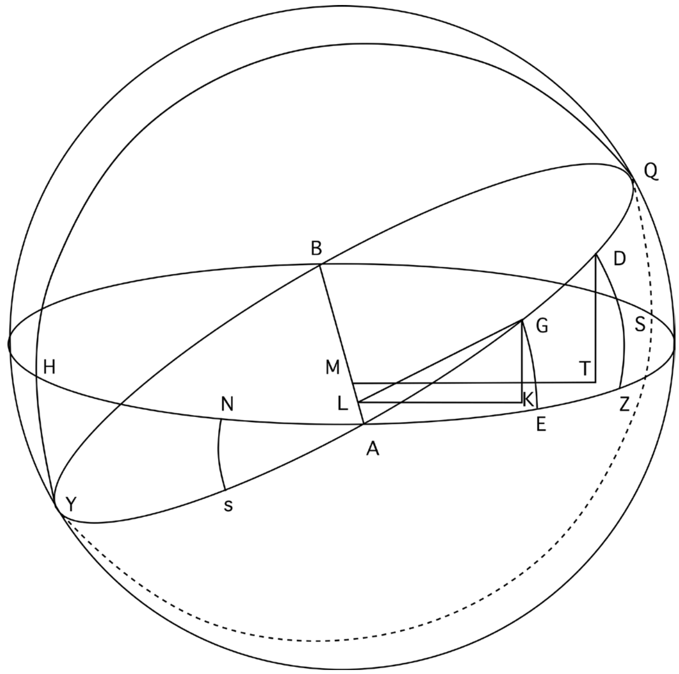

Figure 10. Second part of the Rule of Four Quantities.

Jābir then specifies that with A as a pole one makes arc AQ a quadrant and draws the great circle YHQ through Q intersecting AGB at Y and AEB at H.

The argument then runs as follows:[44]

Sin(AN):Sin(Ns) = Sin(AH):Sin(HY)

by the first part of the Rule applied to triangles ANs and AHY. By the same Rule applied to triangles AGE and AQS it follows that

Sin(AG):Sin(GE) = Sin(AQ):Sin(QS)

But AH = AQ because both arcs are quadrants. And if we remove HQ, the common part, from the two equal semicircles YHQ and HQS we obtain HY = QS, since. So HY = QS and AQ = AH, and therefore

Sin(AQ):Sin(QS) = Sin(AH):Sin(HY).

Hence

Sin(AG):Sin(GE) = Sin(AH):Sin(HY) = Sin(AN):Sin(Ns), which was to be proved.

Jābir now states the Sine Law for spherical triangles as follows:

> [8b, 23] Let there be a triangle ABG of arcs of great circles. Then I say that the ratio of the Sine of each of its sides to the Sine of the arc of the angle subtending it is one and the same ratio.

With reference to Fig. 11, Jābir first takes the case where the triangle has at least one right angle, say B. He extends arc GB to a quadrant, GE, and then considers the great circle through E and

44 Since all pairs XY in this proof denote arcs we shall abbreviate "arc XY" as "XY" and Sine as "Sin."

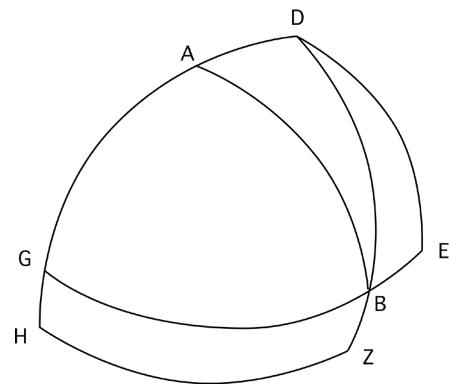

Figure 11. Sine law for spherical triangles.

the pole of GE. He then extends GA to meet the aforesaid great circle at D and then applies the Rule of Four to the two spherical right triangles, GBA and GED, with right angles at B and E and a common angle at G. After straightforward manipulations of proportions he arrives at the result that the ratio of the Sine of side AG to the Sine of the arc of the angle subtending it, B, is as the ratio of the Sine of side AB to the Sine of the arc of the angle subtending it, G.

With a similar construction involving constructing the quadrant AZ and the great circle arc HZ he is able to show that the ratio of the Sine of the side /13/ AG to the Sine of the arc of angle B, which it subtends, is as the ratio of the Sine of side BG to the Sine of the arc of angle A which subtends it.

