Simultaneous Ascending Bid Auctions with Budget Constraints*

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Abstract

We identify and analyze three distinct effects arising from potentially binding budget constraints in multi-unit ascending auctions. First, binding budgets clearly reduce the level of competition among bidders. Second, budget constraints may – at the same time – make it difficult to sustain collusive equilibria when bidders lack sufficient resources to 'punish' defectors. Third, the mere possibility, even if arbitrarily small, of binding budget constraints can reduce competition substantially because bidders can 'pretend' to be constrained, even if they are not. In this cases, measures restricting the participation of low-budget bidders, e.g. reserve prices, can increase social welfare.

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1 Introduction

Most auction theory ignores the possibility that bidders may be willing to pay for an object more than the amount of money they have available, i.e. that bidders may be budget constrained. Yet, budget constraints can play an important role in practice. For example, David Salant (see Salant [20], p. 567), reporting on his experience in the bidding team at GTE during one of the FCC auctions for the sale of spectrum licenses, writes:

> We were very concerned about how budget constraints could affect bidding. Most of the theoretical literature ignores budget constraints. In the MTA [Major Trading Area] auction, budget constraints appeared to limit bids.

Salant also explains how, in order to formulate its strategy, the GTE bidding team used a simulation model in which possible budget levels of the different bidders entered as inputs.

In principle, if the bidders are interested in the objects for investment purposes (this was the case in the spectrum license auctions), and have access to well functioning capital markets, budget constraints should not matter. However, frictions in capital markets often make the amount of available internal funds relevant. Moreover, even when external funding is available at profitable rates, a bidder may be reluctant to borrow from a third party, because this might require disclosure of private information about its valuation for the goods, which in turn may put the bidder at a disadvantage in the auction. Also, a bidder may want to *choose* to be budget-constrained, in order to commit to a less aggressive bidding strategy and thus induce better outcomes in terms of final prices. A recent paper by Benoît and Krishna [5] highlights this effect by showing that in fact, at least in some cases, budget constraints may arise endogenously. Finally, financial constraints may emerge endogenously when bidders act as agents of financing principals (see for example Bolton and Scharfstein [3], or Holmström and Ricart i Costa [12]). These considerations provide good theoretical and empirical reasons to think that budget constraints play important roles in auctions.

The introduction of budget constraints in theoretical models of auctions is fairly recent. Pioneering work in this area is due to Che and Gale, [6] and [7]. They have analyzed single-object environments where each buyer has private information about both her willingness and her (possibly lower) ability to pay. One important insight that emerges from Che and Gale's work is that having a buyer with a budget w and a value v for an object is not the same, in general, as having a buyer with value $\min\{v, w\}$. Single-object second-price auctions with budget constrained bidders have been studied in Fang and Parreiras [10] and [11]. Zheng [23] studies a common value, (single object) first-price auction model in which the bidders can borrow at a given rate and default. Rhodes-Kropf and Viswanathan [18] analyze single object first-price auctions with privately know values and budgets, in which the bidders can finance their bids with cash or securities. Multiple objects auctions with budget constrained are analyzed in Benoît and Krishna [5], but only under the assumption of complete information.

We will study multi-unit simultaneous ascending-bid auctions under Che and Gale's information structure, i.e. under the assumption that each bidder has private information about both her willingness to pay and her budget. Since 1994, multi-unit simultaneous ascending-bid auctions have been used repeatedly by the US government to sell licenses for the use of parts of the electromagnetic spectrum. In a previous paper (Brusco and Lopomo [4]) we have shown that, for a large class of information and preference structures, these auctions provide the bidders with ample opportunities for collusion. The basic idea is that, for many distributions of the bidders' values, trying to win two objects often yields less expected surplus than buying a single object at a relatively low price.[1]

In the present paper we will focus on the effect that the *possibility* (even if small) of binding budget constraints has both on the opportunities for tacit collusion, and on the highest level of competition sustainable in equilibrium. Intuitively, the presence of potentially binding budget constraints can affect the equilibrium set of the auction in at least three conceptually different ways. First, it is clear that bidders with low budget levels cannot place high bids.[2] Thus even "noncollusive" equilibria, which would generate socially efficient outcomes without budget constraints, now yield low levels of both social surplus and seller's expected revenue.

Second, without binding side-contracts among the bidders, collusion in multi-unit simultaneous ascending bid auctions can be sustained only if a credible threat of reverting to non-collusive behavior is available to punish any deviating bidder with higher final prices. But, with sufficiently tight budget constraints, the punishing bidders cannot push prices to sufficiently high levels. Thus the presence of budget constraints may also hinder collusion.

[1] The experimental results in Kwashnica and Sherstyuk [15] corroborate our theoretical results in the case with no complementarities. For a survey on recent experimantal work on collusion in mult-unit ascending bid auctions, see Sherstyuk [21]. There is now general consensus that collusion in mult-unit ascending bid auctions is empirically relevant. See, for example, Cramton and Schwartz [8], or Klemperer [13].

[2] We are assuming that each bid must be backed by "money on the table," hence no bidder can make bids whose total is above her budget. In this respect our model is different from Zheng [23]: we are implicitly assuming that sufficiently severe penalties prevent our bidders from defaulting, so that they never bid above their budget.

Finally, in situations where the bidders' budgets are asymmetric, a third effect, similar to the 'demand reduction' seen in uniform price auctions, arises. Once prices reach levels at which a budget-constrained bidder is unable to buy more than one object, a high-budget opponent can end the auction immediately by simply letting the low-budget bidder win one object. This is more profitable than trying to buy two objects for the high-budget bidder if the willingness to pay for a second object is relatively low.

Demand reduction effects in multiunit sealed-bid auctions with uniform pricing have been noted by Ausubel and Cramton [1] and Englebrecht-Wiggans and Kahn [9]; and the idea is also present in Wilson [22]. We study open ascending bid auctions, in which the prices of the objects need not be equal. In these auctions, without budget constraints, there is an equilibrium in which the bidders simply raise the bid on each object up to their values, hence no demand reduction occurs. Therefore, in our model, the demand reduction effect is entirely attributable to the presence of potentially binding budget constraints.

Significant demand reduction can also occur if the bidders have private information about their budget levels. Suppose that a bidder with a high budget assigns positive probability to the event that her opponent is budget constrained. Then, if her value for the objects is not too high, she will prefer to let the auction end and buy only one object, as soon as the prices reach a level at which her opponent cannot buy more than one object. Thus, even in non-collusive equilibria, a nonempty set of high-budget types, with relatively low values for the objects, will mimic the behavior of budget-constrained types, and accept to split the objects.

We find that for a large class of distributions, even if the probability of having potentially binding budget constraints is arbitrarily small, *all* high-budget types behave as if they were budget constrained, hence the bidders' behavior will be indistinguishable from the case in which it is common knowledge that all bidders are budget constrained. In these cases, imposing a reservation price for each object which is high enough to exclude any low-budget bidder from the auction increases not only the seller's revenue, but also the expected social surplus.[3] Without budget constraints, reservation prices unambiguously reduce social surplus because they prevent potential gains from trade from being realized. With potentially binding budget constraints however, there are distributions for which, even in noncollusive equilibria, the bidders split the objects, thus

[3] Cramton and Schwartz [8] suggest that reservation prices may be used to upset collusion in multi-unit auctions. Their paper contains an example with complete information. We show that reservation prices can increase welfare in noncollusive equilibria when the possibility of binding budget constraints is admitted.

lowering the social surplus. Sufficiently high reservation prices in this case would prevent budget constrained bidders from participating in the bidding, thus making it common knowledge that all active bidders are unconstrained. Therefore, in a noncollusive equilibrium, each object ends up in the hands of a bidder with the highest value. For sufficiently small probabilities of having binding budget constraints, the expected gain in social surplus due to the better allocation of the objects is larger than the expected loss due to the exclusion of budget constrained types.

The insights of our model can also be applied to other situations in which two players compete for multiple 'prizes', and each player's type is characterized by two variables, one measuring the value attached to the prizes, and the other referring to a resource constraint which may or may not preclude the possibility of winning multiple prizes. Our analysis suggests that the mere possibility, no matter how unlikely, that each player may face a tight resource constraint can induce a significant reduction in competition, even in noncollusive equilibria. For example, the presence of capacity constraints in the multi-market contact model developed by Bernheim and Whinston [2] may induce firms to specialize in separate markets, i.e. to behave in a seemingly collusive fashion, even though they are using noncollusive equilibrium strategies.

Going outside the realm of economics, consider a military game in which two armies are trying to occupy two islands. Suppose that each army has private information about its military capacity, e.g. each army may be 'small' or 'large', small meaning able to occupy at most one island. This strategic situation is similar to the one we analyze in this paper, with the small army playing a role similar to the budget constrained bidder. Our results suggest that, even if it is ex ante very unlikely that each army is small, the final outcome can entail a low degree of competition, with each army occupying one island.

The rest of the paper is organized as follows. Section 2 presents the model. For simplicity, we consider only two objects and two bidders, with constant marginal willingness to pay. In Section 3 we analyze the case of commonly known budget levels, focusing first on the symmetric case, and then on the case with different budget levels. In Section 4 we introduce private information about the budget levels. Section 5 concludes. All proofs are relegated to an appendix.

2 The Model

The three models that we analyze in Sections 3 and 4 are special cases of the following environment. There are two objects, and two bidders. Each bidder $i = 1, 2$ is characterized by a type $\theta_i := (v_i, w_i)$, where v_i denotes the utility of each object and w_i is the maximum amount of money that she can spend in the auction. Therefore, the utility of a bidder who obtains n objects paying a total amount of m is $nv_i - m$, and m cannot exceed w_i.

The four variables (v_1, v_2, w_1, w_2) are independently distributed, with support $[0,1]^2 \times W_1 \times W_2$. The c.d.f. F of each variable v_i has a differentiable density f. The sets W_1 and W_2 are singletons in Section 3, and two point sets in Section 4.

The objects are sold using a "simultaneous ascending bid auction", which is a natural extension of the standard one-object English auction to environments with multiple objects. In each round $t = 1, 2, ...$, for each object $j = 1, 2$, each bidder i can either stay silent or raise the highest bid of the previous round by at least a minimum amount $\varepsilon > 0$. Formally, i's bid on object j in round t, denoted by $b_j^i(t)$, can either be $-\infty$, which is to be interpreted as "stay silent", or must be a number in the interval $[b_j(t-1) + \varepsilon, +\infty)$, where $b_j(t-1)$ denotes the "current outstanding bid", defined recursively by:

$$b_j(0) = 0 \quad \text{and} \quad b_j(t) := \max\left\{b_j(t-1),\ b_j^i(t);\ i \in N\right\}.$$

If at least one bidder increases the outstanding bid on at least one object, i.e. if $b_j(t) > b_j(t-1)$ for some j, then for each of these objects the new highest bid is identified, a potential winner is selected among the bidders who have made the new highest bid, and the auction moves to the next round, with the potential winner of all other objects unchanged. If instead all bidders stay silent on all objects, the auction ends, and each object is sold to the winner selected at the end of the previous round, for her last bid.

In our analysis we will consider the minimum bid increment ε negligibly small. This will simplify the statements and proofs of our propositions, essentially by eliminating the need to consider subcases in which a bidder's value is larger than the current outstanding bid but smaller than the current bid plus the minimum increment.

If $2 \leq \min W_i$, $i = 1, 2$, i.e. if each bidder's budget is above the highest total amount that she may be willing to spend in the auction, then the model is a special case of the model studied in Brusco and Lopomo [4]. In that paper we have established the existence of collusive equilibria which are sustained by the threat of reverting to non-collusive continuation strategies. Our focus here is on the effect that the possible presence of budget constraints has on the auction's equilibrium set. Thus we assume, without loss of generality, that $\min W_i < 2$, for some $i = 1, 2$.

To keep the formal statements of our results as simple as possible, we will often write that a given strategy profile σ "forms an equilibrium" to mean that there exists a consistent belief system μ such that the pair (σ, μ) constitutes a perfect Bayesian equilibrium. In most cases, given a strategy profile σ it will be easy to find a consistent belief system which supports σ as an equilibrium. We will be explicit about the belief system that goes together a given strategy profile only in some of our proofs. Finally, we will restrict attention to equilibria in which, on the equilibrium path, *no bidder lets the auction end* (i.e. remains silent if she expect her opponent to remain silent) *if the lowest outstanding bid is below both her value and her budget, and she is losing both objects.*

3 Commonly Known Budgets

In this section we assume that the bidders' budget levels are commonly known and equal, i.e. $W_1 = W_2 = \{w\}$. In this case, the assumption $\min W_i < 2$ specializes to $w < 2$. The analysis is organized as follows. In sub-section 3.1 we identify a "noncollusive" equilibrium that, subject to the restriction stated at the end of the previous section, maximizes the social surplus and, if the hazard rate $\frac{f}{1-F}$ is nondecreasing, also minimizes the bidders' expected surplus and maximizes the seller's expected revenue (when reservation prices are not allowed). In sub-section 3.2, we will use the noncollusive equilibrium to construct a family of collusive equilibria in which the bidders buy one object each, at relatively low prices.

