ISTITUTO DI ANALISI DEI SISTEMI ED INFORMATICA
CONSIGLIO NAZIONALE DELLE RICERCHE

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APPROXIMATING SEPARABLE NONLINEAR FUNCTIONS VIA MIXED ZERO-ONE PROGRAMS

R. 476 Ottobre 1998

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Supported in part by a grant from the Office of Naval Research (N00014-96-0327). Work done in part while visiting IASI-CNR in Rome, Italy, and the Forschungsinstitut für Diskrete Mathematik, Universität Bonn, Germany.

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Abstract

We discuss two models from the literature that have been developed to formulate piecewise linear approximation of separable nonlinear functions by way of mixed-integer programs. We show that the most commonly proposed method is computationally inferior to a lesser known technique by comparing analytically the linear programming relaxations of the two formulations. A third way of formulating the problem, that shares the advantages of the better of the two known methods, is also proposed.

Introduction

Applications of linear programming technology often require the modeling of nonlinearities in the objective function or in some of the constraints of an otherwise linear optimization model. Such nonlinearities may come about due to economies or diseconomies of scale, "kinked" demand or production cost curves, etc. Already in the early 1950's it has been recognized that such occurrences can be dealt with adequately by approximating nonlinearities by piecewise linear functions and modeling these in a mixed-integer framework by introducing new 0-1 variables; see e.g. Balinski and Spielberg (1969) for an overview and historical references. Most textbooks in Operations Research/Integer Programming, see e.g. Nemhauser and Wolsey (1988), Wagner (1969) and others, offer one or two possibilities of expressing piecewise linear approximations of separable nonlinear functions in this manner.

The two classical formulations (Model I and II, below) can be found e.g. in Dantzig (1963). We are, however, not aware of a discussion of the *quality* or *thightness* of the various formulations that have been proposed a long while ago. Such considerations play indeed an essential role when the resulting mixed zero-one program is subsequently solved by branch-and-bound or branch-and-cut using linear programming algorithms in the solution process.

Besides reviewing the two classical formulations, the issue of the quality of the formulation is what we address here. We show that an analytical comparison of the two different formulations of the problem reveals that one of them is always inferior to the other, i.e., that the linear programming relaxation produces *always* worse bounds in one case than in the other. We then propose a third way of formulating piecewise linear approximation *via* a mixed zero-one program that shares the (local) properties of the better of the two classical formulations.

Let $\phi(x_1, \ldots, x_n)$ be any separable, nonlinear function from $\mathbb{R}^n$ into $\mathbb{R}$. Separability means that we assume that the function can be written as

$$\phi(x_1, \ldots, x_n) = \sum_{j=1}^{n} \phi_j(x_j) \ , \tag{1}$$

where each $\phi_j(x_j)$ maps $\mathbb{R}$ into $\mathbb{R}$. Given finite intervals $[a_0^j, a_u^j]$ for each variable x_j where $j \in \{1, \ldots, n\}$ we approximate each $\phi_j(x_j)$ by a piecewise linear function $\widehat{\phi}_j(x_j)$ over this interval. To do so we choose a partitioning $a_0^j < a_1^j < a_2^j < \ldots < a_{k_j}^j = a_u^j$ of the interval $[a_0^j, a_u^j]$ for each $j \in \{1, \ldots, n\}$, see Figure 1. It is well-known that by refining the partitioning, i.e., by choosing k_j large enough and the distance between any two consecutive points of the partitioning small enough, we can –under certain technical conditions– approximate $\phi_j(x_j)$ arbitrarily closely by such piecewise linear functions. We denote by

$$b_\ell^j = \phi_j(a_\ell^j) \quad \text{for } 0 \leq \ell \leq k_j \ , \tag{2}$$

