Divide-and-Choose Games and the Shapley Value

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Abstract

We establish an equivalence between two classic approaches to fairness: the Shapley value and divide-and-choose. We study the allocation of a single indivisible item among N players with heterogeneous values, where the player receiving the item compensates the others. On the cooperative side, we introduce sharing games, in which players are partitioned into syndicates and a coalition's value is determined by the highest-valued player accessible through its members' syndicates. On the procedural side, we introduce an N-player divide-and-choose game inspired by Kuhn's classic cake-cutting procedure. Our main result links the two approaches: for every ordering of proposers in the divide-and-choose game, there is a unique assortative syndicate structure for which the Shapley value of the corresponding sharing game coincides with the payoffs under maxmin-perfect play. Conversely, the Shapley value of every assortative sharing game can be

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implemented in a divide-and-choose game with an appropriate ordering of proposers. The equivalence provides a procedural foundation for the Shapley value and a cooperative interpretation of fairness in divide-and-choose.

1 Introduction

The Shapley value and divide-and-choose are two of the best-known benchmarks for fairness in game theory, yet they originate from different traditions. The Shapley value is an axiomatic solution concept for cooperative games, while divide-and-choose is a strategic game that has been taken to be procedurally fair since biblical times. One begins with axioms characterizing fair allocations, while in the other the participants' actions determine an allocation. This paper introduces a new class of cooperative games, called sharing games, and relates the Shapley values of sharing games to the outcomes of divide-and-choose games.

We study the problem of fairly allocating an indivisible item to one player when several players have equal claims. Such problems arise in partnership dissolution, inheritance disputes, family businesses, jointly owned intellectual property, bidding rings, and common-property assets. Efficiency requires assigning the item to the individual who values it the most, while fairness requires compensating those who relinquish their claim. The central question is not simply who receives the item, but how the accompanying transfers should be determined.

The paper makes two contributions, connected by an equivalence theorem. The first is the introduction of a new family of cooperative games, called sharing games, and a characterization of their Shapley values. In a sharing game, each player belongs to a network of other players called a syndicate.[1] The value to a coalition S of the item is the highest value of any player who belongs to the same syndicate as some member of S. In other

[1] Each player belongs to only one syndicate.

words, if the item is allocated to S, then the members of S can sell the item to the individual in their networks/syndicates who has the highest value for the item. In our results, what we call "assortative" syndicates will play a special role: a syndicate is assortative if, whenever two players belong to the same syndicate, then every other player whose value is between the values of these two players belongs to the same syndicate.

Sharing games with syndicates are motivated by environments in which value depends on networks as well as individuals. In many economic settings, ownership may ultimately be transferred within a family, partnership, consortium, bidding ring, or alliance. Consequently, the value generated by assigning an item to a coalition depends on the network that members of the coalition can access. Sharing games capture this feature by partitioning players into syndicates and defining coalition values through accessible networks. We derive a closed-form expression for the Shapley value of sharing games. We show that these games are concave, implying that their anticores are nonempty and contain the Shapley value. Accordingly, sharing games provide an analytically tractable class of games whose Shapley values are a natural fairness benchmark in the presence of networks.

The second contribution is the introduction of a family of N-player divide-and-choose games and a characterization of their outcomes under maxmin-perfect play. The divide-and-choose game we study is inspired by Kuhn's classic procedure for dividing a divisible cake among N players. Our game, in contrast to Kuhn's, allocates an indivisible item. In subgames in which only two players remain, our game reduces to the well-known Texas Shootout. We show for every divide-and-choose game that a player can guarantee themselves a payoff of at least $1/N$-th of their standalone value for the item, regardless of the actions of the other players. Furthermore, each player has a unique maxmin perfect strategy for every divide-and-choose game.

The main theorem establishes a close connection between Shapley and maxmin-perfect outcomes. Specifically, for each ordering of proposers in a divide-and-choose game, there is a unique syndicate and sharing game such that the maxmin-perfect outcome of the divide-and-choose game and the Shapley outcome of the sharing game coincide.[2] Conversely, for every assortative syndicate of a sharing game, there is an ordering of proposers in the divide-and-choose game such that the Shapley outcome of the sharing game and the maxmin perfect outcome of the divide-and-choose game coincide. This result establishes a bridge between cooperative and procedural notions of fairness and changes how both objects should be viewed. From the cooperative perspective, it provides a procedural foundation for the Shapley value. From the procedural perspective, it explains why the transfers generated by the mechanism possess a compelling cooperative interpretation.

1.1 Related Literature

This paper connects two classic approaches to fair division. The first is the cooperative tradition represented by the Shapley value, which is derived from axioms governing the division of surplus. The second is the tradition of procedural fairness exemplified by divide-and-choose. Both have long served as benchmark notions of fairness, and understanding their relationship has been in an important research program. Our contribution is to establish a novel connection between these two approaches.

THE SHAPLEY VALUE

The Shapley value (Shapley (1953)) is a central solution concept in cooperative game theory and the most influential axiomatic notion of fairness. Moulin (1992) provides an important application of the Shapley value to fair division with monetary transfers, showing that the Shapley value of an appropriately defined cooperative game satisfies desirable fairness properties in allocation problems involving money. Thus, while divide-and-choose is procedurely fair, Moulin's analysis provides an axiomatic justification for the use of the Shapley value as a fair allocation rule. Our paper complements

[2] The syndicate will be assortative.

Moulin (1992). Whereas Moulin begins with a cooperative game and establishes fairness properties of its Shapley value, we establish that the Shapley values of a family of cooperative games are fair since they coincide with the allocations generated by divide-and-choose games.

de Clippel and Serrano (2008) study value allocation in games with externalities, thereby extending the cooperative framework beyond the classical characteristic-function setting. This contribution highlights the continued role of the Shapley value as a benchmark for fairness in increasingly rich allocation problems. Our sharing games contribute to this literature by introducing a network-based cooperative environment in which coalition values are determined by the syndicates accessible to coalition members.

The Nash program provides non-cooperative foundations for the Shapley value by constructing games whose equilibrium outcome is the Shapley allocation (e.g., Gul (1989), Pérez-Castrillo and Wettstein (2001)). We show when players follow maxmin perfect strategies in divide-and-choose games the Shapley outcomes of sharing games are obtained.

Divide-and-Choose

The mathematical study of divide-and-choose began with Steinhaus (1948), who analyzed the classic two-person procedure and proposed generalizations to three and more participants. Dubins and Spanier (1961) introduced a dynamic N-person cake-cutting algorithm that guarantees each participant a proportional share, while Kuhn (1967) developed an alternative N-player divide-and-choose mechanism based on the Frobenius–König theorem. Subsequent work on cake cutting developed a variety of algorithms that achieve fairness benchmarks such as proportionality and envy-freeness (see, for example, Young (1994), Brams and Taylor (1995,1996), Robertson and Webb (1998), and Su (1999)). More recent work has extended the fair-division literature to address welfare guarantees, indivisible goods with monetary transfers, and alternative notions of divisibility (Bogomolnaia and Moulin (2022, 2025) and Bei, Liu, and Lu (2025)).

Our paper studies the allocation of a single indivisible item when monetary transfers are feasible.[3] Inspired by Kuhn's procedure, we propose and study an N-player divide-and-choose game and relate it to the Shapley value.

Connecting Axiomatic and Procedural Fairness

Van Essen and Wooders (2021, 2023) relate maxmin-perfect outcomes of several auction mechanisms to Shapley values. The present paper extends this research program from auctions to divide-and-choose games, and establishes a surprising correspondence between non-cooperative divide-and-choose games and cooperative assortative sharing games.

To our knowledge, this is the first paper establishing a correspondence between divide-and-choose games and Shapley values. Our equivalence result provides a procedural foundation for Shapley values while simultaneously giving a cooperative interpretation to divide-and-choose outcomes. Rather than viewing axiomatic and procedural fairness as competing approaches, our results identify an environment in which they coincide.

2 Model Environment and the Normative Benchmarks

In this section, we introduce a family of cooperative "sharing games" for the problem of assigning a single item to one member of a group, who each have an equal claim. For each sharing game, we identify the game's Shapley value. Later, we show that there is a precise relationship between the Shapley value outcomes of sharing games and the outcomes produced by divide-and-choose under maxmin perfect play.

Cooperative Games

[3] An important application of divide-and-choose with transfers is the dissolution of partnerships. The Texas Shootout is the analogue of the classic two-player divide-and-choose game and is widely used for this purpose in practice. See McAfee (1992).

A *cooperative game with transferable utility* (TU) is a pair $(\mathcal{N}, v)$, where $\mathcal{N} = \{1, \ldots, N\}$ is the set of players and v is the characteristic function that maps each coalition $S \subset \mathcal{N}$ to its value $v(S)$, with $v(\emptyset) = 0$. In a cooperative game $(\mathcal{N}, v)$, the Shapley value is a payoff vector $(\phi_1, \ldots, \phi_n)$, where player i's Shapley payoff ϕ_i is given by

$$\phi_i = \sum_{S \subseteq \mathcal{N}} \frac{(|S| - 1)!(N - |S|)!}{N!} \left[v(S) - v(S \backslash \{i\}) \right].$$

A payoff vector $(\pi_1, \ldots, \pi_N)$ is in the *anti-core* of $(\mathcal{N}, v)$ if (i) $\sum_{i=1}^{N} \pi_i = v(\mathcal{N})$, and (ii) for every coalition $S \in 2^{\mathcal{N}}$, we have that $\sum_{i \in S} \pi_i \leq v(S)$. A payoff vector in the anti-core is (i) efficient and (ii) and is fair in the sense that no coalition S receives more than its value, i.e., no coalition receives a subsidy from the complementary coalition.

A game $(\mathcal{N}, v)$ is *concave* if for all $i \in \mathcal{N}$, we have

$$v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$$

for all $S \subseteq T \subseteq \mathcal{N} \backslash \{i\}$.