He next deals with the case in which the triangle has no right angles by taking two cases, the first in which the perpendicular, AD, from A onto arc BG falls between B and G and the other case in which it falls on an extension of the arc BG, i.e. one of angles B or G is obtuse. With this case he concludes his proof of the Sine Law.[45]

Jābir then uses the Rule of Four Quantities to demonstrate two more rules for spherical trigonometry pertaining to a spherical triangle with a single right angle:

> 3. The ratio of the Sine of the side subtending the right angle to the Sine of one of the sides containing it is as the ratio of the Sine of the complement of the other containing side to the Sine of complement of the arc of the angle subtending it.[46]
>
> 4. Further, the Sine of the complement of the side subtending the right (angle) to the Sine of the complement of one of the two containing it is as the ratio of the Sine of the complement of the third side to the Sine of a quadrant.[47]

Although one can use Jābir's four rules of spherical trigonometry, the last of which is known as "Geber's Theorem" (after the Latinized version of Jābir's name), to find the Sine of a particular arc or angle one could not deduce immediately from that value whether the arc or angle was greater or less than 90°. This is because, as Jābir pointed out in the definitions at the beginning

45 In the Latin edition of *Improvement* published by Peter Apian in 1533 the proof of the Sine Theorem begins with the special cases of spherical triangles with three or two right angles (Lorch 1995, Item VIII, p. 6).

46 The "Sine of the complement" corresponds to the modern cosine.

47 More details on the proof than we have included in our overview may be found in the paper of Lorch, pp. 6 – 8, cited earlier. In particular, the manuscript that Lorch took as the basis for his account of Jābir's trigonometry includes, in the proof of the Sine Law, separate proofs for the case in which the spherical triangle has three or two right angles.

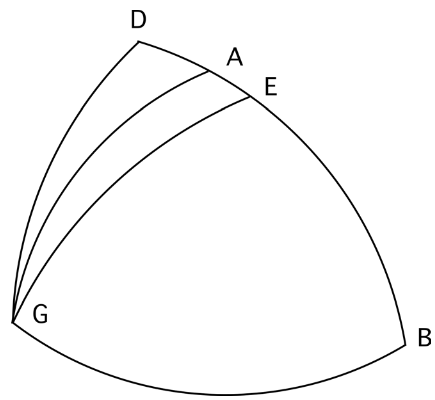

Figure 12. Proposition 1.

of this book, for any given acute angle, Θ, Sin(Θ) = Sin (180°- Θ). Thus, an astronomer would have to have some way of knowing whether a given angle should be the acute or its supplement, and whether a given arc should be less than 90° or the supplement of that arc.

To this end, Jābir followed the above theorems with a section "Some remaining properties of right triangles." The first propositions in this section deal with the two sides, AB and BG, containing the right angle, B, of a spherical triangle ABG. We have numbered the propositions 1 – 5 to aid in referring to them later. We refer to the angles as A, B, G and the sides opposite them as *a, b g*, the side opposite the right angle being *b*. The first theorem states a simple relation between the sides containing the right angle and the angles opposite them, and is tacitly referred to in each of the following three.

> [9a, 1] *Proposition 1.* Either of its two sides containing the right angle, together with the angle subtending it, follow one another. That is, if the side is equal to a quadrant the angle subtending it is right, if it is greater than a quadrant it [the angle] is greater than a right angle, and if it is less it [the angle] is [also] less. And, similarly, the side also follows the angle.
>
> So let the triangle be triangle ABG and let its angle B be right. Then I say that side AB and angle G /5/ subtending it follow one another that is if side AB is equal to a quadrant then angle G subtending it is right, and if it is greater than a quadrant it [the angle] is greater than a right, and, if is less then it, it is less than a right. And likewise, the side also follows the angle. And similarly for side BG with the angle A subtending it.
>
> Proof: If side AB is equal to a quadrant then the point A is a pole of arc BG and so angle G is [also] right. And if it [AB] is greater than a quadrant we cut off from it a quadrant, arc BE, [and] the point E is a pole of arc BG. And so the arc of the great circle passing over the two points, E and G, contains, with arc BG. a right angle. So angle EGB is right and so angle AGB is greater than right. And like that it is proved that if arc AB is less than a quadrant then angle AGB is less than a right angle. And in the same way it is also proved that the very same occurs for side BG and angle A. And that is what we wanted to prove.

