Market Incompleteness and Super Value Additivity: Implications for Securitization*

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1 Introduction

Many of the classic results in modern financial theory were originally derived in the context of a complete financial market in the sense of Arrow-Debreu. Although this framework has yielded rich results for a variety of problems in economics, the fact remains that financial markets are, in fact, incomplete and their incompleteness has implications for many interesting economic issues, in particular the problem of valuation of assets, real and financial. This paper addresses the issue of valuation in incomplete markets and the resultant implications for financial decision making, in general, and securitization, in particular.

The implications of incomplete markets for the valuation of assets can be appreciated by considering the following illustration. In standard financial theory, assets are generally valued on a stand-alone basis. Their prices are derived from the capital market on the assumptions that the cash flows from the real asset can be replicated in the financial market and that all agents are price-takers with respect to financial claims. This raises the question whether the value of a particular asset, say a real asset, owned by an agent such as a firm, can be enhanced by undertaking suitable augmenting or offsetting transactions in the financial markets. It is well-known that in a complete market, in which claims are traded on all future contingencies, it is not possible to improve the valuation of the asset by such transactions (see for example, Diamond (1967), Radner (1972) and Hart (1975)). The answer is not obvious if the market is incomplete, since we can not associate a unique price with every state-contingent claim. We address this question in this paper.

The problem we pose is a fairly common one, since firms often have opportunities that are unique to them, and generate future cash flows that cannot be replicated by existing market transactions. However, firms are typically not large enough to influence the prices of the securities traded in the market. The question is whether a firm with such a unique opportunity can engage in capital market transactions relating to some parts of its future cash flows to enhance its value. We separate this question into two issues. First, what are the conditions that can be imposed on the price of an asset in the context of an incomplete capital market to preclude arbitrage between the prices of the traded claims? Second, given the arbitrage-bounds imposed by the market, how can a price-taking firm "package" the cash flows from its unique investment opportunity so as to take advantage of the incompleteness of the market? The answers to these questions would help isolate the distinct advantage possessed by the firm from owning a particular asset, without abandoning the standard assumptions of price-taking as well as the assumption of no arbitrage in competitive capital markets.

In the analysis of financial markets from a pricing perspective, the base case is clearly one that is complete and has no frictions or transaction costs. There are two points of departure: the first is to introduce some friction, such as a trading cost, the other is to deal explicitly with incompleteness. We explore the latter route. Our focus is on the additional value created by a price-taking firm that takes advantage of the incompleteness of the market and caters to it. It is well known from valuation theory that in an incomplete market, the value of a real asset cannot always be uniquely computed from capital market prices, by arbitrage pricing arguments. Hence, it is often concluded that firms that decide between alternative investments, such as in real assets, cannot easily determine which choices maximize their value. Our research shows that incompleteness of the market puts a premium on those assets that offer hedging possibilities, i.e., on assets that improve the spanning across states. Hence, even though the values of the real assets in question cannot be uniquely determined, this hedging dimension may restrict the bounds on their prices.

A number of typical problems of valuation in the context of incomplete markets can be analyzed within our framework. We present below two such examples, and later, in §6, provide numerical illustrations of each example in the context of our model.

Example A: Securitization

The two aspects of securitization, pooling and tranching, can be explained by value creation in incomplete markets, for instance, in the particular case of the market for collateralized debt obligations (CDOs). The assumption here is that the market is incomplete in the sense that some states associated with poor, mediocre, or good performance by the group of firms whose bonds are being pooled, cannot be spanned by the available securities in the market. This creates an incentive for an intermediary to purchase a portfolio ("pool") of junk bonds and then issue claims against the pool in various categories ("tranches"), e.g., a high-grade AAA tranche, which has a negligible, virtually zero, probability of not meeting its promised payment; a medium-grade tranche, say rated BBB+, which has a low but not negligible probability of such default; and an equity tranche, which is viewed as risky.[1] The question that arises here is what is the optimal pooling and tranching strategy for the financial institution contracting the purchase of CDOs? In other words, what debt instruments should be pooled and what tranches should be created to extract the maximum surplus from the transactions?

Example B. Investment in Real Options

Consider an energy firm that has two alternative investment opportunities in oil exploration, and can undertake only one of them. Suppose the payoffs from these opportunities depend on a benchmark such as West Texas Intermediate (WTI) crude oil prices, and a spread due to the sulfur content of the oil produced. The first project yields oil with low sulfur content, thus its cash flows can be perfectly hedged by selling derivative contracts on oil on the benchmark WTI grade. The second project might yield oil with high sulfur content and there are no matching securities. What option should the firm select? In addition, suppose that the firm can use the oil to feed two existing

[1] Typically, there are more than three tranches in a CDO structure, but three would suffice to explain the essential principles involved.

refineries that it owns, and that are designed to handle crude with different grades of sulfur content. Does the ownership of these refineries change the value of the investment in oil exploration? Are there any synergies involved *across* the states of the economy by combining the cash flows of the project with the existing cash flows?

The common features of these examples are: (1) There is an asset with a given set of payoffs in different states. (2) Since the market is incomplete, the asset cannot be valued in a unique manner. (3) Some of the asset's payoffs are spanned by securities traded in the market. Hence, they can be priced and sold by issuing the corresponding attainable claims in the market. (4) Other payoffs are not spanned by traded securities. To the extent that these payoffs can be sold within the limits of market prices, the value of the asset can be enhanced. (5) The valuation of the asset is greater due to its spanning properties, thus leading to synergistic benefits across states rather than due to conventional economies of scale/scope within the same states. Our research synthesizes these common features to obtain results that are applicable in each of these settings.

This paper is organized as follows. Section 2 reviews the related literature on incomplete markets and securitization. Section 3 presents the model setup and assumptions. Section 4 analyzes the conditions for arbitrage-free trading in the securities market when there exists a 'thin' market for cash flows that are not spanned by the existing securities. Section 5 determines the value of the firm owning the real asset, and §6 concludes with a discussion of the implications of our analysis.

2 Literature Review

We draw upon two strands in the literature in our research. The first is the literature on valuation of assets when markets are incomplete, i.e., when the set of available securities does not span the state-space. The second is the literature on securitization, i.e., the issuance of securities in the capital market that are backed or collateralized by a portfolio of assets.

2.1 Valuation in Incomplete Markets

The Arrow-Debreu framework was originally developed to study the equilibrium valuation of claims in a complete market. Over time, the framework has been modified and developed significantly for analyzing general equilibrium in incomplete markets.[2] Despite some progress in this area, the problem of endogenous asset formation in an incomplete market, such as through making investment decisions in real assets or introducing new securities, and the valuation of such assets, in particular, has not been solved satisfactorily so far. To be precise, for a firm making such decisions, the main issue is how to value an asset, real or financial, that is outside the span of securities that can be priced exactly.

In an Arrow-Debreu economy, when markets are complete, beliefs and attitudes towards risk do not affect the valuation of new assets, given the pricing of state-contingent claims in the existing equilibrium. Therefore, the competitive firm's investment decisions under the objective of value maximization are independent of such attitudes and the standard Fisher separation theorem of valuation of real assets versus their financing holds. However, when markets are incomplete, as pointed out by Radner (1982), we do not have a clear-cut natural way of comparing net revenues at different dates and states. Typically, each investor in a firm has a different attitude to a proposed investment and the unanimity implied by the separation theorem in complete markets no longer holds. In such a situation, the objective of the firm itself is unclear. Various objectives of the firm have been proposed under such a situation such that the actions of the firm are consonant with the value of the firm. A particular objective proposed by Grossman and Hart (1979) (similar to the

[2] Most of the theory is accessible via the surveys by Radner (1982) and Magill and Shafer (1991) and the more detailed book by Magill and Quinzii (2002).

objective studied in Diamond (1967) and Dreze (1974)) is related to our paper. They suggest that each firm's objective function be a weighted sum of the shareholders' private value. If shareholders have to be unanimous about the plan, this objective reduces to maximization of the value of shares. In this context, Radner (1982, p. 981) adds that the market-value maximization hypothesis would require the producer to predict the effect of a choice in production plan on the price equilibrium. Thus, the producer would no longer be a price-taker. Therefore, one would need a theory of general equilibrium in monopolistic competition to determine the optimal plan.

Since the theory of equilibrium under incomplete markets does not yield unambiguous results, research in financial economics has focused on the less restrictive notion of "no arbitrage" to analyze problems of valuation. Harrison and Kreps (1979) is credited with showing that a no arbitrage price process under a suitable change of measure can be set equal to the conditional expectation of the future payoffs. It can also be used as the basis for approximate analysis of equilibrium, as argued by Ross (1976a) and John (1981). The idea of no arbitrage is more primitive than that of equilibrium in the financial markets or even valuation, since the existence of arbitrage opportunities implies that that the economy is not in equilibrium. In fact, the no arbitrage condition helps in searching for equilibrium under incomplete markets (see, for example, Geanokoplos and Polemarchakis (1986)). Despite this, the fact remains that when an asset's payoffs are not spanned by the existing claims in the market, or more particularly, the returns of an asset are not perfectly correlated with marketed assets, it is difficult to value the asset using no-arbitrage pricing.

Three different approaches have been adopted for pricing contingent claims in incomplete markets: through bounds based on no-arbitrage, preference-based approaches that impose restrictions on the utility functions of consumers, and approximate arbitrage-based arguments. Under the arbitrage-based approach of Harrison and Kreps (1979), when the market is incomplete, the pricing kernel is not unique - there are several pricing kernels that price marketed securities correctly.

Since the traded assets do not span the entire state space, there is a multiplicity of stochastic discount factors, under any of which the expected value of future cash flows equals the present price of a traded asset. Hence, for securities than cannot be spanned by the existing market, there is no unique price. However, a lower and an upper bound on the value of the security can be obtained by determining the maximum and minimum prices under the set of pricing kernels—the security cannot be sold above its upper bound, nor can it be purchased below the lower bound price, without presenting an arbitrage opportunity.[3]

In contrast to the above approach, it is possible to restrict investor preferences or return distributions to get exact prices such as in the capital asset pricing model (CAPM) model. Another example of this approach is the literature on option pricing using such preference restrictions.[4] These approaches give bounds that are obviously tighter than the no arbitrage bounds, but are less general, given the nature of the preferences assumed.

Some researchers have criticized both the preference-based and arbitrage-based approaches to pricing. The preference-based approach, which uses subjective probabilities and the preferences of the individual decision-maker, is criticized as being too specific and subject to misspecification error. The arbitrage-based approach, on the other hand, which uses the risk neutral pricing measure, is criticized for yielding price bounds that are too wide - those that rule out arbitrage opportunities, but do not rule out "approximate" arbitrage opportunities. Thus, these bounds are considered too weak to be economically interesting. Recent research has focused on sharpening the price bounds in an incomplete market, either by imposing economic restrictions in the arbitrage pricing theory based on the reward-to-risk or Sharpe ratio, or by combining the arbitrage-based and the

[3] A variant of this approach is to place restrictions using the principle of stochastic dominance, as proposed in the context of options by Merton (1973), and widely used in setting bounds on the prices of derivative instruments.

