Risk Sharing:
Private Insurance Markets or Redistributive Taxes?

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ABSTRACT

We explore the welfare consequences of different taxation schemes in an economy where agents are debt-constrained. If agents default on their debt, they are banned from future intertemporal trade, but retain their private (labor) endowments which are subject to income taxation. We impose individual rationality constraints on agents guaranteeing no default in equilibrium and we solve for efficient consumption distribution across agents. A change in the tax system changes the severity of punishment from default. We demonstrate that a change to a more redistributive tax system leads to a restriction of the set of contracts that are individually rational and that this restriction leads to a limitation of possible risk sharing via private contracts. The welfare consequences of a change in the tax system depend on the relative magnitudes of increased risk sharing forced by the new tax system and the reduced risk sharing in private insurance markets. We quantitatively address this issue by calibrating an artificial economy to US income and tax data. We show that for a plausible selection of the structural parameters of our model, the change to a more redistributive tax system leads to less risk sharing among individuals and, hence, lower ex-ante welfare.

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1. Introduction

A long standing discussion in macroeconomics and public finance evolves around the desirability of redistributive income taxes as a risk sharing device against idiosyncratic income uncertainty.[2] The insights that economic theory provides depend on the assumptions about the structure of private insurance markets. On one hand, if these markets are complete, in the sense that agents can insure against any contingency and thus achieve perfect risk sharing, then redistributive income taxes provide no additional insurance. On the other hand, if private insurance markets are nonexistent or incomplete, redistributive taxes might provide additional insurance not available otherwise and therefore increase ex-ante welfare.

Recent empirical studies (see Hayashi et al. (1996), Attanasio and Davis (1996)) have seriously challenged the complete markets assumption mostly on the ground that risk sharing among individuals is not perfect[3]. As Hayashi et al. conclude: "Our result there is no full insurance even among related households should serve as a final blow to the complete markets paradigm". Attempting to reconcile the theory with the empirical observation of incomplete risk sharing, there is a large body of literature that introduces some form of market incompleteness into the economic environment. One fraction of this literature (see Bewley (1986), Kimball and Mankiw (1989), Huggett (1993), Aiyagari (1994), among others) exogenously assumes that some insurance markets are nonexistent (usually it is assumed that agents can only self-insure through a single uncontingent bond and they are borrowing constrained). Another fraction derives market incompleteness from informational frictions underlying the phenomena of adverse selection and moral hazard (see Cole and Kocherlakota (1998) and their review of this extensive literature).

Both approaches do not explicitly capture what we believe is a crucial aspect of redistributive taxation as a risk sharing device: the fact that a particular tax regime is in place might affect the incentives private agents have to enter into private insurance contracts and thus the form and extent of market incompleteness. It is obvious that the first approach

[2] See Varian (1980) and his review of the related literature. The insurance effect of redistributive income taxes may also invalidate the Ricardian Equivalence theorem, see Chan (1983), Barsky et al. (1986) and Kimball and Mankiw (1989).

[3] As first highlighted by Mehra and Prescott (1985), another empirical dimension along which the complete markets model does not perform well is the explanation of the main stylized facts regarding asset prices.

is silent about this connection since market incompleteness is not derived from underlying primitives but rather exogenously assumed. In the second approach fiscal policy can affect the degree of market incompleteness, but only under the assumption of informational asymmetry between the government and private agents. However, every policy that leads to the same revelation of information to private agents has the same effect on the degree of market incompleteness. In this sense this approach does not provide a natural link between market incompleteness and redistributive income taxation in particular. Also, recent empirical work by Attanasio and Davis (1996) seems to indicate that "evidence against the consumption insurance hypothesis involves publicly observed shocks, hence cannot be rationalized as a consequence of unobserved shocks in environments with informationally constrained insurance" (p. 1259).

We therefore explore an alternative way of deriving endogenous market incompleteness that relies on limited enforceability of private contracts. We follow the approach of Kehoe and Levine (1993, 1998) and assume that private contracts can be enforced only by the threat of exclusion from future credit markets. Tax liabilities, however, are not subject to this enforcement problem as we assume that the penalty for defaulting on tax payments can be made prohibitively large by the government. If agents default on their private debt, they are banned from future intertemporal trade, but retain their private (labor) endowments which are still subject to income taxation. We impose individual rationality constraints on agents guaranteeing no default in equilibrium and we solve for the efficient consumption distribution across agents. A change in the tax system changes the severity of punishment from default by altering the utility an agent can attain without access to credit markets. We demonstrate that a change to a more redistributive tax system leads to a restriction of the set of contracts that are individually rational. In an economy that is characterized by uncertainty with respect to individual endowments, this restriction leads to a limitation of possible risk sharing via private contracts. The welfare consequences of a change in the tax system depend on the relative magnitudes of increased risk sharing forced by the new tax system and the reduced risk sharing in private insurance markets. We quantitatively address this issue by designing an artificial economy calibrated to US income and tax data. We find that for a reasonable selection of the structural parameters of our model, the change to a more redistributive tax system leads to less risk sharing among individuals and, hence, lower ex-ante welfare.

Other authors have studied economies with debt constraints to analyze a variety of issues (See Kocherlakota (1996) and Alvarez and Jermann (1998a,b) among others). These authors consider economies with two (types of) agents in which heterogeneity is somehow limited. The main methodological contribution of this work is the analysis of a debt constrained economy with a *continuum* of agents facing idiosyncratic uncertainty. Therefore a steady state of our economy is characterized by a non-degenerate consumption distribution. We believe that this feature of the model is necessary to analyze issues of income distribution and risk sharing in a quantitatively meaningful way, especially as the quantitative implications for the economy with a continuum of agents turn out to be quite different from the economy with only two agents. But it is also this feature of the model that leads to considerable theoretical and computational complications in solving the model. To this end we draw heavily on the work of Atkeson and Lucas (1992, 1995) who study efficient allocations in an economy with a continuum of agents and private information. We will describe their dual approach to characterize efficient allocations in more detail in the next section.

The paper is organized as follows. In section 2 we lay out the model environment. In section 3 we define an efficient allocation with debt constraints and we show how to make the problem of solving for efficient allocations recursive. We then prove the existence of a stationary solution to the recursive problem. Section 4 contains a discussion of the computational method we employ to solve the recursive problem as well as the parameterization we chose for our quantitative exercises. In section 5 we present our quantitative results and in section 6 we compare our results with those from a model with exogenous incomplete markets. Section 7 concludes and all figures as well as proofs are contained in the appendix.

2. The Environment

A. Consumers

There is a continuum of consumers of measure 1. The consumers have preferences over consumption streams given by

$$(1) \qquad U(\{c_t\}_{t=0}^{\infty}) = (1-\beta)E_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right]$$

The period utility function $u : \Re_+ \to D \subseteq \Re$ is assumed to be strictly increasing, strictly concave, twice differentiable and satisfies the Inada conditions. Its inverse is denoted by $C : D \to \Re_+$. Hence $C(u)$ is the amount of the consumption good necessary to yield period utility u. Let $\bar{D} = \sup(D)$, where it is understood that $\bar{D}$ could be infinity. An individual has a stochastic endowment process $e \in E$, a finite set with cardinality N, that follows a Markov process with transition probabilities $\pi(e'|e)$. For each consumer the transition probabilities are assumed to be the same. We assume a law of large numbers, so that the fraction of agents facing shock e' tomorrow with shock e today in the population is equal to $\pi(e'|e)$. We assume that $\pi(e'|e)$ has a unique invariant measure $\Pi(.)$. We denote by e_t the current period endowment and by $e^t = (e_0, .., e_t)$ the history of realizations of endowment shocks; also $\pi(e^t|e_0) = \pi(e_t|e_{t-1}) \cdots \pi(e_1|e_0)$. We use the notation $e^s|e^t$ to mean that e^s is a possible continuation of endowment shock realization e^t. We also assume that at date 0 (and hence at every date), the measure over current endowment is given by $\Pi(.)$, so aggregate endowment is constant over time. At date 0 agents are distinguished by their initial asset holdings, a_0, and by the their initial shock e_0. Let Φ_0 be the joint measure of initial assets and shocks. Most of the theoretical results presented here depend on the assumption that endowment shocks are uncorrelated across time for each agent. In our numerical exercises we will relax this assumption, although some theoretical results cannot be proved for the more general case in which individual endowment processes are Markov over time.

B. Government

The government uses taxes to finance a constant amount of public spending g in every period that yields no utility to consumers. The government specifies a tax policy $\tau(e_t)$ that is constant over time. We take the government policies $g, \tau(.)$ as exogenously given. For an individual we let $y_t = e_t(1 - \tau(e_t))$ be the after-tax income. Since the function $\tau(.)$ does not depend on time, for a given $\tau(.)$ there is a one to one mapping between pre-tax and after-tax endowments. So from now on we let $y \in Y$, a finite set with cardinality N, denote an individual's generic after-tax endowment, following the Markov process π with invariant distribution Π and denote by $y^t = (y_0, \ldots y_t)$ a history of after-tax endowment shocks. We restrict the government policies $g, \tau(.)$ to satisfy period-by-period budget balance

$$(2) \qquad g = \int e_t \tau(e_t) d\Pi.$$

With this assumption resource feasibility (as defined below) for this economy simply states that the sum of all agents' consumption has to be less or equal than the sum over all individuals' after-tax endowment. Therefore, once $g, \tau(.)$ are fixed and hence the after-tax endowment process is specified, we can carry out the subsequent analysis without explicit consideration of the government.

C. Continuing Participation Constraints

Consumers can trade a full set of state-contingent commodities.[4] A consumption allocation $\{c(a_0, y^t)\}$ specifies how much an agent of type (a_0, y_0) consumes who experienced a history of endowment shocks (net of taxes) y^t. Individuals, at any point in time, have the option to renege on existing contracts. The only punishment for doing so, and hence the only enforcement mechanism for contracts, is that agents that choose to default on their contracts are banned from future intertemporal trade. Defaulters have to live in financial autarky and consume their *after-tax* endowment. This feature of the model requires the existence of a central authority that keeps track of who has defaulted in the past. Individuals have no incentive to revert to the no-risk-sharing, autarkic allocation, at any point in time and any contingency, if and only if an allocation satisfies following continuing participation or debt constraints

$$u\left(c(a_0, y^t)\right) + \sum_{s>t} \sum_{y^s|y^t} \beta^{s-t} \pi(y^s|y^t) u\left(c(a_0, y^s)\right) \geq u(y_t) + \sum_{s>t} \sum_{y^s|y^t} \beta^{s-t} \pi(y^s|y^t) u(y_s) \qquad \forall y^t$$

Since there is no private information, default will not happen in equilibrium as nobody (no financial intermediary) would offer a contract with an individual for a contingency at which this individual would later default with certainty. We first discuss how to compute efficient allocations, then how to decentralize them.

3. Characterization of Efficient Allocations

In order to characterize efficient allocations (for given policies $g, \tau(.)$) we proceed in four steps: we first define what we mean by an efficient allocation. We go on to show that one

[4] Trade occurs with financial intermediaries. See the decentralization section for details.

can find efficient allocations by solving a number of appropriate social planner problems. We then show that solving the social planner problems is equivalent to solving a specific functional equation and finally we discuss how to solve the functional equation. This discussion uses ideas and results developed by Atkeson and Lucas (1992, 1995) extensively. Proofs, however, might differ substantially since these authors analyze a private information economy, and are therefore included in the appendix even when the proof strategy resembles theirs closely.

A. Definition of Efficient Allocations

The key insight of Atkeson and Lucas is to analyze the problem of finding efficient allocation in terms of state contingent *utility* promises rather than state contingent consumption. Instead of being indexed by initial assets and endowment shock, now individuals are indexed by initial entitlements to expected discounted utility at period $0, w_0$ and initial endowment shocks y_0. In the decentralization section we will provide an explicit link between initial wealth a_0 and initial promised utility w_0. Let Φ_0 be the period 0 joint measure over (w_0, y_0). An allocation is then a sequence $\{h_t(w_0, y^t)\}_{t=0}^{\infty}$ that maps initial entitlements w_0 and sequences of shocks y^t into levels of current utility[5] in period t. Here $h_t(w_0, y^t)$ is the current period utility that an agent of type (w_0, y_0) receives if she experienced a history of endowment shocks y^t. Note that $c(a_0, y^t) = C(h_t(w_0, y^t))$ where C is the inverse of the period utility function as defined in Section 2. We will now define the concepts of constrained feasibility and efficiency in this environment.

DEFINITION 1. *An allocation* $\{h_t(w_0, y^t)\}_{t=0}^{\infty}$ *is constrained feasible with respect to a joint distribution over utility entitlements and initial endowments,* Φ_0*, if for each* $(w_0, y_0) \in supp(\Phi_0)$

$$w_0 = (1-\beta)\sum_{t=0}^{\infty}\sum_{y^t|y_0}\beta^t\pi(y^t|y_0)h_t(w_0, y^t) \tag{3}$$

$$\begin{aligned} & h_t(w_0, y^t) + \sum_{s>t}\sum_{y^s|y^t}\beta^{s-t}\pi(y^s|y^t)h_s(w_0, y^s) \\ \geq \; & u(y_t) + \sum_{s>t}\sum_{y^s|y^t}\beta^{s-t}\pi(y^s|y^t)u(y_s) \qquad \forall y^t \end{aligned} \tag{4}$$

[5]The relation between h_t and the period utility derived from consuming c_t is given by $h_t(w_0, y^t) = u(c(a_0, y^t))$.

$$(5) \qquad \lim_{t\to\infty} \beta^t \sup_{y^t} \left(h_t(w_0, y^t) + \sum_{s>t} \sum_{y^s|y^t} \pi(y^s|y^t) h_s(w_0, y^s) \right) = 0$$

$$(6) \qquad \sum_{y^t} \int \left(C(h_t(w_0, y^t)) - y_t \right) \pi(y^t|y_0) d\Phi_0 \leq 0. \quad \forall t$$

We call equation (3) the promise keeping constraint: the allocation delivers utility w_0 to agents entitled to w_0. Equations (4) are the continuing participation constraints, just posed in utility space.[6] Equation (5) is a boundedness condition that assures that continuation utility goes to zero in the time limit. Finally equation (6) is the resource feasibility condition, requiring aggregate consumption in every period to be less or equal than aggregate endowment in that period. Now we can define the concept of efficiency in this environment, due to Atkeson and Lucas (1995).

Definition 2. *An allocation* $\{h_t(w_0, y^t)\}_{t=0}^{\infty}$ *is efficient with respect to* Φ_0 *if*

- It is constrained feasible with respect to Φ_0.
- There does not exist another allocation $\{\hat{h}_t(w_0, y^t)\}_{t=0}^{\infty}$ that is constrained feasible with respect to Φ_0 and such that

$$(7) \qquad \sum_{y^t} \int \left(C(\hat{h}_t(w_0, y^t)) - y_t \right) \pi(y^t|y_0) d\Phi_0 < 0 \text{ for some } t$$

The definition basically says that a utility allocation is efficient if it attains the utility promises made by Φ_0 in an individually rational and resource feasible way and there is no other allocation that does so with less resources.

[6] Note that a Φ_0 that puts positive mass on (w_0, y_0) that satisfies

$$w_0 < u(y_0) + \sum_{s>0} \sum_{y^s|y_0} \beta^s \pi(y^s|y_0) u(y_s)$$

does not permit a constraint feasible allocation as promise keeping and continuing participation constraints are mutually exclusive in this case. We restrict attention to Φ_0 with the property that only initial utility entitlements at least as big as the utilty from autarky have positive mass.

B. A Component Social Planner's Problem (CPP)

Is there an operational way to solve for efficient allocations? Consider now the problem of a social planner faced with a sequence of intertemporal shadow prices $\{R_t\}_{t=0}^{\infty}$, with $R_t > 1$ for every t, that minimizes the value of resources needed to deliver expected discounted utility of w_0 to an individual with initial endowment given by y_0. The planner chooses $\{h_t(w_0, y^t)\}_{t=0}^{\infty}$ to minimize

$$(8) \qquad \left(1 - \frac{1}{R_0}\right) C(h_0(w_0, y_0)) + \sum_{t=1}^{\infty} \left(1 - \frac{1}{R_t}\right) \prod_{s=0}^{t-1} \left(\frac{1}{R_s}\right) \sum_{y^t|y_0} C(h_t(w_0, y^t))\pi(y^t|y_0)$$

subject to (3), (4), and (5). One obtains the following

Theorem 1. *(Atkeson and Lucas (1995)) If there exist allocations $\{h_t(w_0, y^t)\}$, shadow prices $\{R_t\}$ and measure Φ_0 such that:*

1. Given $\{R_t\}_{t=0}^{\infty}$, for each $(w_0, y_0) \in$supp(Φ_0), $\{h_t(w_0, y^t)\}$ solves CPP
2. Feasibility (Equation 4) holds with equality for every t
3. $$1 - \frac{1}{R_0} + \sum_{t=1}^{\infty} \left(1 - \frac{1}{R_t}\right) \prod_{s=0}^{t-1} \left(\frac{1}{R_s}\right) < \infty$$

   then the allocation is efficient with respect to Φ_0.