The remaining theorems deal with how various conditions on *A, G, a, g* affect *b*, the side opposite the right angle. The following is a summary in modern notation. The statements are a bit involved because Jābir felt it necessary to remind the reader in each one of them that conditions on *a* or *g* could be replaced by conditions on A or G.

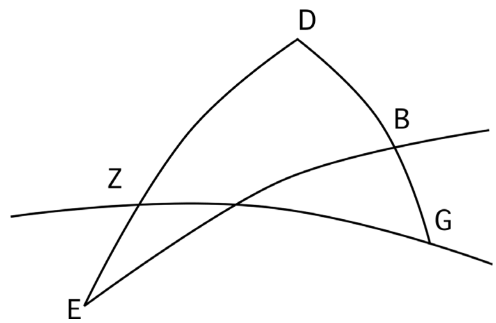

Figure 13. Proposition 3.

Proposition 2

If *a*/*g* is a quadrant or if A/G = 90° then *b* is a quadrant.

Proposition 3

If *g* and *a* are both > or < quadrants or A and G are both greater than or less than 90° then *b* is a quadrant

Proposition 4

If *a* > a quadrant and *g* < a quadrant and one of A or B > 90° and the other is less

Then *b* > 90° and conversely

Proposition 5

If *g* is a quadrant or G = 90° then A is a pole of *a* and *b* is a quadrant.[48]

As one more example of the proofs in this section we give Jābir's statement and proof of the part of Proposition 3 dealing with the case when sides *a* and *g* are both less than quadrants. (We have supplied the necessary figure (Fig. 13) which is lacking in the manuscript.

> I also say that... if each of the two sides containing it [the right angle] is less than or greater than a quadrant, or if each of the two remaining angles is less than or greater than a right [angle] the [side] subtending the right [angle] is less than a quadrant.
>
> If each of the two sides, AB and BG, is less than a quadrant and we make each of GD, BE a quadrant, then the arc of the great circle passing through the two points E, D, namely arc EZD, will cut circle AG beyond point A and its section is, say, at point Z. And because angle B is right and arc BE is a quadrant the point E is a pole of arc DG. So angle D is right. And because

48 Given Propositions 1 (that angles follow sides) and 2 the only thing this proposition adds is that A is a pole.

arc GD is a quadrant point G is a pole of arc ED and so GZ is a quadrant and so arc AG is less than a quadrant.

The following three propositions deal with how the two sides of a spherical triangle ABG containing the right angle, B, depend on side *b*. The first of them is the converse of the second proposition above, the second is the converse of the third proposition above, and the last is the converse of the last.

1. [9b, 23] [Proposition] Let us make the side AG subtending the right [angle] a quadrant. Then one of the two sides, AB and BG, is a quadrant.

Proof: If neither of the two sides, AB, BG is a quadrant then each of the two of them will either be less than a quadrant or greater than a quadrant and the other of the two will be greater than a quadrant or less. So, it follows from what we proved in the preceding, that AG is either greater than a quadrant or less. And that is a contradiction.

2. If side AG [10a, 1] subtending the right angle is smaller than a quadrant then the two sides, AB and BG, are either both larger or both smaller than a quadrant. Its proof is that if the two are not like that then one of them is greater and the second smaller or one of them is a quadrant and the second is smaller. And so it is necessary from what we have proved [earlier] that side AG is [then] greater than a quadrant of a circle. But it was assumed smaller, which is a contradiction, not possible.

And likewise, also, if one of the two is a quadrant it is necessary that side AG is a quadrant according to what we proved /5/ in the preceding, and so it is a contradiction that one of the two is greater and the second is less, or that one of the two of them is a quadrant. And so they will follow each other, i.e. that if one of the two is less than or greater than a quadrant then the second will be like it.

3. And, if the side subtending the right is greater than a quadrant then the two containing the right differ from each other, i.e. one of them is larger than a quadrant and the other is smaller.

Proof: If it is not so, then let the two of them follow one another, so both of them are either greater or less than a quadrant, as it is for when that side AG is less than a quadrant.[49] But it was postulated to be larger. This contradiction is impossible, so the two of them differ one from the other.

/10/ And like that it is also necessary that neither of them is a quadrant because if one of the two of them were a quadrant it is necessary that the one subtending the right is a quadrant according to what has been proved. And this contradiction is impossible.

