Duality Approaches to Economic Lot-Sizing Games*

Xin Chen† and Jiawei Zhang‡

First Version: October 7th, 2006
Revised: May 8nd, 2007

Abstract

We consider the economic lot-sizing (ELS) game with general concave ordering cost. In this cooperative game, multiple retailers form a coalition by placing joint orders to a single supplier in order to reduce ordering cost. When both the inventory holding cost and backlogging cost are linear functions, it can be shown that the core of this game is non-empty. The main contribution of this paper is to show that a core allocation can be computed in polynomial time.

Our approach is based on linear programming (LP) duality and is motivated by the work of Owen [19]. We suggest an integer programming formulation for the ELS problem and show that its LP relaxation admits zero integrality gap, which makes it possible to analyze the ELS game by using LP duality. We show that, there exists an optimal dual solution that defines an allocation in the core.

An interesting feature of our approach is that it is not necessarily true that every optimal dual solution defines a core allocation. This is in contrast to the duality approach for other known cooperative games in the literature.

*Submission to Operations Research. Area of Review: Optimization.

†Department of Industrial and Enterprise Systems Engineering, University of Illinois at Urbana-Champaign. Email: xinchen@uiuc.edu

‡Stern School of Business, IOMS-Operations Management, New York University, New York, NY, 10012. Email: jzhang@stern.nyu.edu

1 Introduction

In the basic economic lot-sizing model (ELS), a retailer is facing a demand for a single product that occurs during each of a consecutive time periods, which can be satisfied by orders at that period or at pervious periods. The objective is to decide the order quantity at each time period so that the demand is satisfied at a minimum total cost, including ordering cost and inventory holding cost.

In a slightly more general model, referred to as *ELS with backlogging*, demand of a period can be backlogged and be fulfilled by orders at later periods. At any period, the unfulfilled demand incurs penalty cost, called backlogging cost. Throughout the paper, we assume that backlogging is allowed.

We consider a situation where multiple retailers sell the same product, which is ordered from a single manufacturer. In a decentralized system, each retailer would solve an ELS problem. However, by exploiting economies of scale, the retailers may find it beneficial to form coalitions and place joint orders. One important issue that we are concerned with is how to allocate the cost or profit in such a way that is considered advantageous by all the retailers, i.e., no retailer(s) gain more by deviating from the cooperation. This naturally gives rise to a cooperative game, referred to as the ELS game, and the cost allocation we are interested in can be studied by using concepts from the cooperative game theory. In this paper, we mainly focus on the core of the ELS game.

A special case of the ELS game, where backlogging is not allowed and the ordering cost includes a fixed setup cost and a linear cost, has been recently analyzed by van den Heuvel et al. [30]. Specifically, they showed for this special case that the core is always nonempty by invoking the well-known Bondareva-Shapley theorem [2, 24]. They also studied the concavity of the ELS game.

In this paper, we focus on core allocations of the ELS game under general conditions. Specifically, our model allows for backlogging and general concave ordering cost. We prove the existence of a core allocation for the general ELS game. Our main contribution is to show that an allocation in the core can be computed in polynomial time by solving a linear program. We would like to point out that this computational issue is not addressed by van den Heuvel et al. [30], as the Bondareva-Shapley theorem, to some extent, provides only an *existence* proof of a core allocation.

Our approach is based on linear programming (LP) duality, and is inspired by the work of Owen [19], who used LP duality to show the non-emptiness of the core for a class of linear production games. Owen's approach has been applied and/or extended to other cooperative games; see, for instance, Granot [10], Tamir [28], and Goemans and Skutella [9]. One scheme commonly seen in all these approaches is that one first formulates the underlying optimization problem of the cooperative game as a linear program, and then use the dual variables to define an allocation that can be proven in the core. In fact, for all these games, the set of allocations defined by optimal dual variables, often referred to as Owen set, is always a subset of the core (see Samet and Zemel [23] for the relationship between Owen set and the core).

One may formulate the ELS problem with concave ordering cost as a facility location problem. The LP relaxation of such a formulation always has an integral solution, i.e., there is no integrality gap between the integer program and its LP relaxation. This fact allows us to show the existence of a core allocation of the ELS game by directly applying Owen's approach. However, the size of the LP relaxation is not necessarily polynomial in the input of the game. Thus, this approach does not provide a polynomial time algorithm for finding a core allocation of the ELS game with general concave ordering cost. Instead, we suggest an *alternative* integer programming formulation for the ELS problem whose LP relaxation always has an integral solution as well.

We would like to point out an interesting feature of our approach, that is in contrast to the duality approach for other known cooperative games in the literature. On the one hand, allocations defined by some optimal solutions to the dual of the LP relaxation may not be in the core of the ELS game. On the other hand, there always exists an optimal dual solution that defines an allocation in the core, which can be found in polynomial time.

Our analysis is based on the fact that there always exists an optimal dual solution that has certain monotonicity. Such a property is quite intuitive and might be interesting on its own.

The topic of this paper also falls into a stream of recent research on applying cooperative game theory in the area of inventory management. One type of such games is related to inventory centralization under demand uncertainty [11, 12]. In the simplest model, which is called the newsvendor game [13], we consider a set of retailers, who face random demands of a single product. The retailers place a joint order before observing the demands, and after the demands are realized, the inventory is optimally allocated to the retailers. Hartman et al. [13] first show that under certain condition of the demand distribution, the newsvendor game has a non-empty core. This result is subsequently generalized by a series of recent papers: by Müller et al. [18], Slikker et al. [25], Slikker et al. [26], and Ozen et al. [20]. Finally, Chen and Zhang [5] unified and generalized these existence results by using duality of stochastic linear programming, and suggest a way to compute an allocation in the core.

Another type of inventory games has been studied by Meca et al. [17], which is closely related to the game studied in our paper. In both games, the retailers cooperate in order to reduce ordering cost. The major difference is that, in the game studied in [17], the underlying optimization problem is the Economic Ordering Quantity (EOQ) model, while in ours it is the ELS model. More general models have recently been studied by Dror and Hartman [7], and Anily and Haviv [1].

2 Preliminaries

2.1 Cooperative Games

Here we briefly introduce some basic concepts of cooperative game theory that will be used in this paper; see Peleg and Sudhölter [21] for more details. Let $N = \{1, 2 \cdots, n\}$ be the set of players. A collection of players $S \subseteq N$ is called a coalition. The set N is sometimes referred to as the grand coalition. A characteristic cost function $F(S)$ is defined for each coalition $S \subseteq N$, which could be the minimum total cost that coalition S should pay if the members of S decide to secede from the grand coalition and cooperate only among themselves. A cooperative game is determined by the pair (N, F). For each subset $S \subset N$, the cooperative game (S, F) is called a subgame.

In cooperative games with transferable cost, the cost of a coalition S can be transferred between the players of S. In such games a coalition S can be completely characterized by $F(S)$. The coalition is allowed to split the cost $F(S)$ among its members in any possible way.

A vector $l = (l_1, l_2, \cdots, l_N)$ is called an allocation for the game (N, F) if $\sum_{j \in N} l_j = F(N)$. The core of a cooperative game is a solution concept which requires that no subset of players has an incentive to secede.

Definition 1. *An allocation l is in the* core *of the game (N, F), if $\sum_{j \in N} l_j = F(N)$ and for any subset $S \subseteq N$, $\sum_{j \in S} l_j \leq F(S)$.*

There are several interesting questions related to the core of a cooperative game (N, F). In particular, we would like to know whether the core of (N, F) is non-empty or not, and if yes, how to design an algorithm to find a core allocation efficiently. The first question may be answered by the Bondareva-Shapely theorem, which we introduce below.

We call any collection $\mathcal{B}$ of coalitions *balanced* if there exists a vector of positive weights $(\lambda_S : S \in \mathcal{B})$ such that

$$\sum_{S \in \mathcal{B}: i \in S} \lambda_S = 1 \quad \forall i \in N.$$

Here $(\lambda_S : S \in \mathcal{B})$ is referred to as a vector of balancing weights. We call the game (N, F) *balanced* if for any balanced collection of coalitions $\mathcal{B}$ and the corresponding balancing weights $(\lambda_S : S \in \mathcal{B})$, we have

$$\sum_{S \in \mathcal{B}} \lambda_S F(S) \geq F(N).$$

It is that, due to Bondareva [2] and Shapley [24], a cooperative game has a non-empty core if and only if it is balanced. Thus, in order to show a cooperative game has a non-empty core, it is sufficient to show the game is balanced by invoking the Bondareva-Shapley theorem. However, in general this approach does not provide an efficient way to compute a core allocation. Finding core allocations efficiently often gives rise to interesting algorithmic problems.

Another important concept in game is concavity. A cooperative game (N, F) is called a concave game if for every pair of subsets $S, T \subseteq N$, $F(S) + F(T) \geq F(S \cup T) + F(S \cap T)$. It is well-known that the core of a concave game is always non-empty. Also, a core allocation of a concave game can be computed in polynomial time by using ellipsoid algorithm for linear programming. Moreover, every concave game has a population monotonic allocation scheme, which is defined as follows.

Definition 2. *(adopted from Sprumont [27]) A vector $l = (l_i^S)_{i \in S, S \subset N}$ is a* population monotonic allocation scheme *of the cooperative game (N, F) if and only if it satisfies the following conditions:*

- *the vector $(l_i^S)_{i \in S}$ is in the core of the subgame (S, V), and*
- *for all $S \subset T \subset N$, $l_i^S \geq l_i^T$.*

Intuitively, a population monotonic allocation scheme gives an allocation vector for each subgame such that the cost allocation to each individual participating the subgame is non-increasing with the set of players participating in the subgame.

For non-concave game, a population monotonic allocation scheme may still exist. But finding such an allocation could be very challenging.