Sharing Games with Syndicates

A sharing game will be a cooperative TU game. For each $i \in \mathcal{N}$, let x_i denote the *stand-alone value* of player i, the value of the item to player i. We order the players so that

$$x_1 > x_2 > \cdots > x_N.$$

Let $\{G_1, \ldots, G_m\}$ be a partition of $\mathcal{N}$, that is, $\cup_{i=1}^{m} G_i = \mathcal{N}$, $G_i \neq \emptyset \ \forall i$, and $G_i \cap G_j = \emptyset$ for $i \neq j$. We call G_k the k^{th} *syndicate* and we call $\{G_1, \ldots, G_m\}$ the *syndicate structure*. For $k \in \{1, \ldots, m\}$, define the value of the k^{th} syndicate to be

$$g_k = \max_{j \in G_k} x_j.$$

We order the syndicates so that

$$g_1 > g_2 > \cdots > g_m.$$

A syndicate structure $\{G_1, \ldots, G_m\}$ is *assortative* if for every $i, j \in \mathcal{N}$, whenever $i < j$ and $i \in G_{k'}$ and $j \in G_{k''}$ it follows that $k' \leq k''$. In other words, if player i has a higher stand-alone value than player j, then player j can not be a member of a higher-valued syndicate than player i. Example 0 gives a syndicate structure which is not assortative.

Example 0: The syndicate structure $G_1 = \{1, 3\}$ and $G_2 = \{2\}$, where $x_1 > x_2 > x_3$, is not assortative. Player 2 has a higher stand-alone value than player 3, but player 3 is a member G_1, with $g_1 = x_1$, while player 2 is a member of G_2, with $g_2 = x_2$, and $g_1 > g_2$. Every other syndicate structure of $\{1, 2, 3\}$, where $x_1 > x_2 > x_3$, is assortative. $\square$

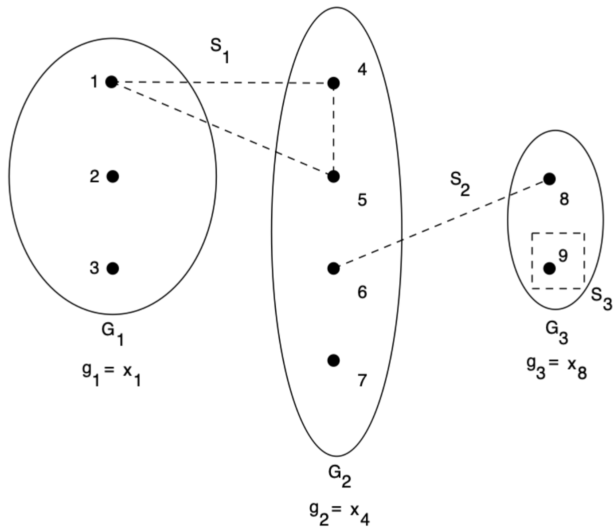
A *sharing game* is a set of players $\mathcal{N}$, stand-alone values $\{x_i\}_{i \in \mathcal{N}}$, a syndicate structure $\{G_1, \ldots, G_m\}$, and the characteristic function given by

$$v(S) = \max_{\{k | S \cap G_k \neq \emptyset\}} g_k$$

for each $S \subseteq \mathcal{N}$. In other words, the value of coalition S is the highest stand-alone value of any player $i \in \mathcal{N}$ who belongs to the same syndicate as some member of S. The economic interpretation of $v(S)$ is that if S obtains the item, then it can sell it to the player with the highest stand alone value who is in the network of S, i.e., $\cup_{\{k | S \cap G_k \neq \emptyset\}} G_k$.[4]

Example 1: Consider a sharing game in which $\mathcal{N} = \{1, \ldots, 9\}$, the stand alone values satisfy $x_1 > \ldots > x_9$, and the syndicate structure is $G_1 = \{1, 2, 3\}$, $G_2 = \{4, 5, 6, 7\}$, and $G_3 = \{8, 9\}$. Then $g_1 = x_1$, $g_2 = x_4$, and $g_3 = x_8$. For coalition $S_1 = \{1, 4, 5\}$ we have $v(S_1) = g_1$, for $S_2 = \{6, 8\}$ we have $v(S_2) = g_2$, and for $S_3 = \{9\}$ we have $v(S_3) = g_3$. Figure 1 illustrates.

[4]This is an optimistic assessment since S captures the entire value of the item.



Syndicates and Coalitions

Proposition 1 provides a simple formula for the Shapley value for sharing games with syndicates.

Proposition 1: *In the sharing game with syndicate structure* $\{G_1, \ldots, G_m\}$, *the Shapley value of player* i *in syndicate* G_j *is given by*

$$\phi_i = \begin{cases} \frac{1}{N-|\overline{G}_j|} g_j - \sum_{k=j+1}^{m} \frac{|\overline{G}_{k-1}|-|\overline{G}_k|}{\left(N-|\overline{G}_{k-1}|\right)\left(N-|\overline{G}_k|\right)} g_k & \text{if } j < m \\ \frac{1}{N} g_m & \text{if } j = m, \end{cases}$$

where $\overline{G}_k = G_{k+1} \cup \cdots \cup G_m$ for $k < m$ and $\overline{G}_m = \emptyset$.

Note that the Shapley value of a player is the same for all players who are members of the same syndicate. The following example illustrates Proposition 1 when there are three players.

Example 2: Let $\mathcal{N} = \{1, 2, 3\}$ be the set of players, with stand-alone values $x_1 > x_2 > x_3$. There are five different syndicate structures and, therefore, five different sharing games with syndicates. The Shapley values for each of these games are provided below.

1. If $G_1 = \{1\}, G_2 = \{2\}, G_3 = \{3\}$, then $g_1 = x_1$, $g_2 = x_2$, $g_3 = x_3$ and

$$\begin{aligned} \phi_1 &= g_1 - \frac{1}{2}g_2 - \frac{1}{6}g_3 \\ \phi_2 &= \frac{1}{2}g_2 - \frac{1}{6}g_3 \\ \phi_3 &= \frac{1}{3}g_3. \end{aligned}$$

2. If $G_1 = \{1\}, G_2 = \{2, 3\}$, then $g_1 = x_1$, $g_2 = x_2$ and

$$\begin{aligned} \phi_1 &= g_1 - \frac{2}{3}g_2 \\ \phi_2 &= \phi_3 = \frac{1}{3}g_2. \end{aligned}$$

3. If $G_1 = \{1, 2\}, G_2 = \{3\}$, then $g_1 = x_1$, $g_2 = x_3$ and

$$\begin{aligned} \phi_1 &= \phi_2 = \frac{1}{2}g_1 - \frac{1}{6}g_2 \\ \phi_3 &= \frac{1}{3}g_2. \end{aligned}$$

4. If $G_1 = \{1, 3\}, G_2 = \{2\}$, then $g_1 = x_1$, $g_2 = x_2$ and

$$\begin{aligned} \phi_1 &= \phi_3 = \frac{1}{2}g_1 - \frac{1}{6}g_2 \\ \phi_2 &= \frac{1}{3}g_2. \end{aligned}$$

5. If $G_1 = \{1, 2, 3\}$, then $g_1 = x_1$ and

$$\phi_1 = \phi_2 = \phi_3 = \frac{1}{3}g_1.$$

$\square$

PROPERTIES OF SHARING GAMES WITH SYNDICATES

Proposition 2 shows that sharing games with sydicates are concave games.

Proposition 2: *Every sharing game is a concave game.*

Every concave game has a non-empty anti-core which contains the Shapley value.[5] The three-player example below illustrates the anti-core and its relationship with the Shapley value.

Example 3: Consider the sharing game with players $\{1,2,3\}$, stand-alone values $x_1 = \frac{3}{4}$, $x_2 = \frac{1}{2}$, and $x_3 = \frac{1}{4}$, and syndicate structure $G_1 = \{1\}, G_2 = \{2\}$, and $G_3 = \{3\}$. The Shapley value of the game is $(\phi_1, \phi_2, \phi_3) = (\frac{11}{24}, \frac{5}{24}, \frac{2}{24})$. A payoff vector (π_1, π_2, π_3) is in the anti-core if

$$\begin{aligned}
\pi_1 + \pi_2 + \pi_3 &= \frac{3}{4} \\
\pi_1 + \pi_2 &\leq \frac{3}{4},\ \pi_1 + \pi_3 \leq \frac{3}{4},\ \pi_2 + \pi_3 \leq \frac{1}{2} \\
\pi_1 &\leq \frac{3}{4},\ \pi_2 \leq \frac{1}{2},\ \pi_3 \leq \frac{1}{4}.
\end{aligned}$$

It is easy to verify that $(\frac{11}{24}, \frac{5}{24}, \frac{2}{24})$ is in the anti-core. □

3 Divide and Choose

The previous section described a class of cooperative games – sharing games with syndicates – and for each syndicate structure, characterized the game's Shapley value. This is a cooperative approach for sharing the surplus created by allocating an item to one member of a group. This section takes a non-cooperative approach to solving the same allocation problem. We study a class of N-player multi-round extensive form divide-and-choose games. In each round, one player (the proposer) offers compensation to the others (the responders) in return for their surrender of their claim to the item. The responders observe the offer and then simultaneously decide to accept or reject.

Rules of the Game

[5] See Shapley (1971).

In round 1, all the players are active. A player remains active until they exit. Let r_{t-1} denote the number of players who at $t-1$ *rejected* the proposer's offer; this is the number of active players at round t. Define $r_0 = N$. Let π be a permutation of the player set $\mathcal{N}$, which orders the players by their priority to propose. If $\pi(i) = 1$ then player i has first priority, if $\pi(i) = 2$ then player i has second priority, and so on.[6] At each round, the active player with priority is the proposer. Each different permutation π of the player set induces a different divide-and-choose game.

At each round t, each active player observes the set of r_{t-1} active players. Play proceeds as follows: The active player with priority is the proposer and the remaining $r_{t-1}-1$ players are responders. The proposer offers an amount p_t to be equally split among the responders, with each receiving $p_t/(r_{t-1}-1)$. The responders observe p_t and then simultaneously choose whether to accept or reject. There are two possible outcomes.

1. If every responder accepts, then each responder receives a payoff of

$$\frac{p_t}{r_{t-1}-1}.$$

   The proposer wins the item and pays a total compensation of

$$\sum\nolimits_{i=1}^{t-1}(r_{i-1}-r_i)\frac{p_i}{r_{i-1}-1}+p_t.$$

   The proposer's payoff is their stand-alone value minus total compensation paid. The game ends.