3.1 Identical Budgets: The Noncollusive Equilibrium

In the auction format that we are considering, in *any* equilibrium in which no bidder lets the auction end when she is losing both objects and has money to pay for at least one object, it must be the case that, whenever $\frac{w}{2} < \min\{v_1, v_2\}$, each bidder wins exactly one object. This is because a bidder can win two objects only if both outstanding bids are higher than her opponent's value. Therefore, if $\frac{w}{2} < \min\{v_1, v_2\}$, winning two objects would require paying more than $2 \times \frac{w}{2} = w$. Letting $q_i(v_i, v_{-i})$ denote the number of objects sold to bidder i in a given equilibrium when his value is v_i and her opponent's is v_{-i}, we have

$$q_1(v_1, v_2) = q_2(v_2, v_1) = 1, \quad \text{if} \quad h < \min\{v_1, v_2\}, \tag{1}$$

where $h := \frac{w}{2}$. The (essentially unique) allocation (q_1^*, q_2^*) which maximizes the social surplus, subject to the equality in (1), is:

$$q_1^*(v_1, v_2) \equiv 2 - q_2^*(v_2, v_1) \equiv \begin{cases} 2, & v_2 < \min\{v_1, h\}; \\ 1, & h \le \min\{v_1, v_2\}; \\ 0, & \text{otherwise.} \end{cases} .$$

Defining $Q_i(v) \equiv \int_0^1 q_i(v, y)\, dF(y)$, and $S_i(v) \equiv v\, Q_i(v) - M_i(v)$, where M_i denotes the equilibrium interim expected payment function of bidder i, standard mechanism design arguments yield the following "envelope condition":

$$S_i(v) = \int_0^v Q_i(t)\, dt, \text{ all } v \in [0, 1],$$

which we can use, together with the definition of Q_i and the restriction in (1), to write the ex-ante expected surplus for bidder i as:

$$\begin{aligned} \int_0^1 S_i(v)\, dF(v) &= \int_0^1 [1 - F(v)]\, Q_i(v)\, dv \\ &= \int_L [1 - F(v)]\, q_i(v, y)\, dF(y)\, dv + [1 - F(h)] \int_h^1 [1 - F(v)]\, dv, \end{aligned}$$

where $L := [0,1]^2 \setminus [h,1]^2$ and we have exploited the property $q_i(v_1, v_2) = 1$ for each $(v_1, v_2) = [h,1]^2$. The total *bidders' expected surplus* can then be written as:

$$\sum_{i=1}^{2} \left(\int_0^1 S_i(v)\, dF(v) \right) = \int_L \left[\sum_{i=1}^{2} \frac{1 - F(v_i)}{f(v_i)}\, q_i(v_i, v_{-i}) \right] dF(v_1)\, dF(v_2) + K_1,$$

where $K_1 = 2(1 - F(h)) \int_h^1 [1 - F(v)]\, dv$; and the *seller's expected revenue* as:

$$\begin{aligned} \sum_{i=1}^{2} \left(\int_0^1 M_i(v)\, dF(v) \right) &= \sum_{i=1}^{2} \int_0^1 v\, Q_i(v)\, dF(v) - \sum_{i=1}^{2} \int_0^1 S_i(v)\, dF(v) \\ &= \int_L \left[\sum_{i=1}^{2} \left(v_i - \frac{1 - F(v_i)}{f(v_i)} \right) q_i(v_i, v_{-i}) \right] dF(v_1)\, dF(v_2) + K_2, \end{aligned}$$

where $K_2 = \int_h^1 \int_h^1 \left[\sum_{i=1}^{2} \left(v_i - \frac{1-F(v_i)}{f(v_i)} \right) \right] dF(v_1)\, dF(v_2)$. It is now immediate to see that the bidders' expected surplus is minimized (pointwise) with respect to $q_1(v_1, v_2)$ and $q_2(v_1, v_2)$, $(v_1, v_2) \in L$, subject to the constraint:

$$q_1(v_1, v_2) + q_2(v_2, v_1) = 2, \quad \text{for all } (v_1, v_2) \in L, \tag{2}$$

by assigning both objects to the bidder with the higher hazard rate, i.e. by setting $q_i(v_i, v_{-i}) = 2$ whenever $\frac{1-F(v_i)}{f(v_i)} < \frac{1-F(v_{-i})}{f(v_{-i})}$, $i = 1, 2$. Similarly, the seller's expected revenue is maximized, subject to the constraint in (2), by assigning both objects to the bidder with the higher "virtual utility", i.e. by setting $q_i(v_i, v_{-i}) = 2$ whenever $v_i - \frac{1-F(v_i)}{f(v_i)} > v_{-i} - \frac{1-F(v_{-i})}{f(v_{-i})}$, $i = 1, 2$. If the hazard rate $\frac{f}{1-F}$ is nondecreasing, the solutions of both programs coincide almost everywhere with the socially efficient allocation q^* defined at the beginning of the section.

The rest of this subsection is devoted to showing that the allocation q^* is obtained in a (symmetric) "noncollusive" equilibrium. We first describe the bidders' behavior on the equilibrium path (ignoring the case in which the bidders have equal values). Then, Proposition 1 will provide the formal definition of the strategies, and establish that they form an equilibrium.

In the proposed equilibrium the auction begins with the bidders increasing both outstanding bids, at the same pace, up to the minimum among the threshold h and the two bidders' values. More precisely, for each $i = 1, 2$, bidder i raises by ε the outstanding bid on object i in any odd round, and on object $3 - i$ in any even round, up to $\min\{v_i, h\}$. Thus the auction progresses with each bidder being the potential winner of one object in each round, until the outstanding bids reach either $\min\{v_1, v_2\}$ or h. In the first case, i.e. if $\min\{v_1, v_2\} < h$, the bidder with the lower value remains silent, and the auction ends with her opponent buying both objects and paying twice the lower value. Otherwise, as soon as the outstanding bids reach h, both bidders remain silent and the auction ends with each bidder buying one object and paying h. The next proposition establishes that this behavior can be supported as a perfect Bayesian equilibrium.

Proposition 1 *(Noncollusive equilibrium with known and equal budgets) If $W_1 = W_2 = \{w\}$, then the following strategy forms a symmetric equilibrium: at any stage $t + 1$, type v_i of bidder i raises by ε the outstanding bid:*

- *of the object with the lowest outstanding bid (breaking ties in favor of object i), if she is not the winner on any object, and*

$$\min\{b_1(t), b_2(t)\} < \min\{v_i, w\}; \tag{3}$$

- *of object j only, if she is the winner on object $3 - j$ only, and*

$$b_j(t) < \min\{v_i, w - b_{3-j}(t), h\}; \tag{4}$$

- *of no object, otherwise.*

Without budget constraints, i.e. if $2 < w$, the equilibrium of Proposition 1 collapses to the "separated English auctions" equilibrium characterized in Brusco and Lopomo [4] — i.e. each bidder bids on each object up to its valuation. The known presence of potentially binding budget constraints affects the noncollusive equilibrium strategies in two ways. First, the sum of the outstanding bids on objects for which a bidder is the potential winner cannot exceed her budget. This constraint, which is captured by the presence of w in inequality (3), and $w - b_{3-j}(t)$ in inequality (4), is binding off the equilibrium path. The effect of budget constraints on the equilibrium path is captured by the presence of h in inequality (4): as we have noted at the beginning of this section, it is never optimal for any bidder to keep trying to buy both objects once the outstanding bids have reached half of her budget. As we will see in the next two sections, this feature is robust to the introduction of asymmetries and private information about budget levels.

3.2 Identical Budgets: Collusive Equilibria

The noncollusive equilibrium of Proposition 1 can be used to construct a family of collusive equilibria, which we label γ-equilibria. For each $\gamma \in [0, h]$, a γ-strategy is defined as follows: each bidder uses the noncollusive strategy of Proposition 1 until either the opponent stays silent, or the outstanding bids reach the level γ; and stays silent in the next round. After any deviation each bidder reverts to the noncollusive strategy.

Clearly, the set of all γ-strategies, $\gamma \in [0, h]$, can be ordered according to their implied degree of collusion. The 0-strategy is the most collusive: it induces all types of each bidder to buy one object at the lowest possible price. At the opposite end of the interval, the h-strategy coincides with the noncollusive strategy defined in Proposition 1.

If both bidders use a given γ-strategy, the interim-expected number of objects sold to a bidder is:

$$Q(v \mid \gamma) \equiv \begin{cases} 2F(v), & v \in [0, \gamma]; \\ \\ 1 + F(\gamma), & v \in (\gamma, 1]. \end{cases}$$

This is because a bidder with value $v \leq \gamma$ buys both objects if her opponent has a lower value, and no object otherwise. Since the probability of the first event is $F(v)$, we have $Q(v \mid \gamma) = 2F(v)$. If instead $v > \gamma$, then the bidder wins at least one object if her opponent's value is above γ, and two objects otherwise. Thus, in this case, $Q(v \mid \gamma) = 1 + F(\gamma)$. If the γ-strategy is incentive compatible, the associated surplus function is:

$$S(v \mid \gamma) = \begin{cases} \int_0^v 2F(t)\ dt, & v \in [0, \gamma]; \\ \\ \int_0^\gamma 2F(t)\ dt + [1 + F(\gamma)](v - \gamma), & v \in (\gamma, 1]. \end{cases}$$

Now let $\Delta(v \mid \gamma, h)$ denote the expected surplus that the γ-strategy profile generates for type v in excess of the surplus obtained in the equilibrium of Proposition 1, i.e.:

$$\Delta(v \mid \gamma, h) \equiv S(v \mid \gamma) - S(v \mid h).$$

Clearly, the γ-strategy forms an equilibrium if and only if:

$$\Delta(v \mid \gamma, h) \geq 0 \quad \text{for all } v \in [0, 1], \tag{5}$$

i.e. each type $v \geq \gamma$ is willing to split the objects when the prices reach γ rather than reverting to the noncollusive equilibrium of Proposition 1. Substituting the expressions for $S(v \mid \gamma)$ and $S(v \mid h)$ we have:

$$\Delta(v \mid \gamma, h) = \begin{cases} 0, & v \in [0, \gamma]; \\ \\ [1 + F(\gamma)](v - \gamma) - \int_\gamma^v 2F(t)\ dt, & v \in (\gamma, h); \\ \\ h - \gamma + F(\gamma)(v - \gamma) - F(h)(v - h) - \int_\gamma^v 2F(t)\ dt, & v \in [h, 1]. \end{cases}$$

The function $\Delta(\cdot \mid \gamma, h)$ is concave in the interval $[\gamma, 1]$, and $\Delta(\gamma \mid \gamma, h) = 0$; hence the equilibrium condition in (5) is equivalent to the single inequality:

$$\Delta(1 \mid \gamma, h) \geq 0.$$

It is thus sufficient to check that the highest type has no incentive to trigger the reversion to the noncollusive strategies, once the prices reach the level γ. We record this conclusion in the next proposition.

Proposition 2 *(Collusive equilibria with known and equal budget levels) A γ-strategy is part of a γ-equilibrium if and only if:*

$$V(\gamma) \geq V(h), \tag{6}$$

where:

$$V(\gamma) \equiv S(1 \mid \gamma) = \int_0^\gamma 2F(t)\ dt + [1 + F(\gamma)](1-\gamma). \tag{7}$$

The inequality in (6) shows that the set of all γ-equilibria is determined by the budget level w via the function $V(\cdot)$, which in turn depends on the distribution F of the bidders' types. Depending on the nature of F, decreasing the budget level w may facilitate or hinder collusion in the sense that a given γ-strategy may form an equilibrium for a budget level $w' \leq 2$, but not for a smaller level w''; or viceversa, it may form an equilibrium for w'' and not for w'. This is because a reduction of the budget level may increase or decrease the expected surplus of the highest type in the noncollusive equilibrium, i.e. the derivative:

$$V'(h) \ = \ (1-h)\ f(h)\ - [1 - F(h)] \tag{8}$$

may be positive or negative. Thus the set:

$$\Gamma(h) := \{\gamma \in [0, h] \mid V(\gamma) \geq V(h)\} \tag{9}$$

of all price levels at which the bidders can split the objects in equilibrium may get larger or smaller as h decreases.

The two terms which make up the derivative $V'(h)$ in (8) have the following familiar interpretation. Consider an increment of the threshold h, say from h to $h + \delta$. This has no effect on the highest type's expected surplus if her opponent's value is below h. Otherwise:

i) if the opponent's value is between h and $h + \delta$, she can now buy two objects instead of one, hence her expected surplus increases by:

$$2[1 - \phi(\delta)] - (1-h) = 1 - 2\phi(\delta) + h,$$

where $\phi(\delta) := E[v \mid h < v < h + \delta]$;

ii) if the opponent's value is above $h + \delta$, she still buys only one object, but her payment increases by δ.

Multiplying each term by its probability and summing yields the overall change in expected surplus:

$$\Delta V \ = \ [F(h+\delta) - F(h)][1 - 2\,\phi(\delta) + h] \ - [1 - F(h+\delta)]\ \delta,$$

from which one obtains the derivative in (8) after dividing by δ and taking the limit for $\delta \to 0$, since $\lim_{\delta\to 0} \phi(\delta) = h$. Thus the first term in (8) is due to the additional surplus deriving from the purchase of a second object; since the second object ends up in the hands of the bidder with the highest value, the efficiency of the allocation is increased. The second term captures the negative effect due to the higher degree of competition; when $v_2 > h + \delta$ then an increase in the budget available to the bidders does not change the allocation, but it increases the price that each bidder pays to the seller. Depending on which of the two effects prevails, a reduction in the bidders' budget level may increase or decrease their expected surplus in the non-collusive equilibrium, and this in turn restricts or enlarges the set of prices at which the bidders can split the objects in equilibrium.

We end this section with a complete characterization of the equilibrium price correspondence Γ, defined in (9), for each of two broad classes of distributions. First, in Proposition 3, we consider the class of all distributions with *single-peaked* densities, i.e. such that f is increasing on $[0, x)$ and decreasing on $(x, 1]$, for some $x \in [0, 1]$. Second, in Proposition 4, we focus on the case in which $-f$ is single-peaked, i.e. f is decreasing on $[0, x)$ and increasing on $(x, 1]$, for some $x \in [0, 1]$.

Proposition 3 *(f single-peaked.) Suppose that there exists a point $x \in [0, 1]$ such that $0 < f'(v)$ for all $v \in [0, x)$ and $f'(v) < 0$ for all $v \in (x, 1]$. In this case,*

(a) *if $f(0) \geq 1$, then $\Gamma(h) = \{h\}$, for each $h \in [0, 1]$;*

(b) *if instead $f(0) < 1$, there are two sub-cases:*

1. *if $\frac{1}{2} \leq E(v)$, then*

$$\Gamma(h) = \begin{cases} [0, h], & h \in [0, h_0], \\ [0, \widehat{\gamma}(h)] \cup \{h\}, & h \in (h_0, 1], \end{cases}$$

where $h_0 := \arg\min_{\gamma \in [0,1]} V(\gamma)$, and the function $\widehat{\gamma} : (h_0, 1] \to [0, h_0]$ is defined by the equality $V(\gamma) = V(h)$;

2. *if $E(v) < \frac{1}{2}$, then*

$$\Gamma(h) = \begin{cases} [0, h], & h \in [0, h_0], \\ [0, \widetilde{\gamma}(h)] \cup \{h\}, & h \in (h_0, h_1], \\ \{h\}, & h \in (h_1, 1]; \end{cases}$$

where the function $\widetilde{\gamma} : (h_0, h_1] \to [0, h_0]$ *is defined by the equality* $V(\gamma) = V(h)$, *and* h_1 *is the unique solution of the equation* $1 = V(h)$ *in* $(0, 1]$.

