Discrete Quantile Estimation

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Abstract

We consider estimation of a quantile from a *discrete* distribution. This gives rise to three new ideas, the confidence set for such a quantile, the notion that the associated confidence level can be increased after the data are collected, and that it is legitimate to strive to obtain a singleton confidence set. We develop properties of the sample quantile noting that the behavior for discrete populations is very different from the behavior for continuous populations. We illustrate the results with simulations and examples.

KEY WORDS: Quantile estimation, Discrete parameter, Discrete population, Ex-post confidence, Singleton confidence set

1 Introduction

We develop statistical inference for a specified quantile based on a random sample from a *discrete* population. Quantile estimation has been well developed for continuous populations (see David (1981), Noether (1967), Scheffé and Tukey (1945), and for more recent contributions, see Hettmansperger and Sheather (1986), Chen (2000), and Ozturk and Deshpande (2006).) But the discrete population case has received very little attention in the literature. Our purpose is to develop quantile estimation methods specifically designed for discrete populations.

We derive an exact distribution and consider asymptotic properties of the sample quantile. We then develop a new large sample confidence set for a population quantile which is based on inversion of a hypothesis test (TI). We also apply the result from David (1981) to construct a confidence set for a population quantile based on order statistics (OS). The OS confidence set is similar to the usual confidence interval for a quantile from continuous population. For a given sample and specified confidence the two methods, TI and OS, may produce different confidence sets.

The distinguishing feature of the estimation of a quantile from a discrete population is that the nominal level confidence set for a population quantile, obtained by either method, is conservative; the observed coverage is more than the nominal confidence. Therefore we propose, once the confidence set is obtained, to enlarge the confidence in this set by examining the data. We refer to the enlarged confidence as *ex-post* confidence. Simulation results reported in this paper overwhelmingly show that *ex-post* confidence gives a more realistic appraisal than the nominal one. The simulations involving the comparison of *ex-post* confidence with the nominal one required a novel simulation approach because the *ex-post* confidence, unlike the nominal one, changes from sample to sample. The *ex-post* and nominal confidence levels are compared using a conceptual betting game, and the closeness of the betting game to the fair game is the criterion.

The other important aspect of the discrete estimation is that it is legitimate to ask for a singleton confidence set, that is, a confidence set consisting of only the sample quantile. We implement a heuristic, based on TI method, procedure to obtain a singleton confidence set. This is a multi-stage procedure such that in each stage the sample size is appropriately increased until the confidence set is a singleton or until a pre-specified bail-out sample size is reached. The simulations and examples for a multi-stage procedure show that a correct singleton set is relatively easy to obtain with large sample sizes. Very large sample sizes may, however, be required when sampling from the populations, in which the population quantile is extremely difficult to identify.

The paper is organized as follows. In Section 2 basic definitions are introduced and background literature is briefly discussed. The exact distribution, illustrated with simulation, and asymptotic properties of sample quantile are obtained in Section 3. In Section 4 we obtain a new large sample confidence set for the population quantile and also discuss confidence set based on order statistics. The ex-post confidence level is defined in Section 5. The characterization of a singleton confidence set and a heuristic procedure for obtaining such a set are discussed in Section 6. Simulation results and examples are presented in Sections 7 and 8. The appendix derives the expression for the probability that the sample quantile is not unique.

2 Notation and Background

For any random variable X, discrete or continuous, and for any value f between 0 and 1, define the population $f-$ quantile as any value ξ for which

$$\begin{aligned} P(X &\leq \xi) \geq f \\ P(X &\geq \xi) \geq 1-f \end{aligned}$$

The value ξ need not be unique. Based on a sample of independent values $X_1, X_2,...,X_n$ the sample $f-$quantile is any $\hat{\xi}$ for which

$$\begin{aligned} \text{number of } \{X_i \ \leq \ \hat{\xi}\} \ &\geq \ nf \\ \text{number of } \{X_i \ \geq \ \hat{\xi}\} \ &\geq \ n(1-f) \end{aligned}$$

If nf is an integer, then $\hat{\xi}$ can be nonunique. If the random variable X is sampled from density g, if ξ is unique, and if $g(\xi)> 0$, then, see David (1981, p. 255), the limiting distribution of $\hat{\xi}$ is asymptotically normal with mean ξ and standard deviation $\frac{1}{g(\xi)}\sqrt{\frac{f(1-f)}{n}}$.