[4] See, for example, Perrakis and Ryan (1984), Levy (1985), Ritchken (1985), Ritchken and Kuo (1989) and Mathur and Ritchken (1999).

preference-based approaches.

The arbitrage-based approach, first formally proposed by Harrison and Kreps (1979), is used as the starting point of the analysis in the approximate arbitrage based approach. Shanken (1992) defines an investment opportunity with a high, but finite Sharpe ratio as an "approximate arbitrage." Hansen and Jagannathan (1991) show that a bound on the maximum Sharpe ratio is equivalent to a bound on the variance of the pricing kernel. Building on this result, Cochrane and Saa-Requejo (2000) derive sharper bounds on the prices of derivative instruments by ruling out the existence of investment opportunities with high Sharpe ratios, which they call "good deals". Many other restrictions on the set of pricing kernels have been studied in the literature. For example, Snow (1992) derives restrictions on the q-th moment of the pricing kernel; and Stutzer (1993) presents an alternative restriction on the entropy of the pricing kernel by restricting the maximum expected utility attained by a CARA agent.

Bernardo and Ledoit (1999, 2000) argue that bounds on the maximum value of the Sharpe ratio are insufficient to rule out approximate arbitrage when returns are not Gaussian because even though such bounds rule out high state prices, they do not rule out low state prices. Thus, Bernardo and Ledoit alternatively define approximate arbitrage as a zero-cost investment opportunity with a high ratio of expected gains to expected losses, where the expectations are taken under a benchmark investor's risk-adjusted pricing measure. By using duality theory they show that a restriction on the maximum gain-loss ratio is equivalent to a restriction on the ratio of state price densities (P/Q) across any two states, and therefore, rules out both high and low state prices, where, P denotes the risk neutral price density, and Q denotes the risk-adjusted price density of a benchmark investor. Thus, Bernardo and Ledoit give an alternative method to show that restricting approximate arbitrage restricts the set of admissible pricing kernels, and thus, gives sharper price bounds than those obtained by a pure arbitrage-based approach.

In contrast to the above approaches, we show that the bounds can be substantially sharpened using no-arbitrage arguments when there are *several* contingent claims to be priced simultaneously. In some sense, our approach is between the large and the small in the following sense: the preference based approach uses the utility function of a single person (small) whereas the competitive no arbitrage approach assumes an efficient market and several players (large). Our approach is that of a single seller who works within the no arbitrage prices but sells to multiple buyers. However, an important assumption of our analysis is related to the previous literature, namely, the price-taking assumption of the individual seller of claims.

The problem of valuing a new asset that is introduced into an incomplete market goes beyond the valuation of assets using prices that prevail before the introduction of the new asset. Even in a complete market, such an action might cause the prices of other assets to change. The assumption made by us and others is that the firm is a price-taker. This may not necessarily be appropriate if the firm's output is a non-negligible proportion of the entire output in the economy.[5] A variant of this assumption is discussed by Grossman and Stiglitz (1980). They examine unanimity of shareholders with regard to a production plan when there is spanning but not complete markets.[6] The set of production plans of a firm is said to be spanned when any of its plans can be written as a linear combination of the production plans of other firms. They show that when markets are incomplete, spanning, in general, does not imply unanimity amongst shareholders when the shares of the firm can be retraded. They show that unanimity can be obtained when firms behave as perfect competitors in the production of commodities that form a basis for the spanned space, an assumption labeled as "competitivity." Another strand of literature on incomplete markets deals with the role of financial markets in using options and "supershares" to augment existing markets

[5] See Hart (1979a), Hart (1979b) and Grossman and Hart (1979) for a discussion.

[6] Unanimity has been studied extensively in the mean-variance setting. See, for example, Merton and Subrahmanyam (1974).

to achieve the allocational efficiency of a complete market.[7]

Our paper also considers the limits to the monopolistic power of an entrepreneur by restricting the entrepreneur to act within the set of observed prices. This approach is similar to using the concept of "viability" as in Harrison and Kreps (1979) and Kreps (1981). A pricing kernel over a set of attainable consumption vectors is said to be viable for a class of conceivable agents if there is one agent that prefers a consumption bundle from the given set to all others at the given prices. Harrison and Kreps show that viability is equivalent to there being an extension of the pricing kernel to the entire set of consumption bundles, i.e., beyond the attainable set. In our setting, the valuation of the asset is undertaken within the set of price kernels that price existing securities correctly. Thus, the owner of the asset behaves monopolistically, but within the restrictions imposed by the no arbitrage criterion. On the other hand, the magnitude of trading possibilities created by the asset is assumed to be insignificant so that the pricing kernels do not change. The approach is also similar to the relative pricing as described by Cochrane and Saa-Requejo (2000), "we are interested only in the value of a specific payoff, we take as given the prices of other assets without questioning their fundamental economic determinants, and we want to make as few economic assumptions as possible. (p. 80)"

2.2 Securitization

Research on securitization has focused on the rationale for the widespread use of pooling and tranching in the asset-backed securities market. This rationale is explained through three types of market imperfections: transaction costs, market incompleteness, and information asymmetry.

[7] Supershares are tranches of the portfolio of all securities in the market. See Ross (1976) and John (1981) for an analysis of how options on a single portfolio of all the primitive securities can achieve such a fully efficient market. Hakansson (1978) provides an argument that supershares issued by an entity invested in the market portfolio may improve the allocational efficiency of the existing market structure.

Specific examples of securitization include the literature on “supershares” (i.e., tranches of the portfolio of all securities in the market), primes and scores (i.e., income and capital gains portions of a stock), and “bull” and “bear” bonds.[8]

Allen and Gale (1991) examine the incentive of a firm to issue a new security when there are transaction costs. Their study is motivated by the observation that the first firm to innovate financially faces a higher cost than imitators. They consider a two-period model with entrepreneurs and investors. Each entrepreneur owns a risky asset that produces a random return in the second period. Asset returns are assumed to have symmetric distributions. In the first period, the entrepreneur issues claims against the returns from the asset. If a firm decides to innovate a new security (which is simply a partition of its asset returns), the *ex-post* values of the firm that decides to innovate and the firms that decide not to innovate will be equal. Thus, there is no value to innovating if the prices remain the same after innovation and when there is a cost incurred for introducing the new securities. On the other hand, if the prices change due to the innovation, then there is a mixed strategy symmetric equilibrium in which each firm computes the value differential from innovating versus not innovating and acts accordingly. Even if one firm decides to innovate, the prices might change to make the innovation worthwhile to the firm. If it decides not to innovate, there is still a probability that one other firm might innovate. Thus, the strategy mixes across these outcomes and trades off the cost of issuing the security against the benefit from innovating. Allen and Gale conclude that competition is necessarily ‘imperfect’ if there is to be any incentive to innovate (this relates back to the observation in Radner (1982)). Our paper shows that this need not be the case even if prices do not change due to the innovation but when there is a single firm. The firm can

[8] See Hakansson (1978), and Jarrow and O’Hara (1989) for details. For example, Hakansson (1978) argues that options or supershares on the market portfolio improve the allocational efficiency of an existing market structure, even if the market portfolio itself is not efficient.

partition its asset returns so that they are sold in the most profitable manner. However, the most profitable manner is constrained by the no-arbitrage condition. We also assume that only the firm can issue these securities and that they have to be backed by the asset returns to be credible.[9] We only assume that the sum total of the payoffs from the securities issued should be less than or equal to the random return in the second period.

Several other researchers examine the rationale for pooling and tranching using information asymmetry between issuers and investors. Pooling is considered beneficial to both an uninformed issuer and an uninformed investor. The benefit to an uninformed issuer is that it reduces the issuer's incentive to gather information (Glaeser and Kallal 1997). The benefit to uninformed investors is that pooling reduces their adverse selection problem when competing with informed investors (DeMarzo 2001). In this context, Subrahmanyam (1991) shows that security index baskets are more liquid than the underlying stocks.

However, DeMarzo (2001) also shows that an informed issuer (or intermediary) does not prefer pure pooling because it destroys the asset-specific information of the informed issuer. Instead, an informed intermediary prefers pooling and tranching to either pure pooling or separate asset sales because pooling and tranching enable the intermediary to design low-risk debt securities that minimize the information asymmetry between the intermediary and uninformed investors. DeMarzo calls this the 'risk diversification effect' of pooling and tranching. Pooling and tranching are also beneficial to uninformed investors. For example, Gorton and Pennachi (1990) show that uninformed investors prefer to split cash flows into a risk-less debt and an equity claim.[10]

[9] Ross (1976b) writes, "Furthermore, in general, it is less costly to market a derived asset generated by a primitive than to issue a new primitive, and there is reason to believe that options will be created until the gains are outweighed by the set-up costs (p. 76) .

[10] There is a vast literature on the role of information asymmetry in securitization. See DeMarzo (2001), DeMarzo and Duffie (1999), and Leland and Pyle (1977) for examples.

Our paper discusses the rationale for securitization from market incompleteness assuming that there are no frictions, such as transaction costs, or information asymmetry. However, the results from our paper are consistent with the results based on information asymmetry. We show that purely from an arbitrage perspective, pooling and tranching are beneficial to the issuer since they enable the issuer to construct tranches that maximize the value of unspanned assets in an incomplete market. We further discuss the issuer's problem of optimal construction of tranches given that there are investors with different preferences in the market. In this regard, our results also relate to the value of options for increasing market efficiency as described in Ross (1976b).

3 Model Setup

We consider a discrete-time Arrow-Debreu economy in which time is indexed as 0 and 1.[11] The set of possible states of nature at time 1 is $\Omega = \{\omega_1, \omega_2, \ldots, \omega_K\}$. All agents have the same informational structure: The true state of nature is unknown at $t = 0$ and is revealed at $t = 1$. Moreover, the K states are a complete enumeration of all possible events of interest, i.e., the subjective probability of any decision maker is positive for each of these states and adds up to one over all the states. There is a set of claims traded in the financial market that can be bought or sold by all agents. The financial market is assumed to be arbitrage-free and frictionless, i.e., there are no transaction costs associated with the sale, purchase or creation of securities. All agents are price-takers in the financial market. To keep the analysis uncluttered, we assume that there is no discounting of cash flows, i.e., the risk-free rate of interest is 0.