2. Otherwise, the proposer and each responder who accepts receives a payoff of

$$\frac{p_t}{r_{t-1}-1},$$

   and exits. If more than one responder rejects, i.e., $r_t > 1$, then the game proceeds to round $t+1$. If only one responder rejects, then the

[6] In the prior section players were indexed the the rank of their stand-alone value, highest to lowest.

lone rejector wins the item and pays a total compensation of

$$\sum\nolimits_{i=1}^{t}(r_{i-1}-r_i)\frac{p_i}{r_{i-1}-1}.$$

This responder's payoff is their stand-alone value minus total compensation paid. The game ends.

4 Maxmin and Maxmin Perfect Play

In this section we characterize the players' maxmin values, their maxmin strategies, and their maxmin perfect strategies for divide-and-choose games.

Fix a divide-and-choose game. Write β^i for player i's strategy in the game. Let $v_i(x_i, x_{-i}, \beta^i, \beta^{-i})$ be the payoff to a player i whose stand-alone value is x_i and who follows β^i when $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ and $\beta^{-i} = (\beta^1, \ldots, \beta^{i-1}, \beta^{i+1}, \ldots, \beta^N)$ are the stand-alone values and strategies of the other players. Player i's *maxmin payoff* when their value is x_i is the largest value, denoted by $\bar{v}_i(x_i)$, for which i has a strategy $\bar{\beta}^i$ such that

$$v_i(x_i, x_{-i}, \bar{\beta}^i, \beta^{-i}) \geq \bar{v}_i(x_i)\ \forall x_{-i}, \beta^{-i}.$$

We say that $\bar{\beta}^i$ is a *maxmin strategy for player* i if for each $x_i \in [0, \bar{x}]$ the strategy guarantees them at least $\bar{v}_i(x_i)$.

In any divide-and-choose game, a player i can guarantee a payoff of at least x_i/N. Proposition 3 establishes that this is player i's maxmin payoff.

Proposition 3: *In any divide-and-choose game (i.e., for any proposer ordering of the players), the maxmin payoff of each player is* $1/N$*-th of their stand-alone value.*

Example 4: Suppose that $\mathcal{N} = \{1, 2, 3\}$, with stand-alone values x_1, x_2, and x_3, a proposer ordering 1, 2, and then 3. The following are maxmin strategies, each guaranteeing a player at least one-third of their stand-alone value, where "A" denotes accept and "R" denotes reject.

Player	**Round 1**	**Round 2**
1	$p_1 = \frac{2}{3}x_1$	NA
2	$\begin{cases} \text{A} & \text{if } p_1 \geq \frac{2}{3}x_2 \\ \text{R} & \text{otherwise} \end{cases}$	$p_2 = \frac{1}{3}x_2$
3	$\begin{cases} \text{A} & \text{if } p_1 \geq \frac{2}{3}x_3 \\ \text{R} & \text{otherwise} \end{cases}$	$\begin{cases} \text{A} & \text{if } p_2 \geq \frac{1}{3}x_3 \\ \text{R} & \text{otherwise.} \end{cases}$

□

In dynamic games, maxmin strategies may be too conservative. After the game has begun, a player can typically guarantee a payoff *more* than their maxmin payoff by adjusting to observed play. The maxmin strategies given in Example 4 illustrate. If in round 1 Player 1 proposes $p_1 = 0$, then both responders reject. In round 2, players 2 and 3 are then playing a two-person divide-and-choose game, with no compensation going to Player 1. Each can guarantee themselves 1/2 of their stand-alone value by updating their strategy: Player 2 can guarantee a payoff of $x_2/2$ by proposing $p_2 = x_2/2$, and Player 3 can guarantee a payoff of $x_3/2$ by accepting any proposal of at least $x_3/2$ and rejecting all others.

We are therefore interested in a refinement of maxmin, a strategy that maximizes the payoff that a player can guarantee themselves for any history of play. We call such a strategy a "maxmin perfect" strategy.

Let β^i be a strategy for player i in the game. Let $\mathcal{N}_t$ be the set of active players at round t, and let $x_{\mathcal{N}_t}$ be the stand-alone value of the proposer when $\mathcal{N}_t$ is the set of active players. A history at the start of round t, denoted h_t, is a sequence of sets of active players $\mathcal{N}_1, \ldots, \mathcal{N}_t$ such that (i) $\mathcal{N}_1 = \mathcal{N}$, and (ii) $\mathcal{N}_j \subset \mathcal{N}_i$ for $i < j$, and proposals $p_0, \ldots, p_{t-1}$, where $p_0 \equiv 0$. The history h_t identifies a proper subgame. We write $\beta^i|_{h_t}$ to be the restriction of β^i to the subgame h_t. A strategy $\bar{\beta}^i$ is a *maxmin perfect strategy* for player i if for every subgame h_t and stand-alone value $x_i \in [0, \bar{x}]$, the restriction $\bar{\beta}^i|_{h_t}$ is a maxmin strategy for player i in subgame h_t.

Proposition 6 characterizes, for every divide-and-choose game, the unique maxmin perfect strategy of each player.

Proposition 4: *The maxmin perfect strategy of a player with stand-alone value x is characterized by the following rule: for each t, and every subgame h_t, when the proposer offer*

$$p_t = \frac{r_{t-1} - 1}{r_{t-1}} \left(x - \sum_{k=0}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \right);$$

when the responder accept any offer of p_t such that

$$p_t \geq \frac{r_{t-1} - 1}{r_{t-1}} \left(x - \sum_{k=0}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \right),$$

and reject otherwise.

A player whose stand-value is x and who is active at subgame h_t, has a maxmin payoff in subgame h_t of

$$v_t(x; \mathbf{r}_{t-1}, \mathbf{p}_{t-1}) = \frac{1}{r_{t-1}} \left(x - \sum_{k=0}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \right).$$

Example 5: Suppose $\mathcal{N} = \{1, 2, 3\}$, and the proposer order is 1, 2, and then 3. Maxmin perfect play in round 1 is given by the first column. The second column describes maxmin perfect play in the subgame $(\{2, 3\}, p_1)$, where the price offer was p_1 at round 1, and player 2 and 3 are active.

Player	**Stage 1**	**Stage 2**
1	$p_1 = \frac{2}{3} x_1$	N/A
2	$\begin{cases} \text{A} & \text{if } p_1 \geq \frac{2}{3} x_2 \\ \text{R} & \text{otherwise} \end{cases}$	$p_2 = \frac{1}{2} \left(x_2 - \frac{p_1}{2} \right)$
3	$\begin{cases} \text{A} & \text{if } p_1 \geq \frac{2}{3} x_3 \\ \text{R} & \text{otherwise} \end{cases}$	$\begin{cases} \text{A} & \text{if } p_2 \geq \frac{1}{2} \left(x_3 - \frac{p_1}{2} \right) \\ \text{R} & \text{otherwise.} \end{cases}$

$\square$

Proposition 5 identifies the payoffs of players who exit the game at round t, when every player follows their maxmin perfect strategy. In the next section we show these payoffs coincide with the Shapley value payoffs of an associated sharing game.

Proposition 5: *Fix a divide-and-choose game. Suppose that each player follows their maxmin perfect strategy. Then, in round t, when the set of active players is $\mathcal{N}_t$, the proposer and any responder that accepts leaves the game with a payoff of*

$$\frac{1}{r_{t-1}} x_{\mathcal{N}_t} - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} r_k} x_{\mathcal{N}_k},$$

whereas any responder with stand-alone value x, whose sole rejection in round t causes the game to end, receives a payoff of

$$x - \sum_{k=1}^{t} \frac{r_{k-1} - r_k}{r_{k-1} r_k} x_{\mathcal{N}_k}.$$

5 Divide-and-Choose and the Shapley Value

In this section, we establish the close relationship between the maxmin perfect payoffs in divide-and-choose games and the Shapley values of sharing games with syndicates. First, we show that maxmin perfect payoffs of any given divide-and-choose game coincide with the Shapley value of an associated sharing game with assortative syndicates. Second, for any sharing game with assortative syndicates, we show that its Shapley value coincides with the maxmin perfect payoffs of an associated divide-and-choose game. In short, the maxmin perfect outcomes of divide-and-choose games span the set of Shapley values of sharing games with assortative syndicates.

The proof of this result is constructive. Given a divide-and-choose game, an algorithm we describe identifies a specific sharing game. We call this the

"Proposer Priority to Syndicate Structure" (hereafter, PP-to-SS) algorithm. We show that maxmin perfect payoffs of the divide-and-choose game coincide with the Shapley value of the identified sharing game. Different proposer priority lists yield different divide-and-choose games. By varying the game's proposer priority list, and applying the PP-to-SS algorithm, we obtain the set of all sharing games with assortative syndicates. Hence, maxmin perfect play in divide-and-choose games gives us the Shapley values of all sharing games with assortative syndicates.

We also provide an inverse algorithm, the SS-to-PP algorithm, which maps sharing games with assortative syndicates to proposer priority lists for divide-and-choose games. This algorithm identifies a divide-and-choose game whose maxmin perfect payoffs coincides with the Shapley value of the sharing game.

In this section, we index players by their stand-alone values, with $x_1 > x_2 > \cdots > x_N$.

PP-TO-SS ALGORITHM

The PP-to-SS algorithm takes a proposer priority list, π, for a divide-and-choose game and identifies a syndicate structure of a sharing game. We will show that the Shapley value payoffs of the identified sharing game coincides with the maxmin perfect payoffs of the divide-and-choose game with proposer priority π. The algorithm is described in two steps.

In step 1, allocate the N players to N sets, $B_1, \ldots, B_N$, as follows: Given a proposer priority list π, start with the highest priority proposer, i.e., player $\pi^{-1}(1)$. Form group B_1 to consist of player $\pi^{-1}(1)$ and every player $j > \pi^{-1}(1)$, i.e., every player who has a lower stand-alone value than player $\pi^{-1}(1)$. Next, go to the second-highest priority proposer, i.e., player $\pi^{-1}(2)$. If this player is already assigned to B_1, then we set $B_2 = \{\emptyset\}$. Otherwise, form group B_2 to consist of player $\pi^{-1}(2)$ and every player $j > \pi^{-1}(2)$ not already assigned. This process repeats. At each iteration $k \leq N$, go to the k-th highest priority proposer, i.e., player $\pi^{-1}(k)$. If this player is already been assigned, that is if $\pi^{-1}(k) \in \cup_{i=1}^{k-1} B_i$, then set $B_k = \{\emptyset\}$. Otherwise, define group B_k to consist of player $\pi^{-1}(k)$ and every player $j > \pi^{-1}(k)$ not already assigned.