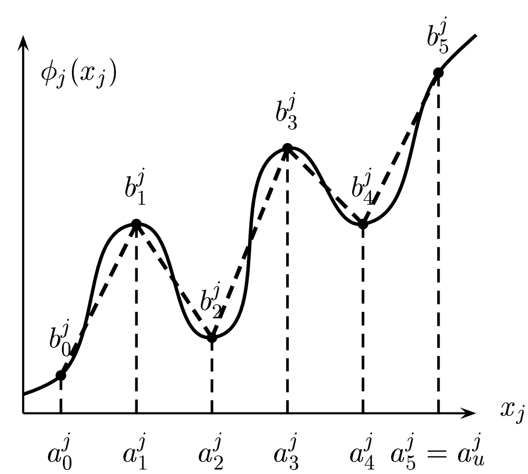

Figure 1: Piecewise linear approximation

the function values at the points a_ℓ^j which we can calculate where $j \in \{1, \ldots, n\}$. In the following we drop the index j for notational convenience because we will consider a single term of the right-hand of (1) only. Of course, as there are typically constraints linking the variables $x_1, \ldots, x_n$

our proceeding is a "local" analysis. What are looking for is a "locally ideal" formulation of the approximation problem, i.e., a formulation that in the absence of other constraints models the problem perfectly; see Padberg and Rijal (1996) for more detail.

1. The first model

In Model I we write for the single, continuous variable x

$$x = a_0 + y_1 + \ldots + y_k \ .$$

We require that each y_ℓ is a continuous variable satisfying

$$(i) \quad 0 \le y_\ell \le a_\ell - a_{\ell-1} \ \text{ for } 1 \le \ell \le k$$

and moreover, the following dichotomy:

$$(ii) \quad \text{either} \quad y_i = a_i - a_{i-1} \ \text{ for } 1 \le i \le \ell \ \text{ or } \quad y_{\ell+1} = 0 \ \text{ for } 1 \le \ell \le k-1.$$

Assuming that this can be "formulated" conveniently, we then get the piecewise linear approximation for any term of the right-hand side of (1) by way of

$$\widehat{\phi}(x) = b_0 + \frac{b_1 - b_0}{a_1 - a_0} y_1 + \frac{b_2 - b_1}{a_2 - a_1} y_2 + \ldots + \frac{b_k - b_{k-1}}{a_k - a_{k-1}} y_k \ .$$

From (i) it follows that (ii) can be replaced by the requirement

$$(ii') \quad \text{either} \quad y_i \ge a_i - a_{i-1} \ \text{ for } 1 \le i \le \ell \ \text{ or } \quad y_{\ell+1} \le 0 \ \text{ for } 1 \le \ell \le k-1.$$

To formulate this in linear inequalities using integer variables we introduce zero-one variables z_ℓ and consider the mixed zero-one model

$$x = a_0 + \sum_{\ell=1}^{k} y_\ell \ , \quad \widehat{\phi}(x) = b_0 + \sum_{\ell=1}^{k} \frac{b_\ell - b_{\ell-1}}{a_\ell - a_{\ell-1}} y_\ell \ , \tag{3}$$

$$y_1 \le a_1 - a_0 \ , \qquad y_k \ge 0 \ , \tag{4}$$

$$y_\ell \ge (a_\ell - a_{\ell-1}) z_\ell \ , \quad y_{\ell+1} \le (a_{\ell+1} - a_\ell) z_\ell \ \text{ for } 1 \le \ell \le k-1 \ , \tag{5}$$

where $z_\ell \in \{0,1\}$ for $1 \le \ell \le k-1$ are the "new" 0-1 variables. For $k = 1$ there is no need for a zero-one variable and (3), (4) describe the linear approximation correctly. For $k = 2$ the correctness follows by examining the two cases where $z_1 = 0$ and $z_1 = 1$, respectively. The correctness of the mixed zero-one model (3), ... , (5) for the piecewise linear approximation of a nonlinear function follows by induction on k.

It follows from (5) that every solution to (4) and (5) satisfies automatically

$$1 \ge z_1 \ge z_2 \ge \ldots \ge z_{k-1} \ge 0,$$

thus the upper and lower bounds on the 0-1 variables are not required in the formulation. In a computer model, however, we would declare the variables z_ℓ to be "binary" variables rather than general "integer" variables. A similar remark applies to the 0-1 variables of the second and third model below. In Theorem 1 (below) we prove that Model I is a locally ideal formulation for piecewise linear approximation.