In this economy, we introduce a firm (decision maker) that owns an asset, say a real asset, which is unique to it and provides an income of $X(\omega_k)$ in each state k at time $t = 1$. We assume that

[11] The model described below can be extended to a multi-period setting with some added complexity in the notation. However, the basic principles and results derived would still obtain.

the firm is small relative to the size of the economy. Thus, the equilibrium prices of financial assets currently traded in the market are unaffected by the introduction of the firm. The firm, therefore, behaves as a price-taker in the financial market. It can undertake the following transactions: (i) buy and sell claims issued against X, (ii) buy or sell securities in the financial market. Claims issued against X should be fully backed by X; in other words, their sum should not exceed the value of X in any state of nature.[12]

We are primarily interested in how the firm can enhance its value by combining X with the securities traded in the financial market. There are three possible cases of interest: (1) when the market is complete and there are no transaction costs, (2) when the market is complete but there are transaction costs or deadweight costs, such as due to bankruptcy, and (3) when the market is incomplete. In the first case, the value of proprietary claims X is unique and cannot be enhanced by transactions in the financial market (see Lemma 3 in §5). In the second case, Allen and Gale (1991) show that the value of the firm cannot be enhanced by offering new securities against X when there are transaction costs of issuing these new securities and prices are unaffected by the issuance of new securities. On the other hand, it is easy to show that when there are deadweight costs such as those associated with bankruptcy, the firm might prefer to insure against loss in certain states of nature. Certainly, risk averse owners might prefer to trade their future cash flows for a time zero profit or even at a small loss. These results establish that interaction between the cash flows from real and financial assets is certainly possible when there are inefficiencies in the securities market. In contrast, our paper focuses on the third case, when the financial market is frictionless but incomplete.

We make the following additional assumption with respect to claims that are not presently

[12] We do not allow for issue of claims that would permit default in some states, since that would involve complex questions relating to bankruptcy and renegotiation, which are outside the purview of this paper.

traded: *There are buyers willing to buy and/or sell contingent claims* $Y_i, i = 1, \ldots, I$ *in the market at prices* p_i.[13] These claims could be made up of packages of other securities and state contingent claims. We refer to them as being traded in a 'thin' market. Thus, even though the market is incomplete, there is demand from individuals who are willing to buy unspanned claims at arbitrage-free prices. Allen and Gale (1991) present an example that demonstrates the existence of demand for such claims in an incomplete market. They show that when the market is incomplete, individuals can have different reservation prices for contingent claims that are not spanned by the existing securities (also see Ross 1976b). This suggests that there are two classes of securities in the market: (a) those that are presently traded, and (b) those that can *potentially* be traded. It should be emphasized that even though securities in the latter group are not presently traded, we model their bid and ask prices to be consistent with the arbitrage bounds implied by the former group.

In §4, we determine necessary and sufficient conditions to prevent arbitrage in an incomplete market when there are prices associated with claims that can potentially be traded. This characterization is essential to determine the distinctive advantage of the subject firm in our model. In §5, we compute the optimal packaging of X such that the value of the firm is maximized.

4 No-Arbitrage Condition in an Incomplete Market

We begin by defining our notation and restating some standard results from the literature in our context. Proofs of these results can be found in the literature, well-synthesized by Pliska (1997).

As stated in §3, the securities market is assumed to be arbitrage free. Therefore, there exists a set, Θ, of risk neutral probability measures over Ω such that all traded claims are uniquely priced.

[13] We presume that arbitrageurs will take advantage of different bid and ask prices for the same claim across investors. Thus, after these transactions are exhausted, no claim Y_i has different bid and ask prices.

It is well known that the set Θ is spanned by a finite set of independent linear pricing measures.[14] Denote the collection of pricing measures that span Θ as $\{q_j, j = 1, \ldots, J\}$. In particular, if the set Θ is a singleton then the market is complete, else it is incomplete.

Consider a claim Z in this market that pays $Z(\omega_k)$ in state k, $k = 1, \ldots, K$. If the expectation of Z under q, $E_q[Z]$, is independent of q for all $q \in \Theta$ (that is, Z is spanned by securities that are traded in the market), then Z is said to be an *attainable* claim, else it is said to be an *unattainable* claim. We shall use the notation $E[Z]$ for the expected value of attainable claims. For use below, note that all attainable claims are uniquely priced regardless of whether the market is incomplete. For any unattainable claim Z, let $V^-(Z) = \max\{E[S] : S \leq Z, S \text{ is attainable}\}$, and let $S^-(Z) = \arg\max\{E[S] : S \leq Z, S \text{ is attainable}\}$. Likewise, let $V^+(Z) = \min\{E[S] : S \geq Z, S \text{ is attainable}\}$, and let $S^+(Z) = \arg\min\{E[S] : S \geq Z, S \text{ is attainable}\}$. $V^-(Z)$ and $V^+(Z)$ are well-defined and finite. Also, $V^-(Z)$ and $V^+(Z)$ may alternatively be defined as $\inf_{q\in\Theta} E_q[Z(\omega_k)]$ and $\sup_{q\in\Theta} E_q[Z(\omega_k)]$, respectively.

We establish the following lemma needed in the sequel. It simplifies the computation of $V^+(Z)$ and $V^-(Z)$ by recognizing that Θ is the interior of a simplex, and that the values of $V^+(Z)$ and $V^-(Z)$ are each realized at an extreme point of this simplex. Since q_j's span Θ, they represent the extreme points of the simplex. Thus, $V^+(Z)$ and $V^-(Z)$ can be computed simply by taking the maximum and the minimum, respectively, of the expected values of Z under q_j's. This Lemma is closely related, but not identical to results in the literature. We state and prove it in the specific form stated below. All proofs are in the Appendix unless otherwise stated.

[14] A linear pricing measure is a probability measure that can take a value equal to zero in some state, whereas a risk neutral probability measure is strictly positive in all states. Thus, the set Θ is the interior of the convex set spanned by the set of independent linear pricing measures. The maximum dimension of this set equals the dimension of the solution set to a feasible finite-dimensional linear program, and thus, is finite.

Lemma 1. $V^+(Z) = \max_{j\in J} E_{q_j}[Z]$. $V^-(Z) = \min_{j\in J} E_{q_j}[Z]$.

The no arbitrage condition for a *single* contingent claim Z in an incomplete market is stated as follows: Let the price of Z at time $t = 0$ be p. Then this contingent claim presents no arbitrage if $V^-(Z) \leq p \leq V^+(Z)$. Notice that for attainable claims, $V^-(Z) = V^+(Z) = E[Z]$. For our analysis in §5, we need to extend this definition to encompass arbitrage-free trading with multiple claims in thin markets, viz., when *several* unattainable claims are priced in the market *simultaneously*. We do this in two steps: Lemma 2 extends the no-arbitrage condition for a single contingent claim to multiple contingent claims by defining a necessary and sufficient condition for no-arbitrage in an incomplete market. Theorem 1 gives a verification technique to check the condition in Lemma 2.

Lemma 2. *To prevent arbitrage in the trading of the Y_i's, it is necessary and sufficient that for all $\alpha \in \Re^I$, $V^-(\sum \alpha_i Y_i) \leq \sum \alpha_i p_i \leq V^+(\sum \alpha_i Y_i)$.*

In words, the lemma states that the price of every portfolio that can be constructed using the available claims should lie between the respective upper and lower bounds to prevent arbitrage.

Example 1A: Consider an incomplete market with $K = 5$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, and $\Theta = \{(x, 0.25 - x, 0.5, y, 0.25 - y) : 0 \leq x \leq 0.25,\ 0 \leq y \leq 0.25\}$. Suppose that there exist agents who are willing to purchase the contingent claims $Y_1 = (1, 0, 0, 0, 0)$ and $Y_2 = (0, 1, 0, 0, 0)$. We have $V^-(Y_1) = 0, V^+(Y_1) = 0.25, V^-(Y_2) = 0$ and $V^+(Y_2) = 0.25$. Thus, according to the standard condition for arbitrage-free trading, the prices of Y_1 and Y_2 must each lie between 0 and 0.25 to avoid an arbitrage. This is because (1,1,0,0,0) is an attainable claim that has value $= x + 0.25 - x = 0.25$. It is also the cheapest attainable claim that is larger than Y_1 or Y_2. However, these conditions alone are not sufficient, taken individually, to prevent arbitrage. For example, if the agents are willing to purchase Y_1 and Y_2 for $p_1 = 0.15$ and $p_2 = 0.2$, respectively, then it is possible to obtain a sure profit by purchasing the attainable claim $(1, 1, 0, 0, 0)$ for 0.25, splitting it into the unattainable claims Y_1 and Y_2, and selling to the respective individuals for a net profit of 0.10. Lemma 2 precludes such prices by stating that the no-arbitrage condition must hold not only for Y_1 and Y_2 individually, but also for portfolios, i.e.,*all linear combinations of* Y_1 *and* Y_2.

The no-arbitrage condition in Lemma 2 is hard to verify because it must hold for every portfolio that can be created by varying α. Theorem 1 makes verification easier by converting the portfolio pricing problem into a pricing problem for the individual Y_i's. It states that the no-arbitrage condition holds if and only if the claims Y_i are priced correctly under at least one pricing measure.

Theorem 1. *The no-arbitrage condition of Lemma 2 holds if and only if there exists* $q \in \Theta$ *such that*

(i) for each claim i *that investors are willing to buy at price* p_i*,*

$$\sum_k q(\omega_k) Y_i(\omega_k) \geq p_i, \tag{1}$$

(ii) for each claim i *that investors are willing to sell at price* p_i*,*

$$\sum_k q(\omega_k) Y_i(\omega_k) \leq p_i, \tag{2}$$

(iii) for each claim i *that investors are willing to both buy and sell at price* p_i*,*

$$\sum_k q(\omega_k) Y_i(\omega_k) = p_i. \tag{3}$$

Given the set of contingent claims Y_i with prices p_i, in order to verify whether these prices satisfy the condition of Theorem 1, we can proceed as follows: first we exhaust any obvious arbitrage opportunities that present themselves when some investors are willing to buy the same claim at a higher price compared to what others are willing to sell the claim at. Next, we partition the set of claims that can be bought, sold, or both bought and sold as S_1, S_2 and S_3, respectively. We can determine whether the prices p_i satisfy Theorem 1 by solving the following problem:

$$\mathbf{P_1}: \quad \exists\, \pi_j,\ j = 1, \ldots, J \text{ such that } \sum_j \pi_j = 1, \quad \pi_j \geq 0 \ \ \forall j \text{ and}$$

$$\begin{array}{cccc} \sum_k \sum_j \pi_j Q_j(\omega_k) Y_i(\omega_k) & \geq & p_i & \forall i \in S_1 \\ & (\leq, \ =) & & (i \in S_2, \ i \in S_3) \end{array}$$

The existence of π_j's can be established using linear programming.