In step 2, use the sets $B_1, \ldots, B_N$ to form a syndicate structure, as follows: Start with the player with the largest stand-alone value, i.e., player 1. Find the set B_k that contains player 1 and label it G_1. Next, go to the player with the highest stand-alone value that is not a member of G_1, say player j, then find the set B_k that contains player j and label that set G_2. Continue in this fashion until all of the players have been assigned to a set G_i, $i = 1, \ldots, m$. Note that $G_1, \ldots, G_m$ is a partition of $\mathcal{N}$.

Example 6: Suppose $\mathcal{N} = \{1, 2, 3, 4, 5\}$, with stand-alone values $x_1 > x_2 > \cdots > x_5$, and proposer priority $\pi = (4, 5, 2, 3, 1)$. Then

$$B_1 = \{4, 5\}, B_2 = \{\emptyset\}, B_3 = \{2, 3\}, B_4 = \{\emptyset\}, \text{ and } B_5 = \{1\},$$

and

$$G_1 = \{1\},\ G_2 = \{2, 3\}, \text{ and } G_3 = \{4, 5\}. \ \square$$

Proposition 6 counts the number of syndicate structures that can be constructed, by varying the proposer priority list, when there are N players.

Proposition 6: *The number of distinct syndicate structures that can be obtained by the PP-to-SS algorithm, for some proposer priority list is* 2^{N-1}*. This is the set of all assortative syndicate structures.*

We now show that for every assortative syndicate structure there is a proposer-priority list π such that the PP-to-SS algorithm applied to π yields the given syndicate structure.

SS-to-PP Algorithm

The SS-to-PP algorithm takes an assortative syndicate structure and returns a proposer priority list. In particular, it gives a proposer priority list which, when input into the PP-to-SS algorithm, returns the original assortative syndicate structure.

The algorithm is described as follows: Given an assortative syndicate structure $\{G_1, \ldots, G_m\}$ construct a proposer priority list in m steps: In step 1, begin with the largest-indexed syndicate G_m. Choose the player in G_m with the largest stand-alone value and assign them the highest proposer priority (i.e., priority 1). Assign the next $|G_m|-1$ highest priorities arbitrarily to the other players in G_m. In step 2, go to the second-largest indexed syndicate, G_{m-1}. Choose the player in G_{m-1} with the largest stand-alone value and assign them proposer priority $|G_m|+1$. Then assign the next $|G_{m-1}|-1$ highest priorities arbitrarily to the other players in G_{m-1}. Proceed in a similar fashion with each syndicate in turn, in decreasing order, that is G_{m-2}, then $G_{m-3}, \ldots$, and, lastly, G_1. The result in m steps is a proposer priority list π.

Example 7: Suppose $\mathcal{N} = \{1,2,3,4,5,6\}$, with stand-alone values $x_1 > x_2 > \cdots > x_6$, and syndicate structure

$$G_1 = \{1\},\ G_2 = \{2,3,4\},\ G_3 = \{5,6\}.$$

This is an assortative syndicate structure. Applied to this syndicate structure, the SS-to-PP algorithm generates two proposer priority lists $\pi = (5,6,2,3,4,1)$ and $\pi = (5,6,2,4,3,1)$. Its easy to verify that the PP-SS algorithm applied to either proposer priority list returns the syndcate structure $\{G_1, G_2, G_3\}$.
□

5.1 Shapley Value and Divide-and-Choose

Proposition 7 shows that the maxmin perfect payoffs of the divide-and-choose game with proposer priority list π are equal to the Shapley value payoffs of the sharing game obtained by applying the PP-to-SS algorithm to π; and the Shapley value payoffs of the sharing game with assortative syndicates

$\{G_1, \ldots, G_m\}$ are equal to the divide-and-choose game with proposer priority list obtained by applying the SS-to-PP algorithm to $\{G_1, \ldots, G_m\}$.

Proposition 7: *The following statements are true:*
(i) *The maxmin perfect payoffs of the divide-and-choose game with proposer priority list π are the Shapley value payoffs in the sharing game with assortative syndicates, where the syndicate structure is given by the PP-to-SS algorithm applied to π.*
(ii) *The Shapley value payoffs of the sharing game with assortative syndicates $\{G_1, \ldots, G_m\}$ are the maxmin perfect payoffs of the divide-and-choose game where the proposer priority list is generated by the SS-to-PP algorithm applied to $\{G_1, \ldots, G_m\}$.*

Example 8: Suppose $\mathcal{N} = \{1, 2, 3\}$, with stand-alone values $x_1 > x_2 > x_3$. There are 3! different proposer priority lists. The table below shows, for each proposer priority list, each player's payoff under maxmin perfect play in the divide-and-choose game. The last column shows the syndicate structure of the sharing game whose Shapley value coincides with the maxmin perfect payoffs.

Proposer List	**1**	**2**	**3**	**Syndicate Structure**
123	$\frac{1}{3}x_1$	$\frac{1}{3}x_1$	$\frac{1}{3}x_1$	$G_1 = \{1, 2, 3\}$
132	$\frac{1}{3}x_1$	$\frac{1}{3}x_1$	$\frac{1}{3}x_1$	$G_1 = \{1, 2, 3\}$
213	$x_1 - \frac{2}{3}x_2$	$\frac{1}{3}x_2$	$\frac{1}{3}x_2$	$G_1 = \{1\}$ $G_2 = \{2, 3\}$
231	$x_1 - \frac{2}{3}x_2$	$\frac{1}{3}x_2$	$\frac{1}{3}x_2$	$G_1 = \{1\}$ $G_2 = \{2, 3\}$
312	$\frac{1}{2}x_A - \frac{1}{6}x_C$	$\frac{1}{2}x_A - \frac{1}{6}x_C$	$\frac{1}{3}x_C$	$G_1 = \{1, 2\}$ $G_2 = \{3\}$
321	$x_1 - \frac{1}{2}x_2 - \frac{1}{6}x_3$	$\frac{1}{2}x_2 - \frac{1}{6}x_3$	$\frac{1}{3}x_3$	$G_1 = \{1\}$ $G_2 = \{2\}$ $G_3 = \{3\}$

Note that no proposer priority list yields the syndicate structure $G_1 = \{1, 3\}$, $G_2 = \{2\}$, as it is not assortative. $\square$

6 Discussion

The results of this paper identify a common foundation for two classic approaches to fair allocation. We study the problem of allocating a single indivisible item among claimants with heterogeneous valuations and show that an axiomatic notion of fairness and a procedural notion of fairness lead to the same outcome.

On the cooperative side, we introduce a new class of cooperative games, called sharing games, in which coalition values are determined by the networks (or syndicates) accessible to coalition members. Sharing games admit a closed-form characterization of their Shapley values and are concave, implying that the Shapley value belongs to the anti-core. Thus, beyond its familiar axiomatic characterization, the Shapley value inherits the normative properties associated with allocations in anti-core.

On the procedural side, we study an N-player divide-and-choose game inspired by Kuhn's cake-cutting algorithm. The mechanism allocates an indivisible item to a single participant and determines the compensation owed to others. As in the classic two-player divide-and-choose game, each participant can guarantee a payoff proportional to their value by following a maxmin strategy. Moreover, each player has a unique maxmin-perfect strategy.

Our main contribution is to show that these two approaches to fairness coincide. For every ordering of proposers in a divide-and-choose game, there exists a unique assortative sharing game whose Shapley allocation is the unique maxmin-perfect outcome of the divide-and-choose game. Conversely, for every assortative sharing game there is a divide-and-choose game whose maxmin perfect outcome is the Shapley allocation of the sharing game.This equivalence offers a new perspective on both concepts.

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Appendix A: Proofs of Main Results

Proof of Proposition 1: First, for a given coalition size s, we count the number of coalitions where (i) a player from syndicate i has the largest value; and (ii) a player from syndicate $k > i$ has the second largest value. This is

$$\sum_{t=0}^{s-2} \binom{|\overline{G}_k|}{t} \binom{|G_k|}{(s-1)-t}.$$

The marginal contribution of i to these coalitions is

$$v(S) - v(S \backslash \{i\}) = g_i - g_k.$$

Since i's marginal contribution is only positive when the second highest value comes from a syndicate $k > i$, we can write the Shapley value of a player in syndicate i as

$$\begin{aligned}
\phi_i &= \frac{g_i}{N} + \sum_{s=2}^{N} \frac{(N-s)!(s-1)!}{N!} \sum_{k=i+1}^{m} \left[\sum_{t=0}^{s-2} \binom{|\overline{G}_k|}{t} \binom{|G_k|}{(s-1)-t} \right] (g_i - g_k) \\
&= \frac{g_i}{N} + \sum_{s=2}^{N} \sum_{k=i+1}^{m} \left[\frac{\sum_{t=0}^{s-2} \binom{|\overline{G}_k|}{t} \binom{|G_k|}{(s-1)-t}}{N \binom{N-1}{s-1}} \right] (g_i - g_k) \\
&= \frac{g_i}{N} + \frac{1}{N} \sum_{k=i+1}^{m} (g_i - g_k) \sum_{s=2}^{N} \left[\frac{\sum_{t=0}^{s-2} \binom{|\overline{G}_k|}{t} \binom{|G_k|}{(s-1)-t}}{\binom{N-1}{s-1}} \right]
\end{aligned}$$

Now, from Vandermonde's combinatorial identity we know that

$$\sum_{t=0}^{s-2} \binom{|\overline{G}_k|}{t} \binom{|G_k|}{(s-1)-t} + \binom{|\overline{G}_k|}{s-1} \binom{|G_k|}{0} = \binom{|\overline{G}_k| + |G_k|}{s-1}.$$

