A Small Sample Study of Goodness-of-fit Tests for Time Series Models

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Abstract

We study the small sample behaviour of two goodness-of-fit tests for time series models which have been proposed recently in the literature. Both tests are generalizations of the popular Box-Ljung-Pierce portmanteau test, one in the time domain and the other in the frequency domain. The tests are found to be oversized under the null of white noise but undersized under other null hypotheses. The cause for this effect is investigated and a finite sample correction proposed which ameliorates this effect. It is found that the corrected versions of the tests have markedly better size properties. The correction is also found to result in an overall increase in power which can be significant in certain alternatives. Furthermore, the corrected tests also have uniformly better power than the Box-Ljung-Pierce portmanteau test, unlike the uncorrected versions.

Keywords: frequency domain, portmanteau test.

1 Introduction

A popular goodness-of-fit test in time series is the Box-Pierce test (1970) given by

$$BP_n = n \sum_{i=1}^{p_n} \hat{\rho}_i^2$$

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and its asymptotically equivalent modified version, the Box-Ljung-Pierce (1978) test

$$B_n = n(n+2)\sum_{i=1}^{p_n}(n-i)^{-1}\hat{\rho}_i^2, \tag{1}$$

where $\hat{\rho}_i$ is the i^{th} sample correlation of the residuals from the fitted model and p_n is such that $p_n \to \infty$ and $p_n/n \to 0$. Hong (1996) proposed a generalization of the Box-Pierce test, given by

$$H_n = n\sum_{i=1}^{p_n} k^2\left(\frac{i}{p_n}\right)\hat{\rho}_i^2,$$

where $k(\cdot)$ is a suitably chosen kernel. The statistic BP_n is a particular version of H_n, obtained by using the truncated uniform kernel $k(x) = I(|x| \leq 1)$. The choice of kernels $k(x)$ which decay for large x will downweight the importance given to correlations at high lags which are estimated less efficiently.

A frequency domain version of the Hong test was proposed recently by Chen and Deo (2000) as follows. Given a kernel $k(\cdot)$, compute the spectral window $W(\cdot)$ as

$$W(\lambda) = \frac{1}{2\pi}\sum_{|j|<n} k(j/p_n)\,e^{-ij\lambda} \qquad -\pi \leq \lambda \leq \pi.$$

The statistic is then

$$T_n = \left\{\frac{2\pi}{n}\sum_{i=0}^{n-1}\hat{f}(\lambda_i)\right\}^{-2}\left\{\frac{2\pi}{n}\sum_{i=0}^{n-1}\hat{f}^2(\lambda_i)\right\},$$

where

$$\hat{f}(\lambda) = \frac{2\pi}{n}\sum_{j=1}^{n-1}\frac{W(\lambda-\lambda_j)\,I(\lambda_j)}{f(\lambda_j)},$$

$f(\cdot)$ is the spectral density of the fitted model and $I(\lambda) = (2\pi n)^{-1}\left|\sum_{t=1}^{n} x_t \exp(-it\lambda)\right|^2$ is the periodogram of the observations x_t. The statistic T_n has the advantage of whitening the data in the frequency domain and does not need an easily obtainable autoregressive representation of the observations x_t to compute the time domain residuals. Chen and Deo (henceforth CD) proved that H_n and $n\pi T_n - 0.5n$ have the same asymptotic null distribution and hence are asymptotically equivalent. However, CD obtained the asymptotic distribution of T_n under null hypotheses which allow the spectral density $f(\cdot)$ of the fitted model to be unbounded at the origin. This encompasses long memory models such as the Autoregressive Fractionally Integrated Moving Average (ARFIMA) models (see Hosking, 1981) which have hyperbolically decaying correlations. The distributions of H_n and BP_n under such long memory null hypotheses are not yet known.

In their simulation study, CD found that the finite sample size and power performance of H_n and T_n were very similar under a variety of null and alternative hypotheses and when using different kernels $k(\cdot)$. However, they found that both the tests were oversized when the null hypothesis was that of white noise. On the other hand, when the null hypothesis was not white noise, both tests were under sized, in some cases quite seriously. In the next section, we investigate the cause of this phenomenon and then propose a small sample correction to rectify it. The effect of our correction on the tests is then studied in section 3 through a Monte Carlo study.

2 Small Sample Behaviour

Hong (1996) established that when $p_n \to \infty$ and $p_n = o(n)$,

$$\left(\frac{H_n - \mu_n(k)}{\sigma_n(k)}\right) \xrightarrow{D} N(0,1), \tag{2}$$

where

$$\mu_n(k) = \sum_{i=1}^{n-1} (1 - i/n)\, k^2(i/p_n) \tag{3}$$

and

$$\sigma_n^2(k) = 2\sum_{i=1}^{n-2} (1 - i/n)(1 - (i+1)/n)\, k^4(i/p_n). \tag{4}$$

Since, as noted earlier, CD have shown that H_n is asymptotically equivalent to $n\pi T_n - 0.5n$, we also have

$$\left(\frac{n\pi T_n - 0.5n - \mu_n(k)}{\sigma_n(k)}\right) \xrightarrow{D} N(0,1). \tag{5}$$

In all of these results, it is assumed that $k(\cdot)$ is continuous with $k(0) = 1$ and that $k(x) \to 0$ as $x \to \infty$.