Proof. See Appendix ■

This theorem gives an operational method for solving for efficient allocations. Given a set of shadow prices one has to solve a minimization problem for allocations $\{h_t(w_0, y^t)\}$. Then one has to check whether the resource constraints are satisfied, and if not, pick different shadow prices. In order to make CPP recursive, however, we have to restrict ourselves to stationary allocations. Define

$$U_t(w_0, y^t) = (1 - \beta) \left(h_t(w_0, y^t) + \sum_{s>t}^{\infty} \sum_{y^s} \beta^{s-t} \pi(y^s) h_t(w_0, y^s) \right)$$

to be the continuation expected discounted utility from history y^t onwards. An allocation is stationary if $\Phi_t = \Phi_0 = \Phi$, where Φ_t is the joint measure over endowment shocks at period t and $U_t(w_0, y^t)$. Stationarity requires the sequence of shadow prices $\{R_t\}$ to be constant at some R. We want to find such R, a corresponding allocation $\{h_t^R(w_0, y^t)\}$ and an initial distribution over utility entitlements and endowment shocks , Φ_R that satisfies the hypothesis of Theorem 1., i.e. $\{h_t^R(w_0, y^t)\}$ is stationary and efficient. In order to do so we now reformulate the problem recursively.

C. Recursive Formulation of the Problem

For constant $R \in (1, \frac{1}{\beta})$, consider the following functional equation (FE) problem. We will show in Theorem 3. that the optimal policies of the functional equation induce an allocation that solves CPP. Individual state variables are the expected discounted utility promises an agent enters the period with, w, and the current income shock. The planner chooses how much current period utility to give to the individual, h, and how much to promise her for the future, $g_{y'}$, conditional on her next periods endowment realization y'. The functional equation the planner solves is:

$$(9) \qquad V(w,y) = \inf_{h, g_{y'}} \left\{ (1-\frac{1}{R})C(h(w,y)) + \frac{1}{R} \sum_{y' \in Y} \pi(y'|y) V(g_{y'}(w,y), y') \right\}$$

$$s.t$$

$$(10) \qquad w = (1-\beta)h(w,y) + \beta \sum_{y' \in Y} \pi(y'|y) g_{y'}(w,y)$$

$$(11) \qquad g_{y'}(w,y) \geq U^{Aut}(y') \qquad \forall y' \in Y$$

where $V(w,y)$ is the per period resource cost for the planner to provide an individual with expected utility w when the individual's endowment is y and the intertemporal shadow price of resources[7] for the planner is $\frac{1}{R}$. Equation (10) is the promise-keeping constraint: an individual that is entitled to w in fact receives utility w through the allocation rules

[7] Let W be the total (not per-period) resource cost of delivering utility w. If the planner faces intertemporal price R then

$$W = C + \frac{1}{R} W$$

Since V (the per-period cost) is equal to $(1 - \frac{1}{R})W$, then

$$V = (1-\frac{1}{R})C + \frac{1}{R} V$$

A similar argument applies for utilities in (10).

$h(.,.), \{g_{y'}(.,.)\}_{y' \in Y}$. The continuing participation constraints in equation (11) state that the social planner for each state tomorrow has to guarantee individuals an expected utility (per period) level at least as high as obtained with the autarkic allocation. The per period value of autarky is given as the solution to the following functional equation:

$$U^{Aut}(y) = (1-\beta)u(y) + \beta \sum_{y' \in Y} \pi(y'|y) U^{Aut}(y'). \tag{12}$$

We now make the following assumptions on the individual endowment process for the rest of this section:

1. $\pi(y'|y) = \pi(y')$ for every $y', y \in Y$, *i.e.* endowment shocks are independently distributed across time.

2. $\pi(y) > 0$, for all $y \in Y$

Remark 1. The fact that π is a stochastic matrix implies that the functional equation (12) has a unique solution $\left\{U^{Aut}(y)\right\}_{y \in Y}$. Assumption 1 immediately implies that if $\bar{y} > y$, then $U^{Aut}(\bar{y}) > U^{Aut}(y)$.

Remark 2. Assumption 1 implies that one can write the functional equation as

$$V(w) = \inf_{h, g_{y'}} \left\{ (1 - \frac{1}{R}) C(h(w)) + \frac{1}{R} \sum_{y' \in Y} \pi(y') V(g_{y'}(w)) \right\} \tag{13}$$

$$s.t$$

$$w = (1-\beta)h(w) + \beta \sum_{y' \in Y} \pi(y') g_{y'}(w) \tag{14}$$

$$g_{y'}(w) \geq U^{Aut}(y') \qquad \forall y' \in Y \tag{15}$$

eliminating the dependence of V on y.

Existence of optimal allocation rules for a given intertemporal price R.

We will first prove existence of optimal allocation rules in a problem with additional constraints. We will then characterize the solution of this problem and we will show that the additional constraints will not be binding so that the solution to the problem with additional constraints is also solution to the original problem.

The modified Bellman equation will be defined on $C(A)$, that is the space of continuous and bounded functions on A where $A = \{w \in \Re | \underline{w} \leq w \leq \bar{w}\} \subseteq D$ is a compact subset of $\Re$ and $\underline{w} := \min_y U^{Aut}(y)$. Consider the operator T_R defined as:

$$
\begin{aligned}
(16) \quad T_R V(w) &= \min_{h, g_{y'}} \left\{ \left(1 - \frac{1}{R}\right) C(h) + \frac{1}{R} \sum_{y' \in Y} \pi(y') V(g_{y'}) \right\} \\
& \quad s.t \\
(17) \quad w &= (1-\beta) h + \beta \sum_{y' \in Y} \pi(y') g_{y'} \\
(18) \quad U^{Aut}(y') &\leq g_{y'} \leq \bar{w} \qquad \forall y' \in Y
\end{aligned}
$$

Notice that the inequalities $g_{y'} \leq \bar{w}$ in (18) are what distinguish this modified problem from the original problem. Also note that (18) and (17) imply that for every w in A

$$
(19) \quad \underline{h}(w) \equiv \frac{w - \beta \bar{w}}{(1-\beta)} \leq h \leq \frac{w - \beta \sum \pi(y') U^{Aut}(y')}{(1-\beta)} \equiv \bar{h}(w)
$$

This in turn implies that $\bar{w}$ must be chosen such that $\underline{h}(w)$ and $\bar{h}(w)$ are in D for every w. We will show below that we can choose $\bar{w} = \max_y U^{Aut}(y) + \varepsilon$, for $\varepsilon > 0$ arbitrarily small. Then under the following assumption 3, (19) is always satisfied.

Assumption 3: Define $\underline{h} = \frac{\underline{w} - \beta \max_y U^{Aut}(y)}{1-\beta} = u(y_{\min}) - \beta u(y_{\max}) + \beta \sum \pi(y) u(y)$ and $\bar{h} = \frac{\max_y U^{Aut}(y) - \beta \sum \pi(y) U^{Aut}(y)}{1-\beta} = u(y_{\max})$. We assume that $(\underline{h}, \bar{h}) \subseteq D$.

Note that this is an assumption purely on the fundamentals of the economy. With the additional constraint we obtain a bounded dynamic programming problem. We can then go on to characterize the operator T_R as a contraction mapping, using Blackwell's sufficient conditions.

LEMMA 1. T_R maps $C(A)$ into itself and is a contraction.

Proof. See Appendix ■

COROLLARY 1. For $R > 1$, the operator T_R has a unique fixed point $V_R \in C(A)$ (i.e. V_R is continuous and bounded) and for all $v_0 \in C(A)$, $||T_R^n v_0 - V_R|| \leq \frac{1}{R^n} ||v_0 - V_R||$, with the norm being the sup-norm.

Characterization of Value and Policy Functions We now proceed to characterize the unique fixed point of T_R, V_R. The fixed point to the functional equation basically inherits all properties of the period cost function C in that it is strictly increasing and has increasing marginal costs.

LEMMA 2. V_R is strictly increasing and strictly convex.

Proof. See Appendix ■

In the next lemma we characterize the properties of the operator T_R further. This will be useful in deducing further properties of the fixed point V_R.

LEMMA 3. For any strictly increasing and strictly convex function $V \in C(A)$, $T_R V$ is continuous, strictly increasing and strictly convex. The optimal policies $h(w), g_{y'}(w)$ are continuous, single-valued functions.

Proof. See Appendix ■

We will use the first order conditions heavily to characterize optimal policies. For this we first have to establish that the value function is differentiable. For any convex and differentiable function $V \in C(A)$ and fixed $w \in A$ the first order conditions characterizing the optimal choices of $h = h(w)$ and $g_{y'} = g_{y'}(w)$ for the problem (16) are:

$$\begin{aligned} C'(h) &\leq \frac{1-\beta}{\beta(R-1)} V'(g_{y'}) \\ &= \frac{1-\beta}{\beta(R-1)} V'(g_{y'}) \qquad if \qquad g_{y'} > U^{Aut}(y') \\ w &= (1-\beta)h + \beta \sum_{y' \in Y} \pi(y') g_{y'} \end{aligned} \tag{20}$$

The envelope condition is:

$$\left(T^R V\right)'(w) = \frac{(R-1)}{R(1-\beta)} C'(h) \tag{21}$$

We now prove that we in fact can use these conditions to characterize the optimal policies by showing that the solution to the functional equation is differentiable.

LEMMA 4. The unique fixed point of T_R is continuously differentiable.

Proof. See Appendix ■

We now turn to the characterization of the optimal policies. First we characterize the behavior of h and $g_{y'}$ with respect to w. It turns out that the social planner reacts to a higher utility promise by increasing current and expected future utility, i.e.. by smoothing the cost as much as possible over time and across states. The only constraint that prevents complete cost smoothing over different states is the continuing participation constraint: certain agents have to be promised more than otherwise optimal in certain states to be prevented from reverting to the autarkic allocation in that state. This is exactly the reason why complete risk sharing may not be constrained feasible in this environment.

LEMMA 5. Suppose $V \in C(A)$ is strictly convex and differentiable. Then the optimal policy h, associated with the minimization problem in (16) is strictly increasing in w.

Proof. See Appendix ■

LEMMA 6. Suppose $V \in C(A)$ is strictly convex and differentiable. Then the optimal policies $g_{y'}$, associated with the minimization problem in 11 are constant in w and equal to $U^{Aut}(y')$ or strictly increasing in w, for all $y' \in Y$.

Proof. See Appendix ■

LEMMA 7. Suppose $V \in C(A)$ is strictly convex and differentiable.

If $g_{y'}(w) > U^{Aut}(y')$ and $g_{\bar{y}'}(w) > U^{Aut}(\bar{y}')$, then $g_{y'}(w) = g_{\bar{y}'}(w)$.

If $g_{y'}(w) > U^{Aut}(y')$ and $g_{\bar{y}'}(w) = U^{Aut}(\bar{y}')$, then $g_{y'}(w) \leq g_{\bar{y}'}(w)$ and $y' < \bar{y}'$

Proof. Follows directly from the first order condition ■

The last lemma states that future promises are equalized across states whenever the continuing participation constraints permit it. Promises are increased in those states in which the constraints bind.

Now we state a result that is central for the existence of an upper bound $\bar{w}$ of utility promises. For promises that are sufficiently high it is optimal to deliver most of it in terms of current period utility, and promise less for the future than the current promises. This, in effect, puts an upper bound on optimal promises in the long run. This is the content of the main result in this section, stated in Theorem 2.

LEMMA 8. Let $\{g_{y'}\}_{y' \in Y}$ be the optimal policies associated with the unique fixed point of T_R, V_R. For every $w \in A$ and every $y' \in Y$, if $g_{y'}(w) > U^{Aut}(y')$, then $g_{y'}(w) < w$. Furthermore, for each y', there exists a unique $w_{y'}$ such that $g_{y'}(w_{y'}) = w_{y'}$ and $w_{y'} = U^{Aut}(y')$.

Proof. See Appendix ■

THEOREM 2. *There exists a $\bar{w}$ such that $g_{y'}(w) < w$ for every $w \geq \bar{w}$ and every $y' \in Y$.*

Proof. See Appendix ■

Note that the preceding theorem implies that whenever $w \in [\underline{w}, \bar{w}]$, then for all $y' \in Y$, the constraint $g_{y'}(w) \leq \bar{w}$ is never binding and assumption 3 guarantees that $h(w) \in D$. We will analyze the Markov process on $A \times Y$ induced by the exogenous π and the endogenous $g_{y'}$. Before this, however, we discuss the relationship between the component planning problem and the solution to the functional equation.

Equivalence between the CPP in sequential and recursive formulation

In this section we show that the allocation $\hat{\sigma} = \{\hat{h}_t(w_0, y^t)\}_{t=0}^{\infty}$ induced by the optimal policies from the recursive formulation, $h(w), g_{y'}(w)$ solve the CPP in sequential formulation, i.e. solve the program CPP for interest rates $\{R_t\}_{t=0}^{\infty}$ constant at R. So for given (w_0, y_0) let $\hat{h}_0(w_0, y_0) = h(w_0)$, $\hat{w}_1(w_0, y_0) = g_{y_1}(w_0)$ and in general $\hat{w}_t(w_0, y^t) = g_{y_t}(\hat{w}_{t-1}(w_0, y^{t-1}))$ and $\hat{h}_t(w_0, y^t) = h(\hat{w}_t(w_0, y^t))$ be the allocation induced by the recursive policy rules. In the appendix we invoke the principle of optimality to argue that the allocation so derived solves CPP for constant interest rates.

THEOREM 3. *Suppose that the sequence $\{R_t\}_{t=0}^{\infty}$ is constant at $R \in (1, \frac{1}{\beta})$. Then the allocation $\hat{\sigma}$ constructed from the optimal policies of the functional equation solves the component planning problem, for every $(w_0, y_0) \in W \times Y$ with $w_0 \geq U^{Aut}(y_0)$.*

Proof. See Appendix ■

Existence and uniqueness of an invariant probability measure

Let $W = [\underline{w}, \bar{w}]$ and $\mathcal{B}(W)$, $\mathcal{P}(Y)$ the set of Borel sets of W and the power set of Y. The function $g_{y'}(w)$, together with the transition function π for the endowment process, defines a Markov transition function on income shock realizations and utility promises $Q : (W \times Y) \times (\mathcal{B}(W) \times \mathcal{P}(Y)) \rightarrow [0,1]$ as follows:

$$(22) \quad Q(w, y, \mathcal{W}, \mathcal{Y}) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y') & if \quad g_{y'}(w) \in \mathcal{W} \\ 0 & else \end{cases}$$

Given this transition function, we define the operator T^* on the space of probability measures $\Lambda((W \times Y), (\mathcal{B}(W) \times \mathcal{P}(Y))$ as

$$(23) \quad (T^*\lambda)(\mathcal{W}, \mathcal{Y}) = \int Q(w, y, \mathcal{W}, \mathcal{Y}) d\lambda = \sum_{y' \in \mathcal{Y}} \pi(y') \int_{\left\{w \in W | g_{y'}(w) \in \mathcal{W}\right\}} d\lambda$$

for all $(\mathcal{W}, \mathcal{Y}) \in \mathcal{B}(W) \times \mathcal{P}(Y)$. Note that T^* maps Λ into itself (see Stokey et. al. (1989), Theorem 8.2). An invariant probability measure associated with Q is defined to be a fixed point of T^*. In this section we address the question of whether such a probability measure exists and is unique. Intuitively, this invariant measure describes the long-run implications of the planners cost minimizing policies.

We first add a property of $g_{y'}$ that will be useful in proving the existence of a unique invariant probability measure associated with the transition function Q.

LEMMA 9. There exists $w^* \in A$ such that $w^* > U^{Aut}(y_{\max})$ and $g_{y_{\max}}(w^*) = U^{Aut}(y_{\max})$

Proof. See Appendix ■

COROLLARY 2. $g_{y_{\max}}(w^*) = U^{Aut}(y)$ for all $w \leq w^*$.