Therefore,

$$\sum_{t=0}^{s-2} \binom{|\overline{G}_k|}{t} \binom{|G_k|}{(s-1)-t} = \binom{|\overline{G}_k| + |G_k|}{s-1} - \binom{|\overline{G}_k|}{s-1}$$

and

$$\phi_i = \frac{g_i}{N} + \frac{1}{N}\sum_{k=i+1}^{m}(g_i - g_k)\sum_{s=2}^{N}\left[\frac{\binom{|\overline{G}_k|+|G_k|}{s-1} - \binom{|\overline{G}_k|}{s-1}}{\binom{N-1}{s-1}}\right].$$

Next, we use a known combinatorial result that if we have integers m and n such that $n \geq m \geq 0$, then we have[7]

$$\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n}{k}} = \frac{n+1}{n+1-m}.$$

So, we have

$$\sum_{k=1}^{m}\frac{\binom{m}{k}}{\binom{n}{k}} = \frac{n+1}{n+1-m} - 1.$$

This implies that

$$\sum_{s=2}^{N}\frac{\binom{|\overline{G}_k|+|G_k|}{s-1}}{\binom{N-1}{s-1}} = \sum_{q=1}^{|\overline{G}_k|+|G_k|}\frac{\binom{|\overline{G}_k|+|G_k|}{q}}{\binom{N-1}{q}} = \frac{N}{N-\left(|\overline{G}_k|+|G_k|\right)} - 1$$

and

$$\sum_{s=2}^{N}\frac{\binom{|\overline{G}_k|}{s-1}}{\binom{N-1}{s-1}} = \sum_{q=1}^{|\overline{G}_k|}\frac{\binom{|\overline{G}_k|+|G_k|}{q}}{\binom{N-1}{q}} = \frac{N}{N-|\overline{G}_k|} - 1.$$

The Shapley value expression simplifies to

$$\begin{aligned}
\phi_i &= \frac{g_i}{N} + \frac{1}{N}\sum_{k=i+1}^{m}(g_i - g_k)\left[\frac{N}{N-\left(|\overline{G}_k|+|G_k|\right)} - \frac{N}{N-|\overline{G}_k|}\right] \\
&= \frac{g_i}{N} + \sum_{k=i+1}^{m}(g_i - g_k)\left[\frac{1}{N-\left(|\overline{G}_k|+|G_k|\right)} - \frac{1}{N-|\overline{G}_k|}\right] \\
&= \frac{g_i}{N} + \sum_{k=i+1}^{m}(g_i - g_k)\left[\frac{|G_k|}{\left(N-|\overline{G}_k|-|G_k|\right)\left(N-|\overline{G}_k|\right)}\right] \\
&= \frac{g_i}{N} + \sum_{k=i+1}^{m}(g_i - g_k)\left[\frac{|G_k|}{\left(N-|\overline{G}_{k-1}|\right)\left(N-|\overline{G}_k|\right)}\right]
\end{aligned}$$

[7] See Graham, Knuth, and Patashnik (1990).

The last step is to combine terms and simplify once more. Notice that the series

$$
\begin{aligned}
& \frac{|G_{i+1}|}{\left(N-|\overline{G}_i|\right)\left(N-|\overline{G}_{i+1}|\right)}+\frac{|G_{i+2}|}{\left(N-|\overline{G}_{i+1}|\right)\left(N-|\overline{G}_{i+2}|\right)}+\ldots+\frac{|G_m|}{\left(N-|\overline{G}_{m-1}|\right)\left(N-|\overline{G}_m|\right)} \\
= & \\
& \frac{|\overline{G}_i|-|\overline{G}_{i+1}|}{\left(N-|\overline{G}_i|\right)\left(N-|\overline{G}_{i+1}|\right)}+\frac{|\overline{G}_{i+1}|-|\overline{G}_{i+2}|}{\left(N-|\overline{G}_{i+1}|\right)\left(N-|\overline{G}_{i+2}|\right)}+\ldots+\frac{|\overline{G}_{m-1}|-|\overline{G}_m|}{\left(N-|\overline{G}_{m-1}|\right)\left(N-|\overline{G}_m|\right)}.
\end{aligned}
$$

Iteratively applying Claim 1 (see below), we conclude

$$
\begin{aligned}
& \frac{|G_{i+1}|}{\left(N-|\overline{G}_i|\right)\left(N-|\overline{G}_{i+1}|\right)}+\frac{|G_{i+2}|}{\left(N-|\overline{G}_{i+1}|\right)\left(N-|\overline{G}_{i+2}|\right)}+\ldots+\frac{|G_m|}{\left(N-|\overline{G}_{m-1}|\right)\left(N-|\overline{G}_m|\right)} \\
= & \\
& \frac{|\overline{G}_i|-|\overline{G}_{i+1}|}{\left(N-|\overline{G}_i|\right)\left(N-|\overline{G}_{i+1}|\right)}+\frac{|\overline{G}_{i+1}|-|\overline{G}_{i+2}|}{\left(N-|\overline{G}_{i+1}|\right)\left(N-|\overline{G}_{i+2}|\right)}+\ldots+\frac{|\overline{G}_{m-1}|-|\overline{G}_m|}{\left(N-|\overline{G}_{m-1}|\right)\left(N-|\overline{G}_m|\right)} \\
= & \\
& \frac{|\overline{G}_i|-|\overline{G}_m|}{\left(N-|\overline{G}_i|\right)\left(N-|\overline{G}_m|\right)} \\
= & \\
& \frac{|\overline{G}_i|}{\left(N-|\overline{G}_i|\right)N}.
\end{aligned}
$$

The last equality follows since $\overline{G}_m = \emptyset$. Thus, the terms in front of g_i are

$$
\begin{aligned}
& \frac{|\overline{G}_i|}{\left(N-|\overline{G}_i|\right)N}+\frac{1}{N} \\
= \quad & \frac{|\overline{G}_i|}{N\left(N-|\overline{G}_i|\right)}+\frac{N-|\overline{G}_i|}{N\left(N-|\overline{G}_i|\right)} \\
= \quad & \frac{1}{\left(N-|\overline{G}_i|\right)}.
\end{aligned}
$$

So,

$$\phi_i = \frac{1}{\left(N-|\overline{G}_i|\right)}g_i - \sum_{k=i+1}^{m}\frac{|\overline{G}_{k-1}|-|\overline{G}_k|}{\left(N-|\overline{G}_{k-1}|\right)\left(N-|\overline{G}_k|\right)}g_k.$$

This gives us the desired result.[8] ■

[8]Note: $\frac{1}{\left(N-|\overline{G}_i|\right)} = \frac{1}{\sum_{k\leq i}|G_k|}$.

Claim 1: The following identity holds for any numbers a, b, and c not equal to n:

$$\frac{a-b}{(n-a)(n-b)}+\frac{b-c}{(n-b)\,(n-c)}=\frac{a-c}{(n-a)\,(n-c)}.$$

Proof of Claim 1:

$$\begin{aligned}
& \frac{a-b}{(n-a)(n-b)}+\frac{b-c}{(n-b)\,(n-c)} \\
= \; & \frac{(a-b)\,(n-c)}{(n-a)(n-b)(n-c)}+\frac{(b-c)\,(n-a)}{(n-a)\,(n-b)\,(n-c)} \\
= \; & \frac{(a-c)\,(n-b)}{(n-a)(n-b)(n-c)} \\
= \; & \frac{a-c}{(n-a)(n-c)}. \; \blacksquare
\end{aligned}$$

Proof of Proposition 2: Suppose player $i \in G_k$. The marginal contribution of player i to S is

$$v(S \cup \{i\}) - v(S) = \begin{cases} g_k - g_l & \text{if } S \subseteq \bar{G}_k \\ 0 & \text{otherwise} \end{cases}$$

where $g_l < g_k$ is the value of the largest syndicate that has a member in S.

If $S \nsubseteq \bar{G}_k$, then $v(T \cup \{i\}) - v(T) = 0$ since $S \subseteq T$. If $S \subseteq \bar{G}_k$, then the $v(T \cup \{i\}) - v(T) \leq g_k - g_l$ since all syndicate memberships of S are still valid in T (i.e., T can do at least as well as g_l) and potentially better. In any case, $v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$. $\blacksquare$

Proof of Proposition 3: Consider the following strategy: In each round t, for a player with stand-alone value x, if in the responder role, accept an offer p_t if

$$p_t \geq \frac{r_{t-1}-1}{r_0}x;$$

and, when in the proposer role, to propose $p_t = \frac{r_{t-1}-1}{r_0}x$.

This strategy guarantees this player a payoff of at least $\frac{x}{N}$. Consider the following cases: (i) if such a player ever accepts, then they receive $\frac{p_t}{r_{t-1}-1}$

$\geq \frac{x}{N}$; (ii) if this player is the proposer and their proposal is rejected, then their payoff is also $\frac{p_t}{r_{t-1}-1} \geq \frac{x}{N}$; (iii) If this player, as the proposer, finds their offer accepted by all active players in some round t, then they win the item and pay

$$x - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k - p_t$$

$$\geq$$

$$x - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} \left(\frac{r_{k-1} - 1}{r_0} x\right) - p_t$$

$$=$$

$$x - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_0} x - p_t$$

$$=$$

$$x - \frac{r_0 - r_{t-1}}{r_0} x - \frac{r_{t-1} - 1}{r_0} x$$

$$=$$

$$x - \frac{r_0 - 1}{r_0} x$$

$$=$$

$$\frac{1}{N} x.$$

Note: If $t = 1$, then $x - p_t = x_i - \frac{N-1}{N} x_i = \frac{1}{N} x_i$. (iv): There is one player that is a responder in all possible rounds. If this player rejects every proposal, then they win the item and achieve at least $\frac{x_i}{N}$ by applying the same argument as the proposer whose offer is accepted by all of the active players above.

Hence, a player can guarantee themselves $\frac{1}{N}$-th of their value by following the proposed strategy.