The proof of Proposition 3 hinges on the following properties of the function V, which hold for any (differentiable) density f:

$$V''(\gamma) = (1-\gamma) f'(\gamma), \tag{10}$$

$$V'(1) = 0, \tag{11}$$

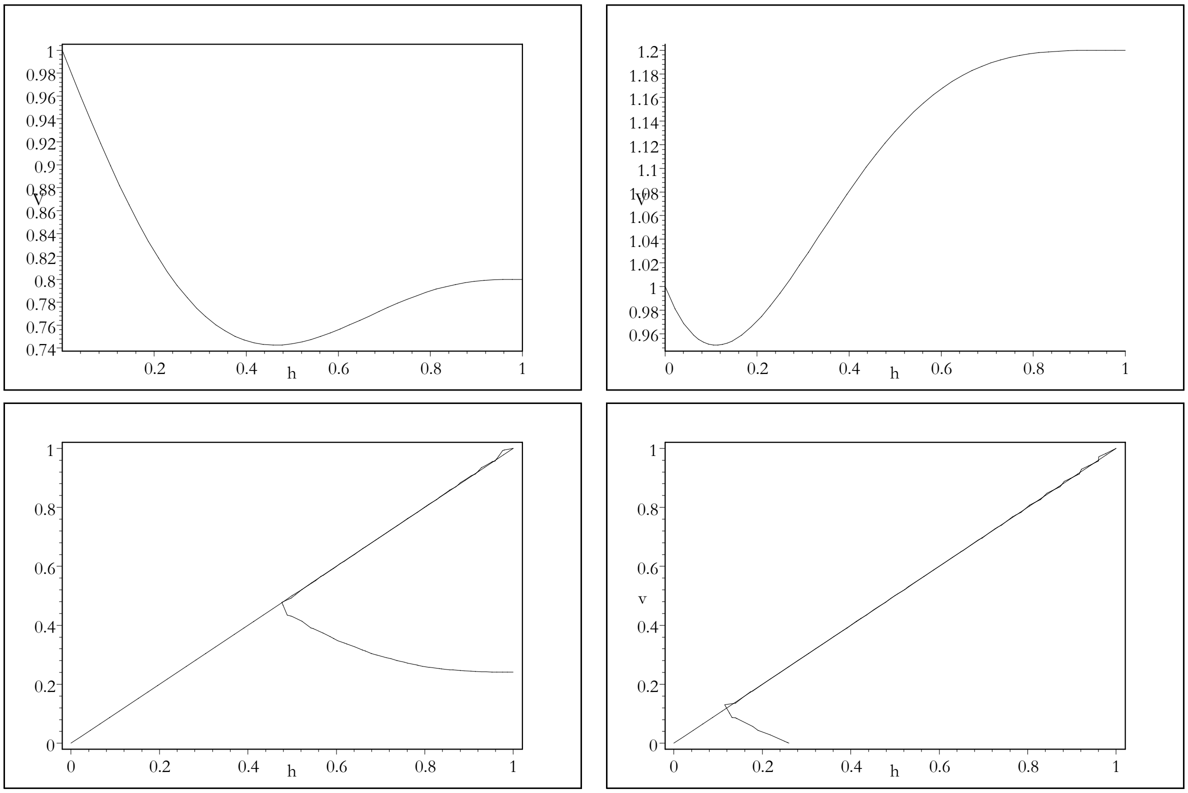
$$V'(0) = f(0) - 1. \tag{12}$$

When f is single-peaked, (10) implies that V is convex on $[0, x)$, and (10) and (11) together imply that V is both concave and increasing on $[x, 1]$. If in addition $f(0) \geq 1$, (case *a* of the Proposition), then by (12) $V'(0) \geq 0$, hence by convexity V is also increasing on $(0, x)$. Thus V is increasing on $(0, 1)$, i.e. the (noncollusive) h-strategy generates more expected surplus for the highest type than any γ-strategy, $\gamma < h$. Thus $\Gamma(h) = \{h\}$ for any $h \in [0, 1]$.

If instead f is single-peaked, but $f(0) < 1$, (case *b*) then, V is first decreasing and then increasing. Thus the (unique) minimizer $h_0 := \arg\min_{\gamma \in [0,1]} V(\gamma)$ is strictly between 0 and 1. For each $h \in [0, h_0]$, all γ-strategies with $\gamma \leq h$ generate more expected surplus that the h strategy, hence $\Gamma(h) = [0, h]$. For h above h_0, the nature of the equilibrium correspondence Γ depends on whether, without budget constraints, it is an equilibrium for the bidders to split the objects immediately, i.e. on how $V(0)$ compares with $V(1)$, or equivalently on how $E(v)$ compares with $\frac{1}{2}$.

If $V(0) \geq V(1)$ — case *b1* — then, for each $h \in (h_0, 1]$, there is a unique $\widehat{\gamma}(h) \in [0, h_0)$ such that $V(\gamma) = V(h)$. Any γ-strategy with $\gamma \leq \widehat{\gamma}(h)$ forms an equilibrium, hence $\Gamma(h) = [0, \widehat{\gamma}(h)] \cup \{h\}$. In particular, splitting the objects immediately (the 0-strategy) is an equilibrium for *any* $h \in [0, 1]$.

If instead $V(0) < V(1)$ — case *b2* — there exists a unique point $h_1 \in (h_0, 1]$ such that $1 = V(h_1)$; and for each $h \in (h_0, h_1]$ there is a unique $\widetilde{\gamma}(h) \in [0, h_0)$ such that $V(\gamma) = V(h)$. Thus any γ-strategy with $\gamma \leq \widetilde{\gamma}(h)$ forms an equilibrium, hence $\Gamma(h) = [0, \widetilde{\gamma}(h)] \cup \{h\}$. Finally, for $h \in (h_1, 1]$, we have $\Gamma(h) = \{h\}$. In this case a reduction of the budget level can lower the seller's expected revenue in two ways: competition in the non-collusive equilibrium decreases, and splitting the objects immediately becomes possible for $h < h_1$.



$\frac{1}{2} < E(v)$ $\qquad$ $E(v) < \frac{1}{2}$

Figure 1: f single peaked,

Figure 1 shows an example for case *b1* (on the left-hand side), and one for case *b2* (on the right-hand side). The two panels above show the graph of V, for $f(v) = 12\,v^2\,(1-v)$ and $f(v) = 12\,v\,(1-v)^2$ respectively. The panels below show the (boundary of) the associated equilibrium correspondences Γ.

Suppose now that $-f$ is *single-peaked*. The characterization of the correspondence Γ is obtained with similar arguments. Now the equalities in (10), (11) and (12) imply that V is first concave, and then both decreasing and convex. As in Proposition 3, we first distinguish two cases: $f(0) \leq 1$ and $f(0) > 1$. In the first case, V is decreasing in $(0,1)$, hence any γ-strategy, $\gamma \leq h$ forms an equilibrium, for any $h \in [0,1]$. In the second case $f(0) > 1$, V is first increasing, up to a point h_*, and then decreasing. If $h \leq h_*$ then no collusive equilibria are possible. If $h > h_*$ (that is, h belongs to the interval on which V is decreasing) then all γ-strategies in a nonempty interval $[\widehat{\gamma}(h), h]$ form an equilibrium for any level of h in $(0,1]$, where $\widehat{\gamma}(h)$ is the lowest value of γ such that $V(\gamma) = V(h)$ (or $\widehat{\gamma}(h) = 0$ if $V(0) \geq V(h)$). The function $\widehat{\gamma}(h)$ is decreasing in h. Therefore, in this case, tightening the bidders' budget constraints can reduces the scope for collusion, in the sense that the lowest price level at which the objects can be split in equilibrium increases as h decreases.

Proposition 4 *(-f single-peaked.) Suppose that there exists a point* $x \in [0,1]$ *such that* $f'(v) < 0$ *for all* $v \in [0,x)$ *and* $0 < f'(v)$ *for all* $v \in (x,1]$. *In this case,*

a) *if* $f(0) \leq 1$, *then* $\Gamma(h) = [0,h]$, *for each* $h \in [0,1]$*;*

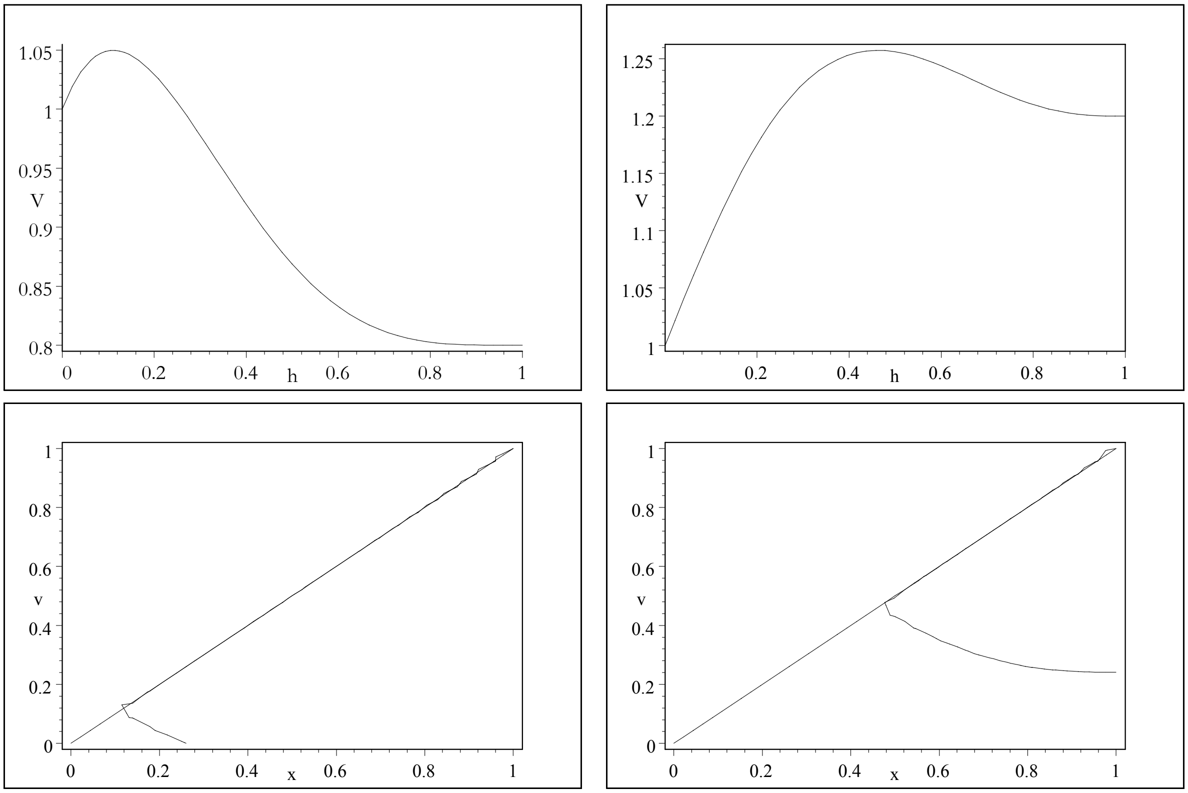
b) *if instead* $f(0) > 1$, *there are two sub-cases:*

1. *if* $E(v) < \frac{1}{2}$, *then*
$$\Gamma(h) = \begin{cases} \{h\} & h \in [0, h_*], \\ [\widehat{\gamma}(h), h] & h \in (h_*, 1], \end{cases}$$
where $h_* := \arg\max V(\gamma)$, *and the function* $\widehat{\gamma} : (h_*, 1] \to [0, h_*]$ *is defined by the equality* $V(\gamma) = V(h)$;

2. *if* $\frac{1}{2} \leq E(v)$, *then*
$$\Gamma(h) = \begin{cases} \{h\} & h \in [0, h_*], \\ [\widehat{\gamma}(h), h] & h \in (h_*, h_1], \\ [0,h] & h \in (h_1, 1]; \end{cases}$$
where $\widehat{\gamma} : (h_*, h_1] \to [0, h_*]$ *is defined by the equality* $V(\gamma) = V(h)$, *and* h_1 *is the unique solution of the equation* $1 = V(h)$ *in* $(0,1]$.

Figure 2 illustrates cases b1 and b2 in Proposition 4. In the graphs on the left, $f(v) = 2 -$

$12v\,(1-v)^2$; and on the right $f\,(v) = 2 - 12v^2\,(1-v)$.



$\frac{1}{2} < E\,(v)$ $\quad$ $E\,(v) < \frac{1}{2}$

Figure 2 $\quad -f$ single peaked

3.3 The Noncollusive Equilibrium with Known and Asymmetric Budget Levels

Before turning to the model with privately known budgets, it is useful to study the case with commonly known, but different, budget levels. Without loss of generality, we set $w_2 < w_1$, and assume $w_2 < 2$, so that bidder 2's budget constraint is potentially binding. We make no assumptions on w_1: bidder 1 may or may not be budget constrained, i.e. w_1 may be greater or smaller than 2. We define $h_i := \frac{w_i}{2}$, $i = 1, 2$.

We begin by observing that, in this case, the noncollusive strategy defined for the symmetric case (Proposition 1) cannot form an equilibrium. In particular, this strategy is not a best reply to itself, for some types of bidder 1 above h_2. To see this, suppose that both bidders have values above h_2, so that, in obedience to the strategy of Proposition 1, the bidders start the auction by raising both outstanding bids up to h_2. Beyond this point, bidder 2 cannot bid on more than one object; but bidder 1 can, and according to the strategy of Proposition 1 she should keep trying to win both objects, until the two bids reach h_1. Under what conditions is it optimal for bidder 1 to do so? Suppose that both outstanding bids are at $b \geq h_2$, each bidder is winning one object, and we are on the equilibrium path, so that bidder 1's beliefs about her opponent's value are represented by the c.d.f. $G(v_2|\, b) \equiv \frac{F(v_2)-F(b)}{1-F(b)}$, with support $[b, 1]$.

At this point, in light of the argument made at the beginning of sub-section 3.1, it is common knowledge that bidder 2 can only "bid defensively," i.e. she will remain silent unless she is losing both objects and both outstanding bids are above her value. Therefore, bidder 1's problem boils down to choosing an optimal "stopping time" s. By bidding on the second object until both bids arrive at s, and then letting the auction end by accepting to split the objects, she wins both objects at unit price v_2, if $v_2 < s$, and one object at price s, otherwise. Her expected surplus (conditional on both outstanding bids having reached b) as a function of the chosen stopping time $s \in [b, h_1]$ is:

$$U(\, v_1, s|\, b) = 2\int_b^s (v_1 - v_2)\, dG(v_2|\, b) + (v_1 - s)\,[1 - G(s|\, b)]\,. \tag{13}$$

The value of s prescribed by the strategy of Proposition 1 is $\min\{v_1, h_1\}$; hence bidder 1 should stop at v_1 if $h_2 < v_1 < \min\{h_1, 1\}$. But the first derivative:

$$\frac{\partial U(v_1, s\,|b)}{\partial s} = (v_1 - s)\; G'(s|\, b) - [1 - G(s|\, b)]\,, \tag{14}$$

evaluated at $s = v_1$ is $-[1 - G(v_1|\, b)] < 0$, for any $v_1 < 1$. Thus, if $h_2 < v_1 < \min\{h_1, 1\}$, bidder 1 has an incentive to stop before the bids arrive at v_1. We conclude that the noncollusive strategy defined in Proposition 1 for the symmetric case does not form an equilibrium if the bidders' budget levels are different.

Once the outstanding bids have reached h_2, bidder 1 faces a single-person decision problem, similar to a standard monopsony profit maximization problem: the essential trade-off is between buying a single object for a given price and buying two objects for a higher unit price. In light of this analogy, one should not be surprised to find that the outcome is in general inefficient; i.e. the stopping time s is in general below bidder 1's value v_1, so that, whenever bidder 2's value is between s and v_1, the objects are split instead of going both to bidder 1, despite the fact that bidder 1 can afford to buy both objects. Similar "demand reduction" effects have been noted before in other auction formats[4], in the absence of budget constraints. In our model, the demand reduction effect

[4] E. g. Ausubel and Cramton [1], or Englebrecht-Wiggans and C. Kahn [9].

is due to the presence of budget constraints; absent such constraints, there is always an equilibrium in which bidders bid up to their values and no demand reduction occurs.