2. The second model

In Model II –which is the only one that one finds e.g. in Nemhauser and Wolsey (1988)– we exploit the fact that given a partitioning $a_0 < a_1 < \ldots < a_k = a_u$ every real $x \in [a_0, a_u]$ can be written uniquely as a convex combination of at most two consecutive points a_ℓ, $a_{\ell+1}$ of the partitioning. Thus we write for the continuous variable x

$$x = a_0\xi_0 + a_1\xi_1 + \ldots + a_k\xi_k \ ,$$

where we require that the continuous variables ξ_ℓ satisfy

$$(i) \quad \sum_{\ell=0}^{k} \xi_\ell = 1 \ , \quad \xi_\ell \geq 0 \ \text{ for } 0 \leq \ell \leq k,$$

$$(ii) \quad \text{at most two consecutive } \ \xi_\ell \ \text{ and } \ \xi_{\ell+1}, \text{ say, are positive.}$$

If the requirement (ii) can be expressed conveniently using integer variables, we then get the piecewise linear approximation for any term of the right-hand side of (1) by way of

$$\widehat{\phi}(x) = b_0\xi_0 + b_1\xi_1 + \ldots + b_k\xi_k \ .$$

To formulate (i) and (ii) as the set of solutions to a mixed zero-one program we introduce 0-1 variables η_ℓ for $0 \leq \ell \leq k-1$ and consider the model

$$x = \sum_{\ell=0}^{k} a_\ell\xi_\ell \ , \quad \widehat{\phi}(x) = \sum_{\ell=0}^{k} b_\ell\xi_\ell \ , \tag{6}$$

$$0 \leq \xi_0 \leq \eta_0 \ , \ 0 \leq \xi_\ell \leq \eta_{\ell-1} + \eta_\ell \ \text{ for } 1 \leq \ell \leq k-1 \ , \quad 0 \leq \xi_k \leq \eta_{k-1} \ , \tag{7}$$

$$\sum_{\ell=0}^{k} \xi_\ell = 1 \ , \quad \sum_{\ell=0}^{k-1} \eta_\ell = 1 \ , \tag{8}$$

$$\eta_\ell \geq 0 \ \text{ for } 1 \leq \ell \leq k-2 \ , \tag{9}$$

where $\eta_\ell \in \{0,1\}$ for $0 \leq \ell \leq k-1$ are the "new" 0-1 variables. Note that the nonnegativity of η_0 and η_{k-1} is implied by (7). For $k = 1$ the formulation (6), ... , (9) of the problem at hand is evidently correct. The correctness of Model II for arbitrary $k \geq 1$ follows inductively.

3. Comparison of Model I and Model II

In the following we **assume** that $k \geq 3$, because for $k \leq 2$ either model is locally ideal. Model I has k real variables and $k-1$ 0-1 variables, while Model II has $k+1$ real variables and k 0-1 variables. To compare the two models we use the equations (8) to eliminate ξ_0 and η_0 from the formulation. Using the variable transformation for the remaining continuous variables

$$y_\ell = (a_\ell - a_{\ell-1}) \sum_{j=\ell}^{k} \xi_j \ \text{ for } 1 \leq \ell \leq k$$

and its inverse mapping that we calculate to be

$$\xi_j = \frac{y_j}{a_j - a_{j-1}} - \frac{y_{j+1}}{a_{j+1} - a_j} \ \text{ for } 1 \leq j \leq k-1 \ , \quad \xi_k = \frac{1}{a_k - a_{k-1}} y_k \ ,$$

we obtain the following equivalent formulation of Model II :

$$x = a_0 + \sum_{\ell=1}^{k} y_\ell \ , \quad \widehat{\phi}(x) = b_0 + \sum_{\ell=1}^{k} \frac{b_\ell - b_{\ell-1}}{a_\ell - a_{\ell-1}} y_\ell \ , \tag{10}$$

$$y_1 \le a_1 - a_0 \ , \quad y_1 \ge (a_1 - a_0) \sum_{\ell=1}^{k-1} \eta_\ell \ , \tag{11}$$