Theorem 1 gives sharper bounds on the prices of unattainable claims than those obtained by valuing each claim individually using Lemma 1 because these prices must not only lie between the values given by the functions $V^-(\cdot)$ and $V^+(\cdot)$ but also satisfy the additional constraints in Theorem 1. For example, suppose we wish to obtain price bounds for claim Y_I given prices $p_i, \ i = 1, \ldots, I-1$ for claims $Y_i, \ i = 1, \ldots, I-1$. We can obtain these price bounds simply by adding the constraints from problem $\mathbf{P_1}$ for claims $i = 1, \ldots, I-1$ to the pricing problem in Lemma 1. We illustrate the computation of these bounds in §6.

It is of interest to relate Theorem 1 to the usual condition for no arbitrage price bounds when there is a single contingent claim-price pair (Y, p) that can be both bought and sold. In that case, Theorem 1 requires the existence of a risk neutral pricing measure q' such that $E_{q'}[Y] \geq p$ as well as $E_{q'}[-Y] \geq -p$. Thus, $E_{q'}[Y] = p$. This is equivalent to the usual condition because $E_{q'}[Y] = p \Rightarrow \inf_{q \in \Theta} E_q[Y] \leq p \leq \sup_{q \in \Theta} E_q[Y]$. Notice that the existence of such a probability measure does not imply that the contingent claim is uniquely priced because its price could be different under different risk neutral pricing measures, say $p'' = E_{q''}[Y]$ and $\inf_{q \in \Theta} E_q[Y] \leq p'' \leq \sup_{q \in \Theta} E_q[Y]$.

Thus, the results in this section can be viewed as an extension to the usual arbitrage pricing theory: the prices in the thin market stay not only within the bounds imposed by the current prices for individual attainable claims in the market, but also within the bounds implied by the prices of other claims in the thin market. Moreover, when operating within the constraints that some contingent claims Y can only be bought (or sold), the prices of such claims must satisfy a one-sided constraint. On the other hand, the new idea here is that even though prices of the contingent claims Y_i's are not unique, they must be correct *simultaneously* to prevent arbitrage.

Example 1B: We now continue with the previous example and examine the pricing bounds for all combinations of securities. According to Theorem 1, to avoid arbitrage with prices $p_1 = 0.15$ and $p_2 = 0.2$, it is necessary that for some x and y, the expectations of Y_1 and Y_2 under $q = (x, 0.25 - x, 0.5, y, 0.25 - y)$ are larger than p_1 and p_2. That is, $x \times 1 \geq 0.15,\ (0.25 - x) \times 1 \geq 0.2$. This is impossible. Further, the range of prices that satisfy the conditions of Theorem 1 is given by: $x \times 1 \geq p_1,\ (0.25 - x) \times 1 \geq p_2$, that is, $p_1 + p_2 \leq 0.25$. Thus, given $p_2 \in [0, 0.25]$, price bounds for Y_1 are $(0, 0.25 - p_2)$. These bounds are sharper than those obtained from Lemma 1 and obviously prevents the arbitrage discussed earlier.

Example 2 shows how the condition of Theorem 1 can be checked using the linear program $\mathbf{P_1}$ in a more complicated situation.

Example 2: Consider an incomplete market with $K = 4$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, and $\Theta = \{(x + 3y, x - y, 0.5 - 2x, 0.5 - 2y)\}$ where

$$\begin{array}{ccccc} 0 & \leq & x + 3y & \leq & 1 \\ 0 & \leq & x - y & \leq & 1 \\ 0 & \leq & 0.5 - 2x & \leq & 1 \\ 0 & \leq & 0.5 - 2y & \leq & 1. \end{array}$$

Figure 1(a) shows the feasible set of values of x and y obtained by solving the above constraints. This set is given by triangle ABC with vertices $(1/4, 1/4), (0, 0)$ and $(1/4, -1/12)$. There is a one-to-one correspondence between these extreme points and the independent pricing measures that span Θ. Thus, Θ is spanned by the pricing measures $q_1 = (1, 0, 0, 0), q_2 = (0, 0, 1/2, 1/2)$ and $q_3 = (0, 1/3, 0, 2/3)$. Figure 1(b) shows the set of admissible pricing kernels (triangle A′B′C′) corresponding to the feasible values of x and y.

Suppose that there exist individuals willing to purchase the contingent claims $Y_1 = (1, 0, 0, 0)$ and $Y_2 = (0, 1, 0, 0)$ at prices p_1 and p_2, respectively. By imposing the arbitrage bounds on these claims one at a time, we have

$$V^-(Y_1) = 0 \quad \le \ p_1 \ \le \quad V^+(Y_1) = 1,$$

$$V^-(Y_2) = 0 \quad \le \ p_2 \ \le \quad V^+(Y_2) = 1/3.$$

Similar to Example 1, these bounds are not sufficient to prevent arbitrage. For example, $p_1 = 1, p_2 = 0.2$ satisfy these bounds but result in an arbitrage. To see this, buy the claim $(1, 1, 1, 1)$ at price 1, sell Y_1 and Y_2 at a total price of 1.2, and throw away the balance of (0,0,1,1).

The necessary and sufficient conditions on p_1 and p_2 to prevent arbitrage are obtained as in $\mathbf{P_1}$ by adding the constraints $x + 3y \ge p_1$ and $x - y \ge p_2$ to set Θ. Notice that in the region ABC, $x - y$ takes a minimum value of 0 on the line segment AB and a maximum value of 1/3 at C. Thus, $0 \le p_2 \le 1/3$. Now, if p_2 is fixed at a value in this range, then, adding the constraint $x - y \ge p_2$ from Theorem 1, we find that the feasible set of pricing measures shrinks from ABC to the triangle DEC (see Figure 1(a)) with vertices D$= (1/4, 1/4 - p_2)$, E$= (3/4p_2, -1/4p_2)$ and C$= (1/4, -1/12)$. Accordingly, the value of p_1 that satisfies the no-arbitrage condition along with p_2 takes a minimum on the segment CE, and a maximum at D. Thus, $0 \le p_1 \le 1 - 3p_2$. If p_1 is also fixed at a value within this range, then, adding the constraint $x + 3y \ge p_1$, we find that the feasible set of pricing measures further shrinks to the triangle DFG, where F$= (3/4p_2 + 1/4p_1, 1/4p_1 - 1/4p_2)$ and G$= (1/4, 1/3p_1 - 1/12)$. Figure 1(b) shows the pricing kernels corresponding to triangles DEC and DFG.

Thus, if the usual bounds are used, the upper bound on the sum of p_1 and p_2 equals 4/3, whereas the arbitrage-free upper bound obtained from the conditions $0 \le p_2 \le 1/3$ and $0 \le p_1 \le 1 - 3p_2$ is 1.

5 Valuation of X

This section considers the problem of determining a matching portfolio of securities Y which, when combined with X, maximizes the value of the firm. We first establish that an enhancement in the value of the firm is feasible only in an incomplete market.

Lemma 3. *If the market is complete, that is, the set Θ contains q as its singleton element, then the value of X can be enhanced by augmenting it with some contingent claim Y if and only if the time 0 value of the combined time 1 cash flows from X and Y is not separable in X and Y.*

The situation is different when the securities market is incomplete. For example,

Lemma 4. *If Θ is not a singleton set then the sum of the minimum value of the asset, X, and the minimum value of the contingent claim, Y, can be enhanced by combining them.*

Proof: The minimum value of the asset is given by $\min_q E[X]$. Similarly, the minimum value of the contingent claim is given by $\min_q E[Y]$. The result follows by noting that

$$\min_q E[X+Y] \geq \min_q E[X] + \min_q E[Y]$$

$\square$

Remark: Notice that the above inequality can be strict only if Θ is not a singleton set. Thus, indirectly it also provides a proof that when cash flows are additive and the set Θ is a singleton, no contingent claim can enhance the value of the asset.

5.1 Determining the Optimal Portfolio

We now consider the problem of maximizing the value of X given that a set of buyers is willing to purchase claims Y_i at prices p_i. The prices p_i obey the conditions defined in Theorem 1, else there would be an arbitrage in the existing market.

The problem is formulated as the following linear program:

$$\mathbf{P_2}: \quad \max\ z + \sum_i \alpha_i p_i \tag{4}$$

subject to

$$\bar{X}(\omega_k) + \sum_i \alpha_i Y_i(\omega_k) \leq X(\omega_k) \qquad k = 1, \ldots, K \tag{5}$$

$$\sum_k q_j(\omega_k)\bar{X}(\omega_k) - z \geq 0 \qquad j = 1, \ldots, J \tag{6}$$

$$z, \bar{X} \text{ unsigned, } \alpha \geq 0. \tag{7}$$

This linear program maximizes the profit obtained by splitting X in such a way that claims $\alpha_i Y_i$ are sold to respective buyers at prices $\alpha_i p_i$ and the remaining portion, $\bar{X}$, is sold in the market at the most conservative price. Here, $\bar{X}$ and $\sum_i \alpha_i Y_i$ denote the components into which X is split, z denotes the price of $\bar{X}$, and $\sum_i \alpha_i p_i$ is the price of $\sum_i \alpha_i Y_i$. Constraint (5) ensures that the sum of $\bar{X}$ and $\sum_i \alpha_i Y_i$ is smaller than X. It enforces the requirement that claims issued against X should be backed by X. Constraint (6) computes the price z of $\bar{X}$. Since z is the most conservative price, it must be less than or equal to $V^-(\bar{X})$. By Lemma 1, this implies that z must be less than or equal to the expectation of $\bar{X}$ under each of the pricing measures q_j. Constraint (6) ensures this condition. By the definition of $V^-(\bar{X})$, this implies that there exists an attainable claim $S^-(\bar{X})$ which is less than or equal to $\bar{X}$ under all states of nature and has price z. Thus it is possible to realize the value z by selling $\bar{X}$. The objective function of $\mathbf{P_2}$ seeks to maximize the sum of the proceeds from $\bar{X}$ and the parts of X sold to the buyers in the thin market, viz., $\sum_i \alpha_i p_i$.

The following theorem gives the optimal solution to this linear program, and thus, the maximum value of X.

Theorem 2. *Let $\Theta_A \subset \Theta$ be the non-empty set of pricing measures that satisfy conditions (i)-(iii) in Theorem 1 for claims $Y_1, \ldots, Y_I$ in the thin market at prices $p_1, \ldots, p_I$, respectively. Also let*

$\{q_{Al}, l = 1, \ldots, L\}$ *denote the set of independent pricing measures that span* Θ_A*. Then the optimal solution to* $\mathbf{P_2}$ *is given by* $\min_{q \in \Theta_A} E_q[X]$*, or equivalently, by* $\min_l E_{q_{Al}}[X]$*.*

According to Theorem 2, the value of X is enhanced by undertaking transactions in the thin market because $\Theta_A \subset \Theta$ so that $\min_{q\in\Theta_A} E_q[X] \geq \min_{q\in\Theta} E_q[X] = V^-(X)$. Further, Theorem 2 does not result in an unbounded solution even though the formulation $\mathbf{P_1}$ does not have an upper bound on α. This is so because the prices p_i satisfy the no-arbitrage condition in Theorem 1.