Finally, no player can guarantee more than $\frac{1}{N}$-th of their value. Suppose some player i could guarantee $\pi_i > \frac{x_i}{N}$ and consider the case where all players have the same value as i. By following the above strategy, all players other than i could guarantee at least $\frac{x_i}{N}$. By assumption, player i can guarantee $\pi_i > \frac{x_i}{N}$. However, if we sum across players, then we see that the total guaranteed payoff is strictly larger than the total surplus x. This is impossible given the rules of the game (the scheme is budget balanced). Hence, the maxmin value for the game must be $\frac{x_i}{N}$. ■

Proof of Proposition 4: The proof is by induction. In the last possible round, there are two players left. Round $N-1$: ($r_0, \ldots, r_{N-2}$ and $p_1, .., p_{N-2}$ are taken as given). We know $r_{N-2} = 2$ since otherwise we would not be in round $N-1$. The liability at the start of the round is

$$L_{N-2} = \sum_{i=1}^{N-2} \frac{r_{i-1} - r_i}{r_{i-1} - 1} p_i.$$

The proposer in round $N-1$ announces p_{N-1}. The responder, with value x_c, accepts if

$$p_{N-1} \geq \frac{1}{2} \left(x_c - L_{N-2} \right).$$

The worst case payoff for the responder is therefore

$$\frac{1}{2} \left(x_c - L_{N-2} \right) = \frac{1}{r_{N-2}} \left(x_c - \sum_{i=1}^{N-2} \frac{r_{i-1} - r_i}{r_{i-1} - 1} p_i \right).$$

On the other hand, if the responder accepts, then the proposer gets $x_p - L_{N-2} - p_{N-1}$; but he gets p_{N-1} if the responder rejects. The maxmin perfect behavior for the proposer is to set

$$p_{N-1} = \frac{1}{2} \left(x_p - L_{N-2} \right),$$

which guarantees a payoff of

$$\frac{1}{r_{N-2}} \left(x_p - \sum_{i=1}^{N-2} \frac{r_{i-1} - r_i}{r_{i-1} - 1} p_i \right).$$

Now, assume that in each round $t, \ldots, N-1$, the player of interest (with stand alone value x) follows the posited maxmin perfect strategy (when they are either the proposer or the responder) and that

$$v_t = \frac{1}{r_{t-1}} \left(x - \sum_{i=1}^{t-1} \frac{r_{i-1} - r_i}{r_{i-1} - 1} p_i \right)$$

is the guaranteed worst case payoff for that player in round t.

In round $t-1$, the proposer announces p_{t-1}. A responder gets

$$\frac{p_{t-1}}{r_{t-2}-1}$$

by accepting. Whereas, if they reject, then they either win the item ($r_{t-1}=1$) and a payoff of

$$x_c - L_{t-2} - p_{t-1}$$

or, if $r_{t-1} > 1$, the game continues to the next round, where (by induction) their maxmin payoff is

$$\begin{aligned} v_t &= \frac{1}{r_{t-1}}\left(x_c - \sum_{i=1}^{t-1} \frac{r_{i-1}-r_i}{r_{i-1}-1} p_i\right) = \\ & \frac{1}{r_{t-1}}\left(x_c - L_{t-2} - \frac{r_{t-2}-r_{t-1}}{r_{t-2}-1} p_{t-1}\right) \end{aligned}$$

So, if they reject the proposer's offer, then their worst case payoff is

$$\begin{aligned} & \min\{x_c - L_{t-2} - p_{t-1}, \frac{1}{r_{t-1}}\left(x_c - L_{t-2} - \frac{r_{t-2}-r_{t-1}}{r_{t-2}-1} p_{t-1}\right)\} \\ = & \\ & \begin{cases} \frac{1}{r_{t-1}}\left(x_c - L_{t-2} - \frac{r_{t-2}-r_{t-1}}{r_{t-2}-1} p_{t-1}\right) & \text{- if } p_{t-1} \leq \frac{r_{t-2}-1}{r_{t-2}}(x_c - L_{t-2}) \\ x_c - L_{t-2} - p_{t-1} & \text{- if } p_{t-1} \geq \frac{r_{t-2}-1}{r_{t-2}}(x_c - L_{t-2}) \end{cases} \end{aligned}$$

This follows, since $r_{t-1} > 1$ (otherwise the game would have already ended), we have

$$\begin{aligned}
\frac{1}{r_{t-1}}\left(x_c - L_{t-2} - \frac{r_{t-2}-r_{t-1}}{r_{t-2}-1}p_{t-1}\right) &\leq x_c - L_{t-2} - p_{t-1} \\
&\rightarrow \\
\left(r_{t-1} - \frac{r_{t-2}-r_{t-1}}{r_{t-2}-1}\right)p_{t-1} &\leq (r_{t-1}-1)(x_c - L_{t-2}) \\
&\rightarrow \\
\left(r_{t-1}\frac{r_{t-2}-1}{r_{t-2}-1} - \frac{r_{t-2}-r_{t-1}}{r_{t-2}-1}\right)p_{t-1} &\leq (r_{t-1}-1)(x_c - L_{t-2}) \\
&\rightarrow \\
r_{t-2}\left(\frac{r_{t-1}-1}{r_{t-2}-1}\right)p_{t-1} &\leq (r_{t-1}-1)(x_c - L_{t-2}) \\
&\rightarrow \\
p_{t-1} &\leq \frac{r_{t-2}-1}{r_{t-2}}(x_c - L_{t-2}).
\end{aligned}$$

Now, we suppose that

$$p_{t-1} \leq \frac{r_{t-2}-1}{r_{t-2}}(x_c - L_{t-2}),$$

then maxmin perfect behavior requires that the responder should accept if

$$\begin{aligned}
\frac{p_{t-1}}{r_{t-2}-1} &\geq \frac{1}{r_{t-1}}\left(x_c - L_{t-2} - \frac{r_{t-2}-r_{t-1}}{r_{t-2}-1}p_{t-1}\right) \\
&\rightarrow \\
p_{t-1} &\geq \frac{r_{t-2}-1}{r_{t-2}}(x_c - L_{t-2}).
\end{aligned}$$

Alternatively, if

$$p_{t-1} \geq \frac{r_{t-2}-1}{r_{t-2}}(x_c - L_{t-2}),$$

then maxmin perfect behavior requires that the responder accept if

$$\begin{aligned}
\frac{p_{t-1}}{r_{t-2}-1} &\geq x_c - L_{t-2} - p_{t-1} \\
&\rightarrow \\
p_{t-1} &\geq \frac{r_{t-2}-1}{r_{t-2}}\left(x_c - L_{t-2}\right).
\end{aligned}$$

Putting these two cases together, we have that the maxmin strategy for a responder is

$$\begin{cases} \text{Accept} & \text{- if } p_{t-1} \geq \frac{r_{t-2}-1}{r_{t-2}} \left(x_c - L_{t-2} \right) \\ \text{Reject} & \text{- otherwise} \end{cases}$$

This gives a maxmin payoff of

$$\frac{1}{r_{t-2}} \left(x_c - L_{t-2} \right) = \frac{1}{r_{t-2}} \left(x_c - \sum_{i=1}^{t-2} \frac{r_{i-1} - r_i}{r_{i-1} - 1} p_i \right).$$

The maxmin choice of the proposer is clearly to set

$$p_{t-1} = \frac{r_{t-2} - 1}{r_{t-2}} \left(x_p - \sum_{i=1}^{t-2} \frac{r_{i-1} - r_i}{r_{i-1} - 1} p_i \right)$$

which equates their payoff from the two possible outcomes: $\frac{p_{t-1}}{r_{t-2}-1}$ and $x - L_{t-2} - p_{t-1}$. This gives them a payoff of exactly

$$\frac{1}{r_{t-2}} \left(x_p - \sum_{i=1}^{t-2} \frac{r_{i-1} - r_i}{r_{i-1} - 1} p_i \right). \quad \blacksquare$$

Lemma 1: *In round 1, in any maxmin strategy, the proposer with value x_i makes proposal $p_1 = \frac{r_0-1}{r_0} x_i$. In any subsequent round $t > 1$, any proposal p_t by the proposer with value x_i prescribed by a maxmin strategy is bounded from below by*

$$\frac{r_{t-1} - 1}{r_0} x_i$$

and above by

$$\frac{r_0 - 1}{r_0} x_i - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k.$$

Proof: If $t = 1$, then a proposal that is different than $\frac{r_0-1}{r_0} x_i$ cannot be maxmin. If the proposal is larger and all responders accept, then the proposer does worse than the maxmin value. Similarly, if the proposal is smaller, and someone rejects, then proposer does worse than their maxmin payoff.

Thus, consider round $t > 1$. The current liability of winning the item is $\sum_{k=1}^{t-1} \frac{r_{k-1}-r_k}{r_{k-1}-1} p_k$. The proposer receives

$$x_i - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k - p_t$$

if their proposal is accepted. They get

$$\frac{p_t}{r_{t-1} - 1}$$

if their offer is rejected. Therefore the smallest price that achieves $\frac{1}{N}$-th of their value is

$$\frac{r_{t-1} - 1}{r_0} x_i \le p_t$$

If the offer is accepted, then they need

$$\begin{aligned} x_i - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k - p_t &\geq \frac{1}{r_0} x_i \\ &\rightarrow \\ p_t &\leq \frac{r_0 - 1}{r_0} x_i - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \end{aligned}$$

Hence,

$$p_t \in \left[\frac{r_{t-1} - 1}{r_0} x_i, \frac{r_0 - 1}{r_0} x_i - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \right].$$

■

Lemma 2: *Suppose for $k = 1, \ldots, t-1$ the proposer at k makes their maxmin perfect bid p_k, then for $2 \le t \le N - 1$, we have*

$$\sum_{i=1}^{t-1} \frac{(r_{i-1} - r_i)}{r_{i-1} - 1} p_i = r_{t-1} \left[\sum_{i=1}^{t-1} \frac{(r_{i-1} - r_i)}{r_{i-1} r_i} x_{\mathcal{N}_i} \right].$$

Proof of Lemma 2: The proof is by induction on t. It's useful to define

$$L_{t-1} = \sum_{i=1}^{t-1} \frac{(r_{i-1} - r_i)}{r_{i-1} - 1} p_i.$$

For $t = 2$, by direct computation, we have

$$L_1 = (r_0 - r_1)\frac{p_1}{r_0 - 1} = (r_0 - r_1)\frac{x_{\mathcal{N}_1}}{r_0} = r_1 \left[(r_0 - r_1)\frac{x_{\mathcal{N}_1}}{r_0 r_1}\right].$$