2.2 The Economic Lot-Sizing Problem

The basic economic lot-sizing model was proposed by Manne [16] and Wagner and Whitin [31]. In this model, demand for a single product occurs during each of T consecutive time periods numbered through 1 to T. The demand of a given time period can be satisfied by orders at that period or at pervious periods, i.e., back-logging is not allowed. The model includes ordering cost and inventory holding cost. The objective is to decide the order quantity at each time period so that the demand is satisfied at a minimum total cost.

Without loss of generality, we assume that the initial inventory is zero and lead time is also zero.

The ELS model considered in this paper was proposed by Zangwill [33], which allows that the demand of a period to be backlogged for some periods, provided that it is satisfied eventually by orders at subsequent periods. At any period, the unfulfilled demand incurs penalty cost refereed to as backlogging cost.

In order to present a mathematical formulation for the economic lot-sizing problem, we introduce the following notations. In particular, for each $t : 1 \le t \le T$, define

- d_t: the amount of demand in period t, which is assumed to be an integer;
- z_t: the order quantity at period t, which is called an ordering period if $z_t > 0$;
- I_t^+: the amount of (non-negative) inventory at the end of period t;
- I_t^-: the amount of backlogged demand at period t;
- $c_t(z_t)$: the cost of ordering z_t units at period t;
- h_t^+: the unit cost of holding inventory at the end of period t;
- h_t^-: the unit cost of having backlogged demand at period t.

The following mathematical formulation is sometimes called the flow formulation for the economic lot-sizing problem:

$$\begin{aligned} C(d) := \quad & \min \quad \sum_{1 \le t \le T} \left\{ c_t(z_t) + h_t^+ I_t^+ + h_t^- I_t^- \right\} \\ & \text{s.t.} \quad z_t + I_{t-1}^+ - I_{t-1}^- = d_t + I_t^+ - I_t^-, \forall\, t = 1, \ldots, T, \\ & \qquad\ \ I_0^+ = I_0^- = 0, \\ & \qquad\ \ z_t \ge 0, I_t^+ \ge 0, I_t^- \ge 0 \ \forall t = 1, \ldots, T, \end{aligned} \tag{1}$$

where $d = (d_1, d_2, \ldots, d_T)^T$ is a vector in R^T, and the first constraint is the inventory balance equation.

We assume that the ordering cost function $c_t(\cdot)$ is non-decreasing and concave with $c_t(0) = 0$. Under the concavity assumption, problem (1) is a concave minimization problem over a polyhedron. Therefore, there exists an optimal solution that is an extreme point of the polyhedron. It follows that, as proved by [33], there exists an optimal solution to (1) with the following properties:

a): $I^+_{t-1} > 0$ implies $z_t = 0$,

b): $z_t > 0$ implies $I^-_t = 0$, and

c): $I^+_{t-1} > 0$ implies $I^-_t = 0$.

Define any period for which $I^+_t = I^-_t = 0$ as a regeneration period. Then the above properties imply that there exists an optimal solution to (1) such that there is always a regeneration period between two ordering periods. An optimal solution with such structure can be found in polynomial time [33].

2.3 The Economic Lot-Sizing Game

We consider a set of retailers $N = \{1, 2, \cdots, n\}$, all of which sell the same product and face an economic lot-sizing problem. For each $i \in N$, let $d^i = (d^i_1, d^i_2, \ldots, d^i_T)$, where d^i_t is the known demand of retailer i in time period $t : 1 \leq t \leq T$. We assume that the ordering cost, inventory holding cost, and backlogging cost are independent of retailers.

The retailers can place orders individually, i.e., each of them solves an economic lot-sizing problem separately to satisfy its own demand. They can also cooperate by placing joint orders and keeping inventory at a central warehouse. For a collection of retailers $S \subset N$, the goal is to minimize the total cost for the coalition, while the aggregated demand being satisfied.

As the ordering cost is a concave function of the ordering quantity, it is not hard to see that it leads to cost reduction if the retailers place joint orders. Therefore, a cooperative game can be defined in this setting. It is called the economic lot-sizing game (N, F).

In this game, the grand coalition is the set of retailers N. For each subset $S \subseteq N$, the characteristic cost function $F(S)$ is defined by $F(S) = C(d^S)$, where $C(\cdot)$ is defined by optimization problem (1), $d^S = (d_1^S, d_2^S, \ldots, d_T^S)$, and $d_t^S = \sum_{i \in S} d_t^i$ for each $1 \leq t \leq T$.

Although our main result for the economic lot sizing game is concerned with the case where $c_t(\cdot)$ is a general concave function, we are also interested in an important special class of the economic lot sizing games, referred to as ELS games with setup cost. In this class of games, the ordering cost function has a fixed cost component and a variable cost component, that is, $c_t(z_t) = K_t\delta(z_t) + c_t z_t$ where

$$\delta(x) = \begin{cases} 1, \text{if } \ x > 0; \\ 0, \text{otherwise.} \end{cases} \tag{2}$$

Notice that this cost function is also commonly used in the literature on inventory theory.

2.4 Concavity of the Economic Lot-Sizing Game

Recall that we are interested in proving the non-emptiness of the core and finding an allocation in the core. It is well known that, for a concave game, the core is always nonempty, there exist polynomial time algorithms to find an allocation in the core, and to check whether an allocation is in the core or not. In other words, we have satisfactory answers to the questions of interest when a cooperative game is concave. Therefore, it is natural to ask whether the economic lot sizing game (N, F) is concave or not.

van den Heuvel et al. [30] has shown that the economic lot-sizing game with setup cost (not general concave ordering cost) is concave when $T = 2$ and is not concave when $T \geq 5$. We strengthen their result in the following theorem.

Proposition 1. *The economic lot-sizing game (N, F) is not concave when $T \geq 3$.*

Proof. It suffices to construct an example with $T = 3$ such that $F(\cdot)$ is not submodular. In the following example, there are three players, i.e., $N = \{1, 2, 3\}$. The demands of the players are given as

$$d^1 = (1, 1, 1), \quad d^2 = (0, \epsilon, 0), \quad \text{and} \quad d^3 = (0, 0, \epsilon).$$

We assume backlogging is not allowed. Also, we let the holding cost for every period be zero, and the ordering costs are defined by (2), where

$$K_1 = K_2 = K_3 = 3/2, c_1 = 2, c_2 = 1, \text{ and } c_3 = 0.$$

Under this cost structure, it is easy to verify that, for any demand (d_1, d_2, d_3) with $d_1 > 0$, the optimal cost is

$$\begin{aligned} & C(d_1, d_2, d_3) \\ = \;& K_1 + c_1 d_1 + \\ & \min\left(c_1(d_2 + d_3), K_2 + c_2(d_2 + d_3), c_1 d_2 + K_3 + c_3 d_3, K_2 + c_2 d_2 + K_3 + c_3 d_3\right) \\ = \;& 3/2 + 2d_1 + \min\left(2(d_2 + d_3), 3/2 + (d_2 + d_3), 3/2 + 2d_2, 3 + d_2\right). \end{aligned}$$

Therefore, for $\epsilon > 0$ sufficiently small, we have

$$\begin{aligned} & F(\{1\}) = C(1, 1, 1) = 7 \\ & F(\{1, 2\}) = C(1, 1 + \epsilon, 1) = 7 + \epsilon \\ & F(\{1, 3\}) = C(1, 1, 1 + \epsilon) = 7 \\ & F(\{1, 2, 3\}) = C(1, 1 + \epsilon, 1 + \epsilon) = 7 + 2\epsilon, \end{aligned}$$

which follows that

$$F(\{1\}) + F(\{1, 2, 3\}) > F(\{1, 2\}) + F(\{1, 3\}).$$

Thus, the game (V, N) is not concave. $\square$

3 Existence of a Core Allocation

A necessary and sufficient condition for the non-emptiness of the core for a cooperative game is given by the famous Bondareva-Shapley Theorem (see Peleg and Sudhölter [21]). van den Heuvel et al. [30] apply this theorem to prove the core of the economic lot sizing game with setup cost is nonempty. Contrary to the approach employed in van den Heuvel et al. [30], we utilize the linear programming (LP) duality to show that the core of the economic lot sizing game with general concave ordering cost function is nonempty.

We first focus on the ELS games with setup cost, that is, the ordering cost function is defined as $c_t(z_t) = K_t\delta(z_t) + c_t z_t$. In this case, it is well-known that the ELS problem can be formulated as a facility location problem (see, for instance, Krarup and Bilde [14] for the case without backlogging, and Pochet and Wolsey [22] for the case with backlogging).

Thus, for the ELS game with setup cost, the characteristic cost function $F(S)$, for any coalition S, is the optimal value of the following integer program:

$$\begin{aligned} F(S) := \quad \min \quad & \sum_{1\le t\le T}\sum_{1\le\tau\le T} d_\tau^S p_{t\tau}\lambda_{t\tau} + \sum_{1\le t\le T} K_t y_t \\ \text{s.t.} \quad & d_\tau^S \sum_{1\le t\le T} \lambda_{t\tau} = d_\tau^S, \quad \forall \tau = 1,\ldots,T, \\ & \lambda_{t\tau} \le y_t, \quad \forall 1 \le t,\tau \le T, \\ & \lambda_{t\tau}, y_t \in \{0,1\}, \quad \forall 1 \le t,\tau \le T, \end{aligned} \tag{3}$$

where the coefficient $p_{t\tau}$ is the cost of satisfying one unit demand at period τ by ordering at period t, i.e., $p_{t\tau} = c_t + \sum_{i=t}^{\tau-1} h_t^+$ if $t \le \tau$, and $p_{t\tau} = c_t + \sum_{i=\tau}^{t-1} h_t^-$ if $t > \tau$, the binary indicator variable $\lambda_{t\tau} = 1$ if and only if the demand at period τ is satisfied by the inventory ordered at period t, and $y_t = 1$ if and only if an order is placed at period t.