We now define a strategy profile which forms the "noncollusive" equilibrium with asymmetric budget levels. For the low-budget bidder (i.e. bidder 2) the strategy is the one described in Proposition 1. The behavior of bidder 1 on the equilibrium path, once the bids reach h_2, is determined by an optimal stopping time $s_1(v_1 \mid b)$, defined for each bid level $b \in [h_2, 1]$ and each type $v_1 \in [h_2, 1]$. Let $U(v_1, s \mid b)$ be the function defined in (13), that is, the utility obtained by a high-budget bidder who has beliefs given by $G(v_2 \mid b)$ on the opponent's value, and sets a stopping time of s; and define the set $R(v_1; b)$ as:

$$R(v_1; b) := \arg \max_{s \in [b, \min\{1, h_1\}]} U(v_1, s \mid b)$$

Since $U(v_1, s \mid b)$ is continuous in s, the set $R(v_1; b)$ is compact, although not necessarily a singleton. Since we are looking for the "most competitive" equilibrium, we select the highest optimal stopping time, i.e. we define:

$$s_1(v_1; b) \equiv \max\ R(v_1; b). \tag{15}$$

The functions $s_1(\cdot; b)$, $b \in [h_2, 1]$ are time-consistent, in the sense that, if $h_2 < s_1(v_1; h_2)$, then $s_1(v_1; b) = s_1(v_1; h_2)$ for any $b \in [h_2, s_1(v_1; h_2)]$.

The function $s_1(\cdot; h_2)$ has the following properties. Its domain $[h_2, 1]$ can be partitioned in three subintervals. First, all types in a lower interval $[h_2, v_1']$ let the auction end at h_2. For these types it is optimal to stop as soon as the opponent starts bidding defensively. The interval $[h_2, v_1']$ is always nonempty: in fact, since f is bounded, we can find $\delta > 0$ such that, for all $v_1 \in [h_2, h_2 + \delta]$, we have $v_1 - s < \frac{1-F(s)}{f(s)}$ whenever $s \in [h_2, v_1]$, hence:

$$\frac{\partial U(v_1, s \mid h_2)}{\partial s} = \frac{(v_1 - s)\ f(s) - [1 - F(s)]}{1 - F(h_2)} < 0, \quad \text{all} \ \ s \in [h_2, v_1].$$

This implies that the optimal s is the lowest one, that is h_2.

At the opposite side of the spectrum, all types in an upper interval $[v_1'', 1]$ push the bids on both objects up to h_1. These are types with a high value for the object, who try to win both objects until they are able to pay for them. Clearly, this upper interval $[v_1'', 1]$ can be nonempty only if $h_1 < 1$, i.e. only if the budget constraint is potentially binding for the high-budget bidder as well.

For all types in the remaining middle interval (v_1', v_1''), the optimal stopping time must satisfy the first order condition:

$$(v_1 - s)\ f(s)\ - [1 - F(s)] = 0.$$

The middle interval can also be empty: in fact, it may be the case that $v_1' = 1$, i.e. all types stop at h_2. For example, if F is uniform, we have

$$\frac{\partial U(v_1, s\,|h_2)}{\partial s} = -\frac{1-v_1}{1-h_2} < 0, \quad \text{all } \ s \in [h_2, v_1], \text{ and all } v_1 \in [h_2, v_1).$$

This completes the description of the bidders' behavior on the equilibrium path. To summarize, the low-budget bidder tries to win both objects until the bids reach $\min\{h_2, v_2\}$. If $h_2 < v_2$, once the bids reach h_2 she adopts the "defensive" strategy of bidding on the lowest priced object as long as the price remains below $\min\{w_2, v_2\}$. The high-budget bidder follows a similar strategy, but she tries to get both objects up to an optimally chosen 'stopping time' $s_1(v_1; h_2)$ — which in general differs from $\min\{h_1, v_1\}$.

To complete the characterization of the equilibrium, we have to specify what happens off the equilibrium path. We use a belief formation rule which is a natural extension of the updating rule adopted on the equilibrium path: at any stage, bidder i looks at the highest bid ever made by her opponent, and simply rules out the possibility that v_{-i} is below it. Thus, beliefs are given by $\widehat{F}(v_{-i}\,|\,v_{-i} \geq \widehat{v}) \equiv \frac{F(v_{-i})-F(\widehat{v})}{1-F(\widehat{v})}$ for $v_{-i} \in [\widehat{v}, 1]$, where $\widehat{v}$ denotes the highest bid ever made by bidder i's opponent.

Off the equilibrium path, the bidders' behavior is as follows. First observe that staying silent is always optimal for any bidder who is winning both objects. Therefore, if bidder i is losing both objects, staying silent would terminate the auction. This is optimal if $\min\{v_i, w_i\}$ does not exceed any of the outstanding bids. Otherwise, it is optimal to increase by ε (one of) the lowest outstanding bids. We are thus left with the task of specifying the bidders' strategies when each bidder is winning one object. The details of this are relegated to the appendix. Here we define the optimal stopping time for bidder 1, when she is winning one object, as the highest maximizer of $U_1(v_1, s\,|\,b_{3-j}, \widehat{v})$, where U_1 is the appropriate extension of the function U defined in (13), and is defined in the appendix. Therefore we let

$$s_1(v_1; b_{3-j}, \widehat{v}) \equiv \max\left\{\arg \max_{s \in \left[\min\left\{b_{3-j}, \widehat{v}\right\}, \min\{h_1, 1\}\right]} U_1(v_1, s|\, b_{3-j}, \widehat{v})\right\}, \tag{16}$$

and note that the stopping function defined along the equilibrium path is obtained as a special case of $s_1(v_1; b_{3-j}, \widehat{v})$ when $b_{3-j} = \widehat{v}$. Also, to save notation in the statement of the equilibrium strategies, we let:

$$s_2(v_2; b_{3-j}, \widehat{v}) \equiv v_2. \tag{17}$$

We are now prepared to formally characterize the equilibrium strategy profile for the case of known and asymmetric budget levels.

Proposition 5 *Let $w_2 < w_1$, and let s_1 and s_2 be defined as in (16) and (17). The following strategy pair forms an equilibrium. Let (b_1, b_2) be the pair of outstanding bids at round t. Then at round $t+1$, each type v_i of bidder i raises by ε the outstanding bid:*

- *on the object with the lowest outstanding bid, breaking ties in favor of object i, if she is not the winner on any object, the current outstanding bids are different, and*

$$\min\{b_1, b_2\} < \min\{v_i, w_i\};$$

- *on object j only, if she is the winner on object $3-j$ only, and if*

$$b_j < \min\{s_i(v_i; b_{3-j}, \widehat{v}), h_i, w_i - b_{3-j}\},$$

  where $\widehat{v}$ is the highest bid ever made on any object by bidder $3-i$.

- *on no object, otherwise.*

The strategy profile described in Proposition 5 is the "most competitive" equilibrium with known and different budget levels. Once it becomes common knowledge that both buyers' values are above h_2, the allocation of the objects is determined solely by the high-budget bidder's behavior. Since we have chosen the highest among all stopping times, the equilibrium of Proposition 5 yields the highest social surplus among all equilibria of the auction with known and different budget levels.

The characterization of the equilibrium in Proposition 5 shows that the presence of budget constraints limits competition not only directly, but also via the fact that a high-budget bidder facing a budget constrained opponent is able to exploit her monopsony power by stopping the bidding on both objects before the bids reach her value. In the next section we show that this effect plays an important role in the case of privately known budgets. In fact, even arbitrarily small probabilities that the bidders may be budget constrained can generate a competition-restraining effect similar to the one described in Proposition 5.

As a last remark, we observe that the equilibrium in Proposition 5 can be used as 'punishment' in constructing collusive equilibria based on $\gamma-$strategies of the type analyzed in subsection 3.2, with the obvious adjustments. Since the analysis is very similar, we don't repeat it here; we just observe that in general it is more difficult to convince a high-budget bidder to collude, since the opponent is weaker. The same remark will apply to the analysis in the next section, where we consider the case of privately known budgets. From this point on we focus exclusively on the 'most competitive' equilibrium.

4 Noncollusive equilibria with Privately Known Budgets

In this section we allow each bidder to have private information about her budget level. Thus a type for bidder $i = 1, 2$ is now identified by the pair $(v_i, w_i) \in [0, 1] \times W$. We analyze the simplest model in which each bidder may or may not be budget constrained: for each $i = 1, 2$, the distribution of w_i is independent of all other random variables, has support $W = \{w_L, w_H\}$, with $1 < w_L < 2 < w_H$, and $\Pr[w_i = w_L] := \lambda \in (0, 1)$. The assumption that budget constrained bidders can always bid up to their value for a single object (i.e. $1 < w_L$) is not essential, but simplifies the analysis.

We begin by observing that there can be no equilibrium where *each* high-budget type bids on both objects until both prices reach her value. Therefore, even in the "most competitive" equilibrium, and even if both bidders are not budget constrained, some demand reduction must occur. To see this, assume that each type behaves as in the noncollusive equilibrium of Proposition 1. Suppose that the outstanding bids have arrived at $h_L := \frac{w_L}{2}$, hence $h_L < \min\{v_1, v_2\}$, and that each bidder is winning one object. At this point, all low-budget types are happy to split the objects, and thus remain silent. All high-budget types instead are supposed to keep trying to win both objects, until both bids reach their value.

But consider a high-budget bidder, say bidder 1, who has just increased the bid on the object that she was not winning and has seen her opponent "offering to split" the objects by remaining silent. Since this reveals that $w_2 = w_L$, bidder 1's problem is now identical to the one discussed at the beginning of the previous section, where we have seen that she will always refrain from pushing the prices up to her value. In particular, a nonempty set of types with value close to h_L will choose to stop immediately.

Thus, in the noncollusive equilibrium, all high-budget types with value below a threshold $v_* > h_L$ remain silent once the bids arrive at h_L. The position of the threshold v_* in the interval $(h_L, 1]$ depends on the distribution F, and on the probability λ of facing a low budget opponent. However, even as λ goes to zero, the value of v_* may remain bounded away from h_L. Therefore, even if the probability of facing a budget constrained opponent is vanishing, there may be first-order effects on expected welfare and the expected revenue for the seller. In some cases the effects are quite dramatic, as we will see shortly by analyzing the case of nondecreasing densities.

The bidders' behavior on the equilibrium path is as follows. The auction starts with both outstanding bids increasing at the same pace, up to $\min\{v_1, v_2, h_L\}$. More precisely, for each $i = 1, 2$, bidder i increases the outstanding bid by the minimum increment ε on object i in any odd round, and on object $3 - i$ in any even round, up to $\min\{v_i, h_L\}$. Thus the auction progresses with each bidder being the potential winner of one object in each round, until the outstanding bids reach either the lowest of the bidders' values, or h_L. In the first case, the bidder with the lowest value stays silent, and the auction ends with her opponent winning both objects. Otherwise, i.e. if $h_L < \min\{v_1, v_2\}$, the behavior of each bidder depends on whether her type is "*tough*," i.e. high-budget and with value above a threshold v_* (which is strictly above h_L), or "*soft*," i.e. either low-budget, or high-budget and with value between h_L and v_*. Once the bids reach h_L, all soft types remain silent. All tough types instead continue to raise the outstanding bid on any object which is assigned to the opponent, up to a threshold which depends on the opponent's behavior.

Thus, if both bidders are soft, they stay silent and the auction ends with each bidder buying one object and paying h_L. If both bidders are tough, the bidding continues as in the initial phase and the bidder with the highest value wins both objects. Finally, if one bidder is tough and the other is soft, the soft bidder starts to bid "defensively," i.e. she bids on one of the objects with the lowest current outstanding bid if she is losing both objects, and stays silent otherwise. The tough bidder instead tries to win both objects until the bids reach an optimally chosen threshold. The auction then ends with the tough bidder buying both objects if her threshold is above her opponent's value, and with the bidders splitting the objects otherwise.

We now provide a formal definition of the strategy just described. As in the previous section, we have to define the "stopping" function $s(v_i, w_i; b_1, b_2)$ which determines the highest price that a bidder of type (v_i, w_i) who has observed a pair of bids (b_1, b_2) is willing to pay in order to get both objects. Except for the definition of the function s, the strategies involved in our equilibrium are identical to the strategies defined in Proposition 5. However, the way in which the function s is determined in the case of privately known budgets is conceptually different from what we have seen in the previous section. In fact, if the bidders' budget levels are commonly known, the high-budget bidder (if one exists, i.e. if $w_1 \neq w_2$) faces a straightforward single person decision problem, as discussed in the previous section. With incomplete information on the budget levels, each bidder's optimal stopping time is also a function of her opponent's behavior. While it is clear that a low-budget opponent will bid defensively, the behavior of a high-budget opponent will depend on her beliefs. Some, but not necessarily all, high-budget types will play "soft," thus mimicking the behavior of low-budget bidders. Therefore each bidder needs to formulate a conjecture about the stopping function used by her opponent in order to compute her own stopping function. The conclusion is that the (symmetric) equilibrium stopping function has to be computed as the solution of a fixed point problem, rather than as the solution of a single-person decision problem.

For all low budget types things are simple. The function $s(v_i, w_L; b_1, b_2)$ prescribes to stop immediately if any of the bids is above $\min\{v_i, h_L\}$, and keep trying to buy both objects otherwise. Since the functions$s$ is defined for $v_i \geq h_L$, we have

$$s(v_i, w_L; b_1, b_2) \equiv v_i.$$

For the high budget types things are more complicated. We first characterize the optimal stopping function on the equilibrium path. Suppose that the current bids are (b, b), with $b > h_L$, and that, when the bids reached the level (h_L, h_L), bidder 1 played 'tough', meaning she raised the bid on the object she was not winning, and bidder 2 played 'soft', i.e. she remained silent. The beliefs of the two bidders are as follows. Bidder 2 believes that $w_1 = w_H$ with probability 1, and that v_1 is distributed according to the posterior c.d.f. determined by the optimal stopping function and by F. Bidder 1's beliefs about her opponent's are as follows:

$$\Pr(w_2 = w_L|\,\text{soft}) = \frac{\lambda[1 - F(b)]}{\lambda[1 - F(b)] + (1-\lambda)\max\{F(v_*) - F(b), 0\}},$$

$$\Pr(w_2 = w_H|\,\text{soft}) = \frac{(1-\lambda)\max\{F(v_*) - F(b), 0\}}{\lambda[1 - F(b)] + (1-\lambda)\max\{F(v_*) - F(b), 0\}};$$

and the conditional densities on v_2 are:

$$g(v_2|\,b, w_L) \equiv \begin{cases} \frac{f(v)}{1-F(b)} & v_2 \in [b, 1], \\ 0 & \text{otherwise;} \end{cases}$$

and

$$g(v_2|\,b, w_H) \equiv \begin{cases} \frac{f(v)}{F(v_*)-F(b)} & v_2 \in [\min\{b, v_*\}, v_*], \\ 0 & \text{otherwise.} \end{cases}$$

Letting $G(\cdot|\,\cdot,\cdot)$ denote the c.d.f. corresponding to the densities g, we can write bidder 1's expected surplus from stopping at s as:

$$U(v_1, s; v_*, b) = \Pr(w_L|\,\text{soft})\left[2\int_b^s (v_1 - y)\; dG(y|\,b, w_L) + (v_1 - s)\;[1 - G(s|\,b, w_L)]\right]$$

$$+\Pr(w_H|\,\text{soft})\left[2\int_b^s (v_1 - y)\; dG(y|\,b, w_H) + (v_1 - s)\;[1 - G(s|\,b, w_H)]\right]$$

After substitutions and ignoring multiplicative constants we can write the objective function as:

- for $b < v_*$:

$$U(v_1, s; v_*, b) = \lambda\left[2\int_b^s (v_1 - y)\; dF(y) + (v_1 - s)[1 - F(s)]\right] \tag{18}$$

$$+(1-\lambda)\left[2\int_b^{\min\{s,v_*\}} (v_1 - y)\; dF(y) + (v_1 - s)[F(v_*) - F(\min\{s, v_*\})]\right];$$

- for $b \geq v_*$:

$$U(v_1, s; v_*, b) = 2\int_b^s (v_1 - y)\, dF(y) + (v_1 - s)\;[1 - F(s)]\,. \tag{19}$$

We now define the set

$$R(v_1, w_H; v_*, b) \equiv \arg\max_s\; U(v_1, s; v_*, b)\,, \tag{20}$$

and, since we are interested in the 'most competitive' equilibrium, we consider the 'stopping rule' given by:

$$r(v_1; v_*, b) \equiv \max\; R(v_1, w_H; v_*, b)\,. \tag{21}$$

When v_* is an equilibrium value, and the bids $b_1 = b_2 = b$ are reached along the equilibrium path, we set:

$$s(v_1, w_H; b, b) \equiv r(v_1; v_*, b)\,.$$

For any given threshold v_* we can compute the expected surplus of each player from playing 'tough' and 'soft' (once the bids reach h_L) when her opponent conjectures that all high-budget types above v_* play tough and all high-budget types below v_* play soft. In equilibrium, v_* must be such that the conjecture is confirmed. We now discuss more in depth the existence of an equilibrium with the characteristics just described.

