# AN INFORMATION VALUE APPROACH TO QUALITY CONTROL

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## ABSTRACT

Quality Control sampling plans are normally chosen by using industrial and military standards. These standards do not involve economic considerations, and usually fail to meet cost considerations.

The Information Economic approach presented in this paper suggests an easy to use methodology which determines the optimal plan for a given situation of a quality control problem. Common quality control attributes like, AQL, LTPD, Operating Characteristics Curves and Quality Control Plans are shown as special cases of Information Economic Models. Theorems involving dominance among various of Quality Control plans are proved. The Blackwell Theorem on the relationship "Generally More Informative" is modified to accommodate to the Quality Control case.

The major results of the paper include new algorithm to optimize the expected utility of decision makers. The value of information in Quality Control systems is assessed, and cost/effective analysis is carried out.

Key Words:

Quality Control, Information Economics, Value of Information, Informativeness

## 1. Introduction

The purpose of Quality Control activities is to provide assurance that goods or services conform to specific standards [Monks, 1982, Ch.14]. The Quality Control process is one of the most important activities in manufacturing as well as in service organizations.

During the last ten years, it was proved that adequate use of Quality Control (hereafter - Q.C.) tools can save incredible amounts of resources [Kaplan, 1983].

Common Q.C. practices such as MIL STD 105D [Duncan, 1965], do not account for the costs of the Q.C. process and sometimes it might yield to waste of resources.

By applying the Information Economic model [McGuire, 1972] to Q.C. we shall be able to answer the following questions:

1) What is the normative value of the information provided by a Q.C. sample?
2) Are there Q.C. plans that predominate other plans?
3) How can one determine the optimal Q.C. plan for a given situation?

Section 2 of this paper reviews the Information Structure Model which will later be modified to solve the problems mentioned above.

In sectionr 3 we define the Q.C. system as a special case of the Information Economic Model.

Section 4 ranks order Information Structures of sample Q.C. plans in a "Generally More Informative" order.

Section 5 shows the relationship between the Operating Characteristics Curves and the Information Structures of plans.

Section 6 derives theorems on the "generally more Informative" relationship of Information Structures.

Section 7 defines the "universally Generally More Informative" order and shows that such an order does not exist in Q.C. plans.

In section 8 we discuss the special and useful case of 2x2 Information Structures, and prove some theorems regarding this case.

Section 9 derives the relationship between a sample matrix and a plan matrix.

Section 10 shows the relationship between the informativeness of a sample and the sample size.

Section 11 proposes a normative process of maximizing the value of information in Q.C. problems.

Section 12 draws the conclusions.

## 2. The Information Structure Model: A Review.

This section briefly reviews the Information Structures Model. For further details the reader is referred to [McGuire 1972], or [Ahituv 1981].

Let E be a finite set of events (states) of nature

$E = \{ e_1, \ldots, e_{nE} \}$

Let $\pi$ be the vector of a priori probabilities associated with the events in E.

$\pi^t = ( \pi_1, \ldots, \pi_{nE} ), \; \sum \pi_i = 1, \; \pi_i \geq 0 \;;$

$i = 1, \ldots, nE$

Where t represents the transpose operation.

Let Z be a finite set of signals

$Z = \{ z_1, \ldots, z_{nz} \}$

An Information Structure or Information Matrix Q is an $n_E \times n_Z$ Markov (stochastic) matrix of the conditional probabilities in which signals of the set Z will be displayed at the occurrence of events in E. Thus, $q_{ij}$ of Q is the probability that for a given event ei, signal $z_j$ will be displayed. (If Q contains only 1 or 0 elements, then it is a noiseless structure).

Let A be a finite set of feasible actions to be taken by the decision maker

$A = \{ a_1, \ldots, a_{nA} \}$

A cardinal payoff function, U, is defined from A x E to the real numbers, R, associating payoffs to pairs of actions and events,

U: AxE--->R

The function U can be depicted by an nA x nE matrix, U, whose elements reflect the payoff gained under any combination of action $a_i$ of A an event $e_j$ of E.

The decision maker does not observe the events, but only the signals, and chooses actions according to signals he or she observes. The decision maker's strategy can be described by an $n_Z$ x $n_A$ Markov Matrix D, which contains the probabilities of taking some actions after being stimulated by some signals.

Thus, $d_{ij}$ of D is the probability that for a given signal $z_i$, action $a_j$ will be taken. (If D contains only 1 or 0 elements, then D is a pure strategy).

Let $\hat{\pi}$ be a square matrix containing the elements of in its main diagonal, and zeros elsewhere

$$\hat{\pi} = \begin{vmatrix} \bar{\pi}_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & \bar{\pi}_{nE} & \end{vmatrix}$$

Then the expected payoff of the combination of an information structure, Q, a decision rule (strategy) D, a payoff matrix U, and a probabilities vector $\bar{\pi}$ would be $tr(QDU\hat{\pi})$, where tr represents the trace operator [McGuire 1972]. Optimization is reached by finding a Markov matrix D* out of all possible $n_Z$ x $n_A$ Markov matrices to maximize tr ( ).

Let us define

$$F(Q,\ U,\ \bar{\pi}) = \max\ \{tr(QDU\hat{\pi})\}$$

Let us define the relationship

$Q_A \succcurlyeq Q_B$ ($Q_B$ is not better than $Q_A$ regarding U and $\overline{\pi}$) if
$F(Q_A, U, \overline{\pi}) \geq F(Q_B, U, \overline{\pi})$

$Q_A$ is regarded as generally more informative than $Q_B$ (denote $Q_A \succcurlyeq Q_B$ ) if $Q_B$ is not better than $Q_A$ for all payoff matrices U and all probability vectors .

The Blackwell Theorem (McGuire 1972) states that $Q_A \succcurlyeq Q_B$ if and only if $Q_A L = Q_B$, where L is a Markov matrix with the appropriate dimensions. This ordering is only a partial ordering of the set of finite information structures operating on a given state-of-the world set.

The gross value of information is always a relative number comparing the expected payoff gained by using different information structures.

For example, assuming that the utility is a linear function of the payoff, and $Q_A$ is not better than $Q_B$, then the value of the information of $Q_A$ over $Q_B$ is $F(Q_A,U,\overline{\pi}) - F(Q_B,U,\overline{\pi})$ with an appropriate calibration.