In the LP relaxation of problem (3), variables $\lambda_{t\tau}$ and y_t are required to be non-negative, rather than integral. It is well known that this LP relaxation has an optimal solution that is integral (see Pochet and Wolsey [22]). This immediately leads to Theorem 1 below. The proof of Theorem 1 is straightforward and is presented here for completeness.

For the grand coalition, i.e., when $S = N$, the dual of the LP relaxation of problem (3) is

$$\begin{aligned} F(N) := \quad \max \quad & \sum_{1\le\tau\le T} d_\tau^N b_\tau \\ \text{s.t.} \quad & \sum_{1\le\tau\le T} d_\tau^N \beta_{t\tau} \le K_t, \quad \forall t = 1,\ldots,T, \\ & b_\tau - \beta_{t\tau} \le p_{t\tau}, \quad \forall 1 \le t,\tau \le T, \\ & \beta_{t\tau} \ge 0, \quad \forall 1 \le t,\tau \le T. \end{aligned} \tag{4}$$

Then we have

Theorem 1. *Assume that $\bar{b} = (\bar{b}_1, \cdots, \bar{b}_T)$ is an optimal solution to dual (4), then the* *allocation $l = (l_1, \cdots, l_N)$ defined by*

$$l_k = \sum_{t=1}^{T} \bar{b}_t d_t^k$$

is in the core of the ELS game (N, F) with setup cost. Therefore, a core allocation of the ELS game (N, F) with setup cost can be computed in polynomial time.

Proof. . By definition,

$$\sum_{k \in N} l_k = \sum_{1 \le \tau \le T} d_\tau^N \bar{b}_\tau = F(N),$$

Also, notice that the optimal solution $\bar{b} = (\bar{b}_1, \cdots, \bar{b}_T)$ to dual (4) is clearly feasible for the dual of the LP relaxation of problem (3), which is the same as problem (4) with d_τ^N being replaced by d_τ^S. Therefore, for any coalition S,

$$\sum_{k \in S} l_k = \sum_{1 \le \tau \le T} d_\tau^S \bar{b}_\tau \le F(S),$$

which implies that $l = (l_1, \cdots, l_N)$ is in the core of the economic lot-sizing game (N, F) with setup cost. Obviously, this core allocation can be computed in polynomial time. $\square$

Now we consider the ELS game with general concave ordering cost functions $c_t(\cdot)$, $1 \le t \le T$. For each $t : 1 \le t \le T$ and each $j = 0, 1, 2, \cdots$, define

$$c_{t(j)} = c_t(j+1) - c_t(j) \quad \text{and} \quad K_{t(j)} = c_t(j) - c_{t(j)} j$$

Then it is clear that, by the concavity of $c_t(\cdot)$,

$$c_t(z) = \min_{j=0,1,2,\cdots} \{K_{t(j)} + c_{t(j)} z\}.$$

Thus, for the ELS game with general concave ordering cost, the characteristic cost function $F(S)$, for any coalition S, is the optimal value of the following integer program:

$$\begin{aligned} F(S) := \quad \min \quad & \sum_{1 \le t \le T} \sum_{j \in OP} \sum_{1 \le \tau \le T} d_\tau^S p_{t(j),\tau} \lambda_{t(j),\tau} + \sum_{1 \le t \le T} \sum_{j \in OP} K_{t(j)} y_{t(j)} \\ \text{s.t.} \quad & d_\tau^S \sum_{1 \le t \le T} \sum_{j \in OP} \lambda_{t(j),\tau} = d_\tau^S, \quad \forall \tau = 1, \ldots, T, \\ & \lambda_{t(j),\tau} \le y_t, \quad \forall 1 \le t, \tau \le T, j \in OP \\ & \lambda_{t(j),\tau}, y_t \in \{0, 1\}, \quad \forall 1 \le t, \tau \le T, j \in OP \end{aligned} \tag{5}$$

where $OP = \{j | j = 1, 2, \cdots, D^N = \sum_{t=1}^{T} \sum_{i \in N} d_t^i\}$, $p_{t(j),\tau} = c_{t(j)} + \sum_{i=t}^{\tau-1} h_t^+$ if $t \leq \tau$, and $p_{t(j),\tau} = c_{t(j)} + \sum_{i=\tau}^{t-1} h_t^-$ if $t > \tau$.

Therefore, we conclude that the ELS game with general concave ordering cost can be reduced to the ELS game with setup cost. (It is known that a facility locaiton problem with concave facility cost can be reduced to a facility location problem with setup cost; see, for instance, Mahdian, Ye, and Zhang [15]). Theorem 1 implies that

Corollary 1. *The core of the ELS game with general concave ordering cost is non-empty.*

Remark 1. We studied the ELS game by using a facility location formulation for the ELS problem. This relates the ELS game to the cooperative facility location game studied by Goemans and Skutella [9]. Although these two games are closely related, we would like to point out one difference between them. Roughly speaking, in the facility location game, each demand point is a player of the game, while in the ELS game, each player may have many demand points (demands of T periods).

Remark 2. Notice that in formulation (5), both the number of variables and the number of constraints depend on the size of OP, which is not necessarily polynomial in the number of periods T. Therefore, formulation (5) only gives us a pseudopolynomial time algorithm for computing a core allocation for the ELS game with general concave cost. The natural question is that is it possible to reduce the size of OP? For the ELS problem with demand $d = (d_1, \cdots, d_T)$, it is known that there exists an optimization solution so that the size of each order is equal to the total demand of a number of consecutive periods. This implies that for each concave function $c_t(z_t)$, we are only interested in its values for $z_t \in V := \{z | z = \sum_{t=i}^{j} d_t \quad \text{for} \quad 1 \leq i \leq j \leq T$. It is clear that the size of $V(d)$ is $O(T^2)$. Thus, for each coalition S of retailers, the size of OP in formulation (5) can be reduced to $O(T^2)$. However, notice that the set $V(d)$ is dependent on the demand vector $d = (d_1, \cdots, d_T)$. Thus, for each coalition S, the set $V(d^S)$ is dependent on S, so is the set OP. That is to say, if we solve the dual problem (5) for the grand coalition N, with the size of OP being polynomial in T, it is not necessarily true that any optimal solution is feasible to the dual problem (5) for a coalition $S \subset N$. Thus, the proof of Theorem 1 does not go through.

To summarize, in this section, we have shown the existence of a core allocation for the ELS game with general concave ordering cost. However, finding a core allocation in polynomial time still remains to be a challenging. This motivates us to take a different approach, which will be discussed in the next section.

4 Computing a Core allocation

The main result of this section is to show that a core allocation for the ELS game with general concave ordering cost can be computed in polynomial time. Our algorithm is based on a new LP formulation for the ELS problem.

Here is the outline of our approach. First, we construct a natural $0-1$ integer programming formulation for problem (1) where $c_t(\cdot)$ is a general concave cost function, and show that an LP relaxation of this integer program has an optimal integral solution. The size of the LP relaxation is polynomial in the size of the input. Second, we construct the dual of this LP relaxation and use the optimal dual solutions of the dual to define allocations of the economic lot sizing game. Third, we illustrate through counterexamples that these allocations may not be in the core of the economic lot sizing game. Fourth, building upon the insight gained from the counterexamples, we strengthen the dual by adding additional inequalities, and show that allocations derived from the optimal solutions of strengthened dual are in the core of the economic lot sizing game. It then follows that a core allocation can be computed by solving an linear program whose size is polynomial in the input of the game.

Before presenting the natural integer programming formulation, we introduce some notations. For any $1 \leq i \leq j \leq T$, we define

$$d_{ij} = \sum_{t=i}^{j} d_t,$$

$$C_{ij}(d) = \min_{l:i\le l\le j}\left\{c_l(d_{ij}) + \sum_{t=i}^{l-1} h_t^- d_{it} + \sum_{t=l+1}^{j} h_{t-1}^+ d_{tj}\right\}, \tag{6}$$

and l_{ij} be an optimal solution to (6). The integer program is then formulated as follows.

$$\begin{aligned} \min \quad & \sum_{1\le i\le j\le T} C_{ij}(d)x_{ij} \\ \text{s.t.} \quad & d_l \sum_{i\le l\le j} x_{ij} = d_l, \forall\, l = 1,\ldots,T, \\ & x_{ij} \in \{0,1\}. \end{aligned} \tag{7}$$

Proposition 2. *If $c_t(\cdot)$ is a concave function for every $1 \le t \le T$, then problem (7) and problem (1) have the same optimal objective values.*

Proof. Assume that (z_t, I_t^+, I_t^-) is an optimal solution to problem (1) with the property that there is always a regeneration period between two ordering periods. Let

$$R = \{t : 0 \le t \le T, I_t^+ = I_t^- = 0\}$$

be the set of regeneration periods and for each $t \in R$, let $n(t)$ be the smallest integer in R such that $n(t) > t$ if it exists, otherwise let $n(t) = T + 1$. By assumption, for each $t \in R$, there exists exactly one ordering period between t and $n(t)$, denoted by $o(t)$. Then it is clear that the optimal objective value of problem (1) is

$$\sum_{i\in R}\left\{c_{o(i)}(d_{i+1,n(i)}) + \sum_{t=i+1}^{o(i)-1} h_t^- d_{it} + \sum_{t=o(i)+1}^{n(i)} h_{t-1}^+ d_{t,n(i)}\right\} \ge \sum_{i\in R} C_{i+1,n(i)}(d).$$

The right hand side of the above inequality can be represented as

$$\sum_{1\le i\le j\le T} C_{ij}(d)x_{ij},$$

where for each $1 \le i \le j \le T$,

$$x_{ij} = \begin{cases} 1, & \text{if } \; i-1 \in R \text{ and } j = n(i); \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $\sum_{i\le t\le j} x_{ij} = 1$ holds for every t. It follows that the optimal value of problem (7) is no more than $C(d)$.