$$(a_\ell - a_{\ell-1}) y_{\ell+1} \le (a_{\ell+1} - a_\ell) y_\ell \ \text{ for } 1 \le \ell \le k-1 \ , \tag{12}$$

$$\frac{y_1}{a_1 - a_0} - \frac{y_2}{a_2 - a_1} \le 1 - \sum_{\ell=2}^{k-1} \eta_\ell \ , \frac{y_\ell}{a_\ell - a_{\ell-1}} - \frac{y_{\ell+1}}{a_{\ell+1} - a_\ell} \le \eta_{\ell-1} + \eta_\ell \ \text{ for } 2 \le \ell \le k-1, \tag{13}$$

$$y_k \ge 0 \ , \quad y_k \le (a_k - a_{k-1}) \eta_{k-1} \ , \tag{14}$$

$$\eta_\ell \ge 0 \ \text{ for } 1 \le \ell \le k-2 \ , \tag{15}$$

where $\eta_\ell \in \{0, 1\}$ for $1 \le \ell \le k-1$. Note that (11) implies that $\sum_{\ell=1}^{k-1} \eta_\ell \le 1$ and thus $\sum_{\ell=j}^{k-1} \eta_\ell \le 1$ for all $1 \le j \le k-1$ and feasible 0-1 values η_ℓ, $1 \le \ell \le k-1$. Using the variable substitution

$$z_j = \sum_{\ell=j}^{k-1} \eta_\ell \ \text{ for } 1 \le j \le k-1 \ ,$$

which is *integrality preserving* because its inverse is given by

$$\eta_j = z_j - z_{j+1} \ \text{ for } 1 \le j \le k-2 \ , \quad \eta_{k-1} = z_{k-1} \ ,$$

the above constraints (11), ... , (15) can be written equivalently as follows:

$$y_1 \le a_1 - a_0 \ , \quad y_1 \ge (a_1 - a_0) z_1 \ , \tag{16}$$

$$(a_\ell - a_{\ell-1}) y_{\ell+1} \le (a_{\ell+1} - a_\ell) y_\ell \ \text{ for } 1 \le \ell \le k-1 \ , \tag{17}$$

$$\frac{y_1}{a_1 - a_0} - \frac{y_2}{a_2 - a_1} \le 1 - z_2 \ , \frac{y_\ell}{a_\ell - a_{\ell-1}} - \frac{y_{\ell+1}}{a_{\ell+1} - a_\ell} \le z_{\ell-1} - z_{\ell+1} \ \text{ for } 2 \le \ell \le k-1, \tag{18}$$

$$y_k \ge 0 \ , \quad y_k \le (a_k - a_{k-1}) z_{k-1} \ , \tag{19}$$

$$z_\ell - z_{\ell+1} \ge 0 \ , \ \text{ for } 1 \le \ell \le k-2 \ , \tag{20}$$

where for $\ell = k-1$ we simply let $z_k = 0$ in (18) and the integer variables z_ℓ are 0-1 valued for $1 \le \ell \le k-1$. It follows that the (equivalently) changed Model II has now the same variable set as Model I and we are in the position to *compare* the two formulations in the context of a linear programming based approach to the solution of the corresponding mixed-integer programming problem. Note that like in Model I the constraints (16), (19) and (20) imply that every feasible solution to (16), ... , (20) automatically satisfies $1 \ge z_1 \ge z_2 \ge \ldots \ge z_{k-1} \ge 0$.

We denote the linear programming (LP) relaxation of Model I by

$$F_{LP}^{I} = \{(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{\mathbf{2k-1}} : (\mathbf{y}, \mathbf{z}) \text{ satisfies } (\mathbf{4}) \text{ and } (\mathbf{5})\}. \tag{21}$$

Likewise we denote the LP relaxation of the (equivalently) changed Model II by

$$F_{LP}^{II} = \{(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{\mathbf{2k-1}} : (\mathbf{y}, \mathbf{z}) \text{ satisfies } (\mathbf{16}), \ldots, (\mathbf{20})\}. \tag{22}$$

It is an immediate consequence of the respective formulations that both F_{LP}^{I} and F_{LP}^{II} are *bounded* subsets and thus polytopes in $\mathbb{R}^{2k-1}$.