Next, suppose the result is true up until $t - 2$, so that

$$L_{t-2} = \sum_{i=1}^{t-2} \frac{(r_{i-1} - r_i)}{r_{i-1} - 1} p_i = r_{t-2}\left[\sum_{i=1}^{t-2} \frac{(r_{i-1} - r_i)}{r_{i-1} r_i} x_{\mathcal{N}_i}\right].$$

Now,

$$L_{t-1} = \sum_{i=1}^{t-1} \frac{(r_{i-1} - r_i)}{r_{i-1} - 1} p_i = L_{t-2} + \frac{(r_{t-2} - r_{t-1})}{r_{t-2} - 1} p_{t-1}.$$

Since all of the proposers in time period $t-1$ (and earlier) have been following their maxmin perfect bid, we have

$$\begin{aligned}
L_{t-1} &= L_{t-2} + \frac{(r_{t-2} - r_{t-1})}{r_{t-2} - 1}\left[\frac{r_{t-2} - 1}{r_{t-2}}\left[x_{\mathcal{N}_{t-2}} - L_{t-2}\right]\right] \\
&= L_{t-2} + \frac{(r_{t-2} - r_{t-1})}{r_{t-2} - 1}\left[\frac{r_{t-2} - 1}{r_{t-2}} x_{\mathcal{N}_{t-2}} - \frac{r_{t-2} - 1}{r_{t-2}} L_{t-2}\right] \\
&= L_{t-2}\left[1 - \frac{(r_{t-2} - r_{t-1})}{r_{t-2}}\right] + \frac{(r_{t-2} - r_{t-1})}{r_{t-2}} x_{\mathcal{N}_{t-2}} \\
&= L_{t-2}\left[\frac{r_{t-1}}{r_{m-2}}\right] + \frac{(r_{t-2} - r_{t-1})}{r_{t-2}} x_{\mathcal{N}_{t-2}} \\
&= r_{t-1}\left[\sum_{i=1}^{t-2} \frac{(r_{i-1} - r_i)}{r_{i-1} r_i} x_{\mathcal{N}_i} + \frac{(r_{t-2} - r_{t-1})}{r_{t-2} r_{t-1}} x_{\mathcal{N}_{t-2}}\right] \\
&= r_{t-1} \sum_{i=1}^{t-1} \frac{(r_{i-1} - r_i)}{r_{i-1} r_i} x_{\mathcal{N}_i}. \qquad \blacksquare
\end{aligned}$$

Proof of Proposition 5: By the rules of the game, the proposer in stage t leaves the game. Since all the players are following their maxmin perfect strategies, the proposer's payoff is

$$\frac{1}{r_{t-1}}\left[x_{\mathcal{N}_t} - L_{t-1}\right] = \frac{1}{r_{t-1}} x_{\mathcal{N}_t} - \sum_{i=1}^{t-1} \frac{(r_{i-1} - r_i)}{r_{i-1} r_i} x_{\mathcal{N}_i},$$

where the equality follows from Lemma 1. A player who accepts the proposal in stage t receives a payoff of $\frac{p_t}{r_{t-1}-1}$ which, by Lemma 2, is equal to

$$\frac{1}{r_{t-1}}\left[x_{\mathcal{N}_t}-L_{t-1}\right]=\frac{1}{r_{t-1}}x_{\mathcal{N}_t}-\sum_{i=1}^{t-1}\frac{(r_{i-1}-r_i)}{r_{i-1}r_i}x_{\mathcal{N}_i}.$$

Finally, if a player with value x rejects the proposal in stage t and this rejection ends the game, then that player receives a payoff of

$$\begin{aligned}
x-p_t-L_{t-1} &= x-\frac{r_{t-1}-1}{r_{t-1}}\left(x_{\mathcal{N}_t}-L_{t-1}\right)-L_{t-1}\\
&= x-\frac{r_{t-1}-1}{r_{t-1}}x_{\mathcal{N}_t}+\frac{r_{t-1}-1}{r_{t-1}}L_{t-1}-L_{t-1}\\
&= x-\frac{r_{t-1}-1}{r_{t-1}}x_{\mathcal{N}_t}+\frac{r_{t-1}-1}{r_{t-1}}L_{t-1}-\frac{r_{t-1}}{r_{t-1}}L_{t-1}\\
&= x-\frac{r_{t-1}-1}{r_{t-1}}x_{\mathcal{N}_t}-\frac{1}{r_{t-1}}L_{t-1}\\
&= x-\frac{r_{t-1}-1}{r_{t-1}}x_{\mathcal{N}_t}-\sum_{i=1}^{t-1}\frac{r_{i-1}-r_i}{r_{i-1}r_i}x_{\mathcal{N}_i}
\end{aligned}$$

where the last equality follows from Lemma 2. Now, since this player's rejection ends the game, $r_t=1$. Hence, we can re-write the above as

$$\frac{1}{r_t}x-\sum_{i=1}^{t}\frac{r_{i-1}-r_i}{r_{i-1}r_i}x_{\mathcal{N}_i}. \qquad \blacksquare$$

Proof of Proposition 6: If $N=1$, then there is only a single coalition. If $N=2$, there are two possible syndicate partitions–(i) the grand coalition and (ii) each person is in their own syndicate. Assume the result is true for $N-1$ (that is, there are 2^{N-2} possible syndicate structures). For simplicity, let's consider the case where the player added is player 1 (the generic argument is identical, but with more notation). Label the existing players 2,..., N. There are two ways we can add player 1. We can add them to the highest valued syndicate G_1 in any of the previous syndicates identified with $N-1$ players– or we can create a new syndicate (G_1) where he is the only member. This happens by placing player 1 either ahead or behind player 2 on the priority list. Hence, we can build a syndicate structure by, first, selecting one of the 2^{N-2} players and, second, by selecting one of the two ways we can add player 1. The total is $2 \times 2^{N-2} = 2^{N-1}$ as desired. ■

Proof of Proposition 7(i): Suppose all players follow their maxmin perfect strategies in divide-and-choose game and that the syndicates are determined according to the PP-to-SS algorithm. The PP-to-SS algorithm arranges players into syndicates based on their order on the proposer list. Suppose for a given permutation π that the algorithm places the players into m different syndicates.

By construction, we know (or can easily deduce) the following characteristics of these syndicates:

(i) The highest valued member of each syndicate makes a proposal before any other member of their syndicate.

(ii) The proposal made by a syndicate leader is accepted by all of their members.

(iii) Proposals made during the game are only made by the highest valued members of each syndicate.

(iv) Proposals are made by syndicate leaders in ascending order of syndicate value. In other words, the first proposal is made by the leader of the lowest valued syndicate. The second proposal is made by the leader of the second lowest syndicate, and so on.

(v) In any given round, the only people that accept the proposal belong to the same syndicate as the proposer.

(vi) Proposers earn the same payoff as the players who accept their proposal. Hence, members of the same syndicate earn the same payoff.

At round t, the highest valued member of syndicate $m-t+1$ makes a proposal that is accepted by all members of G_{m-t+1} and no one else. These players all leave the game. The number of players that reject the proposal in round t is equal to $r_t = N - \sum_{k=1}^{t} |G_{m-k+1}| = N - |\overline{G}_{m-t}|$. Thus, under maxmin perfect play, we have

$$\begin{aligned}
r_{i-1} &= N - |\overline{G}_{m-i+1}| \\
r_i &= N - |\overline{G}_{m-i}| \\
r_{i-1} - r_i &= |\overline{G}_{m-i}| - |\overline{G}_{m-i+1}| \\
x_{\mathcal{N}_t} &= g_{m-t+1} \\
x_{\mathcal{N}_i} &= g_{m-i+1}
\end{aligned}$$

From the maxmin perfect payoff theorem, the payoff to a proposer in round t (or any player that accepts the proposal in round t) is

$$\begin{aligned}
& \frac{1}{r_{t-1}} x_{\mathcal{N}_t} - \sum_{i=1}^{t-1} \frac{(r_{i-1} - r_i)}{r_{i-1} r_i} x_{\mathcal{N}_i} \\
= & \\
& \frac{1}{N - |\overline{G}_{m-t+1}|} g_{m-t+1} - \sum_{i=1}^{t-1} \frac{|\overline{G}_{m-i}| - |\overline{G}_{m-i+1}|}{\left(N - |\overline{G}_{m-i}|\right)\left(N - |\overline{G}_{m-i+1}|\right)} g_{m-i+1} \\
= & \\
& \frac{1}{N - |\overline{G}_i|} g_i - \sum_{k>i} \frac{|\overline{G}_{k-1}| - |\overline{G}_k|}{\left(N - |\overline{G}_{k-1}|\right)\left(N - |\overline{G}_k|\right)} g_k,
\end{aligned}$$

where we suppose that the proposer in round t belongs to syndicate i, then the last inequality follows by changing the summation index, letting $i = m-t+1$. This is identical to the Shapley value of the "sharing game" for a player in syndicate i.

The only remaining case, is when $|G_1| = 1$. In this case, the lone member of syndicate 1 is the only player to reject the proposal in the last round (say round t) of the game. By the maxmin perfect payoff theorem, they earn a payoff of

$$\begin{aligned}
& \frac{1}{r_t} x_{\mathcal{N}_t} - \sum_{i=1}^{t} \frac{(r_{i-1} - r_i)}{r_{i-1} r_i} x_{\mathcal{N}_i} \\
= & \\
& \frac{1}{N - |\overline{G}_1|} g_1 - \sum_{k>1} \frac{|\overline{G}_{k-1}| - |\overline{G}_k|}{\left(N - |\overline{G}_{k-1}|\right)\left(N - |\overline{G}_k|\right)} g_k.
\end{aligned}$$

Proposition 7(ii): The proof is clear by the construction of the SS-to-PP algorithm.

■

Appendix B: Additional Maxmin Results

In general, a player in a divide-and-choose game has many maxmin strategies, as shown by the following proposition. In what follows, it is understood that $p_0 \equiv 0$ and $r_0 \equiv N$.