On the other hand, assume that x_{ij} is an optimal solution to problem (7). For each pair (i,j) with $x_{ij}=1$, we let $\bar{t}=l_{ij}$ and define $z_{\bar{t}}=d_{ij}$, $I_k^-=d_{ik}$, $I_k^+=0$ for each $k\le \bar{t}-1$, and $I_k^-=0$, $I_k^+=d_{k+1,j}$ for each $\bar{t}\le k\le j-1$. It can be easily verified that (z_t, I_t^+, I_t^-) satisfies the constraints of problem (1), and the corresponding objective value is equal to $\sum_{1\le i\le j\le T} C_{ij}(d)x_{ij}$. This shows that $C(d)\le \sum_{1\le i\le j\le T} C_{ij}(d)x_{ij}$. The proof is complete. $\square$

We relax the $0-1$ constraints of problem (7) and obtain its LP relaxation as follows.

$$\begin{aligned} \min \quad & \sum_{1\le i\le j\le T} C_{ij}(d)x_{ij} \\ \text{s.t.} \quad & d_l \sum_{i\le l\le j} x_{ij} = d_l, \forall\ l=1,\ldots,T, \\ & x_{ij}\ge 0, \end{aligned} \tag{8}$$

whose dual is

$$\begin{aligned} \max \quad & \sum_{t=1}^{T} b_t d_t \\ \text{s.t.} \quad & \sum_{t=i}^{j} b_t d_t \le C_{ij}(d), \forall 1\le i\le j\le T. \end{aligned} \tag{9}$$

Here the dual variable b_t can be interpreted as the price of satisfying one unit demand at period t.

It turns out that the dual problem (9) admits a closed-form solution. In order to present this optimal dual solution, we introduce another notation. Given T and the demand vector $d=(d_1,\cdots,d_T)$, for each $1\le t\le T$, we defined a T dimensional vector $g^t(d)$ as follows:

$$g^t(d)_i = \begin{cases} d_i, & \text{if } \ 1\le i\le t; \\ 0, & \text{if } \ t+1\le i\le T. \end{cases}$$

It is clear that $g^T(d)=d$. For each $1\le t\le T$, we define

$$b_t^* = \frac{C(g^t(d)) - C(g^{t-1}(d))}{d_t}, \tag{10}$$

where the function $C(\cdot)$ is defined in the optimization problem (1). By definition, $C(g^t(d))$ is the minimum cost to satisfy the demand from period 1 to period t. It is clear that $b_t^*\ge 0$ for every $1\le t\le T$. The following result essentially implies that there exists an optimal $0-1$ solution to the LP problem (8).

Lemma 1. *The solution* (b_t^*) *defined by (10) is optimal to the dual problem (9), and the corresponding optimal dual value is equal to* $C(d)$*.*

Proof. First of all, we show that (b_t^*) is a feasible solution to problem (9). To that end, we notice that for any $1 \leq i \leq j \leq d$,

$$\begin{aligned} & \sum_{t=i}^{j} b_t^* d_t \\ = & \sum_{t=i}^{j} C(g^t(d)) - C(g^{t-1}(d)) \\ = & C(g^j(d)) - C(g^{i-1}(d)) \\ \leq & C_{ij}(d), \end{aligned}$$

where the last inequality holds since

$$C(g^j(d)) \leq C(g^{i-1}(d)) + C_{ij}(d), \tag{11}$$

which in turn follows from the fact that the right hand side of (11) is the cost of a feasible ordering policy to satisfy the demand from period 1 to period j (i.e., by ordering optimally to satisfy the demand from period 1 and period $i-1$, and ordering at a period in between i and j to satisfy the total demand from period i to period j), while the left hand side of (11) is the minimum cost to achieve that.

Moreover, the objective value of problem (9) associated with the solution (b_t^*) is

$$\sum_{t=1}^{T} b_t^* d_t = C(g^T(d)) = C(d),$$

which is the optimal objective value of the integer program (7). Then it follows from the weak duality of LP, (b_t^*) must be an optimal solution to problem (9) with an objective value of $C(d)$. $\square$

Now we solve the dual problem for the grand coalition, i.e., we solve the following problem, called grand dual problem,

$$\begin{aligned} \max \quad & \sum_{t=1}^{T} b_t d_t^N \\ \text{s.t.} \quad & \sum_{t=i}^{j} b_t d_t^N \le C_{ij}(d^N), \forall 1 \le i \le j \le T. \end{aligned} \tag{12}$$

Assume that $\bar{b} = (\bar{b}_1, \cdots, \bar{b}_T)$ is an optimal solution to problem (12). We define an allocation $l = (l_1, \cdots, l_N)$ as follows:

$$l_k = \sum_{t=1}^{T} \bar{b}_t d_t^k. \tag{13}$$

The grand dual problem may have multiple optimal solutions (see the following examples). Given the conceptual similarity of our approach to the linear programming duality approach to linear production games by Owen [19], it is legitimate to conjecture that l derived from any optimal dual solution $\bar{b} = (\bar{b}_1, \cdots, \bar{b}_T)$ is an allocation in the core. However, the following examples indicate that this is not the case for our economic lot sizing game. Interestingly, our examples illustrate that some of the optimal dual solutions do give rise to core allocations while some other optimal dual solutions do not.

Example 1. We consider the game with three periods and two players, i.e., $T = 3$ and $N = \{1, 2\}$. The demands of the players are given as

$$d^1 = (10, 0, 6), \quad d^2 = (0, 2, 0)$$

We assume that backlogging is not allowed. We let the holding cost for every period be zero, and the ordering costs are defined by $K_t \delta(z_t) + c_t z_t$, where

$$K_1 = 5, K_2 = 9, K_3 = 8, c_1 = 5, c_2 = 0, \text{ and } c_3 = 8.$$

It is easy to verify that $\bar{b} = (5.5, 4.5, 0)$ is an optimal solution to problem (12) with $d^N = (10, 2, 6)$ and the associated allocation $(55, 9)$ defined by (13) is in the core. Unfortunately, it can also be verified that $\bar{b} = (5.5, -23.5, 9\frac{1}{3})$ is an optimal solution to problem (12) which gives an allocation $(111, -47)$. Notice that if these two player does not cooperate, and player one orders at period 1 and period 2, then the cost that player one pays is 64 which is less than 111. Therefore, the allocation $(111, -47)$ is not in the core.

In Example 1, it seems that what makes the allocation $(111, -47)$ not in the core is the negativity of the dual variable $\bar{b}_2$. Lemma 1 suggests that there must exist a non-negative optimal solution to problem (12). However, as illustrated in Example 2, adding non-negativity constraints does not help in finding an allocation in the core.

Example 2. This example is the same as Example 1 except that we assume $c_2 = 1$ here. Then it is easy to verify that $\bar{b} = (5.5, 5, 1\frac{1}{6})$ is optimal to problem (12) and provides an allocation $(62, 10)$, which is in the core. However, similar to Example 1, $\bar{b} = (5.5, 0, 2\frac{5}{6})$ is optimal to problem (12) and defines an allocation $(72, 0)$, which is clearly not in the core.

The above examples show that, unlike many other cooperative games studied in the literature, we can not use an arbitrary dual optimal solution to find an allocation in the core. Fortunately, as we shall prove below, one of the optimal dual solutions to (12) does define an allocation in the core. To derive our main results, we notice that for every $i \leq j$, the dual constraint

$$\sum_{t=i}^{j} b_t d_t \leq C_{ij}(d) \tag{14}$$

is equivalent to

$$\sum_{t=i}^{j} b_t d_t \leq c_k(d_{ij}) + \sum_{t=i}^{k-1} h_t^- d_{it} + \sum_{t=k+1}^{j} h_{t-1}^+ d_{tj} \quad \forall k : i \leq k \leq j. \tag{15}$$

We can further simplify (15) by introducing some notations.

Given a vector $b = (b_1, \cdots, b_T)$, for any $1 \leq i \leq k \leq j \leq T$, define

$$b_t^-(k) = b_t - \sum_{l=t}^{k-1} h_l^- \quad \text{for each} \quad t \leq k,$$

and

$$b_t^+(k) = b_t - \sum_{l=k}^{t-1} h_l^+ \quad \text{for each} \quad t \geq k.$$

It is clear that $b_k^-(k) = b_k^+(k) = b_k$ for any $1 \leq k \leq T$. Then it is straightforward to verify that, (15), and thus (14) is equivalent to

$$\sum_{t=i}^{k-1} b_t^-(k) d_t + \sum_{t=k}^{j} b_t^+(k) d_t \leq c_k(d_{ij}) \quad \forall k : i \leq k \leq j. \tag{16}$$

Lemma 2. *Assume* $b_t - h_t^- \leq b_{t+1} \leq b_t + h_t^+$ *for every* $1 \leq t \leq T-1$. *If (14) holds for* $d = d^N$, *then it also holds for any* $d = d' \leq d^N$.

Proof. Recall that (14), (15), and (16) are equivalent to each other. In order to prove the lemma, it is sufficient to prove that if (16) holds for $d = d^N$, then for every $1 \leq l \leq T$, (16) holds for $d = d' = d^N - \delta e_{l_0}$ for some $\delta > 0$ such that $d' \geq 0$, where e_{l_0} is the unit vector with the l_0th entry being one.