There are few extensions and applications of the Information Structure Model presented by McGuire [McGuire, 1972].

For instance, Ahituv [Ahituv, 1981] introduced some behavioristic aspects into McGuire's model. His assumption was that individuals often tend to stick to rigid decision rules, particulary when they are trained to respond with conditioned reactions. Optimizing the expected utility according to the "Rigid Decision Rule" yields rank ordering of Information Structures different from the Blackwell Theorem.

## 3. Quality Control Systems as Information Structures

Examples of Quality Control problems using the Information Structure model were demonstrated by Wallock and Adams [Wallock and Adams, 1963] and Demski [Demski, 1972]. Further research held by Moskowitz and Berry suggested a method to find an optimal sample [Moskowitz and Berry, 1976]. This paper suggests a complete methodology for selection and comparison of Q.C. plans. This complete methodology was not done so far, using the Information Economic models.

This paper applies Information Economic methodology to Quality Control problems, and uses it to improve the selection of Q.C. plans. In order to do so, we first define some of the Q.C. basics, and the fundamentals of our terminology.

Definition 3.1 - Sample Plan: Decision rule which specifies how large a sample (n) should be and the maximum allowable measurement number, or percentage (c) of defectives in the sample. (Monks, 1982, ch.14). A plan is therefore specified by (n, c).

For example, the plan (50,2) reads as follows: Select a random sample of 50 units and count the number of defectives. If the number of defectives is equal to or lower than 2, accept the lot; otherwise reject it.

Note: in this work we will deal with attribute plans, where items are inspected dichotomically such as good or bad, acceptable or not acceptable (for further details see [Duncan, 1965]).

An attribute plan of n units can display n+1 different results (hereafter called signals) which correspond to the possible numbers of defectives identified by the inspection, i.e., 0,1,2,...n defectives.

Thus, a Q.C. plan can be regarded as an Information Structure whose domain is the real quality of a lot, and whose range is a set of n+1 signals. We will now formulate this in a more rigorous definition.

Definition 3.2 - Information Matrix of a Sample : M is defined as an Information Matrix of an n size sample if it is a Markov matrix as follows:

(3.2.1) $M = \{ m_{ij} \}$ $i=1, \ldots, n_E$

$j=1, \ldots, n+1$

(3.2.2) The number of rows is equal to number of states of nature which are the possible ratios of defective items, i.e., $E=\{ P_1, \ldots, Pn_E \}$

(3.2.3) The of columns is equal to the number of signals, which are the possible amounts of defective items in the sample $y = 0,1, \ldots ,n$

Thus M is a $n_E$ x n+1 matrix as follows:

$$M = \begin{array}{c} \\ P_1 \\ \cdot \\ \cdot \\ P_{nE} \end{array} \begin{array}{c} y=0,\ y=1,\ \ldots,\ y=n \\ \left| \begin{array}{c} \\ m_{ij} \\ \\ \\ \end{array} \right| \end{array}$$

(3.2.4) $m_{ij} = Pr(y=j-1/P=P_i)$ $i=1, \ldots, nE$

$j=1, \ldots, n$

For convenience, the states of nature will be arranged in an ascending order, so that

$$P_1 < P_2 < \ldots < P_{nE}$$

This sorting will not change the generality of the problem (see [Marschack,1971]).

Example 3.1

Let M be a sample matrix representing a 3 item sample and suppose there are two possible states of nature:

$$P_0 = 0.02$$

$$P_1 = 0.05$$

Assuming that the failure probability is Binomial [Duncan, 1965] the information system of this sample, M, is

| M = | | y=0 | y=1 | y=2 | y=3 |
|---|---|---|---|---|---|
| | $P_0$=0.02 | .9412 | .0576 | .0012 | 0 |
| | $P_1$=0.05 | .8574 | .1354 | .0071 | .0001 |

for example, $m_{11}$ was calculated by using the Binomial distribution -

$$m_{11} = \binom{4}{1} .02^1 * .98^3 = .9412$$

$P_0$ and $P_1$ can normally represent AQL and LTPD levels of quality, as defined below:

Definition 3.3: AQL (Accepted Quality Level) is "the quality level of a "good" lot. It is the percentage of defectives that can be considered satisfactory as a process average and represents a level of quality which the producer wants accepted with a high probability of acceptance" [Monks, 1982, ch.14].

$P_0$ can be chosen as the desired AQL.

Definition 3.4: LTPD (Lot Tolerance Percent Defective) is "the quality level of a "bad" lot. It represents a level of quality which the consumer wants accepted with a low probability of acceptance" [Monks, 1982, ch.14].

$P_1$ can be the desired LTPD.

Definition 3.5 - Information Matrix of a Q.C Plan:

An $n_E$ x 2 Markov matrix whose rows represent states of nature $P_i$, $i=1, \ldots, n_E$ and the columns are the aggregated signals $y \leq c$ and $y > c$ (where c is the acceptance number, $0 \leq c \leq n$; n is the sample size) is called the Information Matrix of a Q.C Plan.

The element of an Information Matrix are:

$$q_{i1} = Pr[y\ c/P=P_i] \quad i = 1, \ldots, n_E$$

$$q_{i2} = Pr[y>c/P=P_i] \quad i = 1, \ldots, n_E$$

Example 3.2

We will now show an Information Structure of a Q.C plan having 2 states of nature: AQL and LTPD.

$$Q = \begin{array}{l} \\ P_0=AQL \\ P_1=LTPD \end{array} \begin{array}{c} \begin{array}{cc} y \leq c & y>c \end{array} \\ \left| \begin{array}{cc} 1-\alpha & \alpha \\ \beta & 1-\beta \end{array} \right| \end{array}$$

The matrix represents the relations between the states of nature and the signals; in this example, a lot having a percentage of defects equal to AQL is considered a "good" lot, and a lot having P = LTPD defectives is considered a "bad" lot.

This presentation enables us to elaborate on two more Q.C terms:

Definition 3.6 - Producer's Risk ($\alpha$):

$\alpha$ = Pr(reject a lot / the lot is "good")

$\alpha$ = Pr( $y>c$ / P=AQL ) ,

where y is the number of defectives.

Definition 3.7 - Consumer's Risk ($\beta$):

$\beta$ = Pr(do not reject a lot / the lot is "bad")

$\beta$ = Pr( $y \leq c$ / P=LTPD)

Thus, the presentation in example 3.2 enables us to present all the elements of a Q.C plan (AQL, LTPD, n, c, $\alpha$, and $\beta$ ) in a single matrix by relying on the Information Economics (IE) approach.