The mean and variance of H_n may be obtained by treating all the correlations $\hat{\rho}_i$ as independent normal random variables with mean zero and variance $n^{-2}(n-i)$ and then noting that H_n is a simple quadratic form in them. The pretense in this heuristic argument that $\hat{\rho}_i$ is normally distributed with variance $n^{-2}(n-i)$ follows from the belief that the correlation $\hat{\rho}_i$ of the residuals is asymptotically equivalent to that of the innovations, $\hat{r}_i$, which has mean zero and variance $n^{-2}(n-i)$.

However, Box-Pierce (1970) showed that for fixed h and large n and an idempotent matrix $\mathbf{Q}$,

$$\hat{\boldsymbol{\rho}} \approx (\mathbf{I}-\mathbf{Q})\,\hat{\mathbf{r}}, \tag{6}$$

where $\hat{\boldsymbol{\rho}} = (\hat{\rho}_1, ..., \hat{\rho}_h)'$ and $\hat{\mathbf{r}} = (\hat{r}_1, ..., \hat{r}_h)'$ are the sample correlations of the residuals from the fitted model and the true innovations respectively. The matrix $\mathbf{Q}$ was shown to be of rank p, where p was the number of parameters fitted and its elements depend on the structure of the underlying model. Thus, for any fixed i, the correlation $\hat{\rho}_i$ of the residuals is not asymptotically identical to the corresponding correlation $\hat{r}_i$ of the innovations. If p_n is not very large, it is then easy to see that using the expressions μ_n and σ_n above will result in distortions in size and also affect power. To obtain better approximations to the mean and variance of H_n, we have to thus exploit (6).

For ease of exposition in what follows, we will assume that the variance of $\hat{r}_i$ is n^{-1} rather than $n^{-2}(n-i)$. Then, $Var(\hat{\boldsymbol{\rho}}) \approx n^{-1}\mathbf{I}$ and hence $E(H_n) = tr\left\{\mathbf{K}^2(\mathbf{I}-\mathbf{Q})\right\}$, where $\mathbf{K} = diag(k_1, ..., k_{n-1})$. Letting q_{ii} denote the i^{th} diagonal entry of $\mathbf{Q}$, we get

$$tr\left\{\mathbf{K}^2(\mathbf{I}-\mathbf{Q})\right\} = \sum_i k_i^2 - \sum_i k_i^2 q_{ii} = \sum_i k_i^2 - \sum_i q_{ii} - \sum_i \left(k_i^2 - 1\right) q_{ii}.$$

Since $\mathbf{Q}$ is idempotent with rank p, we have $\sum_i q_{ii} = tr(\mathbf{Q}) = p$. Furthermore, from equation (2.31) of Box-Pierce (1970), we note that

$$q_{ij} \to 0 \qquad as \;\; \max(i,j) \to \infty. \tag{7}$$

Since $k(x) \approx 1$ for x close to 0, it follows that $\sum_i (k_i^2 - 1)\, q_{ii}$ is negligible. Hence,

$$E(H_n) \approx \sum_{i=1}^{n-1} k_i^2 - p. \tag{8}$$

It is immediately apparent from this approximation that the mean μ_n in (3) will overestimate the mean of H_n. This is the cause of the tests based on H_n and T_n being undersized when a model was fit, as found in CD.

Similarly, we have

$$\begin{aligned}
Var\left(H_n\right) &= 2\,tr\left\{\mathbf{K}^2\left(\mathbf{I}-\mathbf{Q}\right)\mathbf{K}^2\left(\mathbf{I}-\mathbf{Q}\right)\right\} \\
&= 2\left\{\sum_i k_i^4 - 2\sum_i k_i^4 q_{ii} + \sum_{i,j} k_i^2 k_j^2 q_{ij}\right\} \\
&= 2\sum_i k_i^4 - \sum_i q_{ii} - 4\sum_i\left(k_i^2-\frac{1}{2}\right)^2 q_{ii} + 2\sum_{i,j}\left(k_i^2-1\right)\left(k_j^2-1\right)q_{ij} \\
&\approx 2\sum_i k_i^4 - p - 4\sum_i\left(k_i^2-\frac{1}{2}\right)^2 q_{ii}, \qquad (9)
\end{aligned}$$

where we have once again used (7) and the fact that $k\left(x\right)\approx 1$ for x close to 0. Note that the last term in (9) is not negligible, since $k_i^2 - 1/2$ will not be close to zero for small i. However, since this term is negative, using $2\sum_i k_i^4 - p$ as an approximation for the variance of H_n will be a conservative measure. From this argument, we also see that the variance σ_n^2 in (4) will overestimate the true variance of H_n causing the test based on H_n to be undersized. The approximation $2\sum_i k_i^4 - p$ that we obtain above for the variance of H_n will also tend to overestimate it but not by as much as σ_n^2.