Let $X_A = \arg\max\{E[W] : W \leq X, W \text{ is attainable}\}$. The structure of the optimal claims issued against X can be obtained as follows.

$$\begin{aligned}
\text{Optimal value of } X &= \min_{q\in\Theta_A} E_q[X] \\
&= \min_{q\in\Theta_A} \{E_q[X - X_A] + E_q[X_A]\} \\
&= \min_{q\in\Theta_A} \{E_q[X - X_A]\} + V^-(X).
\end{aligned}$$

The second equality follows because X_A is attainable. Thus, we find that the value of the firm is maximized when the attainable portion of X, X_A, is stripped away and sold at $E[X_A](= V^-(X))$, and the value of the remaining cash flows, $X - X_A$ is maximized using Y_i's. Notice that X_A need not be unique because the optimal value does not depend on the choice of X_A.[15]

5.2 Application to Securitization

Consider the problem of securitization in an incomplete market. Let there be J originators, X_j denote the cash flows per unit of the debt obligations of originator j, and c_j denote the price at which originator j seeks to sell its cash flows to the intermediary. We define *pooling* as the set of transactions by which a financial intermediary purchases a set of cash flows from one or

[15] To see this, let X'_A be an alternative attainable claim such that $X'_A \leq X$ and $E[X'_A] = V^-(X)$. Then $\min_{q\in\Theta_A} E_q[X] = \min_{q\in\Theta_A}\{E_q[X - X'_A] + E_q[X'_A]\} = \min_{q\in\Theta_A}\{E_q[X - X'_A]\} + V^-(X) = \min_{q\in\Theta_A}\{E_q[X - X_A] + E_q[X_A - X'_A]\} + V^-(X) = \min_{q\in\Theta_A}\{E_q[X - X_A]\} + V^-(X)$.

more originators, and sells attainable securities backed by the combined cash flows in the financial market. Let $X = \sum_j \beta_j X_j$ denote one unit of the pooled asset, where the vector $\beta = (\beta_1, \ldots, \beta_J)$ denotes the proportion in which the debt obligations of the originators are combined together in the pool. Note that $\sum_j \beta_j = 1$ and $\beta_j \geq 0$ for all j. Also note that the value of pooling is, by definition, equal to $V^-(X) - \sum_j \beta_j c_j$.

We define *tranching* as the creation of securities backed by the pooled asset to be sold in the thin market, i.e., to buyers of $Y_1, \ldots, Y_I$. Let α_i denote the number of units of claim Y_i tranched from one unit of the pooled asset X, and let $\alpha = (\alpha_1, \ldots, \alpha_I)$. Applying Theorem 2, the incremental value of tranching is given by $\min_{q \in \Theta_A} E_q[X]$.

We now specify the pooling and tranching strategy as (α, β). The financial intermediary's problem is to determine α and β such that the profit per unit of the pooled asset is maximized. This problem is formulated as the following linear program:

$$\mathbf{P_3}: \quad \max \; z + \sum_i \alpha_i p_i - \sum_j \beta_j c_j \tag{8}$$

subject to

$$\bar{X}(\omega_k) + \sum_i \alpha_i Y_i(\omega_k) \leq \sum_j \beta_j X_j(\omega_k) \qquad k = 1, \ldots, K \tag{9}$$

$$\sum_k q_j(\omega_k) \bar{X}(\omega_k) - z \geq 0 \qquad j = 1, \ldots, J \tag{10}$$

$$\sum_j \beta_j = 1 \tag{11}$$

$$z, \bar{X} \text{ unsigned}, \; \alpha \geq 0, \beta \geq 0. \tag{12}$$

Notice that this problem is similar to problem $\mathbf{P_2}$ defined in §5.1. In $\mathbf{P_2}$, the asset X is given and we seek to maximize its value by splitting it into various tranches. In comparison, in $\mathbf{P_3}$, we seek to both construct the optimal pooled asset and split it into tranches. Constraints (9)-(10) are analogous to constraints (5)-(6) in $\mathbf{P_2}$, and constraint (11) is added to ensure that β corresponds to a single unit of the pooled asset.

Let Θ_A be as defined in Theorem 2 and $\Theta_B \subset \Theta$ be the set of pricing measures satisfying Theorem 1 for claims X_j priced at c_j. Theorems 1 and 2 yield the following conditions characterizing when there is value in pooling by itself (without tranching) and when there is value in pooling followed by tranching.

Corollary 1. *(i) If Θ_B is empty, i.e., if for every $q \in \Theta$, there exists j such that $c_j < E_q[X_j]$, then there is value in pooling by itself.*

(ii) If Θ_B is non-empty and $\Theta_A \bigcap \Theta_B = \emptyset$, then there is no value in pooling, but there is value in pooling and tranching.

(iii) If $\Theta_A \bigcap \Theta_B \neq \emptyset$, then there is value in neither pooling nor tranching.

Here, if Θ_B is empty, then by Theorem 1, some combination of claims X_j can be assembled to provide risk-free profit. Thus, there is value in pooling by itself (Case (i)). This value can be obtained by solving the linear program $\mathbf{P_3}$ setting $\alpha = 0$. Note that Θ_B could be empty even when the prices of the claims X_j are within the bounds obtained by evaluating them individually. When Θ_B is not empty, then there is no combination possible under which an arbitrage can be created. But if Θ_B has no element in common with Θ_A, then once again Theorem 1 can be applied to show that there must be an arbitrage among $X_1, \ldots, X_J$ and $Y_1, \ldots, Y_I$ (Case (ii)). The optimal pooling and tranching strategy can again be obtained by solving $\mathbf{P_3}$. Let X be the weighted combination of $X_1, \ldots, X_J$ that gives the optimal pooled asset. From Theorem 2, the profit per unit of the pooled asset is given by $\min_{q \in \Theta_A} E_q[X] - \sum_j \beta_j c_j$. In Case (iii), the maximum value of X, $\min_{q \in \Theta_A} E_q[X]$, is smaller than the price paid to assemble the pool for all pooling strategies β. Thus, there is no arbitrage opportunity.

6 Numerical Examples

This section presents numerical illustrations of Examples A and B in §1.

6.1 Securitization

Consider a single-period economy with state-space, $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 = \{1, 2, 3\}$ and $\Omega_2 = \{$a, b, c$\}$ are information sets. At time $t = 0$, the market can contract only on the information set Ω_1. The set Ω_2 is verifiable at time $t = 1$ but not contractable at $t = 0$.

Let there be three securities traded in this economy, S_1 with unit payoffs in states {1a, 1b, 1c} and 0 in the remaining states, S_2 with unit payoffs in states {2a, 2b, 2c} and 0 in the remaining states, and S_3 with unit payoffs in states {3a, 3b, 3c} and 0 in the remaining states. Let the prices of these securities be 0.34, 0.36 and 0.30, respectively. These securities determine the set of pricing kernels, Θ, feasible for the economy under no-arbitrage trading. Thus, we have

$$\begin{aligned}
q_{1a} + q_{1b} + q_{1c} &= 0.34, \\
q_{2a} + q_{2b} + q_{2c} &= 0.36, \\
q_{3a} + q_{3b} + q_{3c} &= 0.30.
\end{aligned}$$

Consider two firms, $j = 1, 2$, in this economy with debt obligations, X_1 and X_2. The payoffs per unit of these obligations at time $t = 1$ are as shown in Table 1. The bounds on the prices of each unit of X_1 and X_2 are obtained by using Lemma 1 as follows:

$$V^-(X_1) = 0, \quad V^+(X_1) = 2 \cdot 0.34 + 2 \cdot 0.36 + 2 \cdot 0.30 = 2.00,$$

$$V^-(X_2) = 0, \quad V^+(X_2) = 1 \cdot 0.34 + 4 \cdot 0.36 + 1 \cdot 0.30 = 2.08.$$

Suppose that firm 2 is willing to sell one unit of its debt obligations for $c_2 = 0.15$. Then, sharpened arbitrage bounds for the value of one unit of the debt obligations of firm 1 can be

Firm 1

	a	b	c
1	0	1	2
2	0	2	0
3	0	1	2

Firm 2

	a	b	c
1	1	0	0
2	1	0	4
3	1	0	0

Table 1: Debt obligations of firms 1 and 2 at t=1

obtained as follows:

$$\max(\text{or } \min) \quad q_{1b} + 2q_{1c} + 2q_{2b} + q_{3b} + 2q_{3c}$$

such that

$$\begin{aligned}
q_{1a} + q_{1b} + q_{1c} &= 0.34, \\
q_{2a} + q_{2b} + q_{2c} &= 0.36, \\
q_{3a} + q_{3b} + q_{3c} &= 0.30, \\
q_{1a} + q_{2a} + 4q_{2c} + q_{3a} &\leq 0.15 \qquad \text{(from Theorem 1)}, \\
\sum_{k \in \Omega} q_k &= 1, \\
q_k &\geq 0, \qquad \text{for all } k \in \Omega.
\end{aligned}$$

Thus,

$$V^{-}(X_1|c_2) = 1.06, \quad V^{+}(X_1|c_2) = 2.00. \tag{13}$$

We now illustrate the application of Corollary 1 to pooling and tranching.

Benefits of Pooling. Suppose that firm 1 is willing to sell one unit of its debt obligations for $c_1 = 0.80$. Since this price lies outside the bounds in (13), by Theorem 1, Θ_B, is empty. Therefore, by Corollary 1, there is value in pooling the payoffs of the two firms. A naive pooling strategy is to pool the debt obligations in the ratio 1:1. This strategy yields a profit of 0.025 per unit of the pooled asset. However, this strategy is not optimal. The optimal pooling strategy is obtained by solving the linear program $\mathbf{P_3}$ setting $\alpha = 0$. We find that the optimal strategy involves pooling X_1 and X_2 in the ratio 1:2. The price of the pooled asset, $X = X_1/3 + 2X_2/3$, has a lower bound of $V^-(X) = 0.68$. Thus, it is possible for a financial intermediary to construct one unit of the pooled asset X at a cost of $11/30$ $(= (0.80 + 0.30)/3)$, and sell its attainable portion in the market for $V^-(X)$. (The attainable portion of X consists of $0.5S_1 + S_2 + 0.5S_3$.) Thus, pooling results in a net per unit profit of $94/300$ $(= 0.68 - 11/30)$.