It is obvious to show that every Q.C plan can be presented as an Information Structure.

Definition 3.8 - Decision Matrix of a Sample: D will be a decision matrix of an n item sample if it is an n+1 x 2 Markov matrix that associates signals with decisions.

The signals are the number of defectives (y=0 through y=n) and the decisions are "accept" or "reject" a whole lot.

Example 3.3

Assume a 5 item sample; one of the feasible decision matrices is

| | | Accept | Reject |
|---|---|---|---|
| | y=0 | 1 | 0 |
| | y=1 | 1 | 0 |
| | y=2 | 1 | 0 |
| D = | y=3 | 0 | 1 |
| | y=4 | 0 | 1 |
| | y=5 | 0 | 1 |

The strategy presented in this matrix is - "take a 5 item sample, accept the lot if the number of defectives is two or less; otherwise - reject it". This matrix is , in fact equivalent to the plan (5,2).

<u>Definition 3.9 - Decision Matrix of a Plan:</u>

D will be called a "Decision Matrix of a Q.C Plan" if it is a 2x2 Markov matrix that associates aggregated signals with decisions. The signals are :

$y \leq c$ and $y > c$

The decisions are :

"accept the whole lot" and "reject the whole lot".

<u>Example 3.4</u>

The matrix of example 3.3 , corresponding to the plan (5,2) can be also presented as the following decision matrix

| | | "Accept" | "Reject" |
|---|---|---|---|
| D = | $y \leq c$ | 1 | 0 |
| | $y > c$ | 0 | 1 |

Note that some strategies are not needed to be considered in the first place. For instance, strategies that always accept or reject the lot. Undoubtedly, this case would not require information system (a sample), since the decision is never affected by any of the signals.

Strategies for which $d_{ij} = \{ 0,1 \}$ are labelled "pure strategies". When at least one of the elements is neither one nor zero, the strategy is called a "mixed strategy".

<u>Definition 3.10 - Prior Probability Vector:</u>

The prior probability vector is a vector ,

$$\bar{\Pi} = \begin{vmatrix} \bar{\Pi}_1 \\ \cdot \\ \cdot \\ \bar{\Pi}_{n_E} \end{vmatrix}$$

whose elements are the prior probabilities of the $n_E$ states of nature, P1, ..., PnE, (where $P_1 < P_2 < \ldots < P_{nE}$).

Note: is defined for both matrices - plan and sample.

<u>Definition 3.11 - The Payoff Matrix:</u>

The Payoff Matrix U is an $2xn_E$ matrix whose each element $u_{ki}$ displays the payoff related to a decision k ("accept" or "reject") and an occurrence of state of nature i.

<u>Note:</u> The payoffs can be expressed in terms of costs, that are estimated by the decision maker or derived from historical data stored in the information system of the organization. Some corporations which have adopted the "Quality Costs" concepts [Sullivan, 1983] can construct the payoff matrix almost immediately by extracting data from this data base.

Example 3.5

Suppose a "good" lot is considered a lot having 2% defectives, and a "bad" lot is one having 5% defectives.

Suppose the a-priori probabilities of those states of nature are:

$$\pi^t = (.9 \;\; .1)$$

The payoff matrix is:

$$U = \begin{array}{c} \\ \text{Accept lot} \\ \\ \text{Reject lot} \end{array} \begin{array}{cc} \text{"good" lot} & \text{"bad" lot} \\ \left| \begin{array}{cc} 0 & -1{,}000 \\ & \\ -100 & 0 \end{array} \right| \end{array}$$

Two Q.C plans are considered:

A = (158,4) and B = (184,5).

Assuming that the testing costs are equal for the two plans, which one is preferred?

Solution:

Let $Q_A$ and $Q_B$ be the Information Matrices for plans A and B respectively. According to section 3:

$$Q_A = \begin{array}{c} \\ P=.02 \\ \\ P=.05 \end{array} \begin{array}{cc} y \leq 4 & y > 4 \\ \left| \begin{array}{cc} .79 & .21 \\ & \\ .1 & .9 \end{array} \right| \end{array}$$

$$Q_B = \begin{array}{c} \\ P=.02 \\ \\ P=.05 \end{array} \begin{array}{cc} y \leq 5 & y > 5 \\ \left| \begin{array}{cc} .83 & .17 \\ & \\ .1 & .9 \end{array} \right| \end{array}$$

The optimal decision rule for $Q_A$ and $Q_B$ turns out to be

$$D = \begin{vmatrix} 1 & 0 \\ & \\ 0 & 1 \end{vmatrix}$$

and the expected payoffs are

$$\max_{D} \{tr(RDU\hat{\Pi})\} = -28.9$$

and

$$\max \{tr(QDU\hat{\Pi})\} = -25.3$$

It is seen that under these particular circumstances plan B (represented by Q) is more informative than A, and, therefore, preferred.Suppose plan A was in use, then the switch over to plan B improves the performance by

$$-25.3 - (-28.9) = \underline{3.6}$$

## 4. Ranking Q.C Plans in a "Generally More Informative" Order

Let $Q_A$ and $Q_B$ be the Information Matrices of plans A and B respectively. $Q_A$ will be generally more informative than $Q_B$ if there exists a Markov matrix R that satisfies the equation

(4.1) $Q_B = Q_A L$

This is a result of the Blackwell Theorem [McGuire,1972].

Example 4.1

Suppose the following two Q.C plans are given:

A = (32,1)

B = (200,7)

The states of nature are:

AQL = .02

LTPD = .08

The Information Structures will be:

$$Q_A = \begin{vmatrix} .87 & .13 \\ .26 & .74 \end{vmatrix}$$

$$Q_B = \begin{vmatrix} .95 & .05 \\ .08 & .92 \end{vmatrix}$$

B is generally more informative than A because there exists L,

$$L = \begin{vmatrix} .91 & .09 \\ .21 & .79 \end{vmatrix}$$

that satisfies (4.1).

This will imply that the expected value of the payoffs gained by using plan B will never be less than those obtained by plan A, regardless of the values of the payoff matrix and the a-priori probabilities. Thus, plan B dominates plan A.