Consider a pair of indices $i \leq j$. We show that (16) holds for i, j, k with $i \leq k \leq j$, and $d = d'$. We assume without loss of generality that $d_j^N > 0$ and $d_i^N > 0$. If $j < l_0$ or $i > l_0$, it is straightforward to show that (16) holds for $d = d'$. Hence, we focus on the case with $i \leq l_0 \leq j$. Since $d_i^N > 0$ and $d_j^N > 0$, we can choose $\delta > 0$ such that $\min\{d_i^N, d_j^N\} \geq \delta$. Thus, $d_{ij}^N - \delta \geq \max\{d_{i+1,j}^N, d_{i,j-1}^N\}$. Notice that if (16) holds for $i \leq j$ and $d = d^N$, then the concavity of $c_k(\cdot)$ implies that

$$\sum_{t=i}^{k-1} b_t^-(k) d_t^N + \sum_{t=k}^{j-1} b_t^+(k) d_t^N + b_j^+(k)(d_j^N - \delta) \leq c_k(d_{ij}^N - \delta) \ \ \forall k : i \leq k \leq j.$$

It follows that

$$\sum_{t=i}^{k-1} b_t^-(k) d_t^N + \sum_{t=k}^{j} b_t^+(k) d_t^N - \delta b_j^+(k) \leq c_k(d_{ij}^N - \delta) \ \ \forall k : i \leq k \leq j. \tag{17}$$

Similarly, we have

$$\sum_{t=i}^{k-1} b_t^-(k) d_t^N + \sum_{t=k}^{j} b_t^+(k) d_t^N - \delta b_i^-(k) \leq c_k(d_{ij}^N - \delta) \ \ \forall k : i \leq k \leq j. \tag{18}$$

Now consider any k such that $i \leq k \leq j$. By the assumption that $b_t - h_t^- \leq b_{t+1} \leq b_t + h_t^+$ for every $1 \leq t \leq T-1$, we get $b_{l_0}^+(k) \geq b_j^+(k)$ when $l_0 > k$ and $b_{l_0}^-(k) \geq b_i^-(k)$ when $l_0 \leq k$.

If $l_0 > k$, then,

$$\begin{aligned} c_k(d'_{ij}) &= c_k(d^N_{ij} - \delta) \\ &\geq \sum_{t=i}^{k-1} b^-_t(k) d^N_t + \sum_{t=k}^{j} b^+_t(k) d^N_t - \delta b^+_j(k) \\ &\geq \sum_{t=i}^{k-1} b^-_t(k) d^N_t + \sum_{t=k}^{j} b^+_t(k) d^N_t - \delta b^+_{l_0}(k) \\ &= \sum_{t=i}^{k-1} b^-_t(k) d'_t + \sum_{t=k}^{j} b^+_t(k) d'_t, \end{aligned}$$

where the first inequality follows from (17) and the second inequality follows from the fact that $b^+_{l_0}(k) \geq b^+_j(k)$.

Similarly, if $l_0 \leq k$, then,

$$\begin{aligned} c_k(d'_{ij}) &= c_k(d^N_{ij} - \delta) \\ &\geq \sum_{t=i}^{k-1} b^-_t(k) d^N_t + \sum_{t=k}^{j} b^+_t(k) d^N_t - \delta b^-_i(k) \\ &\geq \sum_{t=i}^{k-1} b^-_t(k) d^N_t + \sum_{t=k}^{j} b^+_t(k) d^N_t - \delta b^-_{l_0}(k) \\ &= \sum_{t=i}^{k-1} b^-_t(k) d'_t + \sum_{t=k}^{j} b^+_t(k) d'_t, \end{aligned}$$

where the first inequality follows from (18) and the second inequality follows from the fact that $b^-_{l_0}(k) \geq b^-_i(k)$.

Therefore, we have shown that (16) holds for $d = d'$, which completes the proof. □

Lemma 3. *Assume that $\bar{b} = (\bar{b}_1, \cdots, \bar{b}_T)$ is an optimal solution to problem (12). If $\bar{b}_t - h^-_t \leq \bar{b}_{t+1} \leq \bar{b}_t + h^+_t$ for every $1 \leq t \leq T-1$, then the allocation $l = (l_1, \cdots, l_N)$ defined by*

$$l_k = \sum_{t=1}^{T} \bar{b}_t d^k_t$$

is in the core of the economic lot-sizing game (N, F).

Proof. Notice that for any $S \subseteq N$,

$$\sum_{k \in S} l_k = \sum_{k \in S} \sum_{t=1}^{T} \bar{b}_t d^k_t = \sum_{t=1}^{T} \bar{b}_t d^S_t.$$

Therefore, by the definition of $\bar{b}$, $\sum_{k \in N} l_k$ is equal to the optimal value of (12), and thus equal to $F(N)$ by using Lemma 1.

It remains to show $\sum_{k \in S} l_k \leq F(S)$ for any $S \subseteq N$. By weak duality of linear programming, it suffices to prove that $\bar{b}$ is feasible to

$$\sum_{t=i}^{j} \bar{b}_t d_t^S \leq C_{ij}(d^S),$$

which is implied by Lemma 2. The proof is complete. $\square$

The following lemma is key to prove our main result. It might be of independent interest as well.

Lemma 4. *Given any feasible solution $b = (b_1, \cdots, b_T)$ to problem (9), we can construct in polynomial time another feasible solution $\hat{b} = (\hat{b}_1, \cdots, \hat{b}_T)$ with the same objective value such that $-h_t^- \leq \hat{b}_{t+1} - \hat{b}_t \leq h_t^+$ for every $t : 1 \leq t \leq T-1$.*

The proof of Lemma 4 is rather long and will be presented in the appendix. Now we provide some intuition on the inequalities in the lemma. Recall that the dual variable b_i may be interpreted as the price of satisfying one unit of demand at period i. Notice that the demand at period $i+1$ can be either satisfied by orders after period i or by an order before period $i+1$. In the latter case, period $i+1$ demand is satisfied by the inventory carried over from period i to period $i+1$ by paying a unit inventory holding cost h_i^+. Thus, it is reasonable to expect that the charge of a unit demand at period $i+1$ is no more than the charge of a unit demand at an period i plus the unit inventory holding cost h_i^+. The inequality $\hat{b}_{i+1} \geq \hat{b}_i - h_i^-$ can be explained in a similar way.

Now we are ready to present the main result of this section.

Theorem 2. *For the economic lot-sizing game (N, F) with backlogging, and with general nondecreasing concave ordering cost, linear inventory holding cost, and linear backlogging cost, the core is always non-empty and an allocation in the core can be found in polynomial time.*

Proof. The non-emptiness of the core is a direct consequence of Lemma 4 and Lemma 3. In order to show that an allocation in the core can be found in polynomial time, we notice that by Lemma 4, an optimal solution to the following linear program

$$\begin{aligned}
\max \quad & \sum_{t=1}^{T} b_t d_t^N \\
\text{s.t.} \quad & \sum_{t=i}^{j} b_t d_t^N \leq C_{ij}(d^N), \quad \forall 1 \leq i \leq j \leq T \\
& b_t \geq b_{t+1} - h_t^+, \quad \forall 1 \leq t \leq T-1 \\
& b_t \leq b_{t+1} + h_t^-, \quad \forall 1 \leq t \leq T-1
\end{aligned} \tag{19}$$

is also optimal to problem (12). In addition, problem (19) can be solved in polynomial time, and any optimal solution of (19) satisfies the conditions of Lemma 3. □

Our approach allows us to compute allocations in the core in polynomial time. An interesting and open question is whether the core allocations derived from our approach allow us to construct a a population monotonic allocation scheme. Unfortunately, since problem (19) may have multiple optimal solutions, it is not clear whether and how one can choose the optimal dual solutions appropriately to construct a population monotonic allocation scheme.

To illustrate the difficulty, we revisit Example 1. It is easy to verify that $\bar{b} = (5.5, 4.5, 0)$ is an optimal solution to problem (19) with $d^N = (10, 2, 6)$ and the associated allocation $(55, 9)$ is in the core. If one more player, player 3 with demand $d^3 = (0, 1, 0)$, joins the game, we can verify that $\bar{b} = (5.5, 1, 1)$ is an optimal optimal solution to problem (19) with $d^N = (10, 3, 6)$ and the associated allocation $(61, 2, 1)$ is in the core. It is interesting to observe that the cost of player 1 actually increases while the cost of player 2 decreases. On the other hand, $\bar{b} = (5.5, 3, 0)$ is another optimal optimal solution to problem (19) with $d^N = (10, 3, 6)$, which gives a core allocation $(55, 6, 3)$. In this case, the cost allocations of both player 1 and player 2 do not increase.

5 Comparison of Different Formulations for the ELS Game

Notice that for the ELS game with setup cost, both optimal solutions to (19) and (4) define allocations in the core. It would be interesting to compare the set of optimal solutions of (19) and (4).

Theorem 3. *Consider the ELS game with setup cost. Let A_1, A_2, and A_3 be the sets of optimal solutions to problems (12), (19), and (4), respectively. Then $A_2 \subsetneq A_3 \subsetneq A_1$.*

Proof. First of all, notice that the constraint of (4) is equivalent to

$$\sum_{1 \le \tau \le T} d_\tau^N \max\{b_\tau - p_{t\tau}, 0\} \le K_t, \forall t = 1, 2, \cdots, T. \tag{20}$$

For any $1 \le i \le j \le T$, (20) implies that

$$\sum_{\tau=i}^{j} d_\tau^N \max\{b_\tau - p_{t\tau}, 0\} \le K_t, \quad \forall t$$

which in turn implies that

$$\sum_{\tau=i}^{j} d_\tau^N b_\tau \le K_t + \sum_{\tau=i}^{j} d_\tau^N p_{t\tau} \quad \forall t.$$

Therefore,

$$\sum_{\tau=i}^{j} d_\tau^N b_\tau \le C_{ij}(d^N).$$

It then follows that $A_3 \subseteq A_1$. Furthermore, Example 1 (where the game is an ELS game with setup cost) shows that there exists $b \in A_1$ that does not define an allocation in core. This fact together with Theorem 1 implies $A_1 \not\subseteq A_3$ and thus $A_3 \subsetneq A_1$.