With this result and the characterization of the functions $g_{y'}$ from previous sections we can prove the main result of this section, namely that for a given $R \in (1, \frac{1}{\beta})$, there is a unique invariant measure over utility promises and endowment shocks associated with transition function Q.

THEOREM 4. *There exists a unique invariant probability measure* Φ *associated with the transition function* Q *defined above and for all* $\Phi_0 \in \Lambda((W \times Y), (\mathcal{B}(W) \times \mathcal{P}(Y))$, $(T^*\Phi_0)^n$ *converges to* Φ *in total variation norm.*

Proof. See Appendix ■

REMARK 3. The argument above also shows that any ergodic set of the Markov process associated with Q must lie within $[U^{Aut}(y_{\min}), U^{Aut}(y_{\max})] \times Y$ and that the support of the unique invariant probability measure is a subset of this set. For any initial measure over shocks and utility entitlements, in the long run the planners' actions result in the invariant cross-sectional measure.

This concludes the discussion of the Markov process on utility promises for a fixed interest rate $R \in (1, \frac{1}{\beta})$. In our subsequent analysis we will index policy functions $(h, g_{y'})$, cost functions V and unique invariant probability measures Φ by R to make clear that these functions and measures were derived for a fixed R. So far we have not imposed any resource feasibility condition. It may be the case that for a fixed R, in order to deliver a distribution over utility entitlements Φ_R, more resources than available have to be used up. In the next section we will analyze how the resource requirements imposed by Φ_R vary with R.

D. Determination of the "market clearing" shadow price R

In the previous section we showed that for a fixed $R \in (1, \frac{1}{\beta})$ there exists a unique stationary joint distribution over (w, y). Define the "excess demand function" associated with R as

$$(24) \quad d(R) = \int V_R(w) d\Phi_R - \int y d\Phi_R$$

In this section we analyze the qualitative features of the function $d(.)$. Since by assumption $\int y d\Phi_R$ does not vary with R, the behavior of d depends on how V_R and Φ_R vary with R. The behavior of Φ_R with respect to R in turn depends on the behavior of $g_{y'}^R$ with respect to R as $g_{y'}^R$ determines the Markov process to which Φ_R is the invariant probability measure. We start by proving that the function V^R varies continuously with R, then show that the optimal policies $g_{y'}^R$ vary continuously in R. These results, in turn, imply continuity of d, as shown in Theorem 5.

LEMMA 10. (Atkeson and Lucas (1995)) Let $R \in (1, \frac{1}{\beta})$ and $\{R_n\}_{n=0}^{\infty}$ be a sequence satisfying $R_n \in (1, \frac{1}{\beta})$ and $\lim_{n\to\infty} R_n = R$. Then the sequence $\{V_{R_n}\}_{n=0}^{\infty}$ converges uniformly to V_R on $[\underline{w}, \bar{w}]$.

Proof. See Appendix ■

LEMMA 11. (Atkeson and Lucas (1995)) Let a sequence $\{R_n, w_n\}_{n=0}^{\infty}$ with $R_n \in (1, \frac{1}{\beta})$ and $w_n \in [\underline{w}, \bar{w}]$ converge to $(R, w) \in (1, \frac{1}{\beta}) \times [\underline{w}, \bar{w}]$. Then for each $y' \in Y$, the sequence $\left\{g_{y'}^{R_n}(w_n)\right\}_{n=0}^{\infty}$ converges to $g_{y'}^{R}(w)$.

Proof. See Appendix ■

The previous two lemmas can be used to prove our first main result about the excess demand function $d(.)$, namely continuity on $(1, \frac{1}{\beta})$.

THEOREM 5. *(Atkeson and Lucas (1995), Lemma 12): $d(R)$ is continuous on $(1, \frac{1}{\beta})$.*

Proof. See Appendix ■

The previous result establishes that the excess demand function varies continuously with R. Now we want to establish a result about the slope of the excess demand function. Intuitively, a higher interest rate R makes resources tomorrow cheaper compared to resources today. This should lead, for given utility entitlement w, to a decrease in utility received today and an increased promised utility from tomorrow onwards. Since the invariant measure over utility is determined by future utility promises, for higher R one expects higher utility promises on average, and hence higher resource costs. In other words, one expects the excess demand function to be increasing in R.

In order to prove this we first establish that future utility promises are indeed increasing in the interest rate R.

LEMMA 12. The optimal policies $g_{y'}^{R}(w)$ are increasing in R and the optimal policy $h^R(w)$ is decreasing in R.

Proof. See Appendix ■

This result enables us to draw conclusions about how the invariant measure over utilities and endowment shocks, Φ_R varies with R. The next result shows that for larger interest rates the invariant measure puts more mass on higher utility entitlements. For every Φ define the probability measures Φ^y on $(W, \mathcal{B}(W))$ by $\Phi^y(B) = \frac{\Phi(B,\{y\})}{\pi(y)}$, for every $B \in \mathcal{B}(W)$. Note that every such measure is well-defined as $\pi(y) > 0$ by assumption, and that $\Phi^y(W) = 1$.

Lemma 13. (Atkeson and Lucas (1995)) Let $R > \hat{R}$. Then for every $y \in Y$, Φ_R^y stochastically dominates $\Phi_{\hat{R}}^y$, i.e.. for every increasing and continuous function f on W,

$$\int f(w) d\Phi_R^y \geq \int f(w) d\Phi_{\hat{R}}^y \tag{25}$$

Proof. See Appendix ■

The previous results can be combined to show that the excess demand function is increasing in the interest rate.

Theorem 6. *(Atkeson and Lucas (1995), Lemma 14) Let $R > \hat{R}$. Then $d(R) \geq d(\hat{R})$.*

Proof. See Appendix ■

The previous results show that the excess demand function is continuous and increasing on $(1, \frac{1}{\beta})$. The behavior of the excess resource function at $R = \frac{1}{\beta}$ (the complete markets shadow price) is easily determined. For such R, $g_{y'}(w) = w$ or $g_{y'}(w) = U^{Aut}(y')$ from the first order conditions. There is a continuum of invariant measures. No $w < g_{y'}(U^{Aut}(y_{\max}))$ is in the support of any of these measures, though, as the probability of leaving such a w is at least $\pi(y_{\max})$ and the probability of coming back (into a neighborhood) is 0. Therefore $g_{y'}(w) = w$ for all points in the support of the invariant measure, and there is complete risk sharing as from the promise-keeping constraint $h(w) = w$. Each individuals' consumption is constant over time. The invariant measure with lowest cost is the one that puts mass $\pi(y)$ on $(U^{Aut}(y_{\max}), y)$. Hence each individual receives the same current utility, hence the same current consumption, which is equal to the average after-tax endowment $\bar{y}$. For this allocation to be incentive feasible one needs

$$(1-\beta)u(y_{\max}) + \beta \sum_{y' \in Y} \pi(y')u(y') \leq u(\bar{y})$$

Since we am interested in efficient allocations that don't feature perfect risk sharing we assume that the above inequality is violated.[8] Then complete risk sharing is not possible with resources $\bar{y}$. By arguments similar to the proof of continuity one can show that for $R \to \frac{1}{\beta}$, the cost of the associated allocation converges to the cheapest complete risk sharing allocation. Hence under the assumption made $\lim_{R \to \frac{1}{\beta}} d(R) > 0$.

[8] The inequality will be violated for β sufficiently small and/or the income process sufficiently variable.

On the other hand, for R close to 1, there are two possible situations. If

$$\beta \frac{u'(y_{\min})}{u'(y_{\max})} < 1 \tag{26}$$

then the autarkic allocation satisfies the first order conditions for some $R > 1$. Since autarky is constrained feasible, it is efficient. If the inequality in (26) is reversed, we *conjecture* that a) autarky is not efficient and b) $d(R) < 0$ for R sufficiently close to, but bigger than 1.

We assume that 26 does not hold.[9] Then under the last two assumptions, there exists an efficient stationary allocation different from complete risk sharing and different from the autarkic solution.[10] This concludes the discussion on the existence of efficient allocations.

E. Qualitative Features of the Efficient Allocation

In figure 1 (see the appendix) we show the typical shape of the policy functions $g_{y'}(w)$. The policies shown here are computed a specific parameterization of the model, in which the endowment process can take on two values, $y_l < y_h$ and a specific interest rate $R \in (1, \frac{1}{\beta})$. We observe the following properties: as shown above $g_{y'}(w)$ is greater or equal to $U^{Aut}(y')$, constant at $U^{Aut}(y')$ for $w \leq U^{Aut}(y')$ and intersects the 45^0-line at $w = U^{Aut}(y')$. For $w > U^{Aut}(y')$ is either constant at $U^{Aut}(y')$ or strictly increasing in w, and lies below the 45^0-line. Furthermore $g_{y_l}(w) \leq g_{y_h}(w)$.

The support of the invariant measure Φ with respect to w is, as shown above, equal to $[U^{Aut}(y_l), U^{Aut}(y_h)]$. The features of the policy functions just described imply that an agent with $y' = y_h$ has $g_{y'}(w) = U^{Aut}(y_h)$ and hence the highest possible future utility entitlements in the support of Φ, regardless of her w. Hence, when an agent receives a high income shock, history is erased in that present utility entitlements w, which summarize the history of past endowment shocks, do not matter for future utility entitlements. For agents with $y' = y_l$, history matters. An agent with $w = U^{Aut}(y_h)$ that receives $y' = y_l$ drops to $g_{y'}(U^{Aut}(y_h))$ $< U^{Aut}(y_h)$, and, upon further bad shocks, works herself downwards through the entitlement distribution. In finite (if $g_{y_l}(w) = U^{Aut}(y_l)$ for some $w > U^{Aut}(y_l)$) or infinite number of steps an agent with a string of bad shocks arrives at $w = U^{Aut}(y_l)$, with any good shock

[9] As is well known, for sufficiently low β only the autarkic allocation satisfies the continuing participation constraints. See Alvarez and Jermann (1998a) for a discussion of the two-agent case.

[10] No claim of uniqueness can be made at this point. The set of efficient shadow prices R is convex. In our computational exercises, however, uniqueness always arises.

putting her immediately back to $U^{Aut}(y_h)$. Note from the theoretical results that the features of immediate jumping up upon receiving the highest shock as well as gradual working down upon receiving lower shocks hold for an arbitrary (finite) number of states for the endowment process.

The properties of the entitlement distribution immediately translate into properties of the stationary consumption measure as the latter is given as

$$\Psi(A) = \int_{\{w|C(h(w))\in A\}} d\Phi$$

Since $h(.)$ and $C(.)$ are strictly increasing functions, consumption reacts to endowment shocks in exactly the same qualitative fashion as utility entitlements. This can be seen from figure 2. All agents with a high endowment shock consume the same, regardless of their history and consume the maximum amount in the support of the consumption distribution. Hence the consumption distribution has a mass point at this consumption level equal to $\Pi(y_h)$. Another mass point occurs at the minimum of the support of the consumption distribution, provided that $g_{y_l}(w) = U^{Aut}(y_l)$ for some $w > U^{Aut}(y_l)$ (which is true in the present example, but need not be true in general). All agents that have a sufficiently long string of bad income realizations will consume the minimum consumption level in the support. In between these two mass points there is a finite number of other mass points consisting of individuals with currently bad shock who are in the process of either falling to the bottom (with additional bad shocks) or jumping again to the top (with a single good shock), but who haven't hit rock bottom yet.

What is the economic intuition for these results? One observes that those individuals that experience an *increase* in income from today to tomorrow are constrained. This can be interpreted as follows. These agents have borrowed against the chance of being lucky in the next period (in utility terms, as they receive more utility today than in autarky) and if the contingency of being lucky tomorrow materializes they are called upon to repay their debt and transfer resources to other agents. Unconstrained agents tend to be those with a drop in income. These agents are unconstrained tomorrow as they will receive transfers (from agents with favorable income shock tomorrow) at that date, which makes defaulting unreasonable. This feature of the model justifies the term "debt constrained" as agents are constrained to borrow today because of their possible incentive to default if called upon to pay back tomorrow (in the contingency of a high income realization).

These results are only suggestive for an economy with more than 2 shocks. However, the phenomena of jumping to the top in one step upon receiving the high endowment shocks and stepwise working through the distribution with consistently low endowment shocks is an inherent property of the model. For higher numbers of exogenous states, of course, a richer consumption distribution "in the middle" arises.

Finally, in figure 3 we show the excess demand function $d(R)$ for a fixed parameterization of the model. As proved in the theoretical part, this function is continuous and increasing in R on $(1, \frac{1}{\beta})$. In the specific example it has a unique $R^* \in (1, \frac{1}{\beta})$ such that $d(R)$ equals to 0.

F. Decentralization

In this section we discuss how we can interpret the solution to the CPP as an equilibrium in an economy in which planners (or if you like HMO's or financial intermediaries), are competing for allocating consumption to a particular large group. Each financial intermediary signs long-term contracts with his clients of type (a_0, y_0) of the following form: the households deliver the claim to their present and future after-tax endowment and their initial wealth to the intermediary, the intermediary assigns them a level of promised utility w_0 and then allocates consumption according to the rule $c(w_0, y^t) = C(h_t(w_0, y^t))$ to his clients in order to fulfill the utility promise w_0. The link between initial wealth of an individual a_0 and associated promised utility w_0 is provided by the zero profit condition for the financial intermediaries, that is

$$(27) \quad V(w_0) = (1 - \frac{1}{R_0})y_0 + \frac{1}{R_0}E(y) + a_0$$

where the right hand side of equation 27 represents the discounted value of the expenses of the intermediary and the left hand side represents the discounted value of the intermediary's receipts. Individual households have the option to renege on the long-term contract with the financial intermediary and revert to autarky, but they can't join another financial intermediary after having defaulted on another.[11] Each financial intermediary attempts to minimize

[11] This assumption of "exclusive contracts" cannot be dispensed with in the current environment.

the cost of delivering the promised utility to his group of individuals by trading the single consumption good with other intermediaries at gross interest rate R. We argue that the gross interest rate among financial intermediaries in this economy, unless all agents are constrained (autarky is the efficient allocation), is given by

$$\begin{aligned} \frac{1}{R} &= \beta \max_{w,y,y'} \frac{C'(h(w,y))}{C'(h(g_{y'}(w,y),y')} \\ (w,y) &\in Supp(\Phi_R^*) \end{aligned}$$

The above condition can also be rewritten in a more familiar way as

$$\begin{aligned} \frac{1}{R} &= \beta \max_{w,y,y'} \frac{u'(c(g_{y'}(w,y),y')}{u'(c(w,y))} \\ (w,y) &\in Supp(\Phi_R^*) \end{aligned}$$

where $c(w,y) = u^{-1}(h(w,y))$.

A simple arbitrage argument is required to show that this conjecture is true. Consider an arbitrarily small $\varepsilon > 0$ and suppose

$$\begin{aligned} \frac{1}{R} &> \beta \max_{w,y,y'} \frac{u'(c(g_{y'}(w,y),y')}{u'(c(w,y))} = \frac{1}{R+\varepsilon} \\ (w,y) &\in Supp(\Phi_R^*) \end{aligned}$$

then an intermediary could make profits (reduce costs) by borrowing one unit at rate R, using the unit to increase current consumption of consumers, reducing future consumption by $R+\varepsilon$ units. This operation would leave uncontrained agents indifferent but would guarantee a profit of ε per unit to the intermediary. Suppose now

$$\begin{aligned} \frac{1}{R} &< \beta \max_{w,y,y'} \frac{u'(c(g_{y'}(w,y),y'))}{u'(c(w,y))} = \frac{1}{R-\varepsilon} \\ (w,y) &\in Supp(\Phi_R^*) \end{aligned}$$

Then an intermediary could make profit (reduce cost) without affecting utility of consumers by taking out one unit of current consumption from consumers with maximal rate of substitution, lending it to other intermediaries at rate R and giving back to consumers $R-\varepsilon$ units of future consumption.

Notice finally that if autarky is the only constrained feasible allocation then

$$\frac{1}{R} = \beta \frac{u'(y_{\min})}{u'(y_{\max})}$$

is an equilibrium interest rate , but so are all $\hat{R} > R$. In this case we will take the equilibrium rate to be R. In the following section we will discuss in more detail the relations between the parameters of the model and the intertemporal price of resources.[12]

4. Computation and Calibration

A. The Computational Procedure

In this subsection we describe how, for a parametric class of our economy, we compute a constant R, policy rules $h^R(w,y), g^R_{y'}(w,y)$ and a distribution over utility entitlements and endowment shocks ,Φ_R as described in the last section.