In real life situations, we usually make incoming inspection on items that are intended for several users to be used in different projects. Thus, payoff matrices cannot be easily assessed. Having one plan that dominates the other, regardless of the payoff matrix, helps in selecting the right plan. In addition, sometimes we test a "brand new" item, for which it is very difficult to estimate the prior probability vector. Choosing a plan that is not dominated by any other plan assures that the test results and payoffs are not inferior to outcomes that could have been obtained by another plan, even if the expected payoff is not accurately computed due to fuzzy a-priori probabilities.

## 5. Operating Characteristic Curves and Information Structures of Plans

An Operating Characteristic Curve (hereafter OCC) is a curve that shows, for a given plan, the probability of acceptance versus the percent of defective parts in the lot [Monks, 1982, ch.14]. Figure 5.1 shows an OCC of the plan (200,7).

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Insert Figure 5.1 about here.

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One can derive many Information Structures from a given OCC, depending on the states of nature selected. For example, for the following states of nature

AQL = 2%

LTPD = 8%

The I.S of (200,7) will be

$$Q_A = \begin{vmatrix} .95 & .05 \\ .08 & .92 \end{vmatrix}$$

If we choose

AQL = 1%

LTPD = 5%

we will get another I.S., $Q_B$ as follows:

$$Q_B = \begin{vmatrix} .9967 & .0033 \\ .0896 & .9104 \end{vmatrix}$$

The components of $Q_A$ and $Q_B$ can be derived from figure 6.1 or can be calculated directly.

<u>Theorem 5.1</u>

(a) For a given Q.C plan (n,c) and a given set of states of nature there exists only one I.S.

(b) For a given set of states of nature different from P=0 and P=1, there is one to one relationship between a Q.C plan and an OCC.

<u>Proof</u>

(a) It can be seen directly that we draw the OCC by using the same calculations that we derive the corresponding I.S. and therefore the formal proof is considered trivial.

(b) The same argument can apply here. We leave the formal proof to the reader.

Thus, we have shown that our terminology and methodology can not only incorporate terms as AQL, LTPD, $\alpha$, and $\beta$, but also the Operating Characteristics Curves. Those relationship will enable us to show in the next section the informatineness of Q.C plans.

## 6. Some Theorems On "Generally More Informative" Of Q.C Plans.

We would like now to identify the conditions that turn a certain Q.C plan $(n_1,c_1)$ to be generally more informative than another one $(n_2,c_2)$.

Definition 6.1: Generally More Informative Order of Q.C Plans:

Plan $(n_1,c_1)$ will be called generally more informative than plan $(n_2,c_2)$ for a given set of states of nature $(P_1, \ldots, P_{nE})$, when the Q.C plan matrix corresponding to $(n_1,c_1)$ is generally more informative than the matrix corresponding to $(n_2,c_2)$.

The following theorem will compare the informativeness of two Q.C plans having the same sample size (n) or the same acceptance number (n):

Theorem 6.1

Let $(n_1,c_1)$ and $(n_1,c_2)$ be two Q.C plans, based on the same sample size $(n_1)$. Neither of the two plans can be considered generally more informative than the other one, for all sets of states of nature.

Proof:

Let A = $(n_1, c_2)$ and B = $(n_1, c_1)$ be two Q.C plans such that $c_2$ $c_1$. Graphically, the OCC of plan A will always be above the OCC of plan B [Juran and Gryna, 1980]. Figure 7.1 shows that relationship.

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Insert Fig. 6.1 about here

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Now we will choose randomly two states of nature $P_1$ and $P_2$. Let $Q_A$ and $Q_B$ be the plan matrices for A and B. Assume $P_1$ and $P_2$ are the states of nature pertaining to this case.

$$Q_A = \begin{array}{c} \\ P_1 \\ \\ P_2 \end{array} \begin{array}{cc} y \leq c & y > c \\ \left| \begin{array}{cc} 1-\alpha_A & \alpha_A \\ & \\ \beta_A & 1-\beta_A \end{array} \right| \end{array}$$

$$Q_B = \begin{array}{c} \\ P_1 \\ \\ P_2 \end{array} \begin{array}{cc} y \leq c_1 & y > c_1 \\ \left| \begin{array}{cc} 1-\alpha_B & \alpha_B \\ & \\ \beta_B & 1-\beta_B \end{array} \right| \end{array}$$

Since $c_2 > c_1$ and A is drawn above B

$$1-\alpha_A > 1-\alpha_B$$

and

$$\beta_A > \beta_B$$

Thus, the plan A cannot be generally more informative than B, and vice versa, according to the Blackwell Theorem.

Theorem 6.2

Let $(n_1,c_1)$ and $(n_2,c_1)$ be two Q.C plans having the same acceptance number $(c_1)$. Neither of the two plans can be considered generally more informative than the other for all sets of states of nature.

The proof follows the same sequence of reasoning as in the previous theorem.

Theorem 6.3

Let $(n_1,c1)$ and $(n_2,c_2)$ be two Q.C plans such that $n_1 \geq n_2$ and $c_1 \leq c_2$. Neither of the two plans can be generally more informative than the other one for all sets of states of nature.

The proof is similar to the one in Theorem 6.1

Corollary

For any given Q.C plans $(n_1,c_1)$ and $(n_2,c_2)$, a necessary condition that $(n_1,c_1)$ will be generally more informative than the other plan is

$$n_1 \geq n_2 \quad \text{and} \quad c_1 \geq c_2.$$

For example, if we have to coose between the following plans:

A = (250,5)

B = (300,4)

there is no way that for any given set of states of nature one plan will be generally more informative than the other, since the conditions stated above ($n_1 \geq n_2$ and $c_1 \geq c_2$) do not exist.

## 7. "Universally Generally More Informative" Order of Q.C plans

So far we have discussed the informativeness of an Information structure for a given set of states of nature. The question is whether there might be an order (defined hereafter as "Universally Generally More Informative") that enables us to say that a certain plan is always more informative than the other, regardless of the prior probabilities and the states of nature. In other words, is it conceivable that a certain plan $(n_1,c_1)$ is generally more informative than another plan $(n_2,c_2)$ for all states of nature? We will call this order "Universally Generally More Informative" order.

Definition 7.1

Q.C plan $(n_1,c_1)$ is called "Universally Generally More Informative" than $(n_2,c_2)$ if plan $(n_1,c_1)$ is generally more informative than plan $(n_2,c_2)$ for every given set of states of nature.