4.1 Existence of the Equilibrium

In order to complete the analysis we have to accomplish two tasks. First, we have to show that a threshold value v_* exists, i.e. we have to show that a fixed point exists. That is, it must be true that when bidder 1 conjectures a threshold value v_* for the opponent, then all types $v_1 < v_*$ are willing to play soft and all types $v_1 > v_*$ are willing to play tough. As we will see, this requires an additional assumption. Second, we have to describe the out of equilibrium behavior.

For the moment, let it be taken for granted that a threshold value v_* exists, so that an optimal stopping function $s(v_i, w_H; b, b)$ can be computed along the equilibrium path. We now proceed to generalize the bidding behavior for any arbitrary pair (b_1, b_2).

As in the previous two sections, we specify that a bidder stays silent when winning both objects, and raises the bid on (one of) the lowest priced object(s) when losing both objects (provided the lowest bid is below her value). Thus, it remains to specify the behavior at pairs (b_1, b_2) at which the two bidders are winning one object each.

Consider the following three cases:

1. $\max\{b_1, b_2\} < h_L$. In this case the strategies are as in the standard 'competitive' equilibrium, i.e. $s(v_i, w_i; b_1, b_2) = v_i$.

2. $\max\{b_1, b_2\} \geq h_L$ <u>and</u> (b_1, b_2) can be reached on the equilibrium path. In this case the beliefs are updated using Bayes' rule. The stopping rule for all types with $v_i \geq v_*$ remains the same. Those with $v_i < v_*$ play defensively and have no interest in triggering the 'competitive' equilibrium, since the opponent has a higher value.

3. $\max\{b_1, b_2\} \geq h_L$ <u>and</u> (b_1, b_2) is out of the equilibrium path. We specify the beliefs so that, whenever a bidder observes the other deviating, she puts probability 1 on $w_i = w_H$, and this belief is maintained in case further deviations are observed. Furthermore, low-budget bidders cannot hope to win both objects, so that they stay silent whenever they win at least one. Note that this implies that any attempt on the part of a bidder to buy both objects signals that the bidder has a high budget. There are 3 sub-cases, depending on how many bidders have deviated.

   (a) Both bidders deviated from the prescribed strategy. In this case both bidders assign probability 1 to the fact that the other bidder has a high budget, and this fact is common knowledge. In this case the 'competitive' equilibrium is triggered. Therefore we set $s(v_i, w_H; b_1, b_2) = v_i$.

(b) If the other bidder deviated then bidder i assigns probability 1 to $v_{3-i} = \widehat{b}$, where $\widehat{b}$ is the highest bid ever made by bidder $3-i$, and assumes that bidder $3-i$ will never make a bid on any object if she becomes convinced that the type of the other bidder is higher. Since by making a bid on the other object bidder i signals that her type is greater than $\widehat{b}$ (remember that $\max\{b_1, b_2\} \geq \widehat{b}$ and i bids on both objects) then it is rational for i to bid myopically on both objects, i.e. assuming that the other bidder will not make any further bid. This in turn justifies a myopic behavior on the part of bidder $3-i$. Notice that this cannot make a deviation profitable, since by deviating bidder $3-i$ only obtains a more aggressive behavior on the part of bidder i. We can therefore set $s(v_i, w_H; b_1, b_2) = v_i$ in this case as well.

(c) The last case we have to deal with is the one in which a deviation occurred only on part of agent i. Since $\max\{b_1, b_2\} > h_L$ any counterbid by $3-i$ signals that she is of type w_H and $v_{3-i} \geq \widehat{b}(i)$, where $\widehat{b}(i)$ is the highest bid ever made by i up to that round. Also, in that case bidder $3-i$ starts bidding myopically. Then bidding myopically is a best reply on part of agent i. The conclusion is that in this case as well we can set $s(v_i, ; v_*, b_1, b_2) = v_i$.

Essentially, out of the equilibrium path the bidders raise the bids whenever the value of the object is superior to their current bids. Along the equilibrium path, the bidders adopt optimal stopping times.

We now come to the issue of the existence of a threshold value v_*. Let $\mu(v_i) \equiv \frac{1-F(v_i)}{f(v_i)}$ be the inverse hazard rate. We make the following assumption.

Assumption 1 *For each $v_i > h_L$, we have:*

$$2f(v_i) \geq f(v_i + \mu(v_i)) \left[1 + \mu'(v_i)\right]$$

whenever $v_i + \mu(v_i) < 1$ and $1 + \mu'(v_i) > 0$.

Assumption 1 is immediately satisfied when either $v_i + \mu(v_i) \geq 1$ or $1 + \mu'(v_i) \leq 0$ for each $v_i \geq h_L$. The latter holds, for example, in the uniform case. More generally, if $\mu'(v_i) \leq 0$ (nondecreasing hazard rate), then a sufficient condition for Assumption 1 is that, for all $v_i > h_L$, we have

$$2f(v_i) \geq f(x), \text{ for each } x > v_i.$$

This is satisfied by any distribution without large peaks. In particular, if $v_{\min}$ and $v_{\max}$ are respectively the points at which the density achieves the maximum and the minimum over the interval $[h_L, 1]$ then a sufficient condition is:

$$2f(v_{\min}) \geq f(v_{\max}).$$

The next proposition characterizes the noncollusive equilibrium.

Proposition 6 *If Assumption 1 is satisfied, then there exists a value v_* and a corresponding stopping function $s(v_i, w_i;\, b_1,\, b_2)$ such that the following strategy profile forms an equilibrium. At any stage in which the current outstanding bids are b_1 and b_2, each type (v_i, w_i) of bidder i increases the bid by the minimum increment:*

- *on the object with the lowest outstanding bid, breaking ties in favor of object i, if she is not the winner on any object, the current outstanding bids are different, and*

$$\min\{b_1, b_2\} < \min\{v_i,\, w_i\};$$

- *on object j, if she is winning object $3-j$ only, and*

$$b_j < \min\left\{s(v_i, w_i;\, b_1,\, b_2),\, \frac{w_i}{2},\, w_i - b_{3-j}\right\};$$

- *on no object, otherwise.*

As in the case with known and asymmetric budgets, a "demand reduction" effect is present in the noncollusive equilibrium of Proposition 6. On the equilibrium path, the stopping function is such that $s(v_i, w_H;\, h_L,\, h_L) = h_L$, for all $v_i \in [h_L, v_*)$. Thus a set of high-budget types with sufficiently low values mimic the behavior of low budget types. Furthermore, $s(v_i, w_H;\, h_L,\, h_L) < v_i$ for $v_i \geq v_*$, hence even high budget types with high values reduce their demand.

We now show that, if the density function f is nondecreasing on $[h_L, 1]$, the demand reduction is actually quite dramatic.

4.2 Nondecreasing Densities

Suppose that the density f is nondecreasing on the interval $[h_L, 1]$. We want to show that in this case the strategy described in Proposition 6 is an equilibrium if, and only if, $v_* = 1$. First, we show that there is no equilibrium with $v_* < 1$.

For any $v_* \in (h_L, 1]$, the problem of bidder 1's type (v_1, w_H), conditional on the bids having reached (h_L, h_L), and the opponent having played soft, can be written as:

$$\max_s \; U(v_1, s; v_*, h_L) = \lambda \left[2 \int_{h_L}^{s} (v_1 - v_2)\; dF(v_2) + (v_1 - s)\; [1 - F(s)] \right]$$

$$+ (1 - \lambda) \left[2 \int_{h_L}^{\min\{s, v_*\}} (v_1 - v_2)\; dF(v_2) + (v_1 - s)\; [F(v_*) - F(\min\{s, v_*\})] \right].$$

Suppose that $v_* < 1$, and consider type $v_1 = v_*$. Since the derivative $\frac{\partial U}{\partial s}$, evaluated at $v_1 = v_*$, and for $s < v_*$, is proportional to:

$$(v_* - s) f(s) - [\lambda + (1 - \lambda) F(v_*) - F(s)],$$

we have that the (left) derivative at $s = v_*$ is strictly negative. Furthermore, the second derivative $\frac{\partial^2 U}{\partial^2 s}$ on the interval (h_L, v_*) is proportional to:

$$(v_* - s)\; f'(s).$$

Given the assumption that $f'(s) \geq 0$ for each $s \geq h_L$, we have that $\frac{\partial U}{\partial s}$ is nondecreasing on (h_L, v_*), and strictly negative at v_*; hence strictly negative over (h_L, v_*). Therefore, the optimal stopping time for v_* must be h_L. This remains true for types $v_* + \delta'$, with δ' small enough. Therefore, a set of types $(v_*, v_* + \delta)$, with $\delta > 0$, will choose a stopping time of h_L. This is a contradiction, since Lemma 5 in the Appendix establishes that the stopping time must be strictly greater than h_L. Thus, it can never be the case that $v_* < 1$.

Thus the only possible candidate for an equilibrium is $v_* = 1$. In fact, we can readily check that we have an equilibrium for $v_* = 1$. In this case, the expected utility of playing soft is $v_1 - h_L$. Playing tough is now an out of equilibrium action, and we specify that, faced with a tough opponent, each bidder plays defensively. (This is optimal for any belief which assigns a high probability to high values of the opponent). Then the highest utility which can be obtained by opening tough and then choosing s is obtained solving:

$$\max_s \left[2 \int_{h_L}^{s} (v_1 - v_2) f(v_2)\, dv_2 + (v_1 - s)(1 - F(s)) \right].$$

Again, it can checked that at $s = v_1$ the derivative is negative, and that, since $f(v_2)$ is nondecreasing, the derivative must be negative over (h_L, v_1). Thus, the optimal stopping time turns out to be $s = h_L$. The deviation is therefore not profitable.

The equilibrium has the remarkable property that it does not depend on λ, the fraction of budget-constrained players. That is, for *any* $\lambda > 0$ the most competitive equilibrium has all the high-budget bidders mimicking the low-budget bidders when the bids reach (h_L, h_L). This implies a discontinuity in the equilibrium set. When $\lambda = 0$ then a "competitive" equilibrium exists in which each bidder pushes up the bid on each object up to their value. However, for any $\lambda > 0$ this equilibrium disappears, and it becomes impossible to induce competition among bidders at prices higher than h_L.

4.3 Increasing Welfare by Excluding Low-budget Bidders

The outcome of the noncollusive equilibrium described above for nondecreasing densities is inefficient. As the probability $(1-\lambda)^2$ that both bidders are not constrained increases toward 1, efficiency requires that the probability with which both objects be assigned to the bidder with the highest value also approach 1. In the limit, the welfare loss is equal to the expected value of $|v_2 - v_1|$, conditional on $\min\{v_1, v_2\} \geq h_L$.

For small values of λ, measures restricting the participation of low-budget bidders can increase the expected social surplus, as well as the seller's expected revenue. For example, the seller may impose a reserve price for each object above w_L, or require each bidder to deposit an amount of w_H at the beginning of the auction. Once the possibility that any participating bidder is budget constrained is ruled out, the high-budget bidders cannot 'hide' behind budget constrained types; hence the noncollusive equilibrium produces the socially efficient outcome. For sufficiently small values of λ, the cost of excluding low-budget bidders is lower than the gain in social surplus obtained by inducing the efficient allocation of the objects.

As an example, suppose that the distribution is $F(v) = v^4$, and take $h_L = 0.35$. First, we check that there is no equilibrium with $v_* < 1$, and that it is an equilibrium for the bidders to split the objects when the two values are above 0.35. This does not follow immediately from the previous analysis because we now have $w_L < 1$. The additional complication in this case is that the low-budget types with value greater than w_L can offer at most w_L for a single object. We now show that, as in the case where $w_L \geq 1$, the equilibrium threshold v_* cannot be strictly less than 1.

First note that, if $v_* < 0.7$, or $s < 0.7$, the analysis of the previous subsection applies immediately. Thus consider $v_* \in (0.7, 1)$ and $s > 0.7$. Then the optimal stopping time for the type v_* is obtained solving:

$$\max_{s} \lambda \left[\int_{0.35}^{0.7} 2\left(v_{*}-y\right)\left(4y^{3}\right) dy + 2\left(v_{*}-0.7\right)\left(1-(0.7)^{4}\right)\right] +$$

$$(1-\lambda)\left[\int_{0.35}^{\min\{s,v_{*}\}} 2\left(v_{*}-y\right)\left(4y^{3}\right) dy + \left(v_{*}-s\right)\left(v_{*}^{4}-s^{4}\right)\right]$$

The derivative is strictly negative for each $s < v_{*}$, so that the optimal stopping time is 0.7. The expected utility of playing tough is therefore:

$$U\left(tough\right) = \lambda \left[\int_{0.35}^{0.7} 2\left(v_{*}-y\right) \frac{4y^{3}}{1-(0.35)^{4}} dy + 2\left(v_{*}-0.7\right) \frac{1-(0.7)^{4}}{1-(0.35)^{4}} \right] +$$

$$(1-\lambda)\left[\int_{0.35}^{0.7} 2\left(v_{*}-y\right) \frac{4y^{3}}{1-(0.35)^{4}} dy + \left(v_{*}-0.7\right) \frac{v_{*}^{4}-(0.7)^{4}}{1-(0.35)^{4}}\right]$$

$$= \left[\int_{0.35}^{0.7} 2\left(v_{*}-y\right) \frac{4y^{3}}{1-(0.35)^{4}} dy\right] +$$

$$+\left[2\lambda\left(1-(0.7)^{4}\right) + (1-\lambda)\left(v_{*}^{4}-(0.7)^{4}\right)\right] \frac{v_{*}-0.7}{1-(0.35)^{4}}$$

The expected utility of opening soft is

$$U\left(soft\right) = \left(\lambda + (1-\lambda) \frac{v_{*}^{4}-(0.35)^{4}}{1-(0.35)^{4}}\right)\left(v_{*}-0.35\right)$$

As λ goes to zero we have:

$$U\left(tough\right) = \left[\int_{0.35}^{0.7} 2\left(v_{*}-y\right) \frac{4y_{2}^{3}}{1-(0.35)^{4}} dy\right] + \left(v_{*}-0.7\right) \frac{v_{*}^{4}-(0.7)^{4}}{1-(0.35)^{4}}$$

$$U\left(soft\right) = \frac{v_{*}^{4}-(0.35)^{4}}{1-(0.35)^{4}} \left(v_{*}-0.35\right)$$

It can now be checked that:

$$U\left(soft\right) > U\left(tough\right)$$

for each $v_{*} \in [0.7, 1]$. Thus, any equilibrium must have $v_{*} = 1$. It is now straightforward to check that $v_{*} = 1$ can in fact be supported in equilibrium.