Now we prove $A_2 \subseteq A_3$. To this end, let $\tilde{b} \in A_2$. It suffices to show that (20) holds for $\tilde{b}$. For any t, we have

$$(\tilde{b}_{\tau+1} - p_{t,\tau+1}) - (\tilde{b}_\tau - p_{t\tau}) = \tilde{b}_{\tau+1} - \tilde{b}_\tau - h_\tau^+ \le 0, \quad \text{if} \quad \tau \ge t$$

and

$$(\tilde{b}_{\tau-1} - p_{t,\tau-1}) - (\tilde{b}_\tau - p_{t\tau}) = \tilde{b}_{\tau-1} - \tilde{b}_\tau - h^-_{\tau-1} \leq 0, \quad \text{if} \quad \tau \leq t$$

where the inequalities follow from the fact that $\tilde{b}_{\tau+1} \leq \tilde{b}_\tau + h^+_\tau$ and $\tilde{b}_{\tau-1} \leq \tilde{b}_\tau + h^-_{\tau-1}$.

Therefore, for each t, if there exists $\tau \geq t$ such that $\tilde{b}_\tau - p_{t\tau} < 0$, then for any $\tau' \geq \tau$, $\tilde{b}_{\tau'} - p_{t\tau'} < 0$; if there exists $\tau \leq t$ such that $\tilde{b}_\tau - p_{t\tau} < 0$, then for any $\tau' \leq \tau$, $\tilde{b}_{\tau'} - p_{t\tau'} < 0$. Now we define $\chi(t)$ be $t-1$ or the largest integer such that $\tilde{b}_\tau - p_{t\tau} \geq 0$, whichever is larger. Similarly, we define $\psi(t)$ be $t+1$ or the smallest integer such that $\tilde{b}_\tau - p_{t\tau} \geq 0$, whichever is smaller.

Thus, for every t,

$$\sum_{1\leq\tau\leq T} d^N_\tau \max\{\tilde{b}_\tau - p_{t\tau}, 0\} = \sum_{\tau=\psi(t)}^{\chi(t)} d^N_\tau \left(\tilde{b}_\tau - p_{t\tau}\right) \leq K_t, \tag{21}$$

where the inequality holds since

$$\sum_{\tau=\psi(t)}^{\chi(t)} d^N_\tau \tilde{b}_\tau \leq C_{\psi(t),\chi(t)}(d^N) \leq K_t + \sum_{\tau=\psi(t)}^{\chi(t)} d^N_\tau p_{t\tau}.$$

It follows that $\tilde{b} \in A_3$ and thus $A_2 \subseteq A_3$. Moreover, as will be seen from Example 3 (presented immediately after the proof), $A_2 \subsetneqq A_3$. This completes the proof. $\square$

The above theorem implies that the dual set given by (19), the set of allocations generated by the optimal dual solutions of problem (19), is a subset (indeed a true subset in the following example) of the dual set given by problem (4). An interesting question is whether the dual set given by problem (4) is the entire core. Unfortunately, the following example illustrates that this is not the case.

Example 3. We consider the game with three periods and two players, i.e., $T = 2$ and $N = \{1, 2\}$. Assume that backlogging is not allowed. Also,

$$K_1 = K_2 = 1, c_1 = c_2 = 1, d^1 = (3/4, 1/4), d^2 = (1/4, 3/4).$$

and the holding costs are zero. In this case, we only place an order at period one. Thus,

$$OPT(\{1\}) = OPT(\{2\}) = 2, OPT(\{1, 2\}) = 3,$$

and the core is $\{(v_1, v_2) : v_1 + v_2 = 3, 0 \leq v_1 \leq 2, 0 \leq v_2 \leq 2\}$. Solving problem (4), we get $\bar{b}_1 = 1 + \delta$ and $\bar{b}_2 = 2 - \delta$ with $\delta \in [0, 1]$. Now, the dual set given by problem (4) is

$$\begin{aligned} \{(\bar{b}_1 d_1^1 + \bar{b}_2 d_2^1, \bar{b}_1 d_1^2 + \bar{b}_2 d_2^2)\} &= \{(5/4 + 1/2\delta, 7/4 - 1/2\delta) : \delta \in [0, 1]\} \\ &= \{(v_1, v_2) : v_1 + v_2 = 3, 5/4 \leq v_1 \leq 7/4\}, \end{aligned}$$

which is a true subset of the core. In this example, for any $\delta \in [0, 1]$, $(1 + \delta, 2 - \delta) \in A_3$. But it is clear that $(1 + \delta, 2 - \delta) \in A_2$ only when $\delta \geq 1/2$. Thus $A_2 \subsetneqq A_3$. In addition, the dual set generated by A_2 is given by

$$\begin{aligned} \{(\bar{b}_1 d_1^1 + \bar{b}_2 d_2^1, \bar{b}_1 d_1^2 + \bar{b}_2 d_2^2)\} &= \{(5/4 + 1/2\delta, 7/4 - 1/2\delta) : \delta \in [1/2, 1]\} \\ &= \{(v_1, v_2) : v_1 + v_2 = 3, 3/2 \leq v_1 \leq 7/4\}, \end{aligned}$$

which is a true subset of the dual set given by problem (4).

6 Conclusion

In this paper, we study ELS games with backlogging and with general concave ordering cost. We show that there exists an allocation in the core, which can be computed in polynomial time by solving a linear program.

For some cooperative games, it may be useful to study other properties of a cost allocation rule, for instance, additivity, aggregate monotonicity, coalitional monotonicity, etc.; see Young [32] for definitions of these properties. But it is our opinion that such properties may not be properly defined for the ELS game. Therefore, we do not discuss these properties in this paper.

There are several directions for future research. First, although we know how to find an allocation in the core, the problem of checking whether a given allocation is in the core or not is still open. In particular, is there a polynomial time algorithm to check the membership of the core or is the problem NP-hard [8]? Moreover, our focus in this paper is the core. However, there are several other important concepts in cooperative game such as the Shapley value [21] and the nucleolus [6]. It would be interesting to design efficient algorithms to compute them.

We have essentially assumed that the retailers have uniform inventory holding costs, we also assumed that the inventory holding cost and backlogging cost are linear functions. It would be interesting if some of these assumptions can be relaxed.

Finally, the demand is assumed to be exogenous in this paper. However, in many practical situations, demand could be a function of the price of the product, and the price can be optimized, together with inventory decisions, in order to maximize profit (see, for instance, [3, 4]). It would be interesting to consider ELS games with joint pricing and inventory decisions.

References

[1] Anily, S. and Haviv, M. (2007). The Cost Allocation Problem for the First Order Interaction Joint Replenishment Model. *Operations Research*, 55 (2), pp. 292-302.

[2] Bondareva, O.N. (1963). Some application of linear programming methods to the theory of cooperative games. *Problemi Kibernitiki*, 10, pp. 119-139.

[3] Chen, X. and D. Simchi-Levi. (2004). Coordinating Inventory Control and Pricing Strategies with Random Demand and Fixed Ordering Cost: The Finite Horizon Case. *Operations Research*, 52 (6), pp. 887-896.

[4] Chen, X. and D. Simchi-Levi. (2004). Coordinating Inventory Control and Pricing Strategies with Random Demand and Fixed Ordering Cost: The Infinite Horizon Case. *Mathematics of Operations Research*, 29 (3), pp. 698-723.

[5] Chen, X. and Zhang, J. (2006). A stochastic programming duality approach to inventory centralization games. *Submitted.*

[6] Deng, X. and Papadimitriou, C.H. (1994). On the complexity of cooperative solution concepts. *Mathematics of Operations Research*, 19, pp. 257–266

[7] Dror, M. and Hartman, B. (2007). Shipment Consolidation: Who pays for it and how much. *Management Science*, 53, pp. 78-87.

[8] Faigle, U., W. Kern, S. P. Fekete and W. Hochstttler (1997). On the complexity of testing membership in the core of min-cost spanning tree games, *Int. J. Game Theory* 26, pp. 361–366.

[9] Goemans, M.X. and Skutella, M. (2004). Cooperative facility location game. *Journal of Algorithms*, 50(2), 194–214.

[10] Granot D. (1986). A generalized linear production model: A unifying model. *Mathematical Programming*, 34(2), 212–222.

[11] Hartman, B. and Dror, M. (1996). Cost allocation in continuous review inventory models. *Naval Research Logistics Quarterly*, 43:549–561.

[12] Hartman, B. and Dror, M. (1997). Inventory centralization costs with varying cost parameters. Working paper, MIS Department, University of Arizona.

[13] Hartman, B., Dror, M. and Shaked, M. (2000). Cores of inventory centralization games. *Games and Economic Behavior*, 31:26–49.

[14] Krarup J. and Bilde, O. (1977). Plant location, set covering and economic lot sizing: an O(mn) algorithm for structural problems. In *Numerische Methoden Bei Optimierungsaufgaben*, volume 3, pp. 155–180.

[15] Mahdian, M., Ye, Y. and Zhang, J. (2006). Approximation Algorithms for Metric Facility Location Problems. *SIAM Journal on Computing*, Vol. 36, No. 2, pp. 411-432.

[16] Manne, A.S. (1958). Programming of dynamic lot sizes. *Management Science*, 4, 115–135.

[17] Meca, A., Timmer, J., Garcia-Jurado, I., and Borm, P. (2004). Inventory games. *European Journal of Operational Research*, 156, 127–139.

[18] Müller, A., Scarsini, M. and Shaked, M. (2002). The newsvendor game has a non-empty core. *Games and Economic Behavior*, 38, 118–126.

[19] Owen, G. (1975). On the core of linear production games. *Mathematical Programming*, 9, 358–370.