/12/ So for that reason it is necessary that one of the two of them is larger than a quadrant and the second is less than a quadrant.

And the statement of the two angles following [the sides they subtend] is the same as the statement of the two sides subtending them. And let it be necessary that the [side] subtending the right is a quadrant or that one of the two remaining angles is right and that it [the side] is less than a quadrant. [Then] each one of them [the angles] is either less than or

49 By the third result above.

greater than a right. And if it is larger than a quadrant [then] one of them is larger than a right and the second is less than a right, and that is what we wanted to prove.

Jābir now leaves spherics and turns to a proof of the latter part of a remark that Ptolemy makes near the end of *Alm*agest I, 3 that "the circle is greater than [all other] surfaces and the sphere greater than [all other] solids." He writes:

/10a, 21/ Among the necessary premises we will prove that the measure of its [a sphere's] body is greater than the measure of every solid of equal surface whose surface is equal to the surface of that sphere.[50]

And it will be proved approximately (*min qurb*) if we prove that the measure of the body of a sphere is equal to[51] the product of half of its diameter by one-third of its surface.[52]

We omit Jābir's proof of the result and go to his conclusion of this section.

/10b, 27/ And now that we have proved this it will be clear that the volume of each sphere [11a,1] is greater than any polyhedral solid whose surface is equal to the surface of the sphere.

The proof of this statement ends at [11a, 13] and we resume our translation at that point where Jābir returns to the topic of spherics (Fig. 14).

[11a, 14] And among the necessary premises also are two cases: Let there be arcs AB, AG of two great circles which contain an angle, A, less than a right, and an arc DBG of a great circle passes through their poles. And let point D be the pole of arc AG. We produce a mean proportional in the ratio of the Sine of arc GBD, which is a quadrant of a circle, to the Sine of the arc DB[53] /20/ and it is the Sine of the arc DE, and we extend it to point L on arc AG. Then point E is the point at which there is the greatest difference between each of the two arcs cut off from arcs AB, AG, which correspond to arcs AE, AL.

So, let us mark two points, Z and H, on the two sides of point E. And let there pass over them and the pole D the two arcs DZT and DHK. Then I say that the difference between the two arcs AE, AL is greater than the differences between the two arcs AZ, AT and between AH, AK.

Proof. [We paraphrase Jābir's proof but use modern symbolism.[54]]

50 Ptolemy states this proposition without proof (and somewhat vaguely) near the end of *Alm*agest I, 4 (Toomer, p. 40.). It is one of a group of theorems on isoperimetry proved by the mathematician Zenodorus early in the second century B.C. (On Zenodorus see Toomer 1972)

51 The MS is somewhat blurred at this point, but instead of "equal to" it seems to say "greater than," which is both untrue and inconsistent with the rest of the text.

52 If *V, S* and *d* are, respectively, the volume, surface area and diameter of a sphere, then Jābir's statement, which we may express as $V = (d/2)(S/3)$ is equivalent to Archimedes' theorem in *Sphere and Cylinder* I, 34 that any sphere is equal to four times the cone which has as its base the greatest circle in the sphere and whose height is equal to the radius of the sphere. The equivalence of Jābir's statement and Archimedes' result is easily seen in light of Archimedes' Proposition *SC* I, 33 that *S* is equal to four times the area of the greatest circle in the sphere. (Archimedes' *Sphere and Cylinder* was known throughout the medieval Islamic world.)

53 The text repeats here "and it is the Sine of the arc DB."

54 When Jābir says the ratio of A to B is as the ratio of C to D we have rendered this as A/B = C/D, and "Sin" abbreviates the medieval Sine. Finally, as remarked earlier, since all magnitudes written XY denote arcs of great circles we omit the usual arc sign over them.

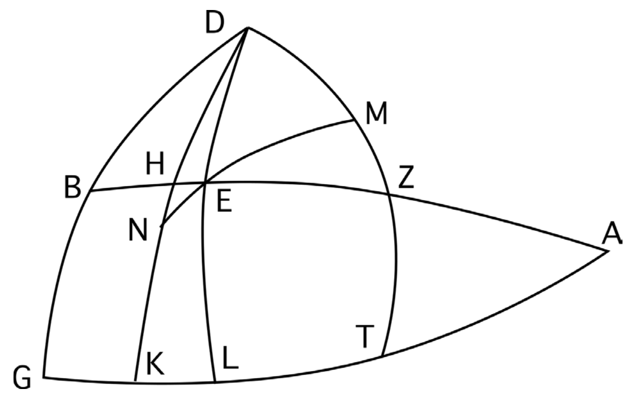

Figure 14.