In the development above if we had assumed, more appropriately, that the variance of $\hat{r}_i$ is $n^{-2}\left(n-i\right)$, then similar though more tedious arguments show that

$$E\left(H_n\right) \approx \mu_{n,f} \equiv \sum_{i=1}^{n-1}\left(1-i/n\right)k_i^2 - p \qquad (10)$$

and

$$Var\left(H_n\right) \approx \sigma_{n,f}^2 \equiv 2\sum_i\left(1-i/n\right)\left(1-\left(i+1\right)/n\right)k_i^4 - p. \qquad (11)$$

We propose that (10) and (11) be used instead of μ_n and σ_n in computing the statistic based on H_n.

As noted above, CD had also found that the tests H_n and T_n were oversized when the null of white noise was being tested. This may be attributed to the fact that these tests were essentially quadratic forms in normal variables and though asymptotically normal, would have finite sample distributions that were right skewed. A simple transformation that would help improve the normal approximation for such variables is the square root transformation, which we suggest be taken before the tests are carried out. A standard delta method argument shows that

$$\frac{2\sqrt{\mu_{n,f}}}{\sigma_{n,f}}\left(\sqrt{H_n}-\sqrt{\mu_{n,f}}\right)\xrightarrow{D}N\left(0,1\right) \tag{12}$$

and since H_n is asymptotically equivalent to $n\pi T_n - 0.5n$, it also follows that

$$\frac{2\sqrt{\mu_{n,f}}}{\sigma_{n,f}}\left(\sqrt{n\pi T_n-0.5n}-\sqrt{\mu_{n,f}}\right)\xrightarrow{D}N\left(0,1\right). \tag{13}$$

In the next section, we study the effects of the mean and variance corrections as well as that of the square root transformation through Monte Carlo simulations.

3 Simulation study

We generated 5000 replications of Gaussian series of length $n = 128$ and 512 from a variety of AR and ARFIMA processes. The AR(1) processes were generated by drawing the initial observation from the marginal stationary distribution of the process. To generate ARFIMA$(1, d, 0)$, we first generated observations from an ARFIMA$(0, d, 0)$ using the algorithm of Davies and Harte (1987). These observations were then used as innovations in the AR(1). In such cases, the AR(1) was initialized from 0 and then the first n observations were discarded.

For each series, we computed five statistics: (i) The Box-Ljung-Pierce statistic B_n given in (1). (ii) Hong's uncorrected statistic H_n given in (2). (iii) The uncorrected Chen-Deo statistic T_n from (5). (iv) The corrected Hong statistic, denoted here by H_n', from (12). (v) The corrected Chen-Deo statistic, denoted here by T_n', from (13). The following three kernels were used in computing the Hong and the Chen-Deo statistics:

(i) Bartlett $k(z) = 1 - |z|$, $|z| \leq 1$,
$\quad = 0$ otherwise,

(ii) Tukey $k(z) = \frac{1}{2}\left(\cos(z\pi) + 1\right)$, $|z| \leq 1$,
$\quad = 0$ otherwise,

(iii) Quadratic Spectral (QS), $k(z) = \frac{25}{12z^2}\left(\frac{\sin(6\pi z/5)}{6\pi z/5} - \cos(6\pi z/5)\right)$, $z \in (-\infty, \infty)$.

In computing the tests we used three bandwidths, $p_n = \left[3n^{0.2}\right]$, $\left[3n^{0.3}\right]$ and $\left[3n^{0.4}\right]$. The sample sizes and bandwidths we have chosen here are identical to those used in CD. In all computations of size and power, the test B_n was compared to a $\chi^2_{p_n-p}$ distribution, where p

was the number of estimated parameters, while the other tests were compared to the standard normal distribution. In computing tests based on residuals from the fitted model, the residuals were computed by truncating the infinite autoregressive representation of the process.

In Tables I, II and III, we report the sizes of all the tests under the null hypothesis of white noise, an AR(1) and an ARFIMA(0,d,0) respectively. The AR(1) parameter was set to 0.8 while the long memory parameter d in the ARFIMA$(0, d, 0)$ was set at 0.4. It is seen from Table I that both the uncorrected tests T_n and H_n as well as B_n are oversized under the white noise null. This effect is significant even at samples as large as 512. Furthermore, the amount by which they are oversized increases as the bandwidth p_n increases. On the other hand, when the null is not white noise, as in Tables II and II, both the tests T_n and H_n are undersized. The B_n test, on the other hand continues to be oversized.