Theorem 1. *(i) Model I is locally ideal, i.e.,* $\mathbf{z} \in \{\mathbf{0}, \mathbf{1}\}^{\mathbf{k-1}}$ *for all extreme points in* F^I_{LP} *.*
(ii) F^I_{LP} *is properly contained in* F^{II}_{LP} *.* F^{II}_{LP} *has extreme points* $(\mathbf{y}, \mathbf{z})$ *with* $\mathbf{z} \notin \{\mathbf{0}, \mathbf{1}\}^{\mathbf{k-1}}$*.*

Proof. We scale the continuous variables of Model I by introducing new variables

$$y'_\ell = y_\ell/(a_\ell - a_{\ell-1}) \quad \text{for } 1 \le \ell \le k. \tag{23}$$

The constraint set defining F^I_{LP} can thus be written as

$$y'_1 \le 1, \;\; y'_k \ge 0, \;\; y'_\ell \ge z_\ell, \;\; y'_{\ell+1} \le z_\ell \quad \text{for } 1 \le \ell \le k-1. \tag{24}$$

It follows that the constraint matrix given by (24) is totally unimodular and hence by Cramer's rule every extreme point of the feasible given by (24) has all components equal to zero or one. This implies (i).

(ii) Let $(\mathbf{y}, \mathbf{z}) \in \mathbf{F^I_{LP}}$, i.e., $(\mathbf{y}, \mathbf{z})$ satisfies (4) and (5). Then $(\mathbf{y}, \mathbf{z})$ satisfies (16) and (19) trivially. From (5) and $a_\ell - a_{\ell-1} > 0$ for all $1 \le \ell \le k$ we calculate

$$(a_{\ell+1} - a_\ell)y_\ell \ge (a_{\ell+1} - a_\ell)(a_\ell - a_{\ell-1})z_\ell \ge (a_\ell - a_{\ell-1})y_{\ell+1}$$

for $1 \le \ell \le k-1$ and thus (17) is satisfied. From (4) and (5) we have $y_1 \le a_1 - a_0$ and $y_2 \ge (a_2 - a_1)z_2$ and thus the first relation of (18) follows. Again from (5) we have $y_\ell \le (a_\ell - a_{\ell-1})z_{\ell-1}$ and $y_{\ell+1} \ge (a_{\ell+1} - a_\ell)z_{\ell+1}$ for all $2 \le \ell \le k-1$, where $z_k = 0$, and thus combining the two inequalities we see that (18) is satisfied. As we have noted in the discussion of Model I every $(\mathbf{y}, \mathbf{z}) \in \mathbf{F^I_{LP}}$ satisfies $1 \ge z_1 \ge z_2 \ge \ldots \ge z_{k-1} \ge 0$ and thus (20) is satisfied as well. Consequently, $(\mathbf{y}, \mathbf{z}) \in \mathbf{F^{II}_{LP}}$ and thus $F^I_{LP} \subseteq F^{II}_{LP}$. Let $(\mathbf{y}, \mathbf{z}) \in \mathbf{F^{II}_{LP}}$ be such that $\mathbf{z} \in \{\mathbf{0}, \mathbf{1}\}^{\mathbf{k-1}}$. It follows from (16), ... , (20) that $y_i = a_i - a_{i-1}$ for $i = 1, \ldots, h$, $0 \le y_{h+1} \le a_{h+1} - a_h$, $y_i = 0$ for $i = h+2, \ldots, k$, $z_i = 1$ for $i = 1, \ldots, h$, $z_i = 0$ for $i = h+1, \ldots, k-1$ where $0 \le h \le k-1$. From (4) and (5) thus $(\mathbf{y}, \mathbf{z}) \in \mathbf{F^I_{LP}}$. Now consider $(\mathbf{y}, \mathbf{z})$ given by $y_1 = (a_1 - a_0)/2$, $y_j = 0$ for $2 \le j \le k$, $z_1 = z_2 = 1/2$, and $z_j = 0$ for $3 \le j \le k-1$. It follows that $(\mathbf{y}, \mathbf{z})$ satisfies (16), ... , (20), i.e., $(\mathbf{y}, \mathbf{z}) \in \mathbf{F^{II}_{LP}}$. But $(\mathbf{y}, \mathbf{z})$ violates the constraint $y_2 \ge (a_2 - a_1)z_2$ of Model I and thus $(\mathbf{y}, \mathbf{z}) \notin \mathbf{F^I_{LP}}$. Since F^{II}_{LP} is a polytope, it follows that it has extreme points $(\mathbf{y}, \mathbf{z})$ with $\mathbf{z} \notin \{\mathbf{0}, \mathbf{1}\}^{\mathbf{k-1}}$. ∎