If $c_2 = 0.15$, pooling has value as long as $c_1 < V^-(X_1|c_2) = 1.06$. When c_1 becomes larger than 1.06, then Θ_B is not empty so that, by Corollary 1, pooling by itself has no value.

Benefits of Pooling and Tranching. Now suppose that there exist customers in the financial market willing to purchase risky claims at prices within the arbitrage bounds. Let there be four classes of investors, $i = 1..4$, willing to purchase the four claims Y_i shown in Table 2 at prices p_i equal to 0.20, 0.50, 0.30 and 0.40, respectively. It can be verified that these prices are consistent according to Theorem 1 to prevent arbitrage.

It is possible for the financial intermediary to tranche the remaining portion of the pooled obligations (after selling the attainable portion) to these investors at the best available price. Thus, we now solve the linear program $\mathbf{P_3}$ to find the optimal pooling and tranching strategy. Here, $\alpha_i, i = 1..4$, represent the number of units of Y_i sold to investor class i, $\alpha_i, i = 5..7$, represent the number of units of traded securities (attainable claims) tranched out to achieve $V^-(X)$, and

β_1, β_2 represent the proportion in which the obligations of the two firms are pooled together.[16]

$$\max 0.2\alpha_1 + 0.5\alpha_2 + 0.3\alpha_3 + 0.4\alpha_4 + 0.34\alpha_5 + 0.36\alpha_6 + 0.3\alpha_7 - 0.80\beta_1 - 0.15\beta_2$$

such that

$$\begin{aligned} \alpha_5 &\leq \beta_2 && \text{state 1a} \\ \alpha_5 &\leq \beta_1 && \text{state 1b} \\ \alpha_2 + \alpha_3 + \alpha_5 &\leq 2\beta_1 && \text{state 1c} \end{aligned}$$

$$\begin{aligned} \alpha_6 &\leq \beta_2 && \text{state 2a} \\ \alpha_1 + \alpha_3 + \alpha_6 &\leq 2\beta_1 && \text{state 2b} \\ \alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 + \alpha_6 &\leq 4\beta_2 && \text{state 2c} \\ \alpha_7 &\leq \beta_2 && \text{state 3a} \\ \alpha_7 &\leq \beta_1 && \text{state 3b} \\ \alpha_2 + 2\alpha_4 + \alpha_7 &\leq 2\beta_1 && \text{state 3c} \\ \beta_1 + \beta_2 &= 1 && \\ \alpha_1, \ldots, \alpha_4 \geq 0, \quad \alpha_5, \ldots, \alpha_7 \text{ unsigned}, \quad \beta_1, \beta_2 &\geq 0. && \end{aligned}$$

If $\beta = (5/9, 4/9)$ is used for pooling, then two candidate feasible solutions to the LP are $\alpha = (2/3, 2/3, 0, 0, 4/9, 4/9, 4/9)$ and $\alpha = (0, 0, 2/3, 2/9, 4/9, 4/9, 4/9)$. The former solution turns out to be optimal. The total per unit profit to the financial intermediary after pooling and tranching

[16]The linear programs $\mathbf{P_1}, \mathbf{P_2}, \mathbf{P_3}$ were formulated in earlier sections assuming that the linear pricing measures that span the set Θ are known explicitly. Alternatively, these linear programs can also be formulated when the set Θ is not known explicitly but the traded securities that determine the set Θ are given. Here, we use the alternative formulation since S_1, S_2, S_3 are given. The variables $\alpha_i, i = 5..7$, give the attainable claims tranched from the pooled asset. They are unsigned since the attainable claims can be both bought and sold.

Investors $i = 1$

	a	b	c
1	0	0	0
2	0	1	1
3	0	0	0

Investors $i = 2$

	a	b	c
1	0	0	1
2	0	0	1
3	0	0	1

Investors $i = 3$

	a	b	c
1	0	0	1
2	0	1	1
3	0	0	0

Investors $i = 4$

	a	b	c
1	0	0	0
2	0	0	3
3	0	0	2

Table 2: Demand for unattainable claims by different investor classes at t=1

is: 4/9 (from the sale of attainable claims) + 14/30 (from the sale to investor classes 1 and 2) $-5/9c_1 - 4/9c_2 = 0.40$, where $c_1 = 0.80$ and $c_2 = 0.15$.

Further, we find that if $c_2 = 0.15$ and $1.06 \leq c_1 < 1.56$, then Θ_B is not empty and $\Theta_A \bigcap \Theta_B$ is empty. In this case, pooling by itself has no value but pooling with tranching has value.

No benefit from pooling and tranching. When $c_2 = 0.15$ and $c_1 \geq 1.56$, then $\Theta_A \bigcap \Theta_B$ is not empty. Thus, there cannot be arbitrage because both prices are high enough to prevent any advantage from pooling and tranching.

Figure 2 shows the ranges of values of c_1 and c_2 illustrating the three cases in Corollary 1. The range where pooling by itself has value is represented by the region below the curve GHI. The range for which pooling by itself has no value but pooling with tranching has value is represented by the region between the curves ABCDEF and GHI. Within this region, tranching of the obligations of firm 1 without pooling has value to the left of the line $c_1 = 0.28$. Tranching of the obligations of firm 2 without pooling never has value. The region above and to the right of the curve ABCDEF represents the range of values for which neither pooling nor tranching has value. The optimal pooling and tranching strategy varies in the region below ABCDEF. The optimal pooling ratio takes the following values depending on c_1 and c_2: (1:0), (2:1), (5:4) or (1:2). The optimal tranching strategy varies accordingly.

6.2 Investment in Real Options

The data for this illustration are identical to those in Example 2 in §4. To the economy in that example, we introduce a firm with a choice of two production plans, for example, two alternative investments in oil exploration as discussed in Example B in §1. The first plan yields cash flows $X_1 = (1, 0, 1/2, 3/2)$ and the second plan yields cash flows $X_2 = (3/2, 1, 1/4, 3/4)$. X_1 is spanned by the securities market and has a unique price of 1 (i.e., the same price under the three pricing kernels, q_1, q_2 and q_3). However, X_2 is not spanned by the securities market. Using the values of q_j's computed earlier and applying Lemma 1, we find that $V^-(X_2) = 0.5$ and $V^+(X_2) = 1.5$. It also follows from Theorem 2 that buying derivative instruments on the financial market will not enhance the value of X_2. Thus, in the absence of any buyers for unattainable claims, the firm might choose plan X_1 over X_2.

Now suppose that there exist individuals willing to buy claims $Y_1 = (1, 0, 0, 0)$ and $Y_2 = (0, 1, 0, 0)$ (defined earlier) at prices $p_1 = 0.34$ and $p_2 = 0.2$, respectively. Note that the prices of Y_1 and Y_2 satisfy the no-arbitrage condition computed in Example 2 in §4 ($0 \leq p_2 \leq 1/3,\ 0 \leq p_1 \leq 1 - 3p_2$). Referring to Figure 1(b), the set of feasible probability measures given p_1 and p_2 correspond to the triangle D′F′G′. Substituting the values of p_1 and p_2, the extreme points of this triangle yield the following probability measures:

$$\begin{aligned}
q_{D'} &= (1 - 3 \cdot 0.2) \cdot q_1 + 0 \cdot q_2 + 3 \cdot 0.2 \cdot q_3 \\
&= 0.4q_1 + 0.6q_3 \\
&= (0.4, 0.2, 0, 0.4), \\
q_{F'} &= (0.34, 0.20, 0.03, 0.43), \\
q_{G'} &= (0.66, 0.34/3, 0, 0.68/3).
\end{aligned}$$

By Theorem 2, the optimal value of X_2 is given by the minimum value of X_2 under the restricted set of probability measures, Θ_A, given by the triangle D$'$F$'$G$'$. Since the extreme points of this triangle are known, we apply Lemma 1 to obtain $\min_{q \in \Theta_A} E_q[X]$. We have,

$$
\begin{aligned}
E_{q_{D'}}[X_2] &= 0.4 \cdot 3/2 + 0.2 \cdot 1 + 0 \cdot 1/4 + 0.4 \cdot 3/4 = 1.1, \\
E_{q_{F'}}[X_2] &= 1.04, \\
E_{q_{G'}}[X_2] &\simeq 1.273.
\end{aligned}
$$

Thus, the optimal value of X_2 is 1.04. It is obtained by stripping the attainable claim $(1/2, 0, 1/4, 3/4)$, selling it for $V^-(X_1)$, and selling one unit each of Y_1 and Y_2 at prices 0.34 and 0.2, respectively. Thus, the value of X_1 is enhanced by finding buyers willing to purchase Y_1 and Y_2. This changes the optimal decision of the firm.

Relating to Example B in §1, note that options X_1 and X_2 could represent the two opportunities for oil exploration. X_1 corresponds to payoffs from oil with low sulfur content, which can be hedged perfectly using derivative instruments on the benchmark WTI grade. X_2 represents payoffs from oil with high sulfur content, which cannot be hedged perfectly in this manner. Y_1 and Y_2 represent investment in oil refineries that are capable of processing oil with different grades of sulfur. The presence of Y_1 and Y_2 puts a premium on oil with high sulfur content and enhances the value of the firm.

7 Discussion and Conclusions

It is well known from the previous literature that when markets are incomplete, it is not possible to compute all asset prices in a unique manner using arbitrage principles since some states are not spanned by traded claims. Attainable claims have unique prices, whereas only bounds can be established for the others. Hence, it is difficult for firms or investors to establish the optimality of their asset choices. Our paper adds to this literature by examining how incompleteness causes the market to place a premium on assets that augment the spanning of the market by existing traded claims.

In particular, we demonstrate that a firm that has an opportunity to invest in an asset can maximize its value by stripping away the maximal attainable portion of the pool of claims for which market prices can be readily established. The remaining part of the cash flows is packaged and sold to willing agents at prices that do not create arbitrage. Our framework and results have several applications to common financial problems relating to the valuation of assets, real and financial. We discuss two such problems, one relating to mergers and the other to securitization.

It has been documented in the academic and practitioner literature that the synergies from a merger of two companies may come from several sources, economies of scale or scope from their operations, improvement in their debt capacity, etc. All these explanations deal with the benefits created in individual states of nature where the cash flow of the combined firm are larger than the sum of the cash flows of the parts taken separately. We provide a different rationale. We argue that if the merger produces synergies *across* states rather *within* states, it may increase the span of the market by creating new claims as described above.

Another application is the area of securitization, discussed in detail in the example in the text. In this problem, our analysis provides guidance in terms of the optimal combination of securities that can be pooled and tranched from a universe of available securities, so as to maximize the benefit to the financial intermediary that designs and implements the structure.