A visual understanding of this phenomenon can be obtained from the plot on the left in Figure 1. We have made boxplots of the 5000 replications of the T_n statistic using the Tukey kernel for $p_n = 13$, $n = 128$. In the white noise case, the distribution of T_n is seen to have a median of roughly 0, but is extremely right skewed which explains why the test is oversized. On the other hand, the distribution of T_n under the AR(1) and ARFIMA$(0, d, 0)$ null, though still right skewed, has a median which is much below zero. This shift in location is significant enough to compensate for the right skewness and cause the tests to be undersized.

On the other hand, the corrected tests based on the Tukey and QS kernel in Tables I, II and III have much better size properties. In the case of white noise, the square root transformation reduces the skewness and hence the size, whereas in the other two cases, the mean and variance adjustment also corrects the bias in the tests. This can be seen visually in the boxplots on the right side of Figure 1. The distribution of the corrected T_n test using the Tukey kernel is seen to be centered almost around zero and the skewness has been drastically reduced.

The corrected tests based on the Bartlett kernel however tend to be oversized in the cases when the null is not white noise. This is due to the fact that the Bartlett kernel is tent shaped and hence drops off rapidly from 1 near the origin. This non-smoothness causes our approximations in (10) and (11) to be poor, resulting in the oversized tests. The Tukey and the QS kernel however do not drop off rapidly from 1 near the origin and the tests based on them are well behaved.

The effect of the square root transformation can be seen independently in Figure 2. We have made normal probability plots of the T_n statistic before and after the square root transformation in the case of white noise. It is seen that though the transformation does not achieve normality, it goes a long way towards reducing the extreme right skewness of the distribution.

To compare the power of the tests, we considered the following four cases: (a) fitting an AR(1) to data generated by an AR(2), $x_t = 0.8x_{t-1} - 0.1x_{t-2} + \varepsilon_t$. (b) fitting an ARFIMA$(1, d, 0)$ to data generated by an ARMA(1,1), $x_t = 0.8x_{t-1} + \varepsilon_t + 0.2\varepsilon_{t-1}$. (c) fitting an ARMA$(1, 1)$ to data generated by an ARFIMA$(0, d, 0)$, $(1 - B)^{0.4}\, x_t = \varepsilon_t$ where B denotes the backshift operator (d) fitting an ARFIMA$(0, d, 0)$ to data generated by an ARFIMA$(1, d, 0)$, $(1 - B)^{0.4}\,(1 - 0.1B)\, x_t = \varepsilon_t$. The results are reported in tables IV, V, VI and VII respectively.

As observed in CD, the power of T_n is similar to that of H_n in all the alternatives considered, irrespective of the choice of kernel. However, the power of these two tests can be quite different from that of B_n, depending on the alternative. Neither of these two tests dominates B_n clearly. However, the use of the corrected tests T_n' and H_n' changes this. The corrected tests have significantly higher power than their uncorrected versions. Though the corrected tests based on the Bartlett kernel show dramatic improvement, this should be discounted since the these tests are oversized as noted above. However, the corrected tests based on the Tukey and QS kernel also show significant improvement in power that cannot be disregarded. Furthermore, these corrected tests now outperform B_n uniformly in all the alternatives considered. The corrections also can have a dramatic effect even for sample size $n = 512$, as seen from Tables V and VI.

The Monte Carlo study seems to suggest that the use of corrected versions T_n' and H_n' based on kernels which decline from 1 near the origin in a gradual manner will have sizes close to the nominal and also be more powerful than the standard Box-Ljung-Pierce portmanteau test. Furthermore, the frequency domain based test T_n' has the added advantage of computational simplicity and also has been theoretically justified for long memory models.

REFERENCES

Box, G. E. P., and Pierce, D. A. (1970), "Distribution of residual autocorrelations in autoregressive-integrated moving average time series models", *Journal of American Statistical Association*, **65**, 1509-26.

Chen, W. W. and Deo, R. S. (2000), "A generalized portmanteau goodness-of-fit test for time series models". Working Paper, New York University.

Davies, R. B. and Harte, D. S. (1987), "Tests for Hurst effect", *Biometrika*, **74**, 95-102.

Hong, Y. (1996), "Consistent testing for serial correlation of unknown form", *Econometrica*, **64**, 837-64.

Hosking, J. R. M. (1981), "Fractional differencing ", *Biometrika*, **68**, 165-76.

Ljung, G. M. and Box, G. E. P. (1978), "On a measure of lack of fit in time series models", *Biometrika*, **65**, 297-303.