By definition of E, and since DG and DL are equal, both being quadrants,

Sin(DL)/Sin(DE) = Sin(DE)/Sin(DB).

By the Rule of Four applied to triangles DEM and DLT, we also have

Sin(DL)/Sin(DE) = Sin(TL)/Sin(TM)[55]

So, *ex aequali*,

Sin(TL)/Sin(EM) = Sin(ED)/Sin(DB).

By the Rule of Four applied to triangles BDZ and MEZ, we also have

Sin(BD)/Sin(DZ) = Sin(EM)/Sin(EZ).

Hence, *ex aequali*,

Sin(TL)/Sin(EZ) = Sin(ED)/Sin(DZ).

But, since the arcs are all less than a quadrant, and ED < DZ it follows that TL < EZ.
From this it follows that AE – EL > AZ – AT.

Jābir proves the other inequality by the same method, using the triangle EHN in Figure 14, which has a right angle at N and an angle at H equal to the vertex angle at H in triangle DHB.

We omit the proof of this proposition, which concludes on 12b/3, where he explains what this is all about:

As has been shown clearly from the previous theorem before this one, we need to know the ecliptic degree for which the difference, between the degrees of the ecliptic and their

55 Here one needs the condition that EM passes through the pole of DT. Theodosius shows how to construct such a great circle, but Jābir takes the existence of such things for granted.

> ascensions in the corresponding[56] sphere, attains a maximum value, for this is something needed to calculate the difference between day and night. He [Ptolemy] mentioned this directly without any proof. We have considered it adequate to give a demonstration. It is also clear from the already previously mentioned second theorem that the variation (*tafāḍul*) in the declination of the degrees of the ecliptic in respect to the equator reaches its maximum at the two points of intersection [between the ecliptic and the equator] and its minimum at the two solstices. This was also mentioned by him in the second book [of the *Almagest*] in a straightforward way, without proof, and we have also decided to demonstrate it with the purpose that nothing in his book remains without demonstration. God (may He be exalted) willing. He is our help, etc.
>
> And this is all we need to premise in order to be free of needing the sector theorem,[57] the book of Menelaus and the book of Theodosius and what was with it that he mentioned, sending it without a proof. So the book will (God, the Exalted willing) stand by itself, without reference to any other, except the book of Euclid.

Jābir concludes his Discourse with, first, an explanation of how to find the lengths of chords in a circle of given radius, beginning with an explanation of how to calculate an approximate value of Crd(1/2°). He follows very closely Ptolemy's presentation of this topic in *Almagest* I, 10.

The final section of the First Discourse in his *Improvement* contains rules showing how to use the table of chords to solve plane triangles,[58] The final case is :

> And if the three angles are known then all three sides are known, on the condition that the diameter of the circle containing the triangle is known. And that follows because each one of its angles is known and so the arcs of that circle that those sides subtend are known[59] and for that [reason] the sides are known, i.e. the ratio of each one of them to the diameter of the circle is known. And for that reason it is necessary that the ratio of each one of them to each of the other two is known. And that is what we wanted to prove.
>
> The First Discourse, On the Premises, has finished with praise to God....

And our survey of the First Discourse of Jābir's work has finished with best wishes to Jim Evans for many more years of health and energy to inform his colleagues with his scholarship and delight his friends with his company.

---

56 *Muttaṣil* the "connected" sphere, which one understands to be the *Sphera recta* and the *Sphera obliqua*. If this interpretation is correct, he refers to the computation of "half the equation of daylight" (*e*), in order to obtain the longest day or night of the year, applying $e = \alpha_{\varphi}(\lambda_{\odot}) - \alpha_{0}(\lambda_{\odot})$. See *Alm.* III, 9. Also Pedersen/Jones 2011.

57 This was the usual name for it in medieval Islam and the Latin west. Today this theorem, upon which Ptolemy built his spherical trigonometry, is usually (and somewhat inaccurately) called Menelaus's Theorem.

58 Details my be found in Lorch Item VIII, pp. 31-4.

59 According to *El.* III, 20 the angle at the center of the circle is twice the angle at the circumference when they subtend the same arc.

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