Our paper also provides a plausible hypothesis for the eventual completion of markets. We show that in an incomplete market, firms have an incentive to produce assets X that are not spanned by the market. The inclusion of X expands the set of agents willing to trade unattainable claims Y_i, and tightens the bounds on their prices. Thus, incrementally, the market is brought closer to completeness. Finally, our approach can be applied within either the preference-based or the approximate arbitrage based pricing approach to further sharpen the price bounds given by those methods.

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Appendix

Proof of Lemma 1. Consider the linear program:

$$\min \; z$$

subject to

$$z \geq \sum_{k=1}^{K} q_j(\omega_k) Z(\omega_k) \qquad j = 1, \ldots, J$$
$$z \text{ unsigned.}$$

If $z \geq \sum_k q_j(\omega_k)Z(\omega_k)$ for all j, then $\sum_j \delta_j z \geq \sum_j \sum_k \delta_j q_j(\omega_k)Z(\omega_k)$ for all $\delta_j \geq 0, \sum_j \delta_j = 1$. Thus, $z \geq \sup_{q\in\Theta} E_q[Z(\omega_k)]$. Therefore, the optimal solution must be greater than or equal to $V^+(Z)$. On the other hand, $z = \max_{j\in J} E_{q_j}[Z]$ is a feasible solution to the linear program. But $\max_{j\in J} E_{q_j}[Z] \leq \sup_{q\in\Theta} E_q[Z(\omega_k)]$. Thus, $V^+(Z) = \max_{j\in J} E_{q_j}[Z]$. Similarly, it can be shown that $V^-(Z) = \min_{j\in J} E_{q_j}[Z]$. □

Proof of Lemma 2. If, for some $\alpha \in \Re^I$, $\sum \alpha_i p_i > V^+(\sum \alpha_i Y_i)$, then it is possible to create an arbitrage position by purchasing the attainable claim $S^+(\sum \alpha_i Y_i)$ at price $V^+(\sum \alpha_i Y_i)$ and selling the portfolio $\sum \alpha_i Y_i$ at price $\sum \alpha_i p_i$. Thus, the inequality is necessary. The proof of necessity of $V^-(\sum \alpha_i Y_i) \leq \sum \alpha_i p_i$ is similar. Moreover, the inequalities are sufficient because for the establishment of an arbitrage position, we must have that for some $\alpha \in \Re^I$, $\sum \alpha_i p_i$ lies outside the interval $[V^-(\sum \alpha_i Y_i), V^+(\sum \alpha_i Y_i)]$. □

Proof of Theorem 1. We assume that $\alpha_i \geq 0$ for all i. This is a technical assumption required to facilitate the proof. It does not result in a loss of generality. In particular, it does not restrict short sales, because if there exists a contingent claim Y_i that agents are willing to both buy and sell at price p_i, then this claim can be represented by two claim-price pairs (Y_i, p_i) and $(Y_j, p_j) = (-Y_i, -p_i)$. Thus, we need to prove Theorem 1 only for condition (i).

Sufficiency: Consider any $\alpha > 0$. If condition (i) holds for some $q \in \Theta$, then we have

$$\sum_i \alpha_i p_i \leq \sum_i \sum_k q(\omega_k) \alpha_i Y_i(\omega_k).$$

Therefore, $\sum \alpha_i p_i \leq V^+(\sum \alpha_i Y_i)$.

An alternative proof of sufficiency is obtained using linear programming. This also provides the setup for establishing the necessity of condition (i). Let $\mathbf{P_L}$ be the linear program

$$\max \sum_{i=1}^{I} \alpha_i p_i - Z \tag{14}$$

subject to

$$Y(\omega_k) - \sum_{i=1}^{I} \alpha_i Y_i(\omega_k) = 0 \qquad k = 1, \ldots, K \tag{15}$$

$$\sum_{k=1}^{K} q_j(\omega_k) Y(\omega_k) - Z \leq 0 \qquad j = 1, \ldots, J \tag{16}$$

$$Y(\omega_k) \text{ unsigned for all } k, \qquad Z \text{ unsigned}, \qquad \alpha_i \geq 0 \text{ for all } i. \tag{17}$$

This linear program maximizes the profit that any agent in the market can make by purchasing an attainable claim, splitting it into components $\alpha_i Y_i$, and selling them to the respective buyers at prices $\alpha_i p_i$. Constraint (15) computes the portfolio Y by adding up the cash flows $\alpha_i Y_i$ for all i. Constraint (16) computes the cost Z of creating Y from the securities traded in the market. Since Y may not be attainable, the minimum cost of creating Y from the securities traded in the market is equal to $V^+(Y)$. This is so because, by the definition of $V^+(Y)$, there exists an attainable claim

$S^+(Y)$ which is greater than or equal to Y in all states of nature and has price $V^+(Y)$. Thus, Z must be greater than or equal to $V^+(Y)$. Applying Lemma 1, this is so if Z is larger than the expected value of Y under each of the pricing measures q_j. Therefore, we get the constraints (16). The objective function of $\mathbf{P_L}$ represents the amount of profit that can be made by purchasing $S^+(Y)$ at price Z and selling $\sum \alpha_i Y_i$ at price $\sum_i \alpha_i p_i$.

Assume to the contrary that the condition in Lemma 2 does not hold, i.e., there exists an α such that $V^+(\sum \alpha_i Y_i) < \sum \alpha_i p_i$. Fix α at this value. Under this assumption, we get an arbitrage by purchasing the attainable claim $S^+(\sum_i \alpha_i Y_i)$ at Z and selling $\sum_i \alpha_i Y_i$ at price $\sum_i \alpha_i p_i$. Therefore, $\mathbf{P_L}$ is unbounded. Thus, by the strong duality theory, the dual of $\mathbf{P_L}$ is infeasible. The dual program, denoted $\mathbf{D_L}$, is shown below. Here, q_k and π_j are the dual variables corresponding to constraints (15) and (16), respectively.

$$\min 0 \tag{18}$$

subject to

$$\sum_{k=1}^{K} q_k Y_i(\omega_k) \geq p_i \qquad i = 1, \ldots, I \tag{19}$$

$$q_k - \sum_{j=1}^{J} \pi_j q_j(\omega_k) = 0 \qquad k = 1, \ldots, K \tag{20}$$

$$\sum_{j=1}^{J} \pi_j = 1 \tag{21}$$

$$q_k \text{ unsigned}, \qquad \pi_j \geq 0. \tag{22}$$

Here, constraints (20)-(22) imply that q is a pricing measure in set Θ that is obtained by taking a convex combination of q_j's with weights π_j. Constraints (19) hold if the expectation of Y_i computed under the measure q is smaller than p_i for each i. Thus, constraints (19)-(22) are equivalent to (1). Thus, the infeasibility of $\mathbf{D_L}$ implies that there does not exist any $q \in \Theta$ such that (1) holds for all i. Therefore, if $V^+(\sum \alpha_i Y_i) < \sum_i \alpha_i p_i$ for any α, then there does not exist any $q \in \Theta$ such that

(1) holds for all i.

Necessity: Note that $\mathbf{P_L}$ is always feasible and $Z = \alpha_i = 0$ is a feasible solution. Therefore, if there is no arbitrage, then $Z \leq \sum_i \alpha_i p_i$ for all α. Thus, $\mathbf{P_L}$ has an optimal solution equal to 0. This implies that $\mathbf{D_L}$ also has an optimal solution equal to 0. The optimal solution of $\mathbf{D_L}$ gives a $q \in \Theta$ such that (1) holds for all i. □

Proof of Lemma 3. The only if part is obvious since it can be easily shown that if the time 0 value of the combined time 1 cash flows from X and Y is separable in X and Y, then the value of X cannot be enhanced by augmenting it with Y.

The if part follows from assuming that the time 0 value of the combined cash flows is a function, $g(X,Y)$. But then, the time zero value of a combination of X and Y must account for the fact that the decision maker has to acquire the contingent claim at time 0. Due to the fact that q is the unique risk neutral probability measure, the time 0 fair price of the contingent claim is $E_q[Y]$. Thus, the "net" value of the cash flows at time zero equals $g(X,Y) - E_q[Y]$ for all Y. In particular,

$$g(X,Y) - E_q[Y] = g(X,0) - E_q[0].$$

Therefore,

$$g(X,Y) = g(X,0) + E_q[Y].$$

□

Proof of Theorem 2. Consider the dual of $\mathbf{P_2}$, $\mathbf{D_X}$:

$$\min \sum_k \lambda_k X(\omega_k) \tag{23}$$

subject to

$$\lambda_k - \sum_j \mu_j q_j(\omega_k) = 0 \qquad k = 1, \ldots, K \tag{24}$$

$$\sum_j \mu_j = 1 \tag{25}$$

$$\sum_k \lambda_k Y_i(\omega_k) \geq p_i \qquad i = 1, \ldots, I \tag{26}$$

$$\lambda_k, \mu_j \geq 0 \qquad j = 1, \ldots, J, \ k = 1, \ldots, K. \tag{27}$$

Here, λ_k and μ_j are the dual variables corresponding to the constraints (5) and (6), respectively.

Observe from constraints (24) and (25) that λ is a pricing measure in Θ. Also, from constraint (26), λ satisfies (1) in Theorem 1. Thus, $\lambda \in \Theta_A$. Since Theorem 1 holds, the dual problem $\mathbf{D_X}$ has at least one feasible solution. Further, the primal problem $\mathbf{P_2}$ is also feasible since $\bar{X} = 0, z = 0, \alpha = 0$ is a feasible solution. Thus, both the primal and dual problems have optimal solutions. These solutions are equal by weak duality theory.