Figure 1. Boxplots of Standard Nomalized T_n

n = 128, p_n= 13, Tukey Kernel

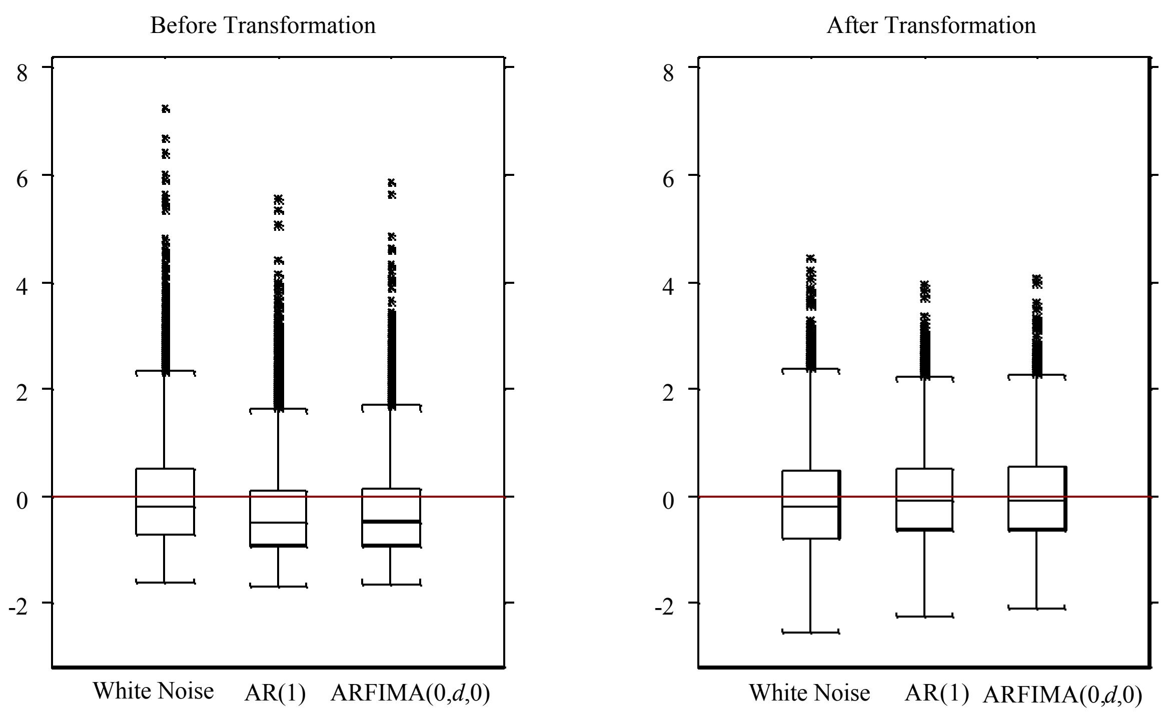

Figure 2. QQ Plots of T_n under Gaussian White Noises

$n = 128$, $p_n = 13$, Tukey Kernel

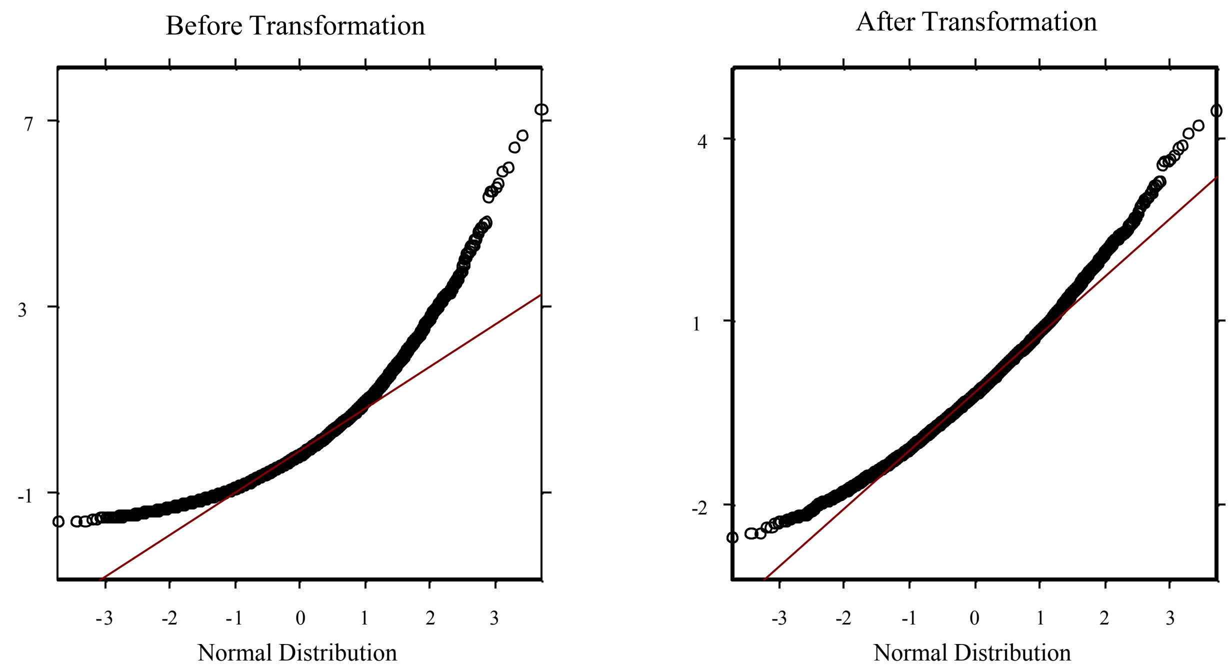

TABLE I

Rejection Rates in Percentage Under Normal White Noises

n		128						512					
p_n		8		13		21		11		20		37	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
T_n	BAR	7.22	10.16	7.56	10.98	8.26	13.04	6.98	10.80	7.46	11.32	7.64	12.16
	TUK	7.30	10.04	7.62	10.72	8.08	12.52	7.10	10.70	7.30	10.98	7.70	12.02
	QS	7.46	10.36	7.88	11.80	9.94	14.64	6.96	11.24	7.56	11.52	7.78	13.02
$\tilde{T}_n$	BAR	4.50	7.82	5.14	8.92	6.22	10.50	4.54	8.06	5.00	9.14	5.86	10.24
	TUK	4.42	7.66	5.00	8.58	5.96	10.08	4.52	8.08	4.76	9.00	5.64	10.32
	QS	4.90	8.26	5.52	9.74	7.22	12.30	4.58	8.42	5.42	9.70	5.98	11.26
H_n	BAR	6.82	9.74	6.90	10.46	7.68	11.04	6.88	10.44	7.14	10.80	7.02	11.48
	TUK	6.78	9.72	6.98	10.28	7.36	11.54	7.00	10.32	6.96	10.56	6.92	11.50
	QS	6.92	10.14	7.22	10.70	7.42	11.42	7.04	10.90	7.10	10.90	6.90	11.70
$\tilde{H}_n$	BAR	4.32	7.50	4.60	8.08	4.94	9.38	4.42	7.92	4.86	8.66	5.20	9.46
	TUK	4.10	7.32	4.40	8.00	4.62	8.92	4.44	7.86	4.80	8.62	5.04	9.68