When λ is close to 1, the expected social welfare is approximately:

$$W^{a} = \int_{0}^{0.35} \int_{0}^{v_{1}} 2v_{1}\left(4v_{2}^{3}\right) dy + \int_{v_{1}}^{1} 2v_{2}\left(4v_{2}^{3}\right) dv_{2}\left(4v_{1}^{3}\right) dv_{1}$$

$$+\int_{0.35}^{1}\left(\int_{0}^{0.35}2v_1\left(4v_2^3\right)dv_2+\int_{0.35}^{1}(v_1+v_2)\left(4v_2^3\right)dv_2\right)\left(4v_1^3\right)dv_1=1.6156.$$

If a reservation price of 0.7 is imposed, then all low budget types, as well as the high budget types with value lower than 0.7, do not participate. In this case, since it is common knowledge that the participants are not budget constrained, there is a competitive equilibrium in which the bidder with the highest value wins both objects, and the expected social welfare is approximately:

$$W^b=(0.7)^4\times\left(\int_{0.7}^{1}2v_2\left(4v_2^3\right)dv_2\right)+$$

$$+\int_{0.7}^{1}\left(\int_{0}^{v_1}2v_1\left(4v_2^3\right)dv_2+\int_{v_1}^{1}2v_2\left(4v_2^3\right)dv_2\right)\left(4v_1^3\right)dv_1=1.706$$

To see how the expression is computed, notice that when $v_1<0.7$, which happens with probability $(0.7)^4$, then the two objects go to bidder 2 iff $v_2>0.7$; this is the first term. When $v_1>0.7$ then the two objects go to the highest bidder (second term). Since $W^b>W^a$, in this case the imposition of a reservation price increases efficiency.

5 Conclusions

We have explored the effects that the possibility of binding budget constraints may have on bidding behavior in simultaneous ascending bid auctions. While it is clear that budget constraints reduce the level of competition because the bidders have a lower ability to pay, we have also seen that competition is further reduced due to strategic reasons. In fact even the slightest possibility of having binding budget constraints may lead to outcomes which appear (but are not) collusive: i.e., the bidders split the objects at low prices. In these cases, measures which exclude budget-constrained bidders from participating can be welfare enhancing, since they stimulate competition and favor a more efficient allocation of the objects.

Appendix

Proof of Proposition 1. We establish the proposition by showing that, if bidder 2 is following the prescribed strategy, then bidder 1 has no profitable deviation. We start by observing that it is never optimal for bidder 1 to raise the bid on any object by more than the minimum increment. This is because bidder 2's behavior at any stage depends only on the current outstanding bids and the assignment of the objects; it does not depend on how the current outstanding bids and assignment have been reached. Therefore jump-bidding can never be optimal for bidder 1. In fact, we can restrict attention to strategies such that at any round bidder 1 raises the bid by the minimum increment on at most one object.

We first show that, on the equilibrium path, the strategies are optimal even ex-post, i.e. for any realization of the bidders' values v_1, v_2. We partition the type space in three subsets.

Case 1: $h \leq \min\{v_1, v_2\}$. If both bidders follow the prescribed strategy, each buys one object and pays h. Because of the budget constraint, bidder 1 can win both objects only if at least one price is below h; but in this case bidder 2 would not let the auction end before pushing each price above h. Also, if each bidder gets one object, say bidder i pays p_i for object i, then we must have that $h \leq p_1$. To see this, observe that since bidder 2 is following the prescribed strategy and the auction ends at (p_1, p_2) assigning object 2 to bidder 2 and object 1 to bidder 1, we must have $\min\{h, w - p_2\} < p_1$; if not, bidder 2 would bid on object 1 as well and the auction would not be over. If $p_1 < h$ then this implies $w - p_2 < h$, i.e. $h < p_2$. This is impossible because, by following the prescribed strategy, bidder 2 cannot push the bid on object 2 above h, unless the bid on the first object goes above h as well.

Case 2: $v_2 < \min\{h, v_1\}$. By following the equilibrium strategy, bidder 1 obtains both objects at the price v_2. Given the strategy of bidder 2, it is impossible for bidder 1 to win any object for a unit price lower than v_2.

Case 3: $v_1 < \min\{h, v_2\}$, or $v_1 = v_2 < h$. By following the prescribed strategy bidder 1 obtains a utility of zero. Whatever the deviation, it is impossible to buy an object for less than v_1, thus implying that a utility of zero is the maximum which can be attained.

We have proved that no deviation is profitable along the equilibrium path. We next show that the strategy is optimal also out of the equilibrium path.

Thus, suppose now that bidder 1 is facing an out-of-equilibrium pair of bids (b_1, b_2) and allocation of objects. If bidder 1 is not winning any object, then it is optimal to stop if $\min\{b_1, b_2\} \geq \min\{v_1, w\}$. If the reverse inequality is true, and given bidder 2's strategy, then it is optimal to make a bid on the object with the lowest value. This is true for any belief on v_2. Similarly, if bidder 1 is winning both objects it is optimal to avoid increasing the bids.

We are left with the case in which bidder 1 is winning object $3-j$ but not object j. If $b_j \geq w - b_{3-j}$ then bidder 1 cannot bid on object j, and given the strategy of agent 2 it doesn't make sense to increase the bid on object $3-j$. Similarly, if $b_j \geq v_1$ then bidding on object j is suboptimal. We conclude that it is optimal to stop bidding whenever $b_j \geq \min\{v_1, w - b_{3-j}\}$. Consider now the case $b_j < \min\{v_1, w - b_{3-j}\}$. If $b_j \geq h$ then bidder 1 believes that the type of agent 2 is in the interval $[b_j, 1]$. Any attempt to buy object j will cause bidder 2 to bid on object b_{3-j} until the price reaches at least h. Therefore bidder 1 cannot get both objects, and it is optimal to stop bidding. Summing up, it is optimal to stop bidding whenever $b_j \geq \min\{v_1, w - b_{3-j}, h\}$. The last case we have to deal with is $b_j < h \leq \min\{v_1, w - b_{3-j}\}$. Notice that this implies $w - b_{3-j} \geq h$, or $h \geq b_{3-j}$.

If $v_2 > h$ then bidder 2 will bid on object $3-j$ as well and the final outcome is that each bidder wins one object at h. Increasing the bid on j is therefore weakly optimal. If $v_2 \in (b_j, h)$ then we distinguish two cases.

If $b_{3-j} \geq v_2$ then bidder 2 will not bid on $3-j$ any longer, and it will bid up to v_2 for object j. Therefore it is optimal for bidder 1 to try to win object j increasing the bid.

If $b_{3-j} < v_2$ then bidder 2 will try to win both objects, always bidding on any object she is not currently winning. Therefore, no object can be bought for less than v_2. The prescribed strategy allows bidder 1 to buy both objects at the lowest possible price, given the strategy of bidder 2. ■

Proof of Proposition 5. First, we characterize the function $U_1(v_1, s|\, b_{3-j}, \widehat{v})$ which is used in (16) to derive the optimal stopping function. Then, we will show that the strategy profile described in the proposition forms a perfect Bayesian equilibrium.

Consider bidder 1 (the high-budget bidder) and suppose that beliefs on the opponent are given by $\widehat{F}(v_2|\, v_2 \geq \widehat{v})$. Suppose that bidder 1 is currently winning object $3-j$, and she has to decide whether to bid on object j. We first observe that if $b_j \geq \min\{h_1, w_1 - b_{3-j}\}$ then the bidder should stop bidding immediately, and try to get a single object. If $b_j \geq w_1 - b_{3-j}$ this is the only feasible action. If $b_j \geq h_1$ then $\widehat{v} \geq h_1$; and this in turn means that it is impossible to win both objects paying less than h_1, which makes it impossible for bidder 1 to win both objects. Therefore, bidder 1 should stop immediately and get one object.

Assume therefore $b_j < \min\{h_1, w_1 - b_{3-j}\}$, so that bidding on j is feasible and possibly optimal. We want to see what is the optimal stopping price in this situation, for each possible pair of prices (b_j, b_{3-j}). In the analysis we can assume that $b_j \leq \widehat{v}$, since the opponent is winning object j at b_j and $\widehat{v}$ is the highest offer made by bidder j.

Case 1. $b_{3-j} \leq \widehat{v}$. In this case it must be $b_{3-j} \leq b_j$. If not then we would have $b_j \leq b_{3-j} \leq \widehat{v}$; since bidder 1 is winning object $3-j$, the highest bid made by 2 on that object must be inferior to b_{3-j}. Since 2 is winning object j, the highest bid made on that object must be b_j. Therefore, $b_{3-j} \leq \widehat{v}$ implies $b_{3-j} \leq b_j$.

As soon as bidder 1 tries to get j, bidder 2 will raise the bid on $3-j$. Therefore, any attempt to get both objects will first increase the price of object $3-j$ to b_j, and then it will increase the bids simultaneously. This creates a discontinuity. By bidding on j, bidder 1 becomes the winner on the highest priced object. The alternative is between winning one object at $s = b_{3-j}$ or both, and in this case it must be $s > \widehat{v}$. The objective function can be written as follows:

$$2\int_{b_{3-j}}^{\max\{s,\widehat{v}\}} (v_1 - v_2)\, d\widehat{F}(v_2 | v_2 \geq \widehat{v}) + (v_1 - s)\left[1 - \widehat{F}(s | v_2 \geq \widehat{v})\right]$$

Notice that $dF(v_2 | v_2 \geq \widehat{v}) = 0$ for each $v_2 \in (b_{3-j}, \widehat{v})$. It is therefore never optimal to choose a stopping time $s \in (b_{3-j}, \widehat{v})$, since such choice is dominated by $s = b_{3-j}$.

Case 2. $\widehat{v} < b_{3-j}$. Again, there is no hope of getting both objects for a price inferior to $\widehat{v}$. When $s \in [\widehat{v}, b_{3-j})$ then giving up means that 1 pays b_{3-j} for the object she is currently winning, while the other bidder pays s. If $v_2 \in (\widehat{v}, s)$ then 1 wins both objects, paying them v_2 and b_{3-j} respectively, while if $v_2 > s$ then bidder 1 wins a single object and pays it b_{3-j}. If $s \geq b_{3-j}$ then bidder 1 wins both objects paying a total of $v_2 + \max\{v_2, b_{3-j}\}$ if $v_2 \leq s$, while if $v_2 > s$ then bidder 1 wins a single object which is paid s. We can summarize the objective function as:

$$\int_{\widehat{v}}^{s} (2v_1 - v_2 - \max\{v_2, b_{3-j}\})\, d\widehat{F}(v_2 | v_2 \geq \widehat{v}) + (v_1 - \max\{s, b_{3-j}\})\left[1 - \widehat{F}(s | v_2 \geq \widehat{v})\right].$$

The analysis of the two cases can be unified by writing the bidder 1's objective function as:

$$U_1(v_1, s | b_{3-j}, \widehat{v}) = 2\int_{\min\{b_{3-j},\widehat{v}\}}^{\max\{s,\widehat{v}\}} (2v_1 - v_2 - \max\{v_2, b_{3-j}\})\, d\widehat{F}(v_2 | v_2 \geq \widehat{v}) +$$

$$+ (v_2 - \max\{s, b_{3-j}\})\left[1 - \widehat{F}(s | v_2 \geq \widehat{v})\right].$$

At last, we observe that the same reasoning applies when any of the two prices is below h_2. It is clear that, if $\max\{b_1, b_2\} < h_2$, then setting an optimal stopping time of v_1 is the only sensible thing to do, since the other bidder tries to go for both objects. If instead $\max\{b_1, b_2\} \geq h_2$, it is optimal the other bidder to stop immediately when winning one object; and the best reply to this strategy is to adopt the optimal stopping time as in the case in which both bids are above h_2.

We can now show that the strategy profile described in the proposition constitutes a perfect Bayesian equilibrium. As in the proof of Proposition 1, in order to prove that a bidder's strategy is a best response, we can restrict attention to strategies in which at each round a bid is made only on (one of) the lowest priced object(s). Optimality for agent 2 basically follows from the arguments used in Proposition 1. It suffices to observe that the strategy is a best reply whenever bidder 1 adopts a 'stopping time' function $s(v_1)$ such that $s(v_1) = v_1$ when $v_1 \leq h_2$.

Assume now that bidder 2 is following the prescribed strategy. The proof that bidder 1's strategy is optimal at any information set is broken in two steps. First, without loss of generality we restrict attention to pure strategies[5], and show that for any type v_1, and for any pair of outstanding bids $(b_1,\ b_2)$, given that bidder 2 is following the noncollusive strategy, any strategy of bidder 1 induces a partition of bidder 2's type space $[0, 1]$ in at most three subintervals: all types with value v_2 below a given threshold τ_L lose both objects, all types with value v_2 above a second threshold τ_H win both objects, and all types with value v_2 between τ_L and τ_H win one object.

The second step consists in showing that the prescribed strategy maximizes bidder 1's expected surplus over all outcomes such that the objects are assigned according to a three-interval partition as specified above and bidder 1's total payments are the lowest among the ones which can be obtained given bidder 2's strategy.

The strategy prescribed for bidder 1 is clearly optimal when she is winning both objects or no object. In the second step we therefore deal only with the case in which bidder 1 is winning only one object.

Step 1. Consider any round t and any pair of outstanding bids b_1 and b_2. Suppose that, given both bidder's (pure) strategies, the outcome entails bidder 2 losing both objects. Since bidder 2 is following the noncollusive strategy, this can happen only if both final outstanding bids are at least as large as v_2. But then any type of bidder 2 with a lower value $v_2' < v_2$ also loses both objects. We conclude that the set of types of bidder 2 which lose both objects is an interval with infimum zero. Let τ_L denote its supremum. Notice also that bidder 1's total payment in this case cannot be

[5]Recall that we are only looking for a best reply for bidder 1 to bidder 2's strategy. Therefore if the best reply correspondence includes a mixed strategy, any pure strategy in its support is also optimal.

lower than $\max\{v_2, b_1\} + \max\{v_2, b_2\}$, since the final price of each object j cannot be lower than $\max\{v_2, b_j\}$.