By Theorem 1 F^{II}_{LP} has extreme points with fractional components for $\mathbf{z}$ and indeed it has many such extreme points. It is not overly difficult to characterize all of them, which we leave as a good exercise for graduate students. It is amazing that most textbooks treat only Model II in the context of using mixed-integer programming to approximate separable nonlinear functions by piecewise linear ones.

Model I, which has been known since the 1950's, is *locally* far better than Model II since all of its extreme points $(\mathbf{y}, \mathbf{z})$ satisfy $\mathbf{z} \in \{\mathbf{0}, \mathbf{1}\}^{\mathbf{k-1}}$. Of course, this does not mean that the "overall" model –of which the piecewise linear approximation is but a part– has the same property. But the proper inclusion $F^I_{LP} \subset F^{II}_{LP}$ shows that the linear programming bound obtained from using Model I must always be equal to or better than the one obtained from Model II in any case, i.e., even in the *worst* case.

It is now an easy exercise to derive *ex post* a formulation of Model II in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ variables that guarantees the same outcome as Model I. This may sometimes be desirable because all variables of Model II assume values between zero and one. (The same effect can, of course, also be obtained by *scaling* the continuous variables of Model I like we did in the proof of Theorem 1; see (23).) We leave it as an exercise to prove that the following Model III is a correct formulation, which is obtained from Model I by reversing the various transformations that we have used to analyze Model II.

$$x = \sum_{\ell=0}^{k} a_\ell \xi_\ell, \quad \widehat{\phi}(x) = \sum_{\ell=0}^{k} b_\ell \xi_\ell, \tag{25}$$

$$\sum_{\ell=0}^{k} \xi_\ell = 1, \quad \sum_{\ell=0}^{k-1} \eta_\ell = 1, \tag{26}$$

$$\sum_{j=\ell}^{k-1} \eta_j \geq \sum_{j=\ell+1}^{k} \xi_j \geq \sum_{j=\ell+1}^{k-1} \eta_j \text{ for } 1 \leq \ell \leq k-2, \tag{27}$$

$$0 \leq \xi_0 \leq \eta_o, \quad 0 \leq \xi_k \leq \eta_{k-1}, \tag{28}$$

where $\eta_\ell \in \{0,1\}$ for $0 \leq \ell \leq k-1$.

Evidently, Model III has at first sight little resemblance to the original Model II except that the same set of variables is used. More precisely, let

$$P_{LP} = \{(\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathbb{R}^{2k+1} : (\boldsymbol{\xi},\boldsymbol{\eta}) \text{ satisfies (7), (8), (9)}\},$$
$$P_{LP}^{\#} = \{(\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathbb{R}^{2k+1} : (\boldsymbol{\xi},\boldsymbol{\eta}) \text{ satisfies (26), (27), (28)}\},$$

be the linear programming relaxation of Model II and III, respectively. It follows that

$$P_{LP}^{\#} \subset P_{LP} \text{ and } P_{LP}^{\#} = conv(P_{LP} \cap (\mathbb{R}^{k+1} \times \mathbb{Z}^k)).$$

By construction, Model III shares *locally* the property of Model I of having all its extreme points $(\boldsymbol{\xi},\boldsymbol{\eta})$ satisfy $\boldsymbol{\eta} \in \{0,1\}^k$. Model III can be used in lieu of Model I, but Model II should definitely be abandoned despite its popularity in the textbooks. Model II just happens to be a poor formulation for the piecewise linear approximation problem when linear programming methods are used.

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