	QS	4.28	7.72	4.64	8.46	5.44	9.66	4.46	8.12	5.06	9.16	5.16	9.94
B_n		5.26	10.20	6.36	11.26	7.42	12.38	5.40	10.56	5.72	10.42	6.12	11.16

Note: Model $x_t \sim N(0,1)$.

TABLE II

Rejection Rates in Percentage Under an AR(1) Model

n		128						512					
p_n		8		13		21		11		20		37	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
T_n	BAR	3.08	5.02	4.04	6.12	4.90	7.80	3.82	5.82	4.32	6.98	5.06	8.02
	TUK	3.04	4.96	4.04	6.12	6.30	9.68	3.98	5.82	4.56	7.10	5.16	8.40
	QS	3.64	5.64	4.52	6.90	5.04	7.74	4.06	6.52	4.74	7.64	5.58	9.26
$\tilde{T}_n$	BAR	6.84	12.40	5.94	10.36	6.54	11.06	6.10	10.50	5.56	9.86	5.90	10.40
	TUK	4.38	8.08	4.68	8.56	5.52	9.86	4.78	8.20	4.98	8.96	5.48	9.90
	QS	4.52	8.40	5.10	9.54	6.88	12.02	4.92	8.86	5.06	9.34	5.86	10.64
H_n	BAR	3.30	5.08	3.82	5.82	4.26	6.76	3.62	5.72	4.20	6.54	4.76	7.34
	TUK	3.16	4.90	3.78	5.92	4.46	6.96	3.76	5.78	4.26	6.84	4.88	7.48
	QS	3.52	5.52	4.22	6.44	4.82	7.40	4.02	6.20	4.36	7.12	5.08	8.36
$\tilde{H}_n$	BAR	6.98	12.34	5.72	9.78	5.56	9.74	5.96	10.58	5.22	9.58	5.58	9.56
	TUK	4.14	8.44	4.38	8.12	4.98	8.70	4.64	8.44	4.86	8.66	5.26	8.98
	QS	4.42	8.30	4.76	8.52	5.56	9.34	4.68	8.82	4.74	8.74	5.48	9.64
B_n		5.98	11.06	6.48	12.08	7.60	12.70	5.44	10.26	5.76	11.08	6.04	11.02

Note: Model $x_t - 0.8\, x_{t-1} = e_t$.