Now suppose that, given both bidder's (pure) strategies, bidder 1 loses both objects. This can happen only if bidder 2's value v_2 is above both final bids. But then all types with value $v_2' > v_2$ behave identically, and win both objects. The set of types of bidder 2 which win both objects is thus also an interval, this time with supremum 1. Let τ_H denote its infimum. Bidder 1's total payment in this case is zero.

All types in the remaining middle interval, with value between τ_L and τ_H, must win only one object. Bidder 1's total payment in this case is at least $\min\{b_1, b_2\}$.

Step 2. Suppose bidder 1 is winning object $3-j$ at b_{3-j}, and the other object has price b_j. We first observe that the belief at round $t+1$ that bidder 2's type is distributed according to $F(v_2 | v_2 \geq \widehat{v})$, where $\widehat{v}$ is the highest bid ever made on any object in the t preceding rounds, is consistent both on and off the equilibrium path. Given this belief and the fact that the types of bidder 2 can be divided in intervals as described in Step 1, the optimal strategy of bidder 1 must involve an optimal stopping time. The function $s_1(v_1; b_{3-j}, \widehat{v})$ is clearly optimal. ■

Proof of Proposition 6. In order to prove that an equilibrium exists, we have to show that a type $v_* \geq h_L$ exists, such that all types $v_i \in [h_L, v_*)$ prefer to play 'soft' when prices reach h_L, while all types $[v_*, 1]$ prefer to play 'tough'. We begin by establishing some preliminary results. Let $U(v_1, s; v_*, b)$ be the function defined by (18) and (19), $R(v_1; b, v_*)$ the correspondence defined by (20) and $s(v_1; b, v_*)$ the function defined by (21). In our first lemma we characterize some properties of the optimal stopping rule.

Lemma 1 *The optimal stopping rule satisfies the following properties:*

- *The correspondence $R(v_1; b, v_*)$ is upper-hemicontinuous in v_1 and v_* and compact valued.*
- *The function $s(v_1; b, v_*)$ is non-decreasing and $s(v_1; b, v_*) < v_1$ whenever $v_1 \in (b, 1)$.*
- *If $s(v_1; b, v_*)$ is constant over an interval $[v_1, v_1 + \delta)$ then either $s(v_1; b, v_*) = b$ or $s(v_1; b, v_*) = v_*$.*
- *Let $K \subset [b, 1]$ be the set of points of discontinuity of $s(v_1; b, v_*)$ and let ψ be a measure defined on $[b, 1]$ which is absolutely continuous with respect to the Lebesgue measure. Then $\psi(K) = 0$.*

Proof. The properties of the correspondence R follow from the Maximum Theorem and the continuity of $U(v_1, s; v_*, b)$.

Since the function $U(v_1, s; v_*, b)$ satisfies increasing differences in $(s; v_1)$, we have that $s(v_1; b, v_*)$ is non-decreasing in v_1 (see e.g. Milgrom and Shannon [16]). It is obvious that $s(v_1; b, v_*) \leq v_1$. To see that $s(v_1; b, v_*) < v_1$ whenever $v_1 \in (b, 1)$ observe that $U(v_1, s; v_*, b)$ is always left-differentiable with respect to s at $s = v_1$, and the left derivative $\left.\frac{\partial^- U(v_1,s;v_*,b)}{\partial s}\right|_{s=v_1}$ is strictly negative whenever $v_1 < 1$. Thus $s = v_1$ cannot be optimal.

Suppose now that $s(v_1; b, v_*)$ is constant over an interval $[v_1, v_1 + \delta)$. When $b \geq v_*$ then the function $U(v_1, s; v_*, b)$ is everywhere differentiable in s. Therefore, for s to be optimal it must be the case that:

$$\frac{\partial U}{\partial s} = (v_1 - s) f(s) - [1 - F(s)] \leq 0.$$

Let $\overline{s} = s(v_1; b, v_*)$. If $\frac{\partial U}{\partial s}(v_1, \overline{s}; b, v_*) = 0$ then, for each $v_1' > 0$ we have $\frac{\partial U}{\partial s}(v_1', \overline{s}; b, v_*) > 0$, so that $\overline{s}$ cannot be the optimal stopping time on an interval $(v_1, v_1 + \delta)$. If $\frac{\partial U}{\partial s}(v_1, \overline{s}; b, v_*) < 0$ then it must be the case that $\overline{s} = b$.

Consider now the case $b < v_*$. The function is differentiable in s except at $s = v_*$. The derivative at $s \neq v_*$ is:

$$\frac{\partial U}{\partial s} = \begin{cases} (v_1 - s) f(s) - [\lambda + (1-\lambda) F(v_*) - F(s)] & \text{if } s < v_* \\ \\ \lambda [(v_1 - s) f(s) - (1 - F(s))] & \text{if } s > v_* \end{cases}$$

Thus, suppose that at v_1 we have $\overline{s} = s(v_1; b, v_*)$. If $\overline{s} < v_*$ then we can apply the same reasoning as above to conclude that the function can only be constant if $\overline{s} = b$. Suppose now $\overline{s} = v_*$. Notice that at v_* the function U is both left and right differentiable, and we have:

$$\left.\frac{\partial^- U}{\partial s}\right|_{s=v_*} = (v_1 - v_*) f(v_*) - \lambda [1 - F(v_*)]$$

$$\left.\frac{\partial^+ U}{\partial s}\right|_{s=v_*} = \lambda [(v_1 - v_*) f(v_*) - [1 - F(v_*)]]$$

Since $\lambda \in (0, 1)$, it is possible to have $\left.\frac{\partial^- U}{\partial s}\right|_{s=v_*} > 0 > \left.\frac{\partial^+ U}{\partial s}\right|_{s=v_*}$ over a set $[v_1, v_1 + \delta)$. In this case v_* can be the optimal stopping time for each v_1 in the set, and the optimal stopping time can therefore be constant. Next, suppose $\overline{s} > v_*$. Then $\overline{s}$ can be optimal only if the derivative is zero, and this in turn implies that $\overline{s}$ cannot be the optimal stopping time if $v_1' > v_1$.

To prove the last point observe that a non-decreasing function defined on a compact set has at most countably many points of discontinuity (Kolmogorov-Fomin, page 316, Theorem 3), and a countable set has Lebesgue measure zero. ■

Suppose now that we are on the equilibrium path, and the bids have just reached the level h_L. Each bidder is winning one object. At this point, each bidder has to signal whether she is "soft," by remaining silent, or "tough," by bidding on the object she is not winning. Fix an arbitrary threshold $v_* \in (h_L, 1]$, and assume that bidder 1 conjectures that her opponent plays soft if and only if $w_2 = w_L$, or $w_2 = w_H$ and $v_2 < v_*$. Let $G(v_2) \equiv \frac{F(v_2)-F(h_L)}{1-F(h_L)}$.

Suppose that bidder 1 plays tough. If bidder 2 is not budget constrained, then with probability $1 - G(v_*)$ she also plays tough: each bidder then bids up to her value for both objects, and bidder 1 earns $2\max\{v_1 - v_2, 0\}$. With probability $G(v_*)$, the high-budget opponent plays soft. In this case, by trying to win both objects until the bids arrive at level s bidder 1 earns $2(v_1 - v_2)$ if $v_2 < s$, and $v_1 - s$ otherwise. Thus the expected payoff for bidder 1 when facing a high-budget opponent is:

$$T_H(v_1; s, v_*) = 2\int_{h_L}^{\min\{v_*,s\}} (v_1 - v_2)\, dG(v_2) + \int_{\min\{v_*,s\}}^{v_*} (v_1 - s)\, dG(v_2)$$
$$+2\int_{v_*}^{1} \max\{v_1 - v_2, 0\}\, dG(v_2).$$

If instead bidder 2 is budget constrained, she will also play soft, and bidder 1 can push both bids up to s, thus earning on average:

$$T_L(v_1; s, v_*) \equiv 2\int_{h_L}^{s} (v_1 - v_2)\, dG(v_2) + (v_1 - s)[1 - G(s)].$$

The overall expected payoff of playing tough, and selecting a stopping time s against a soft opponent is:

$$\lambda T_L(v_1; s, v_*) + (1-\lambda) T_H(v_1; s, v_*).$$

Now let

$$T(v_1; v_*) \equiv \max_{s\in[h_L,1]} \lambda T_L(v_1; s, v_*) + (1-\lambda) T_H(v_1; s, v_*).$$

This function is bidder 1's expected surplus of opening tough when she conjectures that her opponent plays tough if and only if $w_2 = w_H$ and $v_2 > v_*$.

Lemma 2 *The function $T(v_1; v_*)$ is continuous in (v_1, v_*).*

Proof. This follows from the Maximum Theorem and the fact that the function $\lambda T_L(v_1; s, v_*) + (1-\lambda) T_H(v_1; s, v_*)$ is continuous in v_1, s and v_*. ■

Suppose now that bidder 1 plays soft. If $w_2 = w_L$, or if $w_2 = w_H$ and $v_2 < v_*$, bidder 2 also plays soft, and the auction ends immediately with one object going to each bidder. Bidder 1's surplus in this case is $v_1 - h_L$. The probability of this happening is $\lambda + (1-\lambda) G(v_*)$.

If instead $w_2 = w_H$ and $v_* \leq v_2$, bidder 2 plays tough, i.e. bids on her second object, and continues to do so until the bids reach $s(v_2; h_L, v_*)$, if bidder 1 responds by bidding "defensively", i.e. if by bidding on one object only when she is losing both. At any given round however, bidder 1 may decide to bid on a second object. This constitutes an out of equilibrium action, and our equilibrium specifies that in this case the two bidders will simply bid up to their values.

It is clear that, if $v_1 \leq v_*$, no such deviation is profitable for bidder 1, since her opponent has a higher value: $v_1 \leq v_* < v_2$. If instead $v_* < v_1$, suppose that bidder 1 bid defensively until the bids reach level b_a and then try to win both objects. In this case her expected payoff is:

$$S(v_1, b_a; v_*) = [\lambda + (1-\lambda) G(v_*)] \, (v_1 - h_L) + (1-\lambda) \, S_H(v_1, b_a; v_*),$$

where:

$$S_H(v_1, b_a; v_*) \equiv \int_{v_*}^{v(b_a)} [v_1 - s(v_2; h_L, v_*)] \, dG(v_2) + 2 \int_{\min\{v(b_a), v_1\}}^{v_1} (v_1 - v_2) \, dG(v_2).$$

and:

$$v(b_a) := \sup\{ v_2 | \, s(v_2; h_L, v_*) \leq b_a \}$$

is the highest type of bidder 2 with a stopping time inferior to b_a.

To be part of a sequentially rational strategy the point b_a has to be chosen optimally. In order to analyze this problem, it is useful to reformulate it in terms of the choice of an optimal "stopping type" $v_a = v(b_a)$. In this case we write

$$S_H(v_1, v_a; v_*) \equiv \int_{v_*}^{v_a} [v_1 - s(v_2; h_L, v_*)] \, dG(v_2) + 2 \int_{\min\{v_a, v_1\}}^{v_1} (v_1 - v_2) \, dG(v_2).$$

For a given v_* and corresponding function $s(v_2; h_L, v_*)$, define:

$$v^+ = \inf\{ v_2 \in [v_*, 1] | \, s(v_2; h_L, v_*) > h_L \}$$

$$\underline{v} = \sup\{ v_2 \in [v_*, 1] | \, s(v_2; h_L, v_*) < v^* \}$$

$$\overline{v} = \inf \{ v_2 \in [v_*, 1] | \, s(v_2; h_L, v_*) > v^* \}$$

By Lemma 1 the function $s(v_2; h_L, v_*)$ can be flat only over an initial interval $[v_*, v^+]$ and another interval $[\underline{v}, \overline{v}]$ at which $s = v_*$, and it is strictly increasing elsewhere. Therefore choosing a "trigger point" b_a is equivalent to choosing a "trigger type" v_a in the set:

$$A(v_*) = \{v_*\} \cup [v^+, \underline{v}] \cup [\overline{v}, 1].$$

Recall that we are analyzing what happens in the round after the bids have reached (h_L, h_L), bidder 2 has played tough, and bidder 1 has remained silent.

At this point, a choice of v_* can be interpreted as "trigger the fight immediately", by bidding on both objects (this is equivalent to choosing $b_a = h_L + \varepsilon$ as triggering bid). A choice of v^+ can be interpreted as bidding defensively after the opponent has made a bid to $h_L + \varepsilon$, so that the bids reach $(h_L + \varepsilon, h_L + \varepsilon)$, then wait to see if the opponent counterbids and in that case trigger the war (equivalent to choosing $b_a = h_L + 2\varepsilon$ as triggering bid). A choice of $v_a \in (v^+, \underline{v}]$ simply means "trigger the fight as soon as the bids reach $s(v_a; h_L, v_*)$", where $s(v_a; h_L, v_*) \in (h_L, v_*]$. A choice of $\overline{v}$ means "trigger the fight as soon as the bids reach $v_* + \varepsilon$, and so on. Observe that the function $S_H(v_1, v_a; v_*)$ is continuous in v_a and that $A(v_*)$ is compact. Now define:

$$S_H(v_1; v_*) = \max_{v_a \in A(v_*)} \quad S_H(v_1, v_a; v_*).$$

We are now ready to prove the following result.

Lemma 3 *The function $S(v_1; v_*)$ is continuous in (v_1, v_*).*

Proof. It suffices to show that the function $S_H(v_1, v_a; v_*)$ is continuous with respect to $(v_1, v_a; v_*)$, and that the correspondence $A(v_*)$ is continuous. Then the result follows from the Maximum Theorem.

Continuity in v_1 and v_a is immediate. In order to show that $S_H(v_1, v_a; v_*)$ is continuous in v_* it is enough to show that:

$$H(v_1, v_a; v_*) \equiv \int_{v_*}^{v_a} s(v_2; h_L, v_*) \, dG(v_2)$$

is continuous in v_*, which in turn is implied by the fact that s is continuous almost everywhere, since G is atomless. To establish continuity of s almost everywhere, suppose that at a point v_2 we have

$$\lim_{n \to \infty} \quad s(v_2; h_L, v_n) = s^* \neq s(v_2; h_L, v_*),$$

where $\{v_n\}$ is a sequence converging to v_*. It must be $s^* \in R(v_2; h_L, v_*)$, since the correspondence $R(v_2; h_L, v_n)$ is upper-hemicontinuous in v_n (Lemma 1). This in turn implies $s^* < s(v_2; h_L, v_*)$, since $s(v_2; h_L, v_*)$ is the maximum of $R(v_2; h_L, v_*)$. Thus v_2 must be a point of discontinuity of $s(v_2; h_L, v_*)$. But we have already established in Lemma 1 that, since s is nondecreasing in v_2, the set of discontinuity points has measure zero.

We come now to the continuity of $A(v_*)$. The only complications here are created by the 'flat' parts of the stopping function s, which generate 'gaps' in the interval. We will analyze the case in which the only flat part is at v_*, as it is always the case in equilibrium. Extending the analysis to incorporate the possibility of a flat part at h_L is immediate.