[20] Ozen, U., Fransoo, J., Norde, H. and Slikker, M. (2004). Cooperation between multiple newsvendors with warehouses. *Working Paper*, 2004.

[21] Peleg, B. and Sudhölter, P. (2003). *Introduction to the Theory of Cooperative Games.* Kluwer Academic Publishers, Boston.

[22] Pochet, Y. and Wolsey, L.A. (1988). Lot-size models with backlogging: Strong reformulations and cutting planes. *Mathematical Programming*, 40, 317-335.

[23] Samet, D. and Zemel, E. (1984). On the core and dual set of linear programming games, *Mathematics of Operations Research*, 9, 309–316.

[24] Shapley, L.S. (1967). On balanced sets and core. *Naval Res. Logist. Quart.*, 14, 453-460.

[25] Slikker, M., Fransoo, J., and Wouters, M. (2001). Joint ordering in multiple newsvendor situations: a game theoretical approach. *BETA Working Paper*, 64, Eindhoven University of Technology, Eindhoven, The Netherlands.

[26] Slikker, M., Fransoo, J., and Wouters, M. (2005). Cooperation between multiple newsvendors with transshipments. *European Journal of Operational Research*, 167(2), 370–380.

[27] Sprumont, Y. (1990). Population montonic allocation schemes for cooperative games with transferable utility. *Games and Economic Behavior*, **2**, 378–394.

[28] Tamir, A. (1991). On the core of network synthesis games. *Mathematical Programming*, 50, 123–135.

[29] Topkis, D. M. (1998). *Supermodularity and Complementarity.* Princeton University Press, Princeton, NJ.

[30] van den Heuvel, W., Borm, P. E.M., and Hamers, H. (2005). Economic Lot-Sizing Games. *European Journal of Operational Research*, to appear.

[31] Wagner, H.M. and Whitin, T.M. (1958). Dynamic version of the economic lot size model. *Management Science*, 5, 89–96.

[32] Young, H.P. (1994). Cost Allocation. *Handbook of Game Theory*, Vol. 2, Edited by R.J. Aumann and S. Hart, pp. 1193-1235.

[33] Zangwill, W.I. (1969). A Backlogging Model and a Multi-Echelon Model of a Dynamic Economic Lot Size Production System-A Network Approach. *Management Science*, 15(9), 506-527.

Appendix

Proof of Lemma 4

For a given function $f : \Re \to \Re$, define its left-side directional derivative as

$$f'_{-}(x) = \lim_{\delta \to 0-} \frac{f(x+\delta) - f(x)}{\delta}.$$

Then we have,

Lemma 5. *Assume that $u, v : X \subset \Re \to \Re$ are two concave functions on an interval X. In addition, $u(x) \le v(x)$ for any $x \in X$. Then for any x with $u(x) = v(x)$, $u'_{-}(x) \ge v'_{-}(x)$.*

The proof of Lemma 5 is straightforward and thus omitted.

Before presenting the proof for Lemma 4, we make one observation. The inequality that we try to prove, i.e., $b_t - h_t^- \le b_{t+1} \le b_t + h_t^+$ for every $1 \le t \le T-1$, is equivalent to

$$b_t^+(1) \ge b_{t+1}^+(1) \quad \text{and} \quad b_t^-(T) \le b_{t+1}^-(T)$$

for every $1 \le t \le T-1$.

Given a feasible solution $b = (b_1, \cdots, b_T)$ to problem (9), we convert $b = (b_1, \cdots, b_T)$ to another feasible solution $\hat{b} = (\hat{b}_1, \cdots, \hat{b}_T)$ with the same objective value such that $-h_t^- \le \hat{b}_{t+1} - \hat{b}_t \le h_t^+$ for every $1 \le t \le T-1$ in two steps. In the first step, we convert b into a feasible solution $\tilde{b} = (\tilde{b}_1, \cdots, \tilde{b}_T)$ with the same objective value and $\tilde{b}^+_{t+1} \le \tilde{b}^+_t + h^+_t$ for $1 \le t \le T-1$. In the second step, we convert $\tilde{b}$ into a feasible solution $\hat{b}$ satisfying Lemma 4.

Step 1.

Assume that there exists some t such that $b_{t+1} > b_t + h^+_t$. Let l_0 be the smallest such t. Our objective is to show that we can decrease the value of b_{l_0+1} and increase the value of b_{l_0} so that $b_{l_0+1} \le b_{l_0} + h^+_{l_0}$. In addition, the new solution is still feasible with the objective value unchanged, while the smallest index with $b_{t+1} > b_t + h^+_t$ will increase by at least 1. We can repeat the process until $b_{t+1} \le b_t + h^+_t$ for every $1 \le t \le T-1$.

If $d_{l_0+1} = 0$, we can always decrease the value of b_{l_0+1} and keep other dual variables unchanged, which will not affect either the objective value or the feasibility. Thus, we assume $d_{l_0+1} > 0$ in the following analysis.

Let $i_0 = \max\{i \le l_0 : b_{i+1} - b_i < h^+_i\}$ or $i_0 = 0$ if the set is empty. It is clear that $i_0 + 1 \le l_0$. Furthermore, for any $i_0 + 1 \le i < l_0$, $b_{i+1} - b_i = h^+_i$. For a given $\delta > 0$, we define $\tilde{b}$ such that for every j,

$$\tilde{b}_j = \begin{cases} b_j + \delta, & \text{if } \ i_0 + 1 \le j \le l_0; \\ b_j - \frac{d_{i_0+1,l_0}}{d_{l_0+1}}\delta, & \text{if } \ j = l_0 + 1; \\ b_j, & \text{otherwise.} \end{cases}$$

It is clear that $\tilde{b}$ and b have the same objective value. We shall show that there exists some small $\delta > 0$ such that $\tilde{b}$ is feasible. Specifically, we will show that for any $i \le l \le j$,

$$\sum_{k=i}^{l-1} \tilde{b}^-_k(l) d_k + \sum_{k=l}^{j} \tilde{b}^+_k(l) d_k \le c_l(d_{ij}), \tag{22}$$

where $\tilde{b}^-_k(l) = \tilde{b}_k - \sum_{t=k}^{l-1} h^-_k$ for $k \le l-1$ and $\tilde{b}^+_k(l) = \tilde{b}_k - \sum_{t=k}^{j-1} h^+_k$ for $k \ge l$.

If $l > l_0$, or $j \le i_0$, or $l \le l_0$ and $j \ge l_0 + 1$, the inequality (22) obviously holds for $\tilde{b}$. Thus, we assume that $\max(l, i_0 + 1) \le j \le l_0$. In this case, define

$$u^i_l(x) = \sum_{k=i}^{l-1} b^-_k(l) d_k + \sum_{k=l}^{j-1} b^+_k(l) d_k + b^+_j(l)(x - d_{l,j-1}), \text{ for } x \in [d_{l,j-1}, d_{lj}],$$

and

$$v_l^i(x) = c_l(d_{i,l-1} + x).$$

Notice that for any i, $u_l^i(x) \le v_l^i(x)$ for any $x \in [0, d_{lT}]$.

We claim that $u_l^i(d_{lj}) < v_l^i(d_{lj})$ for any $i \le l \le \max(l, i_0+1) \le j \le l_0$. Otherwise, there must exist i, l, j such that $i \le l \le \max(l, i_0+1) \le j \le l_0$ and $u_l^i(d_{lj}) = v_l^i(d_{lj})$. Then by Lemma 5,

$$b_j^+(l) = \left\{\frac{du_{l-}^i(x)}{dx}\right\}|_{x=d_{lj}} \ge \left\{\frac{dv_{l-}^i(x)}{dx}\right\}|_{x=d_{lj}} \ge \frac{dv_{l-}^i(x)}{dx}, \text{ for } x \in [d_{lj}, d_{lT}], \tag{23}$$

where the second inequality follows from the concavity of function $v_l^i(x)$. On the other hand, it is clear that

$$b_{l_0+1}^+(l) > b_{l_0}^+(l) = b_j^+(l). \tag{24}$$

Notice that the function $u_l^i(x)$ is linear in $(d_{l,j}, d_{l,l_0}]$ with a slope of $b_{l_0}^+(l)$ and linear in $(d_{l,l_0}, d_{l,l_0+1}]$ with a slope of $b_{l_0+1}^+(l)$, while $v_l^i(x)$ is concave. Therefore, (23) and (24) imply that $u_l^i(x) > v_l^i(x)$ for $x \in (d_{l,j}, d_{l,l_0+1}]$, which contradicts the feasibility of b. Thus, $u_l^i(d_{lj}) < v_l^i(d_{lj})$ for any $i \le l \le \max(l, i_0+1) \le j \le l_0$. We can now choose δ such that

$$\delta = \min\left\{\min_{i,l,k: i \le l \le \max(l,i_0+1) \le j \le l_0} \frac{v_l^i(d_{lj}) - u_l^i(d_{lj})}{d_{\max(i,i_0+1),j}}, b_{i_0} + h_{i_0} - b_{i_0+1}\right\} > 0,$$

if $i_0 \ge 1$, and

$$\delta = \min_{i,l,k: i \le l \le \max(l,i_0+1) \le j \le l_0} \frac{v_l^i(d_{lj}) - u_l^i(d_{lj})}{d_{\max(i,i_0+1),j}} > 0,$$

if $i_0 = 0$. Then $\tilde{b}$ satisfies (22) and thus is still feasible to problem (9).