A confidence interval for ξ in the continuous case can be formed from the limiting normal distribution, but one can also give an exact nonparametric confidence interval. With a sample of n, and integers r and s with $1 \leq r < s \leq n$, it is shown routinely, see e.g., David (1981, p 15), that

$$P(X_{(r)} \leq \xi \leq X_{(s)}) \ = \ \sum_{i\,=\,r}^{s-1} \binom{n}{i} f^{\,i}\,(1-f)^{n-i}$$

The desired confidence, say $1-\alpha$, is specified in advance, and then r and s are selected so that $\sum_{i\,=\,r}^{s-1} \binom{n}{i} f^{\,i}\,(1-f)^{n-i} \ \geq \ 1-\alpha$, but as close as possible to $1-\alpha$. Then interval $[X_{(r)}, X_{(s)}]$ is to be used as $1-\alpha$ confidence interval. The analyst may have to compromise between "as close as possible to $1-\alpha$" and the length $s-r$.

If the random variable X is discrete, the corresponding result from David (1981, p 16), for the *closed* interval is

$$P(X_{(r)} \leq \xi \leq X_{(s)}) \ \geq \ \sum_{i\,=\,r}^{s-1} \binom{n}{i} f^{\,i}\,(1-f)^{n-i} \tag{1}$$

This had been observed earlier by Scheffé and Tukey (1945) and by Noether (1967). The closed interval $[X_{(r)}, X_{(s)}]$ can always be used with confidence at least $1-\alpha$, but it might be highly conservative when X is discrete. For example, the desire for a 95% confidence interval might lead to a request for the interval $(X_{(106)}, X_{(148)})$. The data could have $X_{(92)} = X_{(106)}$ and $X_{(148)} = X_{(159)}$ and the confidence associated with $(X_{(92)}, X_{(159)})$ would be larger than 0.95, perhaps by a lot. This begs the question as to whether examination of the data could precede selection of the desired confidence coefficient and leads to the definition of the *ex-post* confidence in Section 5.

3 Distribution of a Sample Quantile

In this section we derive the exact distribution of the sample $f-$quantile from a discrete population. Let X be a discrete random variable, and assume for convenience that its support is a subset of nonnegative integers $\{0,1,2,3,\dots\}$. Let $p_i = P(X = i)$, and then define the cumulative probabilities as $p_i^{LE} = P(X \le i)$,and $p_i^{GE} = P(X \ge i)$. Let f be a quantile of interest, so that $f = 0.5$ if we are seeking the median, $f = 0.75$ if we are seeking the upper quartile, and so on. The integer ξ will be the population $f-$ quantile when

$$\begin{aligned} p_\xi^{LE} &= P(X \le \xi) \ \ge \ f \\ p_\xi^{GE} &= P(X \ge \xi) \ \ge \ 1-f \end{aligned} \tag{2}$$

The population $f-$quantile need not be unique. If $p_\xi^{LE} = f$ then both integers ξ and $\xi+1$ qualify as a population quantile. From a random sample $X_1, X_2,\dots, X_n$ define

$$\begin{aligned} Y_j &= \sum_{i=1}^{n} I(X_i = j) \ = \ \text{number of sample values equal to } j \\ Y_j^{LE} &= \sum_{i=1}^{n} I(X_i \le j) \ = \ \text{number of sample values less than or equal to } j \\ Y_j^{GE} &= \sum_{i=1}^{n} I(X_i \ge j) \ = \ \text{number of sample values greater than or equal to } j, \end{aligned}$$

and $\hat{p}_j = Y_j/n$, $\hat{p}_j^{LE} = Y_j^{LE}/n$, $\hat{p}_j^{GE} = Y_j^{GE}/n$. Use $\hat{m}$ to denote the sample quantile. The standard definition is that $\hat{m}$ is equal to any integer k for which

$$\begin{aligned} \hat{p}_k^{LE} &\ge f \\ \hat{p}_k^{GE} &\ge (1-f). \end{aligned} \tag{3}$$