Our computational method is an implementation of the policy function iteration algorithm proposed by Coleman (1990). For a fixed R we search for the optimal policies $g_{y'}(w,y)$ and $h(w,y)$ within the class of piecewise-linear functions in w . We start by specifying a k point grid $G = \{w_0, ..., w_k\} \subseteq D$ and by guessing the values of a function $V_0'(.,.)$ on $G \times Y$. Notice that this defines a function piecewise linear in w for a fixed y. For a given $w, y \in G \times Y$ we then use the first order condition

$$(28) \quad \begin{aligned} C'(h(w,y)) &\leq \frac{1-\beta}{\beta(R-1)} V'(g_{y'}(w,y), y') \\ &= \quad \text{if } g_{y'}(w,y) > U^{Aut}(y') \end{aligned}$$

together with the constraint

$$(1-\beta)h(w,y) + \beta \sum_{y' \in Y} \pi(y'|y) g_{y'}(w,y) = w$$

to solve for solve $N+1$ equations[13] for the $N+1$ optimal policies $g^0_{y'}(w,y)$ and $h^0(w,y)$. Notice that $g^0_{y'}(w,y)$ and $h^0(w,y)$ are not constrained to lie in G. Carrying out this procedure for all $w, y \in G \times Y$ defines $g^0_{y'}(.,.)$ and $h^0(.,.)$ that are piecewise linear functions in w.

[12] See Krueger (1998) for a further discussion of potential equilibrium concepts with respect to which the efficient allocation can be decentralized.

[13] Note that whenever the first order condition does not hold with equality we know that $g_{y'}(w,y) = U^{Aut}(y')$ and we can drop the first order condition for the specific y' as the number of unknowns is reduced by 1.

We then use envelope condition

$$V_1'(w,y) = \frac{(R-1)}{R(1-\beta)}C'(h^0(w,y))$$

to update our guess of V' and repeat the procedure until convergence of $g_{y'}^n(.,.)$, $h^n(.,.)$ and $V_n'(w,y)$ is achieved. This yields policy functions that are piecewise linear in w.

To compute the stationary joint measure over (w,y) we proceed as follows: for a given (w,y) we find $w_{y'}^l(w,y), w_{y'}^h(w,y)$ and $\alpha_{y'}(w,y)$ such that

- $w_{y'}^l(w,y) = \max\{w \in G | w \leq g_{y'}(w,y)\}$
- $w_{y'}^h(w,y) = \min\{w \in G | w > g_{y'}(w,y)\}$
- $\alpha_{y'}(w,y)$ solves $\alpha_{y'}(w,y)w_{y'}^l(w,y) + (1-\alpha_{y'}(w,y))w_{y'}^h(w,y) = g_{y'}(w,y)$.

We then define the Markov transition matrix $Q : (G \times Y) \times (G \times Y) \to [0,1]$ as

$$Q((w,y),(w',y')) = \begin{cases} \pi(y'|y)\alpha_{y'}(w,y) & \text{if} \quad w' = w_{y'}^l(w,y) \\ \pi(y'|y)(1-\alpha_{y'}(w,y)) & \text{if} \quad w' = w_{y'}^h(w,y) \\ 0 & \text{else} \end{cases}$$

Note that the matrix Q has dimension $(K \cdot N) \times (K \cdot N)$. We then solve the matrix equation

$$\Phi = Q^T \Phi$$

for Φ, where Φ has dimension $K \cdot N$ and $\Phi(w,y)$ gives the steady state probability of being in state (w,y). In this way we can find, for a given $R \in (1, \frac{1}{\beta})$, Φ_R, $h^R(w,y)$ and $g_{y'}^R(w,y)$. We then compute the excess demand function

$$d(R) = \sum_{(w,y) \in G \times Y} (C(h^R(w,y)) - y)\Phi_R(w,y)$$

and use a Newton procedure to find R such that $d(R) = 0$. This completes the computation of R, Φ_R and an allocation induced by the policies $h^R(w,y)$ and $g_{y'}^R(w,y)$ that satisfy the hypotheses of theorem 1.

B. Calibration

First we will describe how we set the parameters governing the individual endowment process, together with the government policies (spending and taxes). We will then discuss preference parameters β, the subjective time discount factor, and σ, the coefficient of relative risk aversion, as we will assume that the period utility function is of CRRA-form.

Endowment process

To characterize the Markov chain governing the individual endowment process in the model we need to set the N possible values the endowment e_t can take and estimate the transition matrix $\pi(e_{t+1}|e_t)$. In order to do so we use household level data from the Consumer Expenditure Survey (CEX) for the years 1986-1994. The main reason why we use CEX income data, whose quality is supposedly inferior to PSID data, is because CEX reports also taxes paid by the household members. We try to reduce measurement error by excluding from our sample households classified as incomplete income respondents, as suggested by Nelson (1994) and Lusardi (1996). The CEX quantity we will interpret as e_t, household endowment before taxes, is labor earnings plus net social security receipts. In the data we measure this entity by the sum of labor earnings, plus a fraction[14] of business and farm income plus social security payments net of contributions received by all the members of the household, all divided by the number of adult equivalents[15] in the household.

We first pick N to be equal to 5 . The transition matrix $\pi(e_{t+1}|e_t)$ is computed as follows. For any quarter t in the CEX sample we group households into 5 relative endowment classes delimited by 4 equally spaced quintiles $q_{.2,t}, q_{.4,t}, q_{.6,t}, q_{.8,t}$. We then search for all households for which we have endowment observations at t and $t+4$ (e_t and e_{t+4}) and compute in which relative class they belong in period $t+4$. Notice that here the class delimiters depend upon time to take account of aggregate growth. We repeat this for every quarter in the sample. Then the probability of transiting from class i to class j is given by the number of households transiting from i to j, divided by the total number of households starting in

[14] The fraction of business and farm income we impute to labor income is .864 as reported in Diaz Jimenez, Quadrini, Rios Rull (1997).

[15] The number of adult equivalent is defined as in Deaton and Paxon (1994) as the number of households mebres over age 16 plus .5 times the number of members below age 16.

class i. Once $\pi(e_{t+1}|e_t)$ is computed we set $e_{1,..}, e_N$ to be equal to the median income for each endowment class in the first quarter of 1994.

Table 1. Endowment values (Ratio to 1994.1 median)

e_1	e_2	e_3	e_4	e_5
.21	.63	1	1.57	2.84

Table 2. Transition Matrix (1986.1-1994.4), (26747 obs.)

	Quintile at $t+4$				
Quintile at t	1	2	3	4	5
1	.70	.17	.07	.03	.02
2	.14	.65	.15	.04	.01
3	.05	.16	.59	.17	.03
4	.03	.04	.16	.63	.13
5	.02	.01	.03	.14	.79

Fiscal policy

To characterize the fiscal policy we need to measure the values of $\tau(e_i), i = 1, \cdots N$. In CEX households are asked to report federal state and local taxes deducted from their last paycheck separately from any additional (not deducted form paycheck) federal state and local taxes paid. Since we want a measure of taxes on labor earnings the first measure seems more appropriate for our purposes. From taxes we substract transfers (welfare, unemployment compensation, food stamps). We then set $\tau(e_i)$ equal to the ratio between the total amount of federal state and local taxes deducted from paycheck, net of transfers, in the i-th class and the total endowment in the same class. Once the tax policy is set we can compute the implied level of government spending (net of transfers) such that the budget is balanced in every period. The tax policy we will use in our experiments is the average of the tax policies measured in the four quarters of 1994 and is reported below.

Table 3. Tax Rates, 1994, (8679 obs.)

	Quintiles				
	1	2	3	4	5
Average Tax Rates (%)	-35.8	1.0	5.4	9.2	14.1

The government spending (net of transfers) implied by these tax rates is equal to 8.5% of total output.

Preference parameters

We calibrate the preference parameters (σ, β) so that the solution for the benchmark model delivers an interest rate of 5% per year. In particular we set σ equal to 1 (logarithmic utility) and then choose β so to match the interest rate. The non-standard part of this exercise is that in a debt constrained economy for a given σ there might be multiple β that deliver the same interest rate. In figures 4 and 5 we show the relation between the time discount factor and the interest rate in our economy and in two other economies. To understand the nonmonotonic behavior of the relation in the economy with debt constraints remember that the interest rate is given by

$$\begin{aligned} \frac{1}{R} &= \beta \max_{w,y,y'} \frac{u'(c(g_{y'}(w,y),y')}{u'(c(w,y))} \\ (w,y) &\in Supp(\Phi_R^*) \end{aligned}$$

There are two critical values of the time discount factor β. If $\beta > \beta^{CM}$ then the efficient allocation involves complete risk sharing of idiosyncratic risk, individual consumption is constant, $\frac{u'(c(g_{y'}(w,y),y')}{u'(c(w,y))}$ is always equal to 1 and the interest rate is decreasing in β. If $\beta < \beta^{Aut}$ then autarky is an efficient allocation, $\frac{1}{R} = \beta \frac{u'(y_{\min})}{u'(y_{\max})}$ (as argued is section F) and again the interest rate is decreasing in β. For $\beta \in (\beta^{Aut}, \beta^{CM})$ there is some, but not complete risk sharing. As β moves from β^{CM} to β^{Aut} there are two effects on the interest rate: there is a direct effect due to the decrease of β that raises the interest rate and there is an indirect effect: lower β reduces risk sharing and therefore $\max_{w,y,y'} \frac{u'(c(g_{y'}(w,y),y')}{u'(c(w,y))}$ tends to increase and the interest rate is reduced. As we can see from figures 4-5, when β is close to β^{CM} the first effect dominates and when β is close to β^{Aut} the second effect dominates. From figure 4 we see that for a fixed value of $\sigma = 1$ there are three possible values of β $(\beta = .15, \beta = .17, \beta = .945)$

consistent with an interest rate of 5% . We will discuss the arising allocations for all three cases in the next section.[16]

We summarize the parameter values for our benchmark economy in the table 4

Table 4. Preference Parameters.

Parameter		Value
β	Time Discount Factors	$\{ .945, .17, .15\}$
σ	Risk Aversion	1

5. Results

In tables 5-7 we summarize the results from our numerical experiments with the baseline parameterization. The concept of the interest rate for the economy has been discussed in section F.We define as a measure of private intermediation of idiosyncratic risk

$$PI = \int |y - C(h(w,y)|d\Phi$$

and as a measure of government intermediation

$$GI = \int |\tau(e)e - ge|d\Pi$$

Note that these measures are percentages of aggregate endowment (which has been normalized to 1). The measure for government intermediation is 0 for proportional taxation (as $\tau(e) = g$ for all levels of endowment and for progressive taxation measures the amount of resources that are redistributed by the government as a consequence of a tax regime switch. PI involves only after-tax endowments and measures the extent to which private insurance contracts are present in the economy. We measure welfare as follows: we first determine which of the tax regimes yields higher ex-ante welfare (where ex-ante welfare is measured as $\int h(w,y)d\Phi$). We then reduce the after-tax endowment of every agent by $x\%$ in the tax regime with higher welfare and report that x (in the column of the preferred tax regime) for which ex-ante welfare in the dominated tax regime coincides with that in the preferred tax regime in which part of the resources have been taken away.

[16] In future versions of this paper we will conduct sensitivity analysis with respect to the parameter σ since in many studies values higher than 1 are used.

We see from the tables that the low risk sharing allocation is very similar (in terms of private risk sharing) to the autarkic allocation and it is significantly different from the high risk sharing allocation. Also observe that in the high risk sharing allocation a switch from progressive to proportional tax results in a (small) welfare gain since the decrease in government intermediation (-4.6%) is more than compensated by the increase in private intermediation ($+5.2\%$), while in the low risk sharing and autarchic allocation the same switch results in a (big) welfare loss because now the (absolute) increase in private intermediation is negligible. This suggests that evaluating the level of risk sharing in an economy can be relevant for policy purposes.[17]

Table 5. $\beta = .945$ **(High Risk Sharing)**

	Tax System	
Variable	Progressive	Proportional
Interest rate	5%	5.1%
Govt. Intermediation	4.6%	0%
Private Intermediation	45.9%	51.1%
Ex Ante Welfare	–	+0.1%

Table 6. $\beta = .17$ **(Low Risk Sharing)**

	Tax System	
Variable	Progressive	Proportional
Interest rate	5%	22%
Govt. Intermediation	4.6%	0%
Private Intermediation	.01%	.2%
Ex Ante Welfare	+5.8%	–

[17] This, in fact, motivates the analysis in Krueger (1998).

Table 7. $\beta = .15$ **(Autarky)**

Variable	Tax System: Progressive	Proportional
Interest rate	5%	14.5%
Govt. Intermediation	4.6%	0%
Private Intermediation	0%	.01%
Ex Ante Welfare	+5.8%	–

A. How high is Risk Sharing?

The results just presented naturally lead to question of how high is the level of risk sharing in the US economy. A simple measure of risk sharing, following Mace (1991), is given by the coefficient α_2 in the following regression on CEX panel data

$$\Delta \log c_{it} = \alpha_1 + \alpha_2 \Delta \log y_{it} + \alpha_3 \Delta \log C_t + \varepsilon_{it}$$

where c_{it} is household i's (per-capita) non-durable consumption, y_{it} is household i's (per-capita) after tax endowment and C_t is period t aggregate consumption. Under the full risk sharing hypothesis the coefficient α_2 should be 0 while under autarky the coefficient should be 1. In table 8 below we report the value of the coefficient estimated by Mace (for 1980-84 CEX data) and the coefficient we estimated on a more recent sample together with the coefficient we get by running the same regression on data generated by our model under the two different parameterizations (high and low risk sharing, for the progressive tax system).

Table 8. Measures of Risk Sharing

	Data		Model	
	Mace (91)	CEX (86-94)	High RS	Low RS
α_2	.04	.06	.09	.99
	(.007)	(.004)		

Although other measures of risk sharing should be explored before a final conclusion can be drawn these regression seems to provide at least a *prima-facie* evidence that the high risk sharing model provides a better description of the risk sharing regime of the US economy.

6. Exogenous Incomplete Markets

In this section we compute the equilibrium allocation that arises in an economy in which agents are only allowed to trade a single uncontingent bond and they face an exogenously specified constant borrowing limit. This type of economies has been widely studied (see Huggett (1993) and Aiyagari (1994) among others) and we consider an economy similar to the one studied by Huggett. The household problem in recursive formulation for this model is (see Huggett (1993) for details):

$$
\begin{aligned}
v(a,y) &= \max_{c,a'}(1-\beta)u(c)+\beta\sum_{y'}v(a',y')\pi(y'|y) \\
& \quad s.t. \\
c+a' &= y+(1+r)a \\
a &\geq -\underline{b}
\end{aligned}
$$

where a are holdings of the one-period bond and r is the interest rate on these bonds.

To enable comparison with the endogenous incomplete markets economy we keep the preference, endowment and fiscal policy parameters constant. The borrowing limit $\underline{b}$ is the only additional parameter in the Huggett economy that needs to be specified. We pick $\underline{b}$ so that the equilibrium interest rate is equal to 5% for given (β, σ).

Table 9. Results for $\beta = .945, \underline{b} = 10\bar{y}$

	Tax System	
Variable	Progressive	Proportional
Interest rate	5%	4.7%
Govt. Intermediation	4.5%	0
Private Intermediation	32.7%	33.5%
Ex Ante Welfare	+2.5%	–

In this economy a switch from a progressive to a proportional tax system induces (large) welfare losses. Redistributive taxes act as a partial substitute for private insurance markets that are exogenously assumed to be missing. Removing this partial substitute for private markets leads to negative welfare consequences.

Another important difference between this economy and the one analyzed previously is the impact that the change in tax system has on the interest rate. In the debt-constrained economy a shift from progressive to proportional taxes causes an increase in the interest rate while in the economy with exogenous incomplete markets interest falls in response to the change in the tax system. The intuition for this result is as follows: the change in taxes increases the volatility of the income process. This, by making autarky less attractive, relaxes the borrowing constraints faced by the agents in the debt-constraint economy. Since agents borrow more, the interest rate has to rise to clear the credit markets. In the exogenous incomplete markets, on the other hand, agents, facing higher volatility of output, increase their precautionary saving. Since their borrowing limit is now unaffected this causes a decline in the equilibrium interest rate.