Table III

Rejection Rates in Percentage Under an ARFIMA (0,d,0) Model

n		128						512					
p_n		8		13		21		11		20		37	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
T_n	BAR	2.62	4.08	3.52	5.28	4.90	7.42	2.64	4.60	3.74	6.00	4.80	8.20
	TUK	2.52	4.00	3.46	5.58	4.96	7.50	2.92	4.78	3.86	6.14	5.10	8.42
	QS	3.22	4.98	4.34	6.78	6.62	9.60	3.30	5.74	4.40	7.06	5.58	9.08
$\tilde{T}_n$	BAR	5.44	10.26	5.16	9.70	6.30	10.46	4.92	8.94	5.04	9.16	5.78	10.42
	TUK	3.38	6.72	4.40	7.84	5.54	9.50	3.64	7.02	4.52	8.20	5.32	10.00
	QS	3.96	7.46	5.02	8.94	7.12	11.90	4.08	7.70	4.86	8.76	5.92	10.56
H_n	BAR	2.28	3.76	3.02	4.86	3.54	5.88	2.56	4.42	3.42	5.86	4.22	7.00
	TUK	2.20	3.52	3.20	5.10	3.90	5.88	3.12	5.32	4.14	6.52	4.70	7.86
	QS	2.82	4.46	3.66	5.36	4.10	7.04	2.72	4.54	3.70	5.98	4.44	7.44
$\tilde{H}_n$	BAR	5.08	9.90	4.70	8.24	4.72	8.58	4.62	8.74	4.64	8.56	4.96	9.14
	TUK	3.10	6.50	3.92	6.84	4.26	7.96	3.34	6.84	4.08	7.62	4.74	8.66
	QS	3.74	6.70	4.28	7.46	4.58	8.76	3.76	7.48	4.38	8.18	4.92	9.04
B_n		5.34	10.50	5.70	10.80	6.48	11.52	5.18	9.88	5.32	10.42	5.94	10.78

Note: Model $x_t =$ ARFIMA (0,d,0) with $d = 0.3$.

TABLE IV

Rejections Rates in Percentage under AR(2) Alternative
fitting Model AR(1)

n		128						512					
p_n		8		13		21		11		20		37	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
T_n	BAR	22.48	28.60	22.92	29.04	22.88	29.62	80.18	84.96	76.02	81.64	68.54	75.74
	TUK	21.94	28.16	22.80	28.58	22.40	28.76	79.96	82.18	74.76	81.06	65.66	73.26
	QS	22.44	28.64	22.80	28.96	22.74	29.74	78.20	83.42	70.56	77.68	61.04	69.96
$\tilde{T}_n$	BAR	34.06	44.68	28.50	38.66	26.58	35.62	85.40	90.78	79.24	86.26	71.08	79.36
	TUK	25.88	36.74	25.02	34.68	23.80	32.88	82.18	88.38	76.38	84.10	66.72	76.20
	QS	25.72	35.88	24.44	33.38	24.14	33.26	80.26	86.76	72.02	80.54	61.94	72.00
H_n	BAR	23.58	30.22	23.42	29.66	22.42	28.70	80.62	85.46	75.84	81.86	68.24	75.26
	TUK	23.18	29.36	23.22	29.42	21.98	28.22	80.32	85.24	74.90	80.90	65.02	72.66
	QS	23.28	29.90	22.76	28.54	21.12	27.40	78.34	83.96	70.46	77.14	59.60	68.88
$\tilde{H}_n$	BAR	35.74	46.58	29.14	38.96	26.08	34.64	85.86	90.86	79.28	86.34	70.52	78.56
	TUK	27.50	38.90	25.40	35.04	23.38	31.72	82.48	88.64	76.16	84.20	66.20	75.32
	QS	26.34	36.52	24.44	33.62	22.28	30.68	80.48	86.76	71.84	80.28	60.70	71.00
B_n		22.46	32.40	20.08	28.72	18.70	26.84	64.92	75.34	52.24	64.34	41.26	53.00

Note: Model $x_t - 0.8\,x_{t-1} + 0.15\,x_{t-2} = u_t$.

TABLE V

Rejections Rates in Percentage Under ARMA(1,1) Alternative
fitting Model ARIMA(1,*d*,0)

n		128						512					
p_n		8		13		21		11		20		37	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
T_n	BAR	9.50	13.38	8.44	12.58	8.96	13.04	31.84	40.74	28.36	36.56	25.10	34.24
	TUK	7.24	11.28	8.04	12.06	8.80	12.80	31.34	40.54	26.94	35.12	23.36	32.48
	QS	8.74	12.26	8.04	12.02	10.04	14.74	29.20	37.80	25.00	33.62	22.78	31.50
$\tilde{T}_n$	BAR	50.50	68.44	25.38	37.72	19.66	29.88	66.30	78.92	47.06	60.68	36.68	48.88
	TUK	16.00	30.34	16.70	27.26	14.76	23.62	50.30	65.78	38.18	52.64	31.08	43.76
	QS	18.10	29.72	15.68	25.34	15.46	24.98	44.82	59.56	34.50	48.02	28.96	40.46
H_n	BAR	12.68	17.04	11.28	15.52	8.92	13.20	33.02	42.28	28.94	37.36	24.82	33.70
	TUK	7.98	12.12	8.20	12.32	8.66	12.96	32.66	41.88	27.70	35.72	23.12	31.88
	QS	11.48	15.70	10.18	14.40	9.06	13.68	30.32	38.70	25.30	33.72	21.78	30.10
$\tilde{H}_n$	BAR	52.92	70.82	26.82	38.20	19.18	28.84	67.60	79.72	47.62	61.68	36.12	48.40
	TUK	17.26	32.70	17.24	27.92	14.74	23.22	52.04	67.00	38.76	52.94	30.88	43.06
	QS	18.86	30.92	15.76	24.84	14.46	22.08	45.54	60.58	34.84	43.38	27.78	39.44
B_n		15.12	24.58	14.08	22.30	13.00	20.56	30.10	44.14	23.40	36.46	19.28	29.44

Note: Model $x_t = 0.8x_{t-1} + u_t + 0.2\,u_{t-1}$.