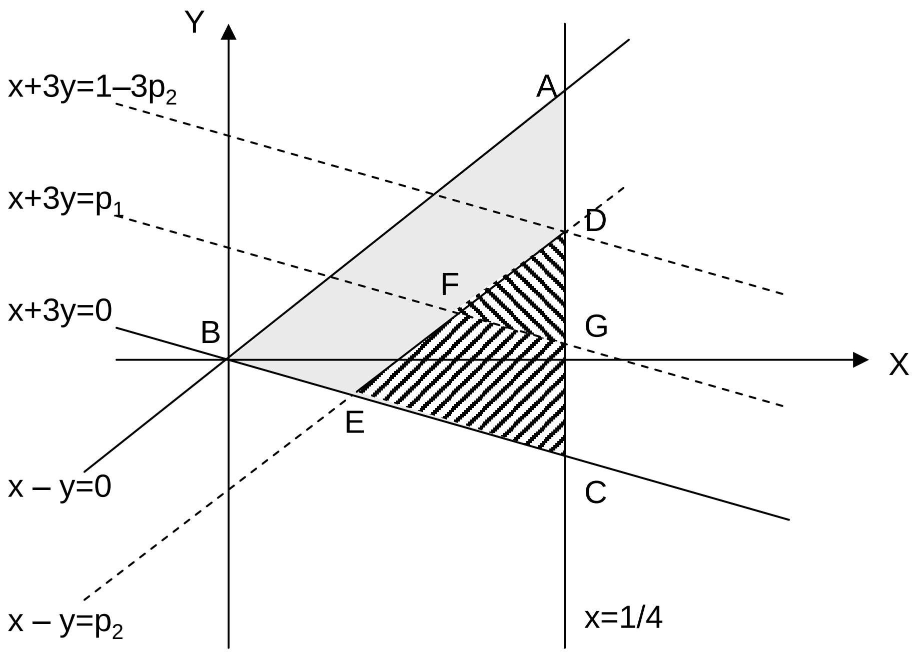
Also note that the objective function of $\mathbf{D_X}$ is the expectation of X under λ. Therefore, the optimal solution of $\mathbf{D_X}$ (and of $\mathbf{P_2}$) is equal to $\min_{q \in \Theta_A} E_q[X]$. By Lemma 1, the optimal solution is further equal to $\min_l E_{q_{Al}}[X]$. $\square$

Proof of Corollary 1. (i) From Theorem 1, if Θ_B is empty, then there exist weights $\beta_j \geq 0$ such that $\sum_j \beta_j c_j < V^-(\sum_j \beta_j X_j)$. Therefore, $V^-(\sum_j \beta_j X_j) - \sum_j \beta_j c_j > 0$ so that there is value to pooling claims X_j in the proportion given by β_j for all j. Conversely, if Θ_B is not empty, then for all $\beta_j \geq 0$, we have that $\sum_j \beta_j c_j \geq V^-(\sum_j \beta_j X_j)$, so that there is no value to pooling without tranching.

(ii) If $\Theta_A \bigcap \Theta_B = \emptyset$, then applying Theorem 1 to claims $\{X_j\}$ and $\{Y_i\}$, there exist (α, β) such that $\sum_j \beta_j c_j - \sum_i \alpha_i p_i < V^-(\sum_j \beta_j X_j - \sum_i \alpha_i Y_i)$. Thus, there is value in pooling the claims $\{X_j\}$ in ratio β, selling tranches $\{\alpha_i Y_i\}$, and selling the remaining payoffs at $V^-(\sum_j \beta_j X_j - \sum_i \alpha_i Y_i)$.

(iii) Let $q \in \Theta_A \bigcap \Theta_B$. Then, from Theorem 1, $\sum_j \beta_j c_j \geq E_q[\sum_j \beta_j X_j]$ for all $\beta \geq 0$. Further, from Theorem 2, the value of $\sum_j \beta_j X_j$ is less than or equal to $E_q[\sum_j \beta_j X_j]$. Thus, there is no value in pooling or tranching. $\square$

Figure 1(a): Graph of feasible values of *x* and *y* for Example 2

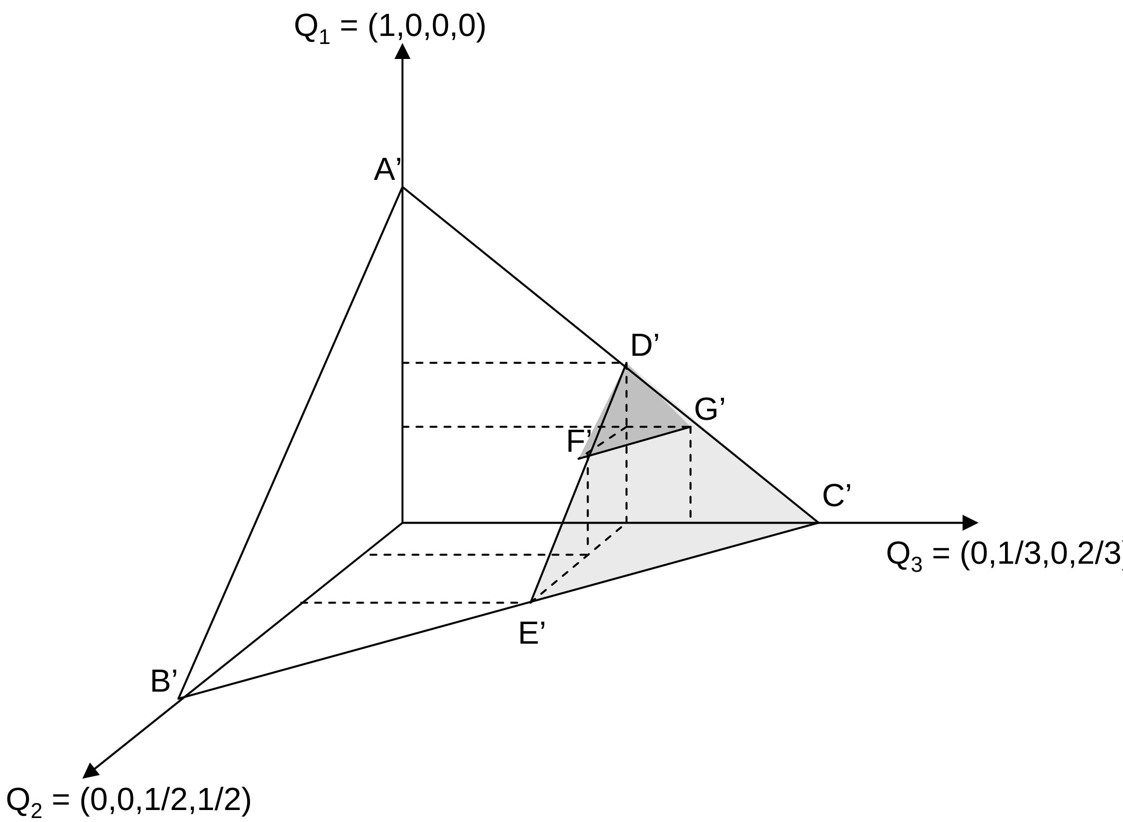


The figure illustrates the feasible set of values of x and y in Example 2. The triangle ABC represents the feasible set in the absence of information about thinly traded claims. When individuals are willing to buy claims Y_2 (or Y_1 and Y_2), the set shrinks to DEC (or even further to DFG).

Coordinates of the labeled vertices:

A = (1/4, 1/4), B = (0, 0), C = (1/4, -1/12), D = (1/4, 1/4 – p_2),
E = (3/4 p_2, – 1/4 p_2), F = (3/4 p_2 + 1/4 p_1, 1/4 p_1 – 1/4 p_2), G = (1/4, 1/3 p_1 – 1/12).

Figure 1(b): Graph of feasible pricing measures for Example 2



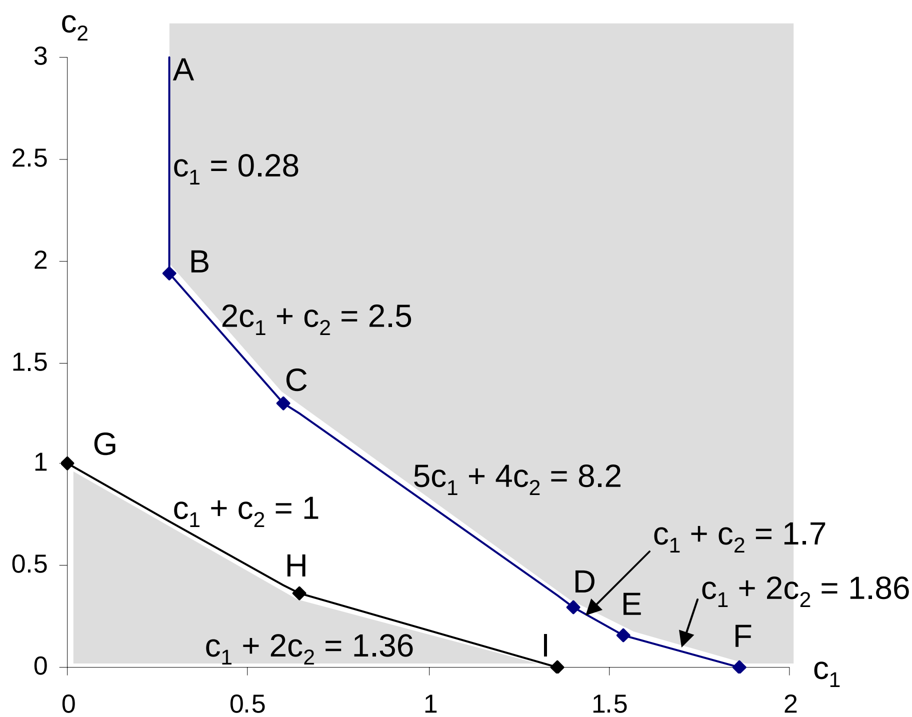
The figure illustrates the feasible set of pricing kernels in Example 2. The axes represent the independent pricing kernels that span Θ. Points in the region A'B'C' represent all feasible pricing kernels as linear combinations of the independent pricing kernels when there is no information about thinly traded claims. This set shrinks to D'E'C' when individuals are willing to buy claim Y_2 at price p_2, and further shrinks to D'F'G' when individuals are willing to buy Y_1 and Y_2 at prices p_1 and p_2, respectively. Sets A'B'C', D'E'C' and D'F'G' correspond to ABC, DEC and DFG, respectively, in Figure 1(a).

The coordinates of the labeled vertices are:

A' (1, 0, 0),	B' (0, 1, 0),	C' (0, 0, 1),	D' ($1-3p_2$, 0, $3p_2$),
E' (0, $1-3p_2$, $3p_2$),	F' (p_1, $1-p_1-3p_2$, $3p_2$),	G' (p_1, 0, $1-p_1$).	

For example, F' corresponds to the pricing kernel:
$p_1 * Q_1 + (1-p_1-3p_2) * Q_2 + 3p_2 * Q_3 = (p_1, p_2, 1/2-p_1/2-3p_2/2, 1/2-p_1/2+p_2/2)$.

Figure 2: Value of securitization in the example in §6.1 as a function of c_1 and c_2



A (0.28, ∞)	B (0.28, 1.94)	C (0.6, 1.3)	D (1.4, 0.3)	E (1.54, 0.16)
F (1.86, 0)	G (0, 0.68)	H (0.64, 0.36)	I (1.36, 0).	

In the region below the curve GHI, pooling by itself has value. In the region between ABCDEF and GHI, pooling by itself has no value but pooling with tranching has value. Tranching of the obligations of firm 1 without pooling has value to the left of the line $c_1 = 0.28$. Tranching of the obligations of firm 2 without pooling never has value. In the region above and to the right of the curve ABCDEF, neither pooling nor tranching has value.