We will show now that such a relationship cannot exist.

Theorem 7.1

Let $(n_1,c_1)$ and $(n_2,c_2)$ be Q.C plans. $(n_1,c_1)$ can never be unversally generally more informative than $(n_2,c_2)$ and vice versa.

Proof

The relationship between $(n_1,c_1)$ and $(n_2,c_2)$ can be as follows:

(a) OCC of $(n_1,c_1)$ is always above that of (n2,c2).

In this case, according to Theorems 6.1 and 6.2, the relationship of "Generally More Informative" does not hold.

<u>(b) OCCs of $(n_1,c_1)$ and $(n_2,c_2)$ intersect in the open interval (0,1).</u>

Figure 7.1 portrays this situation .

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Insert Figure 7.1 about here.

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For the states of nature bounded by the interval $[e_1,e_2]$, the OCC of (n1,c1) will always be above that of (n2,c2), therefore (as implied by part (a) of this theorem) the order of universally generally more informative cannot apply. The same argument disables the states of nature bounded by $[r_1,r_2]$,to be universally generally more informative. Since one can always find two states of nature that are in those intervals, the relationship of Universally Generally More Informative does not hold.

Q.E.D

## 8. Dominance of Q.C Plans Having Two States of Nature.

The most common use of Q.C plans is in the case of two states of nature: AQL and LTPD [Duncan,1965]. The conditions for one Q.C plan to be generally more informative than the other one is provided in the following theorem.

Theorem 8.1

Let Q and R be two information structures of Q.C plans operating on a common set of states of nature (AQL and LTPD). Let $q_{ij}$ and $r_{ij}$ be the elements of those matrices. R is generally more informative than Q if and only if

$$r_{11} \geqslant q_{11} \quad \text{and} \quad r_{21} \leqslant q_{21}$$

Proof

According to the Blackwell Theorem (McGuire, 1972), $Q_A$ is generally more informative than $Q_B$ if and only if there exists a Markov matrix L that satisfies

$$Q_A L = Q_B$$

thus,

$$\begin{vmatrix} q_{A11} & 1-q_{A11} \\ & \\ q_{A21} & 1-q_{A21} \end{vmatrix} \begin{vmatrix} l_{11} & 1-l_{11} \\ & \\ l_{21} & 1-l_{21} \end{vmatrix} = \begin{vmatrix} q_{B}l_{1} & 1-q_{B11} \\ & \\ q_{B21} & 1-q_{B21} \end{vmatrix}$$

Solving those equations yields

$$l_{11}-l_{12} = \frac{q_{B11}-q_{B21}}{q_{A11}-q_{B21}}$$

based on the properties of a Q.C plan, $q_{B11}$ $q_{B21}$, and $q_{A11}$ $q_{A21}$ (since the states of nature are arranged in a descending order). Thus, it is easy to show that the conditions

$$q_{A11} \geq q_{B11} \quad \text{and} \quad q_{A21} \leq q_{B21}$$

yield

$$0 \leq l_{11} \leq 1 \quad \text{and} \quad 0 \leq l_{12} \leq 1,$$

which means that L can be presented as a Markov matrix, and the Blackwell Theorem is in effect.

Q.E.D

<u>Interpretations of Theorem 8.1</u>

Theorem 8.1 can be interpreted in two ways:

<u>1) Interpretation by "error types" considerations.</u>

The matrices Q and R can be rewritten as

$$Q_B = \begin{vmatrix} 1-\alpha_{qB} & \alpha_{qB} \\ \beta_{qB} & 1-\beta_{qB} \end{vmatrix}$$

$$Q_A = \begin{vmatrix} 1-\alpha_{qA} & \alpha_{qA} \\ \beta_{qA} & 1-\beta_{qA} \end{vmatrix}$$

where $\alpha_{qB}$, $\alpha_{qA}$ are the "producer's risk" of plans $Q_B$ ana $Q_A$ respectively (or statistically speaking, they are the errors of type I).

The condition $q_{A11} \geq q_{B11}$ yields that $\alpha_{qA} \leq \alpha_{qB}$, which means that the "producer's risk" in $Q_A$ plan is less than the one in $Q_B$ plan. The second condition ($q_{A21} \leq q_{B21}$) implies that the "customer's risk" of plan $Q_A$ is smaller than the one of plan $Q_B$. Thus, $Q_A$ is generally more informative than $Q_B$, if and only if both the "producer's risk" and the "customer's risk" of $Q_A$ are less than those of $Q_B$. Thus plan $Q_A$ will always yield a greater expected payoff value than $Q_B$, regardless of the prior probabilities or the payoff matrix.

2) Interpretation by OCC

The graphical interpretation of Theorem 8.1 is shown in Fig. 8.1

---

Insert Figure 8.1 about here

---

The relationship of generally more informative between $Q_A$ and $Q_B$ occurs if two conditions are satisfied:

(a) The two corresponding OCC's must intersect.

(b) The point of intersection is between the values of AQL and LTPD.

Result: The relationship of Generally More Informative order can exist only if there are two states of nature.

Whenever there exist three or more states of nature, it can be shown that either two or three states are on one side of the intersection, thus by Theorem 8.1 they cannot be rank ordered under the relationship generally more informative.

## 9. The Relationship Between a Sample Matrix and a Plan Matrix

In the previous sections we have defined two different Information Structures: An Information Structure of a Sample (a Sample Matrix) and an Information Structure of a Plan (a Plan Matrix). In this section we will show and prove the relationship between those matrix.

Theorem 9.1

Let $E = \{e_1, \ldots, e_{nE}\}$ be a given set of states of nature. Let M be an n item sample matrix, and let Q be an information structure of the plan (n.c); both matrices are defined on the same set of states of nature. Then, there exists a matrix D such that

$$Q = MD$$

and D is an nE+2 matrix, $D = \{d_{ij}\}$

$$dij = \begin{cases} 1 & i \leq c,\ j = 1 \\ & i \geq c,\ j = 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$0 \leq c \leq n$$

Proof

By definition, Q is an $n_E$ x 2 matrix, and M is an $n_E$ x (n+1) matrix. Therefore, any D that satisfies the equation

$$Q = MD$$

must be an (n+1) x 2 matrix.