TABLE VI

Rejections Rates in Percentage Under ARFIMA(0,*d*,0) Alternative
fitting Model ARMA(1,1)

n		128						512					
p_n		8		13		21		11		20		37	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
T_n	BAR	6.46	9.38	7.50	11.10	8.90	13.30	37.28	44.78	38.54	46.16	37.02	45.06
	TUK	6.54	9.10	7.50	11.32	8.68	13.14	38.06	45.20	39.14	46.74	36.20	44.42
	QS	7.20	10.22	8.24	12.56	10.44	15.34	39.70	46.54	37.90	45.72	34.36	43.00
$\tilde{T}_n$	BAR	42.30	60.50	23.14	33.64	19.02	28.46	63.04	73.32	53.82	63.56	46.92	57.46
	TUK	12.76	23.94	15.32	24.90	15.14	24.00	52.08	63.30	48.88	59.20	43.44	53.58
	QS	15.84	25.80	15.54	24.68	16.08	24.52	51.60	61.72	46.62	56.54	40.70	51.50
H_n	BAR	5.26	7.54	6.22	8.90	6.84	10.34	36.14	43.38	37.28	44.80	35.00	42.88
	TUK	5.32	7.48	6.46	9.12	6.92	10.54	37.16	44.08	37.90	45.46	34.22	42.36
	QS	6.04	8.64	6.68	10.18	7.32	10.78	38.42	45.56	36.72	44.56	32.26	40.10
$\tilde{H}_n$	BAR	38.40	56.44	19.50	30.36	15.10	23.28	61.78	72.44	52.56	62.12	45.08	55.22
	TUK	10.54	21.32	12.60	21.60	12.30	19.66	51.02	62.18	47.92	57.86	41.20	52.26
	QS	12.82	22.78	12.34	20.54	11.32	18.74	50.16	60.60	45.28	55.08	38.04	48.78
B_n		13.30	22.04	11.80	19.16	10.96	17.90	42.24	53.80	33.26	44.92	27.02	36.54

Note: Model x_t = ARFIMA (0,*d*,0) with $d = 0.4$.

TABLE VII

Rejections Rates in Percentage Under ARFIMA(1,d,0) Alternative
fitting Model ARFIMA(0,d,0)

n		128						512					
p_n		8		13		21		11		20		37	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
T_n	BAR	8.52	12.48	8.76	12.68	9.68	14.16	16.92	22.42	14.94	21.14	13.32	19.42
	TUK	8.16	12.10	8.10	12.14	9.10	13.60	16.26	21.78	14.50	20.50	12.80	18.32
	QS	8.24	11.74	8.82	12.86	10.88	15.54	15.76	21.34	13.16	19.22	12.62	18.24
T'_n	BAR	16.46	25.96	12.20	20.12	11.92	18.96	22.94	32.92	18.24	26.98	15.08	23.76
	TUK	10.72	18.90	9.64	16.56	10.22	16.58	18.90	27.82	15.84	23.92	13.44	21.18
	QS	9.96	16.94	9.82	16.70	11.80	18.06	17.40	26.24	14.36	22.34	13.14	20.54
H_n	BAR	7.54	10.84	7.54	11.42	7.98	11.56	16.22	21.78	14.22	20.14	12.32	17.74
	TUK	7.36	10.68	7.26	11.06	7.60	11.36	15.28	20.38	12.52	18.14	10.88	16.32
	QS	7.32	10.70	7.32	11.34	8.12	11.53	15.84	20.98	13.88	19.68	11.60	16.82
H'_n	BAR	14.42	23.78	11.22	18.00	9.92	14.78	22.22	32.30	17.50	26.08	13.64	21.74
	TUK	9.34	16.80	8.82	14.68	8.32	14.10	17.90	27.16	15.08	23.26	12.20	19.38
	QS	9.22	15.08	8.42	14.12	8.82	13.56	16.72	25.16	13.70	21.56	11.30	18.28
B_n		8.60	15.74	8.80	15.10	9.90	15.50	12.38	21.60	10.30	17.66	9.26	15.82

Note: Model $x_t - 0.1\,x_{t-1} = u_t$, u_t = ARFIMA (0,d,0) with $d = 0.4$.