Proposition 8: *Any strategy for player i* constructed as follows is a maxmin strategy. *In round 1, if the proposer, then $p_1 = \frac{r_0-1}{r_0}x_i$; and, if a responder they accept proposal p_1 if $p_1 \geq \frac{r_0-1}{r_0}x_i$ and reject otherwise. In every subsequent round $t > 1$ and history of play for which i is a responder they accept or reject the proposal p_t according to the rule*

$$\begin{cases} \text{A} & \text{if } p_t \geq \lambda_t(x_i, \mathbf{r}_{t-1}, \mathbf{p}_{t-1}) \\ \text{R} & \text{otherwise,} \end{cases}$$

where $\mathbf{r}_{t-1} = (r_0, \ldots, r_{t-1})$ and $\mathbf{p}_{t-1} = (p_0, \ldots, p_{t-1})$, where the threshold λ_t satisfies

$$\lambda_t(x_i, \mathbf{r}_{t-1}, \mathbf{p}_{t-1}) \in \left[\frac{r_{t-1}-1}{r_0}x_i, \frac{r_0-1}{r_0}x_i - \sum_{k=1}^{t-1}\frac{r_{k-1}-r_k}{r_{k-1}-1}p_k\right].$$

For any history of play for which player i is the proposer in round t, they select a proposal

$$p_t(x_i, \mathbf{r}_{t-1}, \mathbf{p}_{t-1}) \in \left[\frac{r_{t-1}-1}{r_0}x_i, \frac{r_0-1}{r_0}x_i - \sum_{k=1}^{t-1}\frac{r_{k-1}-r_k}{r_{k-1}-1}p_k\right].$$

Next, we need to show that if a player follows a strategy specified in Proposition 8, the specified intervals are well-defined.

Lemma 3: *The construction of a strategy according to the guidelines prescribed by Proposition 8 is feasible.*

Proof of Lemma 3: The result is clear in round 1 since the recommendation is a point.

Suppose the result is true for rounds $1, \ldots, t-1$ leading up to t. We prove the result by iteratively using the following inequality

$$
\begin{aligned}
& \frac{r_0 - 1}{r_0} x_i - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \\
\geq \; & \frac{r_{t-1} - 1}{r_{t-2} - 1} \left[\frac{r_0 - 1}{r_0} x_i - \sum_{k=1}^{t-2} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \right]
\end{aligned}
$$

This inequality is true since in round t, by hypothesis we have

$$
\begin{aligned}
p_1 &< \lambda_1^i \\
&\vdots \\
p_{t-1} &< \lambda_{t-1}^i,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
& \frac{r_0 - 1}{r_0} x_i - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \\
= \; & \left[\frac{r_0 - 1}{r_0} x_i - \sum_{k=1}^{t-2} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \right] - \frac{r_{t-2} - r_{t-1}}{r_{t-2} - 1} p_{t-1} \\
\geq \; & \left[\frac{r_0 - 1}{r_0} x_i - \sum_{k=1}^{t-2} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \right] - \frac{r_{t-2} - r_{t-1}}{r_{t-2} - 1} \lambda_{t-1}^i
\end{aligned}
$$

Since λ^i_{t-1} is in the specified interval for round $t-1$, we have

$$\begin{aligned}
&\left[\frac{r_0-1}{r_0}x_i - \sum_{k=1}^{t-2}\frac{r_{k-1}-r_k}{r_{k-1}-1}p_k\right] - \frac{r_{t-2}-r_{t-1}}{r_{t-2}-1}\lambda^i_{t-1} \\
\geq\; & \\
&\frac{r_0-1}{r_0}x_i - \sum_{k=1}^{t-2}\frac{r_{k-1}-r_k}{r_{k-1}-1}p_k - \frac{r_{t-2}-r_{t-1}}{r_{t-2}-1}\left(\frac{r_0-1}{r_0}x_i - \sum_{k=1}^{t-2}\frac{r_{k-1}-r_k}{r_{k-1}-1}p_k\right) \\
=\; & \\
&\left(\frac{r_0-1}{r_0} - \frac{r_{t-2}-r_{t-1}}{r_{t-2}-1}\frac{r_0-1}{r_0}\right)x_i - \left(\frac{r_{t-1}-1}{r_{t-2}-1}\right)\left(\sum_{k=1}^{t-2}\frac{r_{k-1}-r_k}{r_{k-1}-1}p_k\right) \\
=\; & \\
&\frac{r_{t-1}-1}{r_{t-2}-1}\left[\frac{r_0-1}{r_0}x_i - \sum_{k=1}^{t-2}\frac{r_{k-1}-r_k}{r_{k-1}-1}p_k\right].
\end{aligned}$$

Putting together the inequality we have

$$\begin{aligned}
&\frac{r_0-1}{r_0}x_i - \sum_{k=1}^{t-1}\frac{r_{k-1}-r_k}{r_{k-1}-1}p_k \\
\geq\; &\frac{r_{t-1}-1}{r_{t-2}-1}\left[\frac{r_0-1}{r_0}x_i - \sum_{k=1}^{t-2}\frac{r_{k-1}-r_k}{r_{k-1}-1}p_k\right].
\end{aligned}$$

Similar inequalities for smaller rounds are easily deduced. In general, for $t = q \geq 2$ we have

$$\begin{aligned}
&\frac{r_0-1}{r_0}x_i - \sum_{k=1}^{q-1}\frac{r_{k-1}-r_k}{r_{k-1}-1}p_k \\
\geq\; &\frac{r_{q-1}-1}{r_{q-2}-1}\left[\frac{r_0-1}{r_0}x_i - \sum_{k=1}^{q-2}\frac{r_{k-1}-r_k}{r_{k-1}-1}p_k\right].
\end{aligned}$$

Now repeatedly nesting these inequalities, we have

$$
\begin{aligned}
& \frac{r_0 - 1}{r_0} x_i - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \\
\geq & \\
& \frac{r_{t-1} - 1}{r_{t-2} - 1} \left[\frac{r_0 - 1}{r_0} x_i - \sum_{k=1}^{t-2} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \right] \\
\geq & \\
& \vdots \\
\geq & \\
& \frac{r_{t-1} - 1}{r_{t-2} - 1} \left[\frac{r_{t-2} - 1}{r_{t-3} - 1} \left[\cdots \left[\frac{r_1 - 1}{r_0 - 1} \right] \frac{r_0 - 1}{r_0} x_i \right] \right] \\
= & \\
& \frac{r_{t-1} - 1}{r_0} x_i.
\end{aligned}
$$

This is the desired result.

Hence, as long as the dropout prices for the previous rounds $1, \ldots, t-1$ are contained in the specified interval for that round, the interval of interest in round t is non-empty. ■

Proof of Proposition 8: First, consider a round t where i is an active responder. If player i accepts a proposal p_t, then $p_t \geq \lambda_t$ and i's payoff is

$$\frac{p_t}{r_{t-1} - 1} \geq \frac{\lambda_t}{r_{t-1} - 1} \geq \frac{1}{r_{t-1} - 1} \frac{r_{t-1} - 1}{r_0} x_i = \frac{1}{N} x_i.$$

Second, in any round t where i is called upon to be the proposer, their payoff must be at least $\frac{1}{N} x_i$, by the proof of Lemma 3. Finally, if i rejects all proposals, then $p_j < \lambda_j$ for all $k < t$. It follows, at the start of round t, the interval

$$\left[\frac{r_{t-1} - 1}{r_0} x_i, \frac{r_0 - 1}{r_0} x_i - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \right]$$

is non-empty (by a proof akin to the proof of Lemma 3). Since i is playing the strategy constructed with the proposition, we have $\lambda_t \in \left[\frac{r_{t-1}-1}{r_0} x_i, \frac{r_0 - 1}{r_0} x_i - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1}-1} p_k \right]$.

Since i rejected, it must be that $p_t < \lambda_t$. Hence, i ends up with the item and pays $\sum_{k=1}^{t-1} \frac{r_{k-1}-r_k}{r_{k-1}-1} p_k + p_t$ in compensation to the others for a payoff of

$$\begin{aligned}
& x_i - \sum_{k=1}^{t-1} \frac{r_{k-1}-r_k}{r_{k-1}-1} p_k - p_t \\
\geq \; & x_i - \sum_{k=1}^{t-1} \frac{r_{k-1}-r_k}{r_{k-1}-1} p_k - \lambda_t \\
\geq \; & x_i - \sum_{k=1}^{t-1} \frac{r_{k-1}-r_k}{r_{k-1}-1} p_k - (\frac{r_0-1}{r_0} x_i - \sum_{k=1}^{t-1} \frac{r_{k-1}-r_k}{r_{k-1}-1} p_k) \\
= \; & x_i - \frac{r_0-1}{r_0} x_i \\
= \; & \frac{x_i}{r_0}. \quad \blacksquare
\end{aligned}$$

Proposition 8 identifies a family of maxmin strategies. Proposition 9 shows that that every maxmin strategy is observationally equivalent to a strategy given in Proposition 8.

Proposition 9: *Suppose for some history that a player takes an action inconsistent with Proposition 8. Then the player has not followed a maxmin strategy.*

Proof of Proposition 9: There are three cases to consider. Case 1: Suppose the proposer makes a proposal outside the interval. This case is handled by Lemma 3 and therefore not maxmin. *Case 2:* Suppose the responder accepts a proposal that is strictly less than the lower bound of the interval. In this case, if i accepts a $p_t < \frac{r_{t-1}-1}{r_0} x_i$, then, their payoff is $\frac{p_t}{r_{t-1}-1} < \frac{x}{r_0}$. Hence, it is not maxmin. Finally, in case 3, suppose the responder rejects a proposal that is greater than the upper bound of the interval. Imagine the scenario where all of the other responders accept this proposal. Then i's payoff is

$$x - p_t - \sum_{k=1}^{t-1} \frac{r_{k-1}-r_k}{r_{k-1}-1} p_k$$

where $p_t > \frac{r_0-1}{r_0}x - \sum_{k=1}^{t-1} \frac{r_{k-1}-r_k}{r_{k-1}-1}p_k$. Therefore

$$\begin{aligned} x - p_t - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k &< x - (\frac{r_0 - 1}{r_0} x - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k) - \sum_{k=1}^{t-1} \frac{r_{k-1} - r_k}{r_{k-1} - 1} p_k \\ &= \frac{x}{r_0}. \end{aligned}$$

Hence, in each case, we can show there is some outcome that gives this player a payoff smaller than the maxmin. Since the cases are exhaustive, we have shown the desired result. ■