With no flat part at h_L, we have $A(v_*) = [h_L, \underline{v}] \cup [\overline{v}, 1]$. Consider a sequence $\{v^n\}$ converging to v_*, and let $A(v^n) = [h_L, \underline{v}^n] \cup [\overline{v}^n, 1]$. To prove the continuity of the correspondence $A(\cdot)$ it suffices to show that $\underline{v}^n \to \underline{v}$ and $\overline{v}^n \to \overline{v}$.

As a preliminary result, we first prove that if $\underline{v} < \overline{v}$, so that there is an open set $(\underline{v}, \overline{v})$ of types having v_* as optimal stopping time, then v_* is the unique optimal stopping time for all types in the set. Suppose not, so that $s^* \neq v_*$ is also an optimal stopping time for a type $v' \in (\underline{v}, \overline{v})$. This means $U(v', s^*; h_L, v_*) = U(v', v_*; v_*, h_L)$, since both s^* and v_* are optimal stopping times. Furthermore, since v_* is the highest stopping time, it must be $s^* < v_*$, so that:

$$U(v', s^*; v_*, h_L) = 2\int_{h_L}^{s^*} (v' - y)\, dF(y) + (v' - s)\left[\lambda + (1-\lambda) F(v_*) - F(s^*)\right].$$

Now observe that:

$$\frac{\partial U(v', s^*; v_*, h_L)}{\partial v'} = 2F(s^*) - 2F(h_L) + \left[\lambda + (1-\lambda) F(v_*) - F(s^*)\right].$$

On the other hand, when the optimal stopping time is v_* we have:

$$U(v', v_*; v_*, h_L) = 2\int_{h_L}^{v_*} (v' - y)\, dF(y) + \lambda (v' - s)\left[1 - F(v_*)\right],$$

so that:

$$\frac{\partial U(v', v_*; v_*, h_L)}{\partial v'} = 2F(v_*) - 2F(h_L) + \lambda\left[1 - F(v_*)\right].$$

Thus

$$\frac{\partial U(v', s^*; v_*, h_L)}{\partial v'} < \frac{\partial U(v', v_*; v_*, h_L)}{\partial v'}.$$

Since $U(v', s^*; h_L, v_*) = U(v', v_*; v_*, h_L)$, this implies that there is a type v'' sufficiently close to v' such that $v'' \in (\underline{v}, \overline{v})$ and $U(v'', s^*; h_L, v_*) > U(v'', v_*; v_*, h_L)$, a contradiction.

We now come back to proving the continuity of the correspondence $A(\cdot)$. Suppose first that $\underline{v} = \overline{v}$. This happens when there is a single value $\widehat{v}$ such that $s(\widehat{v}; h_L, v_*) = v_*$ (that is, no flat part), so that we have $A(v_*) = [h_L, 1]$. Suppose now that $\lim_{n\to\infty} \underline{v}^n = \widehat{\underline{v}} < \lim_{n\to\infty} \overline{v}^n = \widehat{\overline{v}}$. For each type $v' \in \left(\widehat{\underline{v}}, \widehat{\overline{v}}\right)$ it must be the case that $\lim_{n\to\infty} s(v'; v^n, h_L) = k$, a constant, and $k \in R(v'; v_*, h_L)$. Since no open interval of types can have a common optimal stopping time other than v_*, we conclude that v_* is an optimal stopping time for types in $\left(\widehat{\underline{v}}, \widehat{\overline{v}}\right)$. Then $\widehat{v} < \widehat{\underline{v}}$, since $s(\widehat{v}; h_L, v_*) = v_*$ and s is non-decreasing. Moreover, all types in $v' \in \left(\widehat{\underline{v}}, \widehat{\overline{v}}\right)$ must have both v_* and another (higher) point as optimal stopping times. But this cannot be true since, as proved above, if v_* is an optimal stopping time for an open interval of types then it has to be unique.

We now come to the case in which there is an open set $(\underline{v}, \overline{v})$ of types having v_* as optimal stopping time when the threshold is v_*. We will prove that for each $v' \in (\underline{v}, \overline{v})$, there is N large enough such that v' has v^n as optimal stopping time for each $n > N$. Furthermore, if v' and v'' have v^n as optimal stopping time then all types in the set (v', v'') have v^n as optimal stopping time. This is turn implies that $\underline{v}^n \to \underline{v}$ and $\overline{v}^n \to \overline{v}$.

Since v_* is the unique optimum for v', it must be the case that:

$$
\begin{aligned}
\left.\frac{\partial^- U(v', s; h_L, v_*)}{\partial s}\right|_{s=v_*} &= (v' - v_*) f(v_*) - \lambda (1 - F(v_*)) > 0, \\
\left.\frac{\partial^+ U(v', s; h_L, v_*)}{\partial s}\right|_{s=v_*} &= \lambda \left[(v' - v_*) f(v_*) - (1 - F(v_*))\right] < 0.
\end{aligned}
$$

This in turn implies that

$$
\begin{aligned}
\left.\frac{\partial^- U(v', s; h_L, v^n)}{\partial s}\right|_{s=v^n} &= (v' - v^n) f(v^n) - \lambda (1 - F(v^n)) > 0, \\
\left.\frac{\partial^+ U(v', s; h_L, v^n)}{\partial s}\right|_{s=v^n} &= \lambda \left[(v' - v^n) f(v^n) - (1 - F(v^n))\right] < 0,
\end{aligned}
$$

for n large enough, and it also implies that if the inequalities hold for two types v' and v'' then they must hold for all types in the interval (v', v''). This in turn implies that v^n is a local maximum for an open interval of types (v', v''). Now suppose that there is a different global maximum $s(v'; h_L, v^n)$. Since v^n is the only point of non-differentiability with respect to s, it must be the case that the derivative with respect to s computed at the optimal point $s(v'; h_L, v^n) \neq v^n$ must be zero.

Fix now a neighborhood of $I(v_*)$ such that $\frac{\partial U(v', s; h_L, v_*)}{\partial s} \neq 0$ for each $s \in I(v_*)$ at which U is differentiable. For n large enough, we also have $\frac{\partial U(v', s; h_L, v^n)}{\partial s} \neq 0$ for each $s \in I(v_*)$ such that U is differentiable. Now observe that, given the upper-hemicontinuity of the best response correspondence, it must be the case that $s(v'; h_L, v^n) \to v_*$, since v_* is the unique optimal stopping time $s(v'; h_L, v_*)$. This in turn implies that for n large enough, we have $s(v'; h_L, v^n) \in I(v_*)$. This is a contradiction, since at $s(v'; h_L, v^n)$ the derivative is supposed to be zero. ■

Lemma 4 *For any $v_* > h_L$ there exists a $\delta > 0$ such that all types $v_1 \in (h_L, h_L + \delta)$ prefer playing soft to playing tough.*

Proof. For any v_* we have $T(h_L; v_*) = S(h_L; v_*) = 0$. Assume $v_* > h_L$ and consider δ such that $h_L + \delta < v_*$. Then the utility of playing soft is:

$$S(v_1; v_*) = (\lambda + (1-\lambda) G(v_*))(v_1 - h_L) + (1-\lambda) S_H(v_1; v_*).$$

The utility of playing tough for a type v_1 is:

$$T(v_1; s, v_*) = \lambda \left[2 \int_{h_L}^{s} (v_1 - v_2)\, dG(v_2) + (v_1 - s)(1 - G(s)) \right]$$
$$+ (1-\lambda) \left[2 \int_{h_L}^{s} (v_1 - v_2)\, dG(v_2) + (v_1 - s)(G(v_*) - G(s)) \right].$$

Now observe that, for a small enough δ and for all types $v_1 < h_L + \delta$,

$$\frac{\partial T}{\partial s} = \lambda \left[(v_1 - s)\, g(s) - (1 - G(s)) \right]$$
$$+ (1-\lambda) \left[(v_1 - s)\ g(s) - (G(v_*) - G(s)) \right] < 0,$$

for each $s \in [h_L, h_L + \delta]$. Then the utility of playing tough is exactly:

$$H_L(v_1; s, v_*) = [\lambda + (1-\lambda) G(v_*)] [(v_1 - h_L)].$$

This is less than or equal to $S(v_1; v_*)$. ■

Lemma 5 *Suppose that v_* is an equilibrium threshold, and let $s(v_1; h_L, v_*)$ be the corresponding stopping function defined for $v_1 \in [v_*, 1]$. Then $\lim_{v_1 \downarrow v_*} s(v_1; h_L v_*) > h_L$.*

Proof. Since $s(v_1; h_L v_*)$ is monotone a limit exists. Suppose that the claim of the lemma is not true, so that $\lim_{v_1 \downarrow v_*} s(v_1; h_L, v_*) = h_L$. It must be the case that all types $v_1 > v_*$ prefer playing tough to playing soft. When we consider the utility of playing soft we have:

$$\lim_{v_1 \downarrow v_*} S(v_1; v_*) = [\lambda + (1-\lambda) G(v_*)]\ (v_* - h_L)$$
$$+ (1-\lambda) \int_{v_*}^{1} \max\{v_* - s(v_2; h_L, v_*), 0\}\, dG(v_2),$$

while when we look at the utility of playing tough we have:

$$\lim_{v_1 \downarrow v_*} T(v_1; v_*) = [\lambda + (1-\lambda) G(v_*)] \, (v_* - h_L).$$

Since there is a set with positive measure such that $s(v_2; h_L, v_*) < v_*$, we conclude that:

$$\lim_{v_1 \downarrow v_*} S(v_1; v_*) > \lim_{v_1 \downarrow v_*} T(v_1; v_*),$$

a contradiction. ■

The last lemma implies that, when v_* is actually an equilibrium value, then the corresponding stopping function $s(v_1; h_L; v_*)$ cannot take the value h_L over an interval. Combined with Lemma 1, it implies that the only value at which the stopping function can be constant is v^*.

Lemma 6 *For any equilibrium threshold value v_*, there is a set $[h_L, h_L + \delta]$ such that for each $v_1 \in [h_L, h_L + \delta]$ we have*

$$S(v_1; v_*) = T(v_1; v_*)$$

Proof. By Lemma 5, for every possible equilibrium threshold v_* there is $\delta' > 0$ such that $\lim_{v_1 \downarrow v_*} s(v_1; v_*) = h_L + \delta'$. This implies that all types $v_1 \in [h_L, h_L + \delta']$, when playing soft or tough can possibly win something only if they meet a soft type. Therefore

$$S(v_1; v_*) = (\lambda + (1-\lambda) G(v_*)) (v_1 - h_L).$$

When we look at the utility of playing tough, we know by Lemma 4 that for a set of types $[h_L, h_L + \delta]$ the optimal stopping time is h_L. Therefore, for this set $T(v_1; v_*) = S(v_1; v_*)$, thus yielding the result. ■

Define now:

$$D = \{\delta \in [h_L, 1] \mid S(v_1; \delta) \geq T(v_1; \delta) \text{ for all } v_1 \in [h_L, \delta]\}.$$

We know by Lemma 6 that the set D is non-empty. Since by Lemma 3 the functions $S(v_1; \delta)$ and $T(v_1; \delta)$ are continuous in δ, the set D is a closed interval. Then we define:

$$v_* = \max \quad D. \tag{22}$$

If $v_* = 1$ then we are done.

Suppose now $v_* < 1$. By definition of v_*, all types $v_1 < v_*$ prefer to play soft. The final step is to show that, for all types $v_1 \geq v_*$ we have $T(v_1; v_*) \geq S(v_1; v_*)$. This is done in the next lemma, which makes use of Assumption 1.

Lemma 7 *Let v_* be defined by (22), and suppose $v_* < 1$. Then, if assumption 1 is satisfied, we have $T(v_1; v_*) \geq S(v_1; v_*)$ for each $v_1 > v_*$.*

Proof. Consider the function

$$\Psi(v_1) = T(v_1; v_*) - S(v_1; v_*).$$

Since both T and S are continuous, so is Ψ. Also, we know that there exists $\varepsilon > 0$ such that $T(v_1; v_*) > S(v_1; v_*)$ for each $v_1 \in (v_*, v_* + \varepsilon)$. Thus it is enough to show that

$$\frac{\partial \Psi(v_1)}{\partial v_1} = \frac{\partial T(v_1; v_*)}{\partial v_1} - \frac{\partial S(v_1; v_*)}{\partial v_1} \geq 0$$

at each point of differentiability of Ψ. We start observing that, since $T(v_*; v_*) = S(v_*; v_*)$, then for any $\varepsilon > 0$ there must be some $v_1 \in (v_*, v_* + \varepsilon)$ such that $\frac{\partial \Psi(v_1)}{\partial v_1} > 0$.

By the previous analysis we have

$$\frac{\partial T}{\partial v_1} = \lambda(1 + G(s)) + (1-\lambda)[2G(v_1) - G(v_*) + G(\min\{s, v_*\})],$$

and

$$\frac{\partial S}{\partial v_1} = \lambda + (1-\lambda)[G(v_a) + 2\max\{G(v_1) - G(v_a), 0\}].$$

Suppose first $v_a \leq v_1$. Then

$$\frac{\partial \Psi(v_1)}{\partial v_1} = \lambda G(s) + (1-\lambda)[G(v_a) - G(v_*) + G(\min\{s, v_*\})],$$

which is positive since $v_a \geq v^*$.

Consider next $v_a > v_1$ (in which case $v_a = \sup\{v_2 \,|s(v_2; h_L, v_*) \leq v_1\}$). Now

$$\frac{\partial \Psi(v_1)}{\partial v_1} = \lambda G(s) + (1-\lambda)[2G(v_1) - G(v_*) - G(v_a) + G(\min\{s, v_*\})]$$

Since $v_a > v_1 > v_*$ then v_1 must be on a strictly increasing part of the stopping function, hence the following first order condition must hold :

$$(v_a - v_1)\, g(v_1) = 1 - G(v_1).$$

This is the condition ensuring that v_1 is the optimal stopping time for type v_a. (Since $v_1 > v_*$, the FOC that we apply is the one relative to the case $s > v_*$). The FOC can be rewritten as

$$v_a = v_1 + \mu(v_1).$$

Substituting into the expression above for $\frac{\partial\Psi(v_1)}{\partial v_1}$, we have

$$\begin{aligned}\frac{\partial\Psi(v_1)}{\partial v_1} &= \lambda G(s) + (1-\lambda)\left[G(\min\{s, v_*\}) - G(v_*)\right] \\ &\quad + (1-\lambda)\left[2G(v_1) - G(v_1 + \mu(v_1))\right].\end{aligned}$$

Now observe that, since s is increasing, the function

$$\lambda G(s) + (1-\lambda)\left[G(\min\{s, v_*\}) - G(v_*)\right]$$

is increasing. Furthermore, assumption 1 implies that

$$2G(v_1) - G(v_1 + \mu(v_1))$$

is increasing. We can then conclude that the expression of the derivative in this case is increasing. Thus, in order to prove that $\frac{\partial\Psi(v_1)}{\partial v_1}$ is positive, it is enough to show that at any $v_1' < v_1$ the expression is positive. Now remember that in a right neighborhood of v_* the function Ψ is strictly increasing. Furthermore, for v_1' sufficiently close to v_* it must be the case that $v_a > v_1'$ (it would not make sense to trigger a war). Then, there must be some point v_1' at which $\frac{\partial\Psi(v_1')}{\partial v_1} > 0$. This completes the proof. ■

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