7. Conclusions

We have presented a model that can be used to analyze the effects of different taxation schemes on private financial markets. We have shown that, when private insurance markets are active, risk sharing provided through taxes always crowds out private risk sharing, but the magnitude of the crowding out can vary a great deal. In particular we have shown that this magnitude depends on the level of private risk sharing. When risk sharing is low, crowding out is small and redistribution through taxes is welfare improving, while when in high risk sharing regimes redistribution through taxes crowds out private financial market more than one to one and is welfare reducing. According to a simple measure of risk sharing the US economy seems closer to the latter case.

In contrast, if private insurance markets are exogenously assumed to be missing, as in the exogenous incomplete markets model, a tax reform that reduces the variance of after-tax income serves as a partial substitute for private insurance markets and leads to unambiguous welfare gains.

This, to us, demonstrates that an analysis of a tax policy reform that does not take into account the interaction between tax policy and the functioning of private insurance markets may lead to biased welfare conclusions.

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Appendix

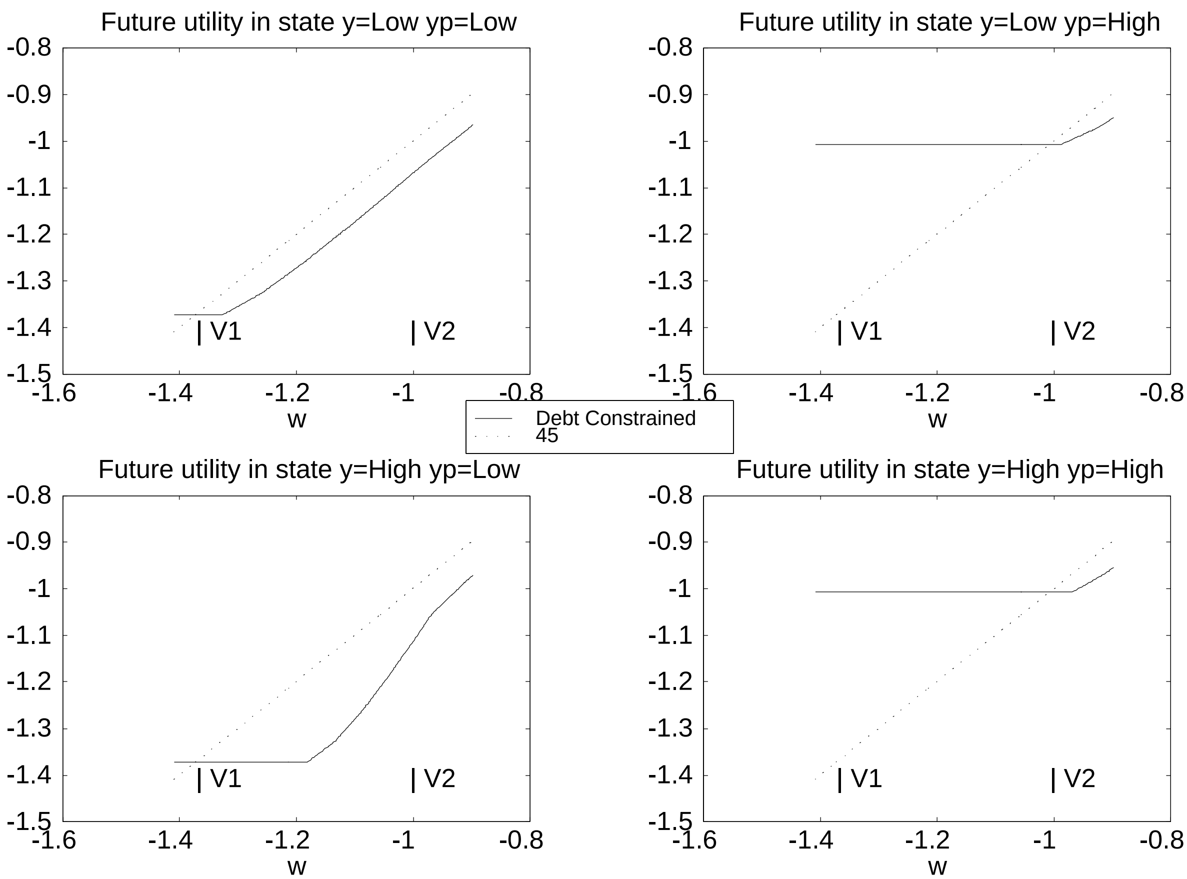

Figure 1:

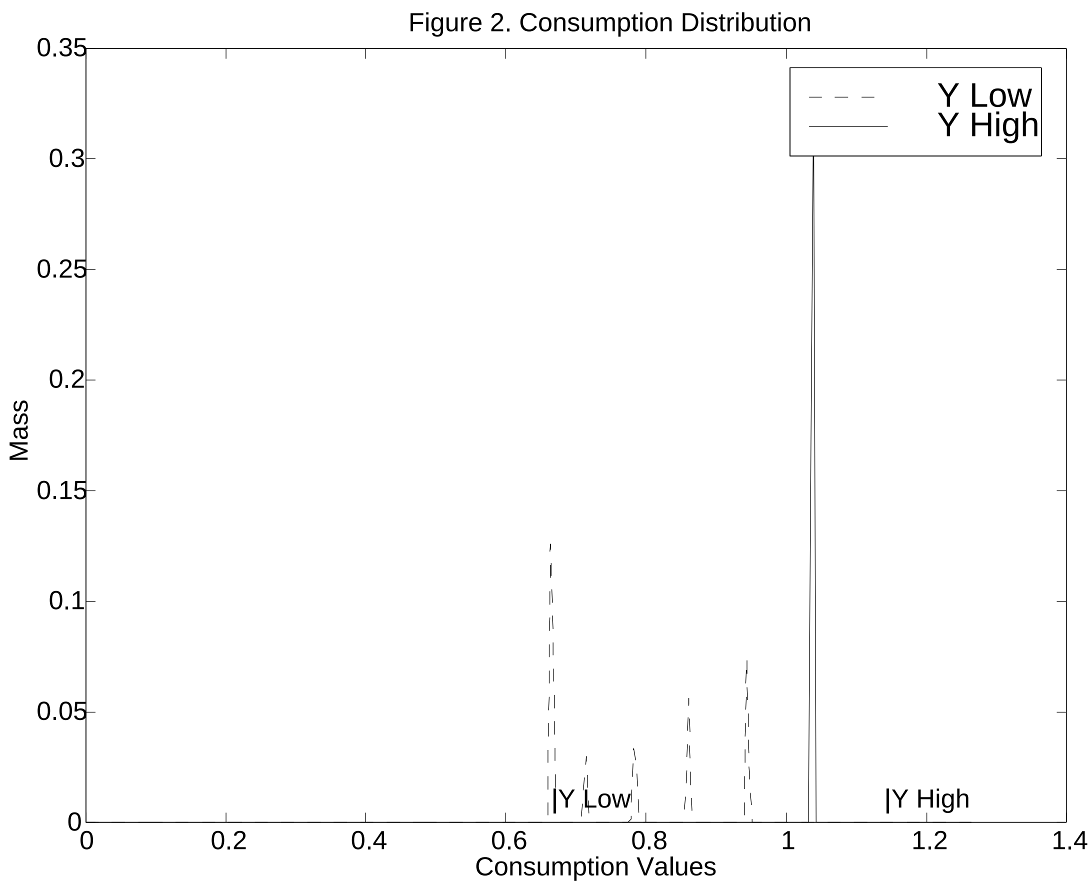
Figure 2. Consumption Distribution
Y Low
Y High
Mass
Consumption Values
|Y Low
|Y High
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.2
0.4
0.6
0.8
1
1.2
1.4

Figure 2:

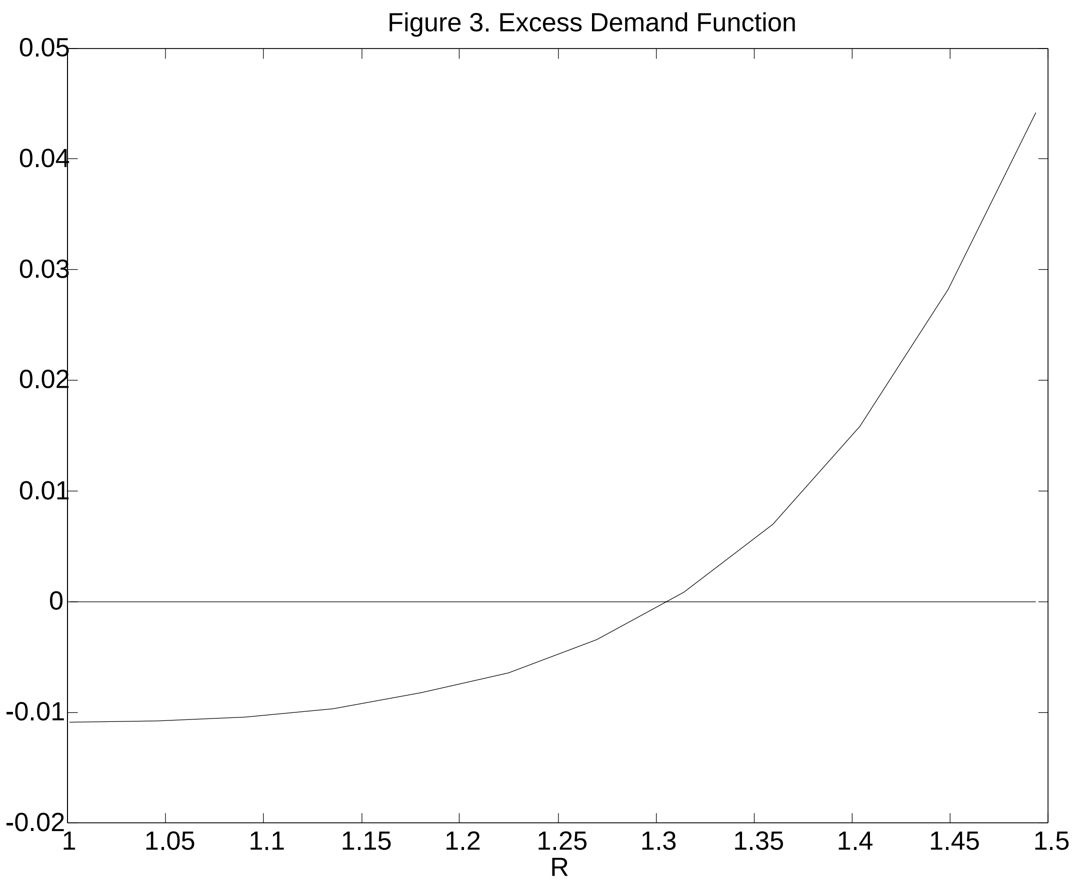
Figure 3. Excess Demand Function
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
1
1.05
1.1
1.15
1.2
1.25
1.3
1.35
1.4
1.45
1.5
R

Figure 3:

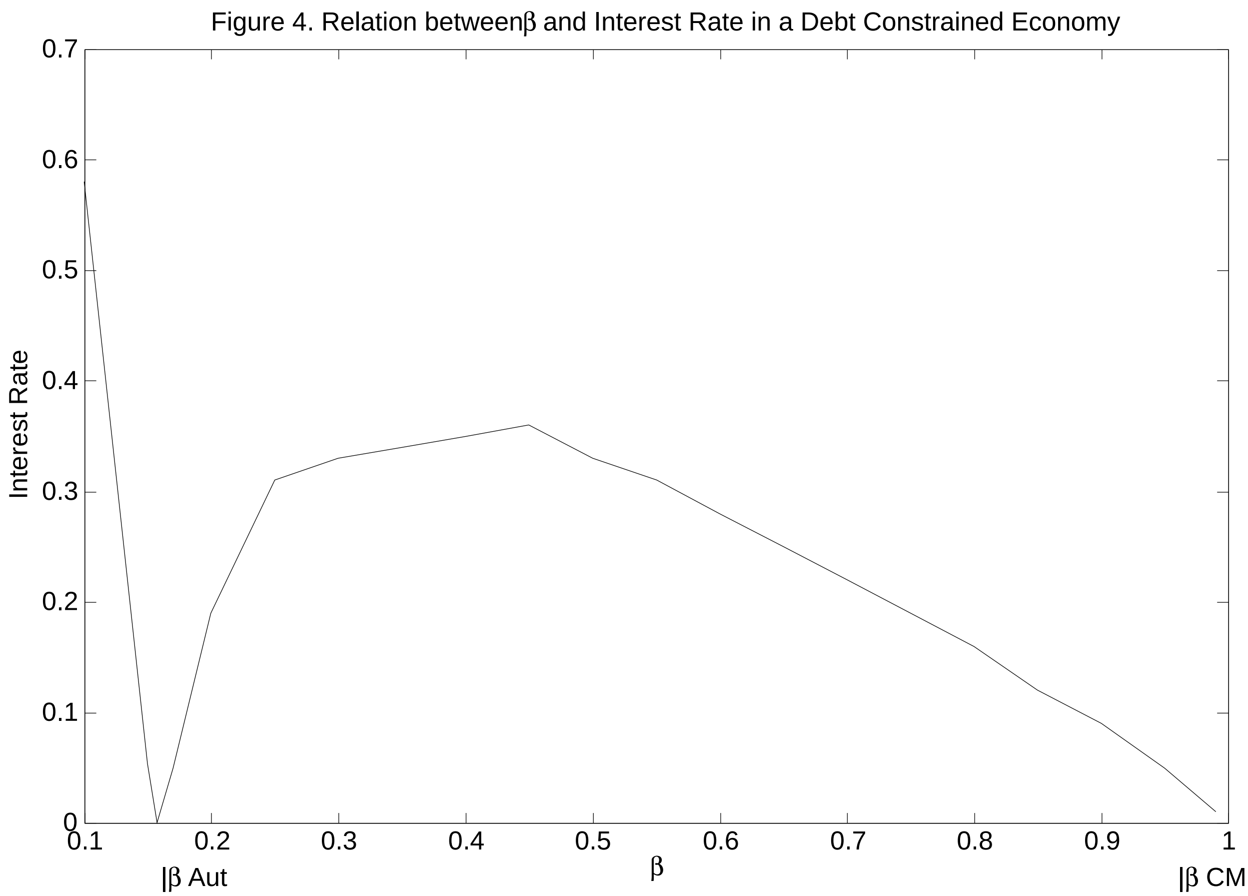
Figure 4. Relation betweenβ and Interest Rate in a Debt Constrained Economy
Interest Rate
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
|β Aut
β
|β CM

Figure 4:

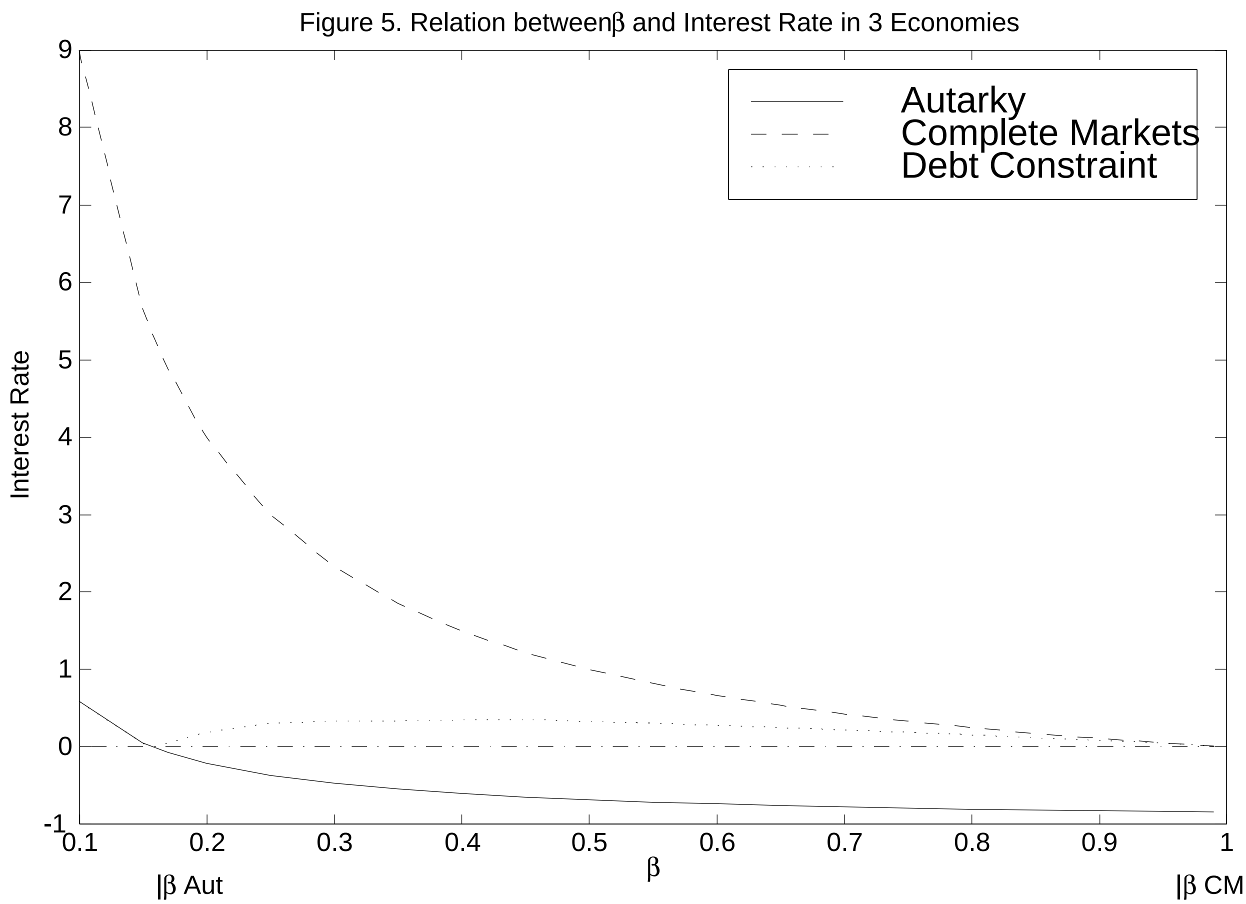
Figure 5. Relation betweenβ and Interest Rate in 3 Economies
Autarky
Complete Markets
Debt Constraint
Interest Rate
9
8
7
6
5
4
3
2
1
0
-1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
β
|β Aut
|β CM