The general element of M (the sample matrix) is

$$m_{ie} = \Pr( y=e \;/\; P=P_i) = \sum_{e-1}^{n} P_i^{e-1} * (1-P_i)^{n-e-1} \qquad e=1, \ldots, n$$

The general element of Q is

$$q_{i1} = \Pr( y \leq c \;/\; P=P_i) = \sum_{y=0}^{c} {}_{y}^{n} P^{iy} * (1-P_i)^{n-y}$$

changing the indeces e=y+1 yields

$$q_{i1} = \sum_{e=1}^{c} m_i e$$

and

$$q_{i2} = 1-\sum_{e=1}^{c} m_{ie} = \sum_{e=c+1}^{n} m_{ie}$$

and therefore D (as designated above) satisfies

Q = MD

Q.E.D

Results

1. D is the Decision Matrix of the Sample Matrix M, and will look like the following structure:

| | Accept | Reject |
|---|---|---|
| y=0 | 1 | 0 |
| . | 1 | 0 |
| . | 1 | 0 |
| y=c | 1 | 0 |
| y=c+1 | 0 | 1 |
| . | 0 | 1 |
| . | 0 | 1 |
| y=n | 0 | 1 |

2. Every Q.C plan matrix can be presented as a multiplication of a sample matrix and a decision matrix. A Q.C plan aggregates the signals (y) into two groups : $y \leq c$ and $y > c$. We saw now that this aggregation was done by multiplying the sample matrix (M) with the decision matrix (D).

3. THis theorem covers also the two degenerated strategies: "Accept Always" and "Reject Always". Those strategies can be presented by appropriate decision matrices. For instance, the "Accept Always" strategy can be presented by the following Decision matrix:

| | |
|---|---|
| 1 | 0 |
| . | . |
| . | . |
| 1 | 0 |

## 10. The Informativeness of a Q.C Sample and the sample Size

Is an n+1 sample lot generally more informative than an n sample lot? Intuitively, the bigger the lot is, the more information we get. But, according to Theorem 10.1, certain Q.C plan matrices can be derived from the same sample matrix. Thus, should the expected payoff that we can get from the optimal plan derived from the n+1 sample will yield a better expected payoff?

For example, assume that we have a 100 item sample. Assume also that for a given payoff matrix and given prior probabilities the optimal plan is (100,7). Will a 101 item sample yield more information? Should the Q.C plan (101,7) be better than (100,7)? Should (101,6), (101,8), or (101,7) yield better expected payoff than (100,7)?

The next theorem will show that an n+1 item sample is always generally more informative than an n item sample, and the optimal plan derived from it (for a given situation) will yield an expected payoff not smaller than the maximal expected payoff derived from an n size sample.

Theorem 10.1

An Information Structure of an n+1 size sample is generally more informative than an Information Structure of an n size sample.

Proof

Let E be a set of states of nature

$E = \{ e_1, \ldots, e_{nE} \}$

and $M_{n+1}$ be an information matrix of an n+1 item sample defined on E. Let $M_n$ be an information matrix of an n item sample defined on E. According to the Blackwell Theorem [McGuire, 1972] $M_{n+1}$ will be generally more informative than $M_n$ if and only if there exists a Markov Matrix R that satisfies

$$M_n = M_{n+1} * R$$

Where R is an (n+2)x(n+1) Markovian matrix.

Using the Bayes Theorem it can be shown that the general element of R, $r_{ij}$ is

$$r_{ij} = \begin{cases} 0 & j>i \\ 0 & j<i-1 \\ \dfrac{n+2-j}{n+1} & i=j \\ \dfrac{j}{n+1} & j=i-1 \end{cases}$$

$r_{ij} = 1$ for every j , so R is a Markov matrix.

Q.E.D

## 11. A Normative Process of Maximizing the Value of Information in Q.C Problems.

We will devise now the normative process to maximize the value of the information provided by a Q.C plan. We will show how to choose the most appropriate plan:

### 11.1 Step 1 - Effectiveness of Sampling

11.1.1 Choose appropriate $n_E$ states of nature (as a default, unless otherwise specified, determine $n_E = 2$, where $P_0 = AQL$ and $P_1 = LTPD$).

11.1.2 Assess payoff matrix U.

11.1.3 Check whether $u_11 \leqslant u_{21}$. If $u_{11} \leqslant u_{21}$ - do no sample. Reject the lot. End.

11.1.4 Check if $u_{1nE} \leqslant u_{2nE}$. If true - do not sample. Accept the lot. End.

### 11.2 Step 2 - Determining the lot size

11.2.1 Determine prior probabilities for all states of nature.

11.2.2 Determine function for the cost of testing the sample.

$$f = f(n)$$

where f is a non-decreasing monothonic function of the sample size.

11.2.3 Determine the value of the maximal expected payoff without sampling ($EMV_{max}$).

$$EMV_{max} = MAX \{ \sum_{i=1}^{n_E} \pi_i u_{1i} , \sum_{i=1}^{n_E} \pi_i u_{2i} \}$$

11.2.4 Calculate the Expected Value of Perfect Information (EVPI, see [Raiffa, 1968]) by subtracting the $EMV_{max}$ from the Expected Profit with Perfect Information (EPWPI, see [Raiffa, 1968]).

$$EVPI = EPWPI - EMV_{max}$$

11.2.6 Find the maximal value of n (n=n*) that solves the equation

$$n^* = f^{-1}(EVPI)$$

n* is the upper bound for the sample size.

11.2.7 To find the optimal sample size optimize

$$\underset{n',D}{MAX} \{tr( M_{n'}DU) - f(n')\} \qquad 0 \leq n' \leq n^*$$

where $M_{n'}$ is an n size sample matrix over the given states of nature.

Optimization can be obtained by exercising several searching methods (i.e., Fibonacci, see [Avriel, 1976, Ch.8]).

## 12. Conclusions

In this paper we have dealt with the normative value of information derived from Q.C sampling.

The major results and contribution of this paper can be summarized as follows:

1. Information Economic model was applied to Q.C. problems providing a complete and closed application to the theory developed. A full terminology was defined to fit Q.C. problems into the model. Terms as Structures of Plans and Samples, Decision Matrix of Q.C. Plans and Samples, Payoff Matrix and Prior Vector were defined and analyzed. All the existing terms (e.g., AQL, LTPD, OCC etc.) can be identified and used in the proposed terminology.

2. The new terminology enables simply and clearly to apply a Baysian approach to the Q.C. problem. Thus it is convenient to incorporate costs considerations into sampling.