If $\delta < b_{i_0} + h_{i_0} - b_{i_0+1} = b_{i_0}^+(1) - b_{i_0+1}^+(1)$, we claim that $\tilde{b}_1^+(1) \ge \ldots \ge \tilde{b}_{l_0}^+(1) \ge \tilde{b}_{l_0+1}^+(1)$. If this is not true, it is clear that l_0 is the smallest l such that $\tilde{b}_{l+1}^+(1) > \tilde{b}_l^+(1)$. Then, similar to the argument we used before, we can show that for $i \le l \le \max(l, i_0+1) \le j \le l_0$, $u_l^i(d_{lj}) < v_l^i(d_{lj})$, where in the definition of $u_l^j(\cdot)$ and $v_l^j(\cdot)$, b is replaced by $\tilde{b}$. But this contradicts the definition of δ.

If $\delta = b_{i_0}^+(1) - b_{i_0+1}^+(1)$, then $\tilde{b}_{i_0}^+(1) = \tilde{b}_{i_0+1}^+(1) = \ldots = \tilde{b}_{l_0}^+(1)$. Therefore, the value of $\max\{i \le l_0 : b_{i+1} - b_i < h_i^+\}$ is decreased by at least one. In this case, if $\tilde{b}_{l_0}^+(1) < \tilde{b}_{l_0+1}^+(1)$, we repeat the above argument by at most i_0 times and each time the value of i_0 will be reduced by at least one. Then we end up with $\tilde{b}_1^+(1) \geq \ldots \geq \tilde{b}_{l_0}^+(1) \geq \tilde{b}_{l_0+1}^+(1)$.

Therefore, repeating the above argument for each l_0 (at most $T-1$ different values) we end up with $\tilde{b}$ with $\tilde{b}_{t+1} \leq \tilde{b}_t + h_t^+$ for $t \leq T-1$.

Step 2.

We now perform the second step by converting $\tilde{b}$ to $\hat{b}$ satisfying the conditions of the lemma. The procedure is similar to Step 1. Assume that there exists $\tilde{b}_{t+1} < \tilde{b}_t - h_t^-$. Let l_0 be the largest such t.

If $d_{l_0} = 0$, we can always decrease $\tilde{b}_{l_0}$ while maintaining the same objective value until $\tilde{b}_{l_0+1} = \tilde{b}_{l_0} - h_{l_0}^-$. In this case, we still have $\tilde{b}_{t+1} \leq \tilde{b}_t + h_t^+$ for $t \leq T-1$. Thus, we assume that $d_{l_0} > 0$.

Let $j_0 = \min\{j \geq l_0 : \tilde{b}_{j+1} - \tilde{b}_j > -h_j^-\}$ or $j_0 = T$ is the set is empty. It is clear that $j_0 \geq l_0 + 1$ and for any $l_0 + 1 \leq i \leq j_0$, $\tilde{b}_{i+1} - \tilde{b}_i = -h_i^-$. For a given $\delta > 0$, define $\hat{b}$ such that for every j,

$$\hat{b}_j = \begin{cases} \tilde{b}_j + \delta, & \text{if } \; l_0 + 1 \leq j \leq j_0; \\ \tilde{b}_j - \frac{d_{l_0+1,j_0}}{d_{l_0}}\delta, & \text{if } \; j = l_0; \\ b_j, & \text{otherwise.} \end{cases}$$

It is clear that $\hat{b}$ and $\tilde{b}$ have the same objective value. We shall show that there exists some small $\delta > 0$ such that $\hat{b}$ is feasible, i.e., for any $i \leq l \leq j$,

$$\sum_{k=i}^{l-1} \hat{b}_k^-(l) d_k + \sum_{k=l}^{j} \hat{b}_k^+(l) d_k \leq c_l(d_{ij}), \tag{25}$$

where $\hat{b}_k^-$ and $\hat{b}_k^+$ are defined similarly to $\tilde{b}_k^-$ and $\tilde{b}_k^+$, respectively.

If $l \leq l_0$, or $i > j_0$, or $l > l_0$ and $i \leq l_0$, the inequality (25) obviously holds for $\hat{b}$. Thus, we assume that $l_0 + 1 \leq i \leq \min(j_0, l)$. In this case, we define,

$$u_l^j(x) = \sum_{k=i+1}^{l-1} \tilde{b}_k^- d_k + \sum_{k=l}^{j} \tilde{b}_k^+ d_k + \tilde{b}_i^-(x - d_{i+1,l-1}), \text{ for } x \in [d_{i+1,l-1}, d_{i,l-1}],$$

and

$$v_l^j(x) = c_l(x + d_{lj}).$$

Then we know that $u_l^j(x) \leq v_l^j(x)$ for any $x \in [0, d_{1,l-1}]$. We claim that $u_l^j(d_{i,l-1}) < v_l^j(d_{i,l-1})$ for any $l_0 + 1 \leq i \leq \min(j_0, l)$. Otherwise, assume that there exists i, l, and j such that $u_l^j(d_{i,l-1}) = v_l^j(d_{i,l-1})$. Then we can show, similar to (23) and (24),

$$\tilde{b}_{l_0}^- > \tilde{b}_{l_0+1}^- = \tilde{b}_i^- = \left\{ \frac{du_{l-}^j(x)}{dx} \right\} |_{x=d_{il-1}} \geq \left\{ \frac{dv_{l-}^j(x)}{dx} \right\} |_{x=d_{i,l-1}} \geq \frac{dv_{l-}^j(x)}{dx}, \text{ for } x \in [d_{i,l-1}, d_{1,l-1}].$$

This implies that $u_l^j(x) > v_l^j(x)$ for $x \in (d_{i,l-1}, d_{l_0+1,l-1}]$, which contradicts the feasibility of $\tilde{b}$. Thus, $u_l^j(d_{i,l-1}) < v_l^j(d_{i,l-1})$ for any $l_0 + 1 \leq i \leq \min(j_0, l)$. We can now choose δ such that

$$\delta = \min\left\{ \min_{i,l,j: l_0+1 \leq i \leq \min(j_0,l) \leq l \leq j} \frac{v_l^j(d_{i,l-1}) - u_l^j(d_{i,l-1})}{d_{l_0+1,\min(j_0,l)}}, (\tilde{b}_{l_0+1} - \tilde{b}_{l_0} + h_{l_0}^-)\frac{d_{l_0}}{d_{l_0,j_0}}, \tilde{b}_{j_0+1} - \tilde{b}_{j_0} + h_{j_0}^- \right\} > 0,$$

if $j_0 < T$, and

$$\delta = \min\left\{ \min_{i,l,j: l_0+1 \leq i \leq \min(j_0,l) \leq l \leq j} \frac{v_l^j(d_{i,l-1}) - u_l^j(d_{i,l-1})}{d_{l_0+1,\min(j_0,l)}}, (\tilde{b}_{l_0+1} - \tilde{b}_{l_0} + h_{l_0}^-)\frac{d_{l_0}}{d_{l_0,j_0}} \right\} > 0$$

if $j_0 = T$. Then $\hat{b}$ satisfies (25) and thus is still feasible to problem (9).

Now we consider two cases.

Case 1. $\delta = (\tilde{b}_{l_0+1} - \tilde{b}_{l_0} + h_{l_0}^-)\frac{d_{l_0}}{d_{l_0,j_0}}$. In this case, we have that $\hat{b}_{l_0+1} = \hat{b}_{l_0} - h_{l_0}^-$. Therefore, the largest integer such that $\hat{b}_{t+1} < \hat{b}_t - h_t^-$ has decreased by at least 1. This is exactly what we need.

Case 2. $\delta < (\tilde{b}_{l_0+1} - \tilde{b}_{l_0} + h_{l_0}^-)\frac{d_{l_0}}{d_{l_0,j_0}}$. In this case, $\hat{b}_{l_0+1} < \hat{b}_{l_0} - h_{l_0}^-$. It implies that l_0 is the largest integer such that $\hat{b}_{t+1} < \hat{b}_t - h_t^-$. Then by the same argument as above, $u_l^j(d_{i,l-1}) < v_l^j(d_{i,l-1})$ for any $l_0 + 1 \leq i \leq \min(j_0, l)$, where in the definition of $u_l^j(\cdot)$ and $v_l^j(\cdot)$, we replace $\tilde{b}$ with $\hat{b}$. This would imply that $\delta < \frac{v_l^j(d_{i,l-1}) - u_l^j(d_{i,l-1})}{d_{l_0+1,\min(j_0,l)}}$. It then follows that $\delta = \tilde{b}_{j_0+1} - \tilde{b}_{j_0} + h_{j_0}^-$. Hence we have $\hat{b}_{j_0+1}^- = \cdots = \hat{b}_{j_0}^-$. Therefore, the value of $\min\{j \geq l_0 + 1 : \hat{b}_{j+1} - \hat{b}_j > -h_j^-\}$ is no less than $j_0 + 1$. We can repeat the procedure by at most $T - j_0$ times until Case 2 will not happen (and thus eventually we are in Case 1).

The only thing that is left to verify is that $\hat{b}_{t+1} \leq \hat{b}_t + h_t^+$ for every $1 \leq t \leq T-1$. This inequality can possibly be violated if during the process one of the following situations happens. (Notice that for the variables increased, they are increased by the same amount.)

(1) $\tilde{b}_{t+1}$ is increased by δ, but $\tilde{b}_t$ is unchanged, or $\tilde{b}_t$ is unchanged, but $\tilde{b}_{t-1}$ is decreased by a positive amount. It is clear that this can never happen in our procedure.

(2) $\tilde{b}_{t+1}$ is increased by δ, but $\tilde{b}_t$ is decreased by a positive amount. This happens only if $t = l_0$. But after the change, $\hat{b}_{l_0+1} = \tilde{b}_{l_0+1} + \delta$, and $\hat{b}_{l_0} = \tilde{b}_{l_0} - \frac{d_{l_0+1,j_0}}{d_{l_0}}\delta$. By the definition of δ, $\hat{b}_{l_0+1} = \hat{b}_{l_0} - h_{l_0}^-$ and thus $\hat{b}_{l_0+1} \leq \hat{b}_{l_0} + h_{l_0}^+$.