Figure 5:

Proof of Theorem 1.:

To show efficiency we first need to show that $\{h_t(w_0, y^t)\}$ is constrained feasible with respect to Φ_0. By assumption the allocation satisfies feasibility (equation 4) and since it solves CPP also satisfies 3, 4, and 5. It is therefore constrained feasible. Now we need to show that there does not exist another allocation $\{\hat{h}_t(w_0, y^t)\}_{t=0}^{\infty}$ that is constrained feasible with respect to Φ_0 and such that

$$\text{(A1)} \quad \sum_{y^t} \int \left(C(\hat{h}_t(w_0, y^t)) - y_t \right) \pi(y^t|y_0) d\Phi_0 < 0 \text{ for some } t$$

Suppose this is the case. Since $\{h_t(w_0, y^t)\}$ solves CPP for all w_0, y_0 we have

$$\text{(A2)} \quad \begin{aligned} &\left(1 - \frac{1}{R_0}\right) C(h_0(w_0, y_0)) + \sum_{t=1}^{\infty} \left(1 - \frac{1}{R_t}\right) \prod_{s=0}^{t-1} \left(\frac{1}{R_s}\right) \sum_{y^t|y_0} C(h_t(w_0, y^t))\pi(y^t|y_0) \\ \leq &\left(1 - \frac{1}{R_0}\right) C(\hat{h}_0(w_0, y_0)) + \sum_{t=1}^{\infty} \left(1 - \frac{1}{R_t}\right) \prod_{s=0}^{t-1} \left(\frac{1}{R_s}\right) \sum_{y^t|y_0} C(\hat{h}_t(w_0, y^t))\pi(y^t|y_0) \end{aligned}$$

where the left hand side of equation A2 is finite[18]. Integrating both sides of A2 with respect to Φ_0 One has

$$\text{(A3)} \quad \begin{aligned} &\int \left\{ \left(1 - \frac{1}{R_0}\right) C(h_0(w_0, y_0)) + \sum_{t=1}^{\infty} \left(1 - \frac{1}{R_t}\right) \prod_{s=0}^{t-1} \left(\frac{1}{R_s}\right) \sum_{y^t|y_0} C(h_t(w_0, y^t))\pi(y^t|y_0) \right\} d\Phi_0 \\ \leq &\int \left\{ \left(1 - \frac{1}{R_0}\right) C(\hat{h}_0(w_0, y_0)) + \sum_{t=1}^{\infty} \left(1 - \frac{1}{R_t}\right) \prod_{s=0}^{t-1} \left(\frac{1}{R_s}\right) \sum_{y^t|y_0} C(\hat{h}_t(w_0, y^t))\pi(y^t|y_0) \right\} d\Phi_0 \end{aligned}$$

From the fact that for $\{h_t(w_0, y^t)\}$ feasibility holds with equality for all t, that $\left\{\hat{h}_t(w_0, y^t)\right\}$ is constrained feasible and from A1 one obtains:

$$\text{(A4)} \sum_{y^t|y_0} \int C(\hat{h}_t(w_0, y^t))\pi(y^t|y_0) d\Phi_0 \leq \sum_{y^t|y_0} \int y_t \pi(y^t|y_0) d\Phi_0 = \sum_{y^t|y_0} \int C(h_t(w_0, y^t))\pi(y^t|y_0) d\Phi_0$$

for all t with the inequality being strict for some t. Multiplying each inequality by the appropriate term $\left(1 - \frac{1}{R_t}\right) \prod_{s=0}^{t-1} \left(\frac{1}{R_s}\right) > 0$ and summing over all t we obtain A3, but with the inequality reversed and strict, a contradiction ■

[18] This is guaranteed since we can always pick a constant $h_t(w_0, y^t) = \max(w_0, \max_y V^{Aut}(y))$. Such a policy satisfies all the constraints of CPP and since $\max(w_0, \max_y V^{Aut}(y)) \in D$ and condition 3. in Theorem 1 is satisfied the value of the minimization problem is finite.

Proof of Lemma 1.:

For every $(w, y) \in A$ the objective function in 16 is continuous in $h, g_{y'}$ and the constraint set is compact and non-empty; therefore the minimum exists. V is bounded and since $\underline{h}(w, y) \leq h \leq \bar{h}(w, y)$, $C(h)$ is bounded as well. It follows that $T_R V$ is a bounded function. The fact that $T_R V$ is continuous follows from the Theorem of the Maximum (note that the constraint set is continuous in w). It is also easy to show that since $R > 1$ the operator T_R satisfies the hypotheses of Blackwell's theorem and thus is a contraction with modulus $\frac{1}{R}$ ■

Proof of Lemma 2.:

For the first part we note that $C(A)$ (together with the sup-norm) is a complete metric space and that the set of bounded continuous nondecreasing (in its first argument) functions on A, $C'(A)$, is a closed subset of $C(A)$ and that the set of bounded continuous strictly increasing functions, $C''(A)$, satisfies $C''(A) \subset C'(A)$. By Lemma 1. T_R is a contraction mapping. Hence by Corollary 1 of Stokey et al., p. 52, it is sufficient to show that, whenever $V_R \in C'(A)$, then $T_R V_R \in C''(A)$. Fix $w, \hat{w}$ with $\underline{w} \leq w < \hat{w} \leq \bar{w}$. we need to show that $(T_R V_R)(w) < (T_R V_R)(\hat{w})$. Let $\hat{h}, \hat{g}_{y'}$ be the optimal choices for $\hat{w}$. The choices $g_{y'} = \hat{g}_{y'}$ and $h = \hat{h} - \hat{w} + w < \hat{h}$ are feasible for w and therefore

$$\begin{aligned}
(T_R V_R)(\hat{w}) &= \left(\frac{R-1}{R}\right) C(\hat{h}) + \frac{1}{R} \sum_{y' \in Y} \pi(y') V_R(\hat{g}_{y'}) \\
&> \left(\frac{R-1}{R}\right) C(h) + \frac{1}{R} \sum_{y' \in Y} \pi(y') V_R(g_{y'}) \\
&\geq (T_R V_R)(w) \qquad \text{(A5)}
\end{aligned}$$

To prove that V_R is convex we note that the set of bounded continuous convex functions, $C'''(A)$ is a closed subset of $C(A)$. Again by Corollary 1 of Stokey et al., p. 52, it is sufficient to show that if $V_R \in C'''(A)$, then $(T_R V_R)$ is convex in its first argument. So we have to show that for all $w, \hat{w} \in A$ with $w \neq \hat{w}$, and all $\lambda \in (0, 1)$, $(T_R V)(\lambda w + (1-\lambda)\hat{w}) \leq \lambda (T_R V)(w) + (1-\lambda)(T_R V)(\hat{w})$. Let $\hat{h}, \hat{g}_y$ be the optimal choices for $\hat{w}$ and h, g_y be the optimal choices for w and define $h^\lambda = \lambda h + (1-\lambda)\hat{h}$, $g_y^\lambda = \lambda g_y + (1-\lambda)\hat{g}_y$.

Since h^λ, g_y^λ are feasible for $(\lambda w + (1-\lambda)\hat{w}, y)$, and

$$\begin{aligned}
& (T_R V_R)(\lambda w + (1-\lambda)\hat{w}) \\
\leq\ & \left(\frac{R-1}{R}\right) C(h^\lambda) + \frac{1}{R}\sum_{y' \in Y} \pi(y') V(g_{y'}^\lambda) \\
\leq\ & \left(\frac{R-1}{R}\right) \left(\lambda C(h) + (1-\lambda) C(\hat{h})\right) + \frac{1}{R}\sum_{y' \in Y} \pi(y') \left(\lambda V(g_{y'}) + (1-\lambda) V(\hat{g}_{y'})\right) \\
=\ & \lambda (T_R V_R)(w) + (1-\lambda)(T_R V_R)(\hat{w}) \qquad \text{(A6)}
\end{aligned}$$

by convexity of V in its first argument and strict convexity of C.

We want to show that the fixed point of T_R, V, is strictly convex on A. We know that V is convex, continuous and strictly increasing. These facts imply that V is differentiable almost everywhere on A and that for the countable number of points at which V is not differentiable, right hand derivatives V'_+ and left hand derivatives V'_- exist (although need not coincide).

Now suppose that V is not strictly convex on A. Then there exists an interval $I \subseteq A$ such that V is linear on I. Then by the envelope theorem, for all $w \in I$,

$$a = V'(w) = \frac{R-1}{R(1-\beta)} C'(h(w))$$

for some $a > 0$. Hence for all $w \in I$, $w - \sum_{y'} \pi(y') g_{y'}(w)$ is constant. It follows that there exists $\bar{y}'$ such that $g_{\bar{y}'}(w)$ is strictly increasing in w on an interval $I' \subseteq I$. Then from the first order condition

$$\begin{aligned}
V'_-(g_{\bar{y}'}(w)) &\leq \beta R a \\
V'_+(g_{\bar{y}'}(w)) &\geq \beta R a
\end{aligned}$$

for all $w \in we'$. Since $g_{y'}$ is strictly increasing on I' and V is convex, this implies that on $I'' = \{w' \in A | g_{\bar{y}'}(w) = w'$, for some $w \in I'\}$, V' is constant at $b < a$ (as $\beta R < 1$). Since V is convex, for every $w' \in I''$ and every $w \in I$ we have $w' < w$. Repeating this argument one can show that for every $\varepsilon > 0$ there has to exist an interval $I(\varepsilon)$ such that V' is constant at $c < \varepsilon$. Now let $d = \frac{R-1}{R(1-\beta)} C'(\underline{h}(\underline{w}))$, and pick ε such that the associated c satisfies $c < d$. Then for all $w \in we(\varepsilon)$, the envelope condition

$$c = V'(w) = \frac{R-1}{R(1-\beta)} C'(h(w)) \geq \frac{R-1}{R(1-\beta)} C'(\underline{h}(\underline{w})) = d$$

a contradiction. This proves that there does not exist an interval $I \subseteq A$ such that V is linear on I. Since V is convex on A it then follows that V is strictly convex ■

Proof of Lemma 3.:

The fact that $T_R V$ is strictly increasing and strictly convex follows from the properties of V. The choice variables h and $g_{y'}$ are constrained to lie in compact and convex intervals, and by assumption the objective function is strictly convex. Hence the minimizers are unique. Since the constraint set is continuous in w, the theorem of the maximum applies and $T_R V$ is continuous and $h(w), g_{y'}(w)$ are upper hemicontinuous correspondences. Since $h(w), g_{y'}(w)$ are functions, they are continuous ■

Proof of Lemma 4.:

Consider the following sequence of functions $\{V^n\}_{n=0}^{\infty}$, defined recursively as:

$$\begin{aligned} V^0(w) &= C(w) & \forall w \in A \\ V^{n+1}(w) &= \left(T^R V^n\right)(w) & \forall w \in A \end{aligned}$$

From corollary 1. we know that this sequence converges uniformly to the unique fixed point V_R of T_R. Also Lemma 3. assures that each V^n is continuous, strictly increasing and strictly convex (as by assumption C possesses these properties) and that the associated policies $h^n(w)$ and $g_{y'}^n(w)$ are continuous functions. From 21 we have (as C is continuously differentiable by assumption) that each V^n is differentiable and that this derivative is continuous, since $h^{n-1}(w)$ is a continuous function. Now we will establish that V_R is continuously differentiable.

From Lemmas 3., 2. and corollary 1. we know that each V^n as well as V_R are strictly convex and continuous and that the sequence $\{V^n\}_{n=0}^{\infty}$ converges to V_R uniformly. Also A is compact. Then by theorem 3.8 of Stokey et al., p. 64, the sequences $\left\{h^n(w), g_{y'}^n(w)\right\}_{n=1}^{\infty}$ converge uniformly to the optimal policies associated with V_R, $h^R(w)$ and $g_{y'}^R(w)$, respectively. Therefore from 21 $(T_R V^n)'$ converges to $\frac{(R-1)}{R(1-\beta)} C'(h^R(w))$ uniformly. Since $\{V^n\}_{n=0}^{\infty}$ converges to V_R uniformly, we have that V_R is differentiable, with

$$(V_R)'(w) = \frac{(R-1)}{R(1-\beta)} C'(h^R(w))$$

■

Proof of Lemma 5.:

We want to show that for all $\underline{w} \leq w < \hat{w} \leq \bar{w}$, $h(w) < h(\hat{w})$. Suppose, to the contrary, $h(w) \geq h(\hat{w})$. Then from 20

$$V'(g_{y'}(w)) \geq V'(g_{y'}(\hat{w}))$$

for all y' such that $g_{y'}(\hat{w}) > U^{Aut}(y')$, and hence $U^{Aut}(y') < g_{y'}(\hat{w}) \leq g_{y'}(w)$ for all those y' by strict convexity of V. But then from the promise keeping constraint there must exist $\bar{y}'$ such that $g_{\bar{y}'}(w) < g_{\bar{y}'}(\hat{w}) = U^{Aut}(\bar{y}')$, a violation of the debt constraint. We obtain a contradiction to the assumption that $h(w,y) > h(\hat{w},y)$ ■

Proof of Lemma 6.:

Again let $\underline{w} \leq w < \hat{w} \leq \bar{w}$. Under the assumptions made h is strictly increasing in w. Therefore $C'(h(w)) < C'(h(\hat{w}))$. First suppose that $g_{y'}(w) > U^{Aut}(y')$. Then from 20 we have

$$V'(g_{y'}(w)) < V'(g_{y'}(\hat{w}))$$

and from the strict convexity of V it follows that $g_{y'}(\hat{w}) > g_{y'}(w)$. Obviously, if $g_{y'}(w) = U^{Aut}(y')$ then $g_{y'}(\hat{w}) \geq g_{y'}(w)$, i.e. either $g_{y'}(\hat{w}) > g_{y'}(w)$ or $g_{y'}(w) = g_{y'}(\hat{w}) = U^{Aut}(y')$ ■

Proof of Lemma 8.:

From Lemmas 2. and 4.V_R is strictly convex and differentiable. By assumption $g_{y'}(w) > U^{Aut}(y')$. Then combining 20 and 21 we obtain

$$\text{(A7)} \quad \beta R\,(V_R)'\,(w) = (V_R)'\,(g_{y'}(w))$$

Since $R < \frac{1}{\beta}$ we have $V'(w) > V'(g_{y'}(w))$, and by strict convexity of V_R the first result follows. Hence $g_{y'}(.)$ are always strictly below the 45^0 line in their strictly increasing part. On the other hand $g_{y'}(w) \geq U^{Aut}(y')$ for all w. Hence for $w < U^{Aut}(y')$, $g_{y'}(w) = U^{Aut}(y') > w$. By continuity of $g_{y'}(.)$, $g_{y'}(U^{Aut}(y')) = U^{Aut}(y')$ and from the first result $g_{y'}(w) < w$ for all $w > U^{Aut}(y')$ ■

Proof of Theorem 2.:

Take $\bar{w} = \max_y U^{Aut}(y) + \varepsilon$, for $\varepsilon > 0$ arbitrarily small. If $g_{y'}(w) > U^{Aut}(y')$, then the previous Lemma yields the result. If $g_{y'}(w) = U^{Aut}(y')$, then $g_{y'}(w) = U^{Aut}(y') \leq \max_y U^{Aut}(y) < \bar{w}$

Proof of Theorem 3.:

For any allocation $\sigma = \{h_t(w_0, y^t)\}_{t=0}^{\infty}$ define

$$(A8) \quad U_t(w_0, y^t, \sigma) = (1-\beta)\left(h_t(w_0, y^t) + \sum_{s>t}^{\infty}\sum_{y^s}\beta^{s-t}\pi(y^s)h_t(w_0, y^s)\right)$$

$$(A9) \quad U_t^{Aut}(y_t) = (1-\beta)\left(u(y_t) + \sum_{s>t}^{\infty}\sum_{y^s}\beta^{s-t}\pi(y^s)u(y_s)\right)$$

By theorem 4.3 in Stokey et al. (the assumption of which are satisfied as $C(w) \geq 0$, all w), for all $w_0 \in W$ and all $y_0 \in Y$, the solution to the functional equation, V_R satisfies

$$V_R(w_0) = \inf_{\{h_t(w_0,y^t), w_t(w_0,y^t)\}} \left(1 - \frac{1}{R}\right) C(h_0(w_0, y_0)) + \sum_{t=1}^{\infty}\left(1 - \frac{1}{R}\right)\frac{1}{R^t}\sum_{y^t} C(h_t(w_0, y^t))\pi(y^t)$$

$$(A10) \quad s.t\ w_t(w_0, y^t) = (1-\beta)h_t(w_0, y^t) + \beta\sum_{y^{t+1}}\pi(y_{t+1})w_{t+1}(w_0, y^{t+1}) \qquad \text{all } t$$

$$(A11a) \quad U_t^{Aut}(y_t) \leq w_t(w_0, y^t) \leq \bar{w} \qquad \text{all } t \geq 1$$

$$(A11b) \quad w_0 \geq U_0^{Aut}(y_0) \text{ given}$$

By theorem 4.4 and 4.5 (which are applicable as V_R is bounded on W and the sequence $\{\hat{w}_t(w_0, y^t)\}_{t=1}^{\infty}$ defined above never leaves W), the allocation $\{\hat{h}_t(w_0, y^t)\}_{t=0}^{\infty}$ together with $\{\hat{w}_t(w_0, y^t)\}_{t=1}^{\infty}$ defined above uniquely attains the minimum of the above problem. In order to argue that $\{\hat{h}_t(w_0, y^t)\}_{t=0}^{\infty}$ solves CPP we have to show that any allocation $\{h_t(w_0, y^t)\}_{t=0}^{\infty}$ together with some $\{w_t(w_0, y^t)\}_{t=1}^{\infty}$ satisfies A10 and A11a if and only if $\{h_t(w_0, y^t)\}_{t=0}^{\infty}$ satisfies 3, 4 and 5, i.e. if

$$(A12a) \quad w_0 = U_0(w_0, y_0, \sigma)$$

$$(A12b) \quad U_t^{Aut}(y_t) \leq U_t(w_0, y^t, \sigma) \leq \bar{w} \qquad \text{all } t$$

$$(A12c) \quad \lim_{t\to\infty}\beta^t \sup_{y^t} U_t(w_0, y^t, \sigma) = 0$$

Step 1: Pick any allocation $\sigma = \{h_t(w_0, y^t)\}_{t=0}^{\infty}$ that satisfies A12a to A12c. Define $w_t(w_0, y^t) = U_t(w_0, y^t, \sigma)$. It is immediate from A12b that A11a is satisfied. From the definition of $U_t(w_0, y^t, \sigma)$ it follows that A10 is satisfied as well.

Step 2: Pick any allocation $\sigma = \{h_t(w_0, y^t)\}_{t=0}^{\infty}$ and $\{w_t(w_0, y^t)\}_{t=1}^{\infty}$ that satisfies A10 and A11a. Since for all t, $w_t(w_0, y^t) \leq \bar{w}$ from A11a, by using A10 we see that the allocation satisfies A12c. Now for all allocations satisfying A12c, and for all t

$$
\begin{aligned}
\left|w_t(w_0,y^t) - U_t(w_0,y^t,\sigma)\right| &= \beta\left|\sum_{y^{t+1}} \pi(y_{t+1})\left(w_{t+1}(w_0,y^{t+1}) - U_t(w_0,y^t,\sigma)\right)\right| \\
&\leq \beta \sup_{y^{t+1}} \left|w_{t+1}(w_0,y^{t+1}) - U_t(w_0,y^t,\sigma)\right| \\
&\leq \beta^s \sup_{y^{t+s}} \left|w_{t+s}(w_0,y^{t+s}) - U_{t+s}(w_0,y^{t+s},\sigma)\right| \\
&\leq \beta^s \sup_{y^{t+s}} \left(\left|w_{t+s}(w_0,y^{t+s})\right| + \left|U_{t+s}(w_0,y^{t+s},\sigma)\right|\right) \qquad \text{(A13)}
\end{aligned}
$$

This inequality is valid for all t and all s. Taking limit with respect to s yields (by A12c and A11a) that $w_t(w_0, y^t) = U_t(w_0, y^t, \sigma)$ for all t. Hence A10 implies that

$$
\begin{aligned}
w_0 &= (1-\beta)h_0(w_0,y_0) + \beta\sum_{y^1}\pi(y_1)w_1(w_0,y^1) \\
&= (1-\beta)h_0(w_0,y_0) + \beta\sum_{y^1}\pi(y_1)U_1(w_0,y^1,\sigma) \\
&= U_0(w_0,y_0,\sigma) \qquad \text{(A14)}
\end{aligned}
$$

and hence A12a is satisfied. For $t \geq 1$ A12b is obviously satisfied, and it is satisfied for $t = 0$ by the assumption that $w_0 \geq U_t^{Aut}(y_t)$.

This proves that the allocation constructed from the policies of the functional equation solves the component planning problem with the additional constraint $U_t(w_0, y^t, \sigma) \leq \bar{w}$. By Theorem 2. $\hat{w}_t(w_0, y^t) < \bar{w}$ and hence, as $\hat{w}_t(w_0, y^t) = U_t(w_0, y^t, \hat{\sigma})$, the constraint is never binding. Since the constraint set associated with the CPP is convex, this implies that the allocation $\hat{\sigma}$ indeed solves the original component planning problem for constant interest rates.

Proof of Lemma 9.:

Suppose $g_{y_{\max}}(w) > U^{Aut}(y_{\max})$ for all $w \in A, w > U^{Aut}(y_{\max})$. Then by Lemma 7. $g_{y'}(w) = g_{y_{\max}}(w)$, for all $y' \in Y$ and $w > U^{Aut}(y_{\max})$. By continuity of $g_{y'}$ and Lemma 8., $g_{y'}(U^{Aut}(y_{\max})) = U^{Aut}(y_{\max})$, for all $y' \in Y$. But since $U^{Aut}(y_{\max}) > U^{Aut}(y')$ for all $y' \neq y_{\max}$, by Lemma 8. $g_{y'}(U^{Aut}(y_{\max})) < U^{Aut}(y_{\max})$ for all $y' \neq y_{\max}$, a contradiction ■

Proof of Theorem 4.:

The proof is an application of Stokey et al., theorem 11.12. We have to prove that there exists an $\varepsilon > 0$ and an $N \geq 1$ such that for all sets $(\mathcal{B}, \mathcal{Y}) \in \mathcal{B}(W) \times \mathcal{P}(Y)$, either $Q^N((w, y, \mathcal{B}, \mathcal{Y}) \geq \varepsilon$ or $Q^N((w, y, (\mathcal{B}, \mathcal{Y})^C) \geq \varepsilon$, for all $(w, y) \in (W, Y)$. For this it is sufficient to prove that there exists an $\varepsilon > 0$ and an $N \geq 1$ such that for all $(w, y) \in (W, Y)$, $Q^N((w, y, U^{Aut}(y_{\max}), y_{\max}) \geq \varepsilon$. If $w^* \geq \bar{w}$ this is immediate, as for all $(w, y) \in (W, Y)$, $Q((w, y, U^{Aut}(y_{\max}), y_{\max}) \geq \pi(y_{\max})$, since $g_{y_{\max}}(w) = U^{Aut}(y_{\max})$ for all $w \in W$.

Suppose $w^* < \bar{w}$. Define

$$d = \min_{w \in [w^*, \bar{w}]} w - g_{y_{\max}}(w)$$

Note that d is well-defined as $g_{y_{\max}}$ is a continuous function and that $d > 0$ from Lemma 8.. Define

$$N = \min \{n \in N | \bar{w} - nd \leq w^*\}$$

and $\varepsilon = \pi(y_{\max})^N$. We will show that for N and ε so defined the result follows. Suppose an individual receives $y_{\max}$ for N times in a row, an event that occurs with probability ε. For (w, y) such that $w \leq w^*$ the result is immediate as for those $w, g_{y_{\max}}(w) = U^{Aut}(y_{\max})$ and $g_{y_{\max}}(U^{Aut}(y_{\max})) = U^{Aut}(y_{\max})$. For any $w \in (w^*, \bar{w}]$ we have $g_{y_{\max}}(w) \leq w - d$, $g_{y_{\max}}(g_{y_{\max}}(w)) \leq w - 2d$, and so on. Then the result follows by construction of N and ε ■

Proof of Lemma 10.:

we have to show that

$$\lim_{n \to \infty} ||V_{R_n} - V_R|| = 0$$

where $||V_{R_n} - V_R|| = \sup_{[\underline{w}, \bar{w}]} |V_{R_n} - V_R|$. By the triangle inequality

$$\text{(A15)} \quad ||V_{R_n} - V_R|| \leq ||V_{R_n} - T_{R_n}^n V_R|| + ||T_{R_n}^n V_R - V_R||$$

Now the operator T_{R_n} is a contraction mapping on $[\underline{w}, \bar{w}]$ with unique fixed point V_{R_n} (see corollary 1). Hence

$$\lim_{n \to \infty} ||V_{R_n} - T_{R_n}^n V_R|| = 0$$

For the second term in the sum we note that

$$\text{(A16)}\quad ||T^n_{R_n}V_R - V_R|| \leq \sum_{k=1}^{n} ||T^k_{R_n}V_R - T^{k-1}_{R_n}V_R|| \leq \sum_{k=1}^{n} \frac{1}{(R_n)^k}||T_{R_n}V_R - V_R||$$

Here the first inequality again follows from the triangle inequality and the second from the fact that T_{R_n} is a contraction mapping on $[\underline{w}, \bar{w}]$ with modulus $\frac{1}{(R_n)^k}$. Hence

$$\begin{aligned}
\lim_{n\to\infty} ||T^n_{R_n}V_R - V_R|| &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(R_n)^k}||T_{R_n}V_R - V_R|| \\
&\leq \lim_{n\to\infty} \frac{R_n}{R_n - 1}||T_{R_n}V_R - V_R|| \\
\text{(A17)}\qquad &= \frac{R}{R-1} \lim_{n\to\infty} ||T_{R_n}V_R - T_RV_R||
\end{aligned}$$

where we used the fact that V_R is the unique fixed point of T_R. Hence $\lim_{n\to\infty} ||T^n_{R_n}V_R - V_R|| = 0$ if and only if $\lim_{n\to\infty} ||T_{R_n}V_R - T_RV_R|| = 0$, i.e. if the operator T_{R_n} is continuous in R_n. To see that T_{R_n} is in fact continuous in R_n consider the following argument: for arbitrary $\hat{w} \in [\underline{w}, \bar{w}]$ by the theorem of the maximum

$$\lim_{n\to\infty} |T_{R_n}V_R(\hat{w}) - T_RV_R(\hat{w})| = 0$$

Since $[\underline{w}, \bar{w}]$ is a compact set and $T_{R_n}V_R$, T_RV_R are continuous functions in w, we have

$$\lim_{n\to\infty} \max_{\hat{w}\in[\underline{w},\bar{w}]} |T_{R_n}V_R(\hat{w}) - T_RV_R(\hat{w})| = \lim_{n\to\infty} ||T_{R_n}V_R - T_RV_R|| = 0$$

Hence both terms on the right hand side of A15 converge to 0, which proves the result ■

Proof of Lemma 11.:

We have to show that for each $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$,

$$|g^{R_n}_{y'}(w_n) - g^R_{y'}(w)| < \varepsilon$$

We note that by the triangle inequality

$$|g^{R_n}_{y'}(w_n) - g^R_{y'}(w)| \leq |g^{R_n}_{y'}(w_n) - g^R_{y'}(w_n)| + |g^R_{y'}(w_n) - g^R_{y'}(w)|$$

Since the function $g^R_{y'}$ is continuous, for each $\varepsilon_1 > 0$ there exists $N(\varepsilon_1)$ such that $|g^R_{y'}(w_n) - g^R_{y'}(w)| < \varepsilon_1$ for all $n \geq N(\varepsilon_1)$. By Lemma 2. V_R as well V_{R_n} are strictly convex, for each $n \in N$. Also $\{V_{R_n}\}_{n=0}^{\infty}$ converges uniformly to V_R by Lemma 10. on the compact set $[\underline{w}, \bar{w}]$.

Then by theorem 3.8, Stokey et al. (1989), for each $\varepsilon_2 > 0$ there exists $N(\varepsilon_2)$ such that $|g_{y'}^{R_n}(w) - g_{y'}^{R}(w)| < \varepsilon_2$ for all $m \geq N(\varepsilon_2)$ and *all* $w \in [\underline{w}, \bar{w}]$. So fix $\varepsilon > 0$ and choose $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$ and $N(\varepsilon) = \max\{N(\varepsilon_1), N(\varepsilon_2)\}$. Then for all $n \geq N$

$$|g_{y'}^{R_n}(w_n) - g_{y'}^{R}(w)| \leq |g_{y'}^{R_n}(w_n) - g_{y'}^{R}(w_n)| + |g_{y'}^{R}(w_n) - g_{y'}^{R}(w)| < \varepsilon_2 + \varepsilon_1 = \varepsilon$$

■

Proof of Theorem 5.:

Consider a sequence $\{R_n\}_{n=0}^{\infty}$ with $R_n \in (1, \frac{1}{\beta})$ converging to $R \in (1, \frac{1}{\beta})$. With each R_n and with R there is associated an operator $T_{R_n}^*$ and T_R^*, respectively. By Theorem 4. there exist a unique sequence of probability measures $\{\Phi_{R_n}\}_{n=0}^{\infty}$ such that $\Phi_{R_n} = T_{R_n}^* \Phi_{R_n}$ and a unique Φ_R such that $\Phi_R = T_R^* \Phi_R$. we will argue that the sequence $\{\Phi_{R_n}\}_{n=0}^{\infty}$ converges weakly to Φ_R.

First, the state space $[\underline{w}, \bar{w}] \times Y$ is compact. Now consider the sequence of transition functions $\{Q_{R_n}\}_{n=0}^{\infty}$ associated with $\{R_n\}_{n=0}^{\infty}$. For any sequence $\{w_n\}_{n=0}^{\infty}$ in $[\underline{w}, \bar{w}]$ converging to $w \in [\underline{w}, \bar{w}]$, for all $y' \in Y$, $g_{y'}^{R_n}(w_n)$ converges to $g_{y'}(w)$ by Lemma 11.. Now consider the sequence of probability measures $\{Q_{R_n}((w_n, y), .)\}_{n=0}^{\infty}$ and the probability measure $Q_R((w, y), .)$. If we can show that for each set $B \in \mathcal{B}(W) \times \mathcal{P}(Y)$ for which $Q_R((w, y), \partial B) = 0$,

$$\text{(A18)} \quad \lim_{n \to \infty} Q_{R_n}((w_n, y), B) = Q_R((w, y), B)$$

then the sequence $Q_{R_n}((w_n, y), .)$ converges weakly to $Q_R((w, y), .)$ by theorem 12.3, Stokey et al. Here ∂B denote the boundary of B, i.e.. the set of points that are limit points of B as well as B^C. Take an arbitrary such set B. By definition of Q_R, for all w' such that $g_{y'}^{R}(w) = w'$ for some $y' \in Y$, we have that w' is in the interior of B (otherwise $Q_R((w, y), \partial B) > 0$). But then, since $g_{y'}^{R_n}(w_n)$ converges to $g_{y'}(w)$, $Q_{R_n}((w_n, y), B) = Q_R((w, y), B)$ for n sufficiently big. Hence A18 is satisfied and the sequence $Q_{R_n}((w_n, y), .)$ converges weakly to $Q_R((w, y), .)$.