3. An easy to use algorithm for determining the optimal sample size is proposed.

4. The methods and terminology defined in the paper enables to assess in a quick way the value of information derived by any sampling in any given situations.

5. The theorems on the dominance of one sampling plan over the other (in a generally more informative order) may lead to better and more cost effective sampling.

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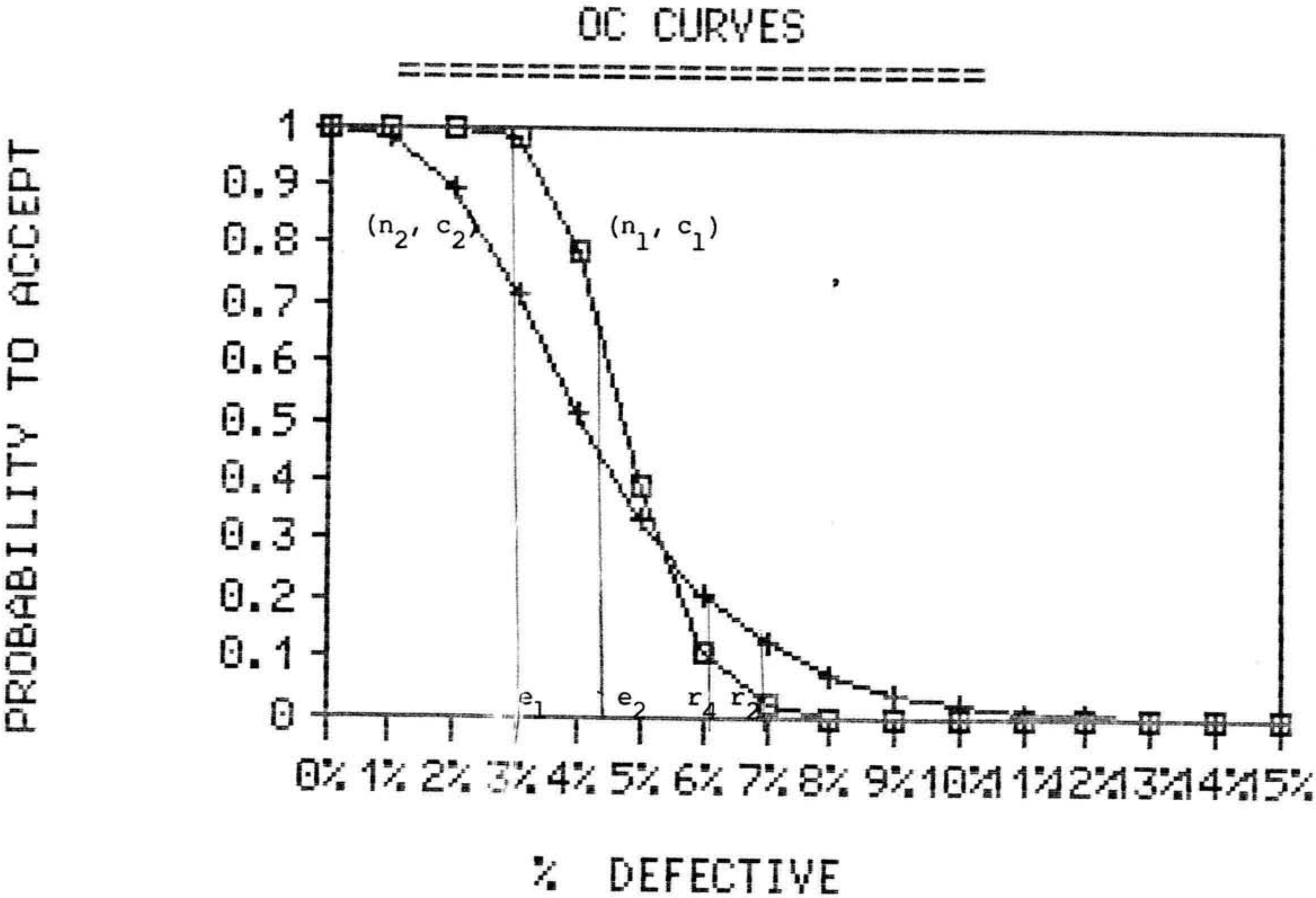
OC CURVES
PROBABILITY TO ACCEPT
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
(n2, c2)
(n1, c1)
e1
e2
r4
r2
0% 1% 2% 3% 4% 5% 6% 7% 8% 9% 10% 11% 12% 13% 14% 15%
% DEFECTIVE

FIGURE 7.1

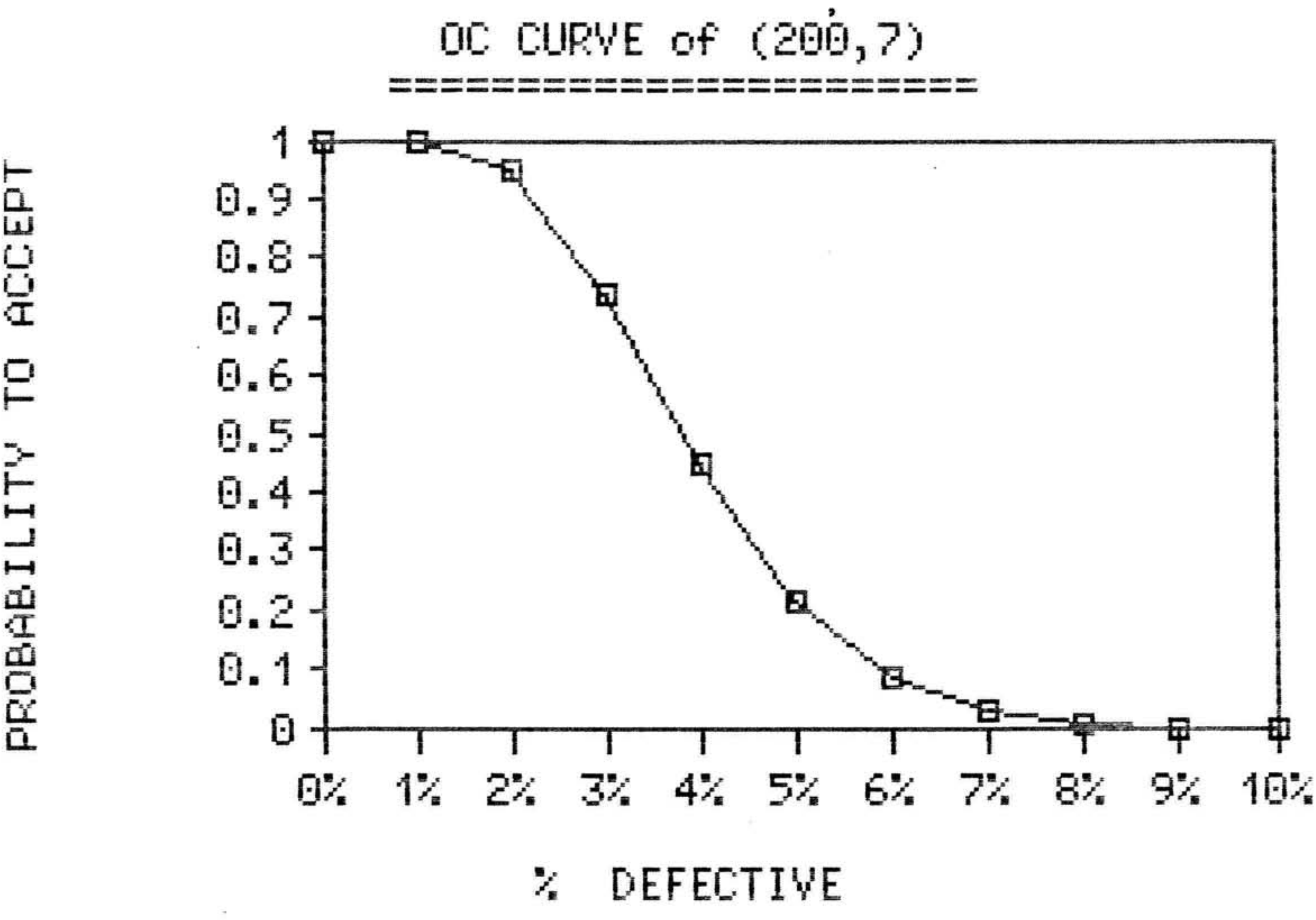


FIGURE 5.1:

OPERATING CHARACTERISTIC CURVE OF PLAN (200, 7)

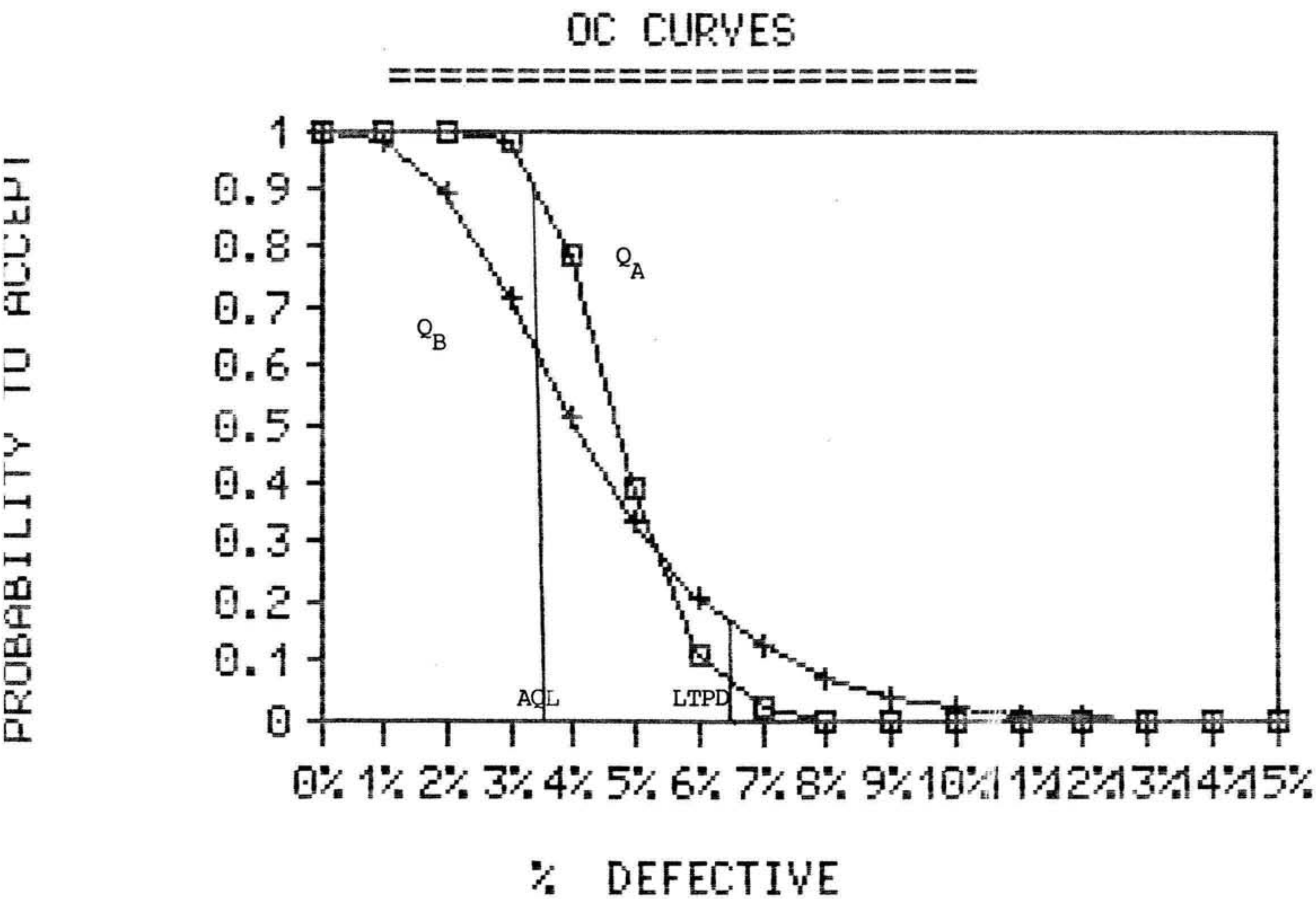
OC CURVES
PROBABILITY TO ACCEPT
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$Q_A$
$Q_B$
AQL
LTPD
0% 1% 2% 3% 4% 5% 6% 7% 8% 9% 10% 11% 12% 13% 14% 15%
% DEFECTIVE

FIGURE 8.1