This result enables us to apply theorem 12.13 of Stokey et al. to conclude that the sequence $\{\Phi_{R_n}\}_{n=0}^{\infty}$ converges weakly to Φ_R. By Lemma 10. $\{V_{R_n}\}_{n=0}^{\infty}$ converges uniformly to V_R. To show continuity of $d(.)$ we note that

$$\begin{aligned} |d(R_n) - d(R)| &= \left| \int V_{R_n}(w) d\Phi_{R_n} - \int V_R(w) d\Phi_R \right| \\ \text{(A19)} \qquad &\leq \left| \int V_{R_n}(w) d\Phi_{R_n} - \int V_R(w) d\Phi_{R_n} \right| + \left| \int V_R(w) d\Phi_{R_n} - \int V_R(w) d\Phi_R \right| \end{aligned}$$

by the triangle inequality. The first term converges to zero (as $n \to \infty$) as $\{V_{R_n}\}_{n=0}^{\infty}$ converges uniformly to V_R, the second term converges to zero as $\{\Phi_{R_n}\}_{n=0}^{\infty}$ converges weakly to Φ_R and V_R is a continuous and bounded function ■

Proof of Lemma 12.:

Let $R > \hat{R}$. We want to show that $h^R(w) \leq h^{\hat{R}}(w)$ and $g^R_{y'}(w) \geq g^{\hat{R}}_{y'}(w)$, for all $y' \in Y$ and all $w \in [\underline{w}, \bar{w}]$. Define the sequence $\{V^n\}_{n=1}^{\infty}$ by $V^n = (T_{\hat{R}})^n V_R$. Note that as V_R is strictly convex and differentiable (by the argument in the proof to Lemma 4.), so are all V^n (by the argument in the proof to Lemma 4.). Let $(h^n, g^n_{y'})$ be the optimal policies associated with V^n, i.e.

$$\text{(A20)}\quad V^n(w) = \left(1 - \frac{1}{R}\right) C(h^n(w)) + \frac{1}{R}\sum_{y'} \pi(y') V^{n-1}(g^n_{y'}(w)).$$

We prove by induction that for all $n \geq 1$,

$$\begin{aligned} &\text{(A21)}\quad g^R_{y'}(w) \quad \geq \quad g^n_{y'}(w) \\ &\text{(A22)}\quad h^R(w) \quad \leq \quad h^n(w) \\ &\text{(A23)}\quad \frac{V'_R(w)}{R-1} \quad \leq \quad \frac{(V^n)'(w)}{\hat{R}-1} \end{aligned}$$

for all $y' \in Y$ and all $w \in [\underline{w}, \bar{w}]$. Since $\{V^n\}_{n=1}^{\infty}$ converges to $V_{\hat{R}}$ uniformly (by corollary 1.) and $\left\{h^n, g^n_{y'}\right\}_{n=1}^{\infty}$ converge uniformly to $(h^{\hat{R}}, g^{\hat{R}}_{y'})$ (again see Lemma 4.), it then follows that $g^R_{y'}(w) \geq g^{\hat{R}}_{y'}(w)$ (and the other two relations also hold for n replaced with $\hat{R}$).

Step 1: Let $n = 1$ and fix $w \in [\underline{w}, \bar{w}]$

Suppose, to obtain a contradiction, that there exists y' such that $g^1_{y'}(w) > g^R_{y'}(w) \geq U^{Aut}(y')$. Then from the respective first order conditions (note that $V^1 = T_{\hat{R}} V_R$)

$$\begin{aligned} &\text{(A24)}\quad V'_R(g^1_{y'}(w)) \quad = \quad \frac{\beta(\hat{R}-1)}{1-\beta} C'(h^1(w)) \\ &\text{(A25)}\quad V'_R(g^R_{y'}(w)) \quad \geq \quad \frac{\beta(R-1)}{1-\beta} C'(h^R(w)) \end{aligned}$$

Since V_R is strictly convex $V'_R(g^1_{y'}(w)) > V'_R(g^R_{y'}(w))$ and hence (as $R > \hat{R}$), $h^1(w) > h^R(w)$. From the promise keeping constraint there must exist $\bar{y}'$ such that $g^R_{\bar{y}'}(w) > g^1_{\bar{y}'}(w) \geq U^{Aut}(\bar{y}')$.

But then (using A24 and A25)

$$V_R'(g_{\bar{y}'}^R(w)) = \frac{\beta(R-1)}{1-\beta}C'(h^R(w)) < \frac{\beta(\hat{R}-1)}{1-\beta}C'(h^1(w)) \leq V_R'(g_{\bar{y}'}^1(w))$$

which implies $g_{\bar{y}'}^R(w) < g_{\bar{y}'}^1(w)$, a contradiction. Hence $g_{y'}^1(w) \leq g_{y'}^R(w)$, for all $y' \in Y$. Then from the promise keeping constraint $h^1(w) \geq h^R(w)$. The envelope conditions are

$$\begin{aligned} \frac{(V^1)'(w)}{\hat{R}-1} &= \frac{C'(h^1(w))}{\hat{R}(1-\beta)} \\ \frac{V_R'(w)}{R-1} &= \frac{C'(h^R(w))}{R(1-\beta)} \end{aligned} \tag{A26}$$

It follows from the previous result that $\frac{(V^1)'(w)}{\hat{R}-1} \geq \frac{V_R'(w)}{R-1}$.

Step 2: Suppose that A21 to A23 are true for $n-1$. We want to show that A21 to A23 are true for n. Again suppose, to obtain a contradiction, that there exists y' such that $g_{y'}^n(w) > g_{y'}^R(w) \geq U^{Aut}(y')$. From the first order conditions we have

$$\begin{aligned} \frac{(V^{n-1})'(g_{y'}^n(w))}{\hat{R}-1} &= \frac{\beta}{1-\beta}C'(h^n(w)) \\ \frac{V_R'(g_{y'}^R(w))}{R-1} &\geq \frac{\beta}{1-\beta}C'(h^R(w)) \end{aligned}$$

Since V_R and V^{n-1} are convex, $g_{y'}^n(w) > g_{y'}^R(w)$ and A23 holds for $n-1$, we have that $h^n(w) > h^R(w)$. Again by the promise keeping constraints there exists $\bar{y}'$ such that $g_{\bar{y}'}^R(w) > g_{\bar{y}'}^n(w) \geq U^{Aut}(\bar{y}')$. But by Lemma 7.

$$\begin{aligned} g_{y'}^n(w) &\leq g_{\bar{y}'}^n(w) \\ g_{y'}^R(w) &\geq g_{\bar{y}'}^R(w) \end{aligned}$$

and hence

$$g_{y'}^n(w) \leq g_{\bar{y}'}^n(w) < g_{\bar{y}'}^R(w) \leq g_{y'}^R(w) < g_{y'}^n(w)$$

a contradiction. It follows that for all $y' \in Y$, $g_{y'}^n(w) \leq g_{y'}^R(w)$. From promise keeping we have $h^n(w) \geq h^R(w)$. As before the envelope conditions imply that $\frac{(V^n)'(w)}{\hat{R}-1} \geq \frac{V_R'(w)}{R-1}$ ■

Proof of Lemma 13.:

Define the sequence of measures $\{\Phi_n\}_{n=1}^{\infty}$ by $\Phi_n = \left(T^*_{\hat{R}}\right)^n \Phi_R$. We shall prove by induction that for each $n \geq 1$, and each $y \in Y$, Φ^y_R stochastically dominates Φ^y_n. Since by Theorem 4. $\{\Phi_n\}$ converges to $\Phi_{\hat{R}}$ in total variation norm, the result then follows.

It will be convenient to define the distribution function associated with any probability measure Φ^y_n, $F^y_n : W \to [0,1]$, as $F^y_n(w) = \Phi^y_n([\underline{w}, w]) = \Phi_n([\underline{w}, w], \{y\})/\pi(y)$. Since the domain of these functions is a subset of $\Re^1$, in order to prove that Φ^y_R stochastically dominates Φ^y_n it is sufficient to prove that for all $w \in W$, $F^y_R(w) \leq F^y_n(w)$.

Step 1: Let $n = 1$

By definition $\Phi_1 = T^*_{\hat{R}}\Phi_R$ whereas $\Phi_R = T^*_R\Phi_R$. Fix an arbitrary $y \in Y$, $w \in W$. Then

$$\begin{aligned} F^y_R(w) &= \frac{\Phi_R([\underline{w}, w], \{y\})}{\pi(y)} \\ &= \int_{\{v \in W | g^R_y(v) \leq w\}} d\Phi^y_R \\ &\leq \int_{\{v \in W | g^{\hat{R}}_y(v) \leq w\}} d\Phi^y_R \\ &= \frac{\Phi_1([\underline{w}, w], \{y\})}{\pi(y)} \\ &= F^y_1(w) \end{aligned} \tag{A27}$$

where the inequality is due to the fact that $g^R_y(w) \geq g^{\hat{R}}_y(w)$, for all $w \in W$.

Step 2: Suppose $F^y_R(w) \leq F^y_{n-1}(w)$, for all $w \in W$, all $y \in Y$. We want to show that the same is true for n. Note that

$$\begin{aligned} F^y_n(w) &= \frac{\Phi_n([\underline{w}, w], \{y\})}{\pi(y)} \\ &= \int_{\{v \in W | g^{\hat{R}}_y(v) \leq w\}} d\Phi^y_{n-1} \\ &= \sum_{\bar{y} \in Y} \pi(\bar{y}) F^{\bar{y}}_{n-1}(v_n) \end{aligned} \tag{A28}$$

where $v_n := \max\{v \in W | g^{\hat{R}}_y(v) \leq w\}$. Note that the last equality requires $g^{\hat{R}}_y$ to be increasing in v as shown in Lemma 6. Continuity of $g^{\hat{R}}_y$ ensures that v_n is well-defined. Similarly $F^y_R(w) = \sum_{\bar{y} \in Y} \pi(\bar{y}) F^{\bar{y}}_R(v_R)$ with $v_R := \max\{v \in W | g^R_y(v) \leq w\}$. Lemma 12. implies that $v_R \leq v_n$. Then the induction hypothesis implies that for all $\bar{y} \in Y$, $F^{\bar{y}}_R(v_R) \leq F^{\bar{y}}_{n-1}(v_n)$, and hence $F^y_R(w) \leq F^y_n(w)$ ■

Proof of Theorem 6.:

By definition of $d(R)$

$$d(R) = \int V_R(w) d\Phi_R - \int y d\Phi_R$$

Since for all R, $\int y d\Phi_R$ is a constant, we focus on the analysis of the first part of the excess demand function. From the functional equation

$$\text{(A29)} \quad \int V_R(w) d\Phi_R = \left(1 - \frac{1}{R}\right) \int C(h^R(w)) d\Phi_R + \frac{1}{R} \sum_{y'} \pi(y') \int V(g^R_{y'}(w)) d\Phi_R$$

we note that by stationarity and the definition of Φ^y_R,

$$\text{(A30)} \quad \int V_R(w) d\Phi_R = \sum_{y \in Y} \pi(y) \int V_R(w) d\Phi^y_R$$

$$\text{(A31)} \quad \int V(g^R_{y'}(w)) d\Phi_R = \int V(w) d\Phi^{y'}_R$$

so that

$$\text{(A32)} \quad \int V_R(w) d\Phi_R = \sum_{y'} \pi(y') \int V(g^R_{y'}(w)) d\Phi_R$$

It follows that

$$\text{(A33)} \quad \int V_R(w) d\Phi_R = \int C(h^R(w)) d\Phi_R.$$

We want to prove that

$$\int V_R(w) d\Phi_R \geq \int V_{\hat{R}}(w) d\Phi_{\hat{R}}$$

By the previous lemma for all $y \in Y$, Φ^y_R stochastically dominates $\Phi^y_{\hat{R}}$, and since $V_{\hat{R}}$ is strictly increasing it follows (using A30 that

$$\text{(A34)} \quad \int V_{\hat{R}}(w) d\Phi_R \geq \int V_{\hat{R}}(w) d\Phi_{\hat{R}}$$

So if we can prove that

$$\text{(A35)} \quad \int V_R(w) d\Phi_R \geq \int V_{\hat{R}}(w) d\Phi_R$$

we are done. Define the sequence $\{V^n\}_{n=1}^{\infty}$ by $V^n = (T_{\hat{R}})^n V_R$. We will prove by induction that for all $n \geq 1$

$$\int V_R(w) d\Phi_R \geq \int V^n(w) d\Phi_R$$

Since the sequence $\{V^n\}_{n=1}^{\infty}$ converges uniformly to $V_{\hat{R}}$ (by corollary 1.), this proves A35.

Let $\{h^n, g^n_{y'}\}_{n=1}^{\infty}$ be the optimal policies associated with $\{V^n\}_{n=1}^{\infty}$ and $(h^R, g^R_{y'})$ be the optimal choices associated with V_R.

Step 1: Let $n = 1$.

By definition $V^1 = T_{\hat{R}} V_R$. Hence

$$\begin{aligned} V^1(w) &= \left(1 - \frac{1}{\hat{R}}\right) C(h^1(w)) + \frac{1}{\hat{R}} \sum \pi(y') V_R(g^1_{y'}(w)) \\ &\leq \left(1 - \frac{1}{\hat{R}}\right) C(h^R(w)) + \frac{1}{\hat{R}} \sum \pi(y') V_R(g^R_{y'}(w)) \end{aligned} \tag{A36}$$

since $(h^1, g^1_{y'})$ are the minimizing choices associated with V^1. Integrating with respect to Φ_R and using A33 and A32 yields

$$\begin{aligned} \int V^1(w) d\Phi_R &= \left(1 - \frac{1}{\hat{R}}\right) \int C(h^1(w)) d\Phi_R + \frac{1}{\hat{R}} \sum \pi(y') \int V_R(g^1_{y'}(w)) d\Phi_R \\ &\leq \left(1 - \frac{1}{\hat{R}}\right) \int C(h^R(w)) d\Phi_R + \frac{1}{\hat{R}} \sum \pi(y') \int V_R(g^R_{y'}(w)) d\Phi_R \\ &= \left(1 - \frac{1}{\hat{R}}\right) \int V_R(w) d\Phi_R + \frac{1}{\hat{R}} \sum \pi(y') \int V_R(g^R_{y'}(w)) d\Phi_R \\ &= \int V_R(w) d\Phi_R \end{aligned} \tag{A37}$$

Step 2: Suppose $\int V_R(w) d\Phi_R \geq \int V^{n-1}(w) d\Phi_R$. We want to show that the same is true for n. By definition $V^n = T_{\hat{R}} V^{n-1}$, hence

$$\begin{aligned} V^n(w) &= \left(1 - \frac{1}{\hat{R}}\right) C(h^n(w)) + \frac{1}{\hat{R}} \sum \pi(y') V^{n-1}(g^n_{y'}(w)) \\ &\leq \left(1 - \frac{1}{\hat{R}}\right) C(h^R(w)) + \frac{1}{\hat{R}} \sum \pi(y') V^{n-1}(g^R_{y'}(w)) \end{aligned} \tag{A38}$$

by the same reason as in step 1. Again integrating with respect to Φ_R and using A33 and A32 we obtain

$$\begin{aligned} \int V^n(w) d\Phi_R &= \left(1 - \frac{1}{\hat{R}}\right) \int C(h^n(w)) d\Phi_R + \frac{1}{\hat{R}} \sum \pi(y') \int V^{n-1}(g^n_{y'}(w)) d\Phi_R \\ &\leq \left(1 - \frac{1}{\hat{R}}\right) \int C(h^R(w)) d\Phi_R + \frac{1}{\hat{R}} \sum \pi(y') \int V^{n-1}(g^R_{y'}(w)) d\Phi_R \\ &= \left(1 - \frac{1}{\hat{R}}\right) \int V_R(w) d\Phi_R + \frac{1}{\hat{R}} \int V^{n-1}(w) d\Phi_R \end{aligned}$$

$$
\begin{aligned}
&\leq \left(1-\frac{1}{\hat{R}}\right)\int V_R(w)d\Phi_R + \frac{1}{\hat{R}}\int V_R(w)d\Phi_R \\
&= \int V_R(w)d\Phi_R \qquad \text{(A39)}
\end{aligned}
$$

where the last inequality uses the induction hypothesis ■