The sample quantile is guaranteed to be unique whenever nf is non-integer. To avoid confusion about the possible non-uniqueness of the sample quantile when nf is an integer, we make this new definition: $\hat{m}$ = the smallest integer qualifying as sample quantile and $U = 1$ if sample quantile is unique and $U = 0$ if sample quantile is non-unique. Also note that the sample quantile k is nonunique iff one of the inequalities in (3) is an equality.

This definition of $\hat{m}$ implies that it is always a value in the population distribution. The situation $\{U = 0\}$ involving three or more consecutive integers is exceedingly unlikely, especially when the sample size is large. We will nonetheless obtain exact calculations related to $\{U = 0\}$. The probability distribution of $\hat{m}$ will be given in terms of f, the p_i's, and the sample size n. Define B as the integer below nf meaning

$$\begin{aligned} B &= \lfloor nf \rfloor \ \text{ the integer part of } nf, \text{ when } nf \text{ is not an integer} \\ &= nf - 1 \text{ when } nf \text{ is an integer.} \end{aligned}$$

3.1 Exact distribution of sample quantile

The event $\{\hat{m} > k\}$ is exactly equivalent to $\left\{Y_k^{LE} \le B\right\}$. A short example illustrates the concern with whether or not nf is an integer. Suppose that n=100 and that $f = \frac{1}{3}$. Then $nf = 33\frac{1}{3}$, and $B = 33$. The $\frac{1}{3}-$quantile is bigger than 18 provided that no more than 33 values are at or below 18. In symbols, $\{\hat{m} > 18\} \equiv \{Y_{18}^{LE} \le 33\}$. Now suppose that $n = 100$ and that $f = 0.30$. Then $nf = 30$, and $B = 29$. The 0.30 quantile is bigger than 16 provided that no more than 29 values are at or below 16. In symbols, $\{\hat{m} > 16\} \equiv \{Y_{16}^{LE} \le 29\}$.

Certainly

$$P(Y_k^{LE} \le B) \;\; = \;\; \sum_{j\,=\,0}^{B} \binom{n}{j} \left(p_k^{LE}\right)^j \; (1 - p_k^{LE})^{n-j}.$$

Thus

$$\begin{aligned} P(\hat{m} \;\; = \;\; k) \;\; &= \;\; P(\hat{m} > k-1) - P(\hat{m} > k) = \;\; P(Y_{k-1}^{LE} \le B) - P(Y_k^{LE} \le B) \\ &= \;\; \sum_{j\,=\,0}^{B} \binom{n}{j} \left(p_{k-1}^{LE}\right)^j \; (1 - p_{k-1}^{LE})^{n-j} \;\; - \;\; \sum_{j=0}^{B} \binom{n}{j} \left(p_k^{LE}\right)^j \; (1 - p_k^{LE})^{n-j}. \quad (4) \end{aligned}$$

This result can be rearranged in a number of ways, but the resulting forms are not nearly as computationally useful. An enumeration of $P(Y_0^{LE} \le B)$, $P(Y_1^{LE} \le B)$, $P(Y_2^{LE} \le B)$, ... will permit the immediate calculation of $P(\hat{m} = 0), P(\hat{m} = 1), P(\hat{m} = 2)$, and so on.

An extension of this logic will permit the calculation of $P(\hat{m} = k, U = 0)$, the probability that k is the smallest of two or more sample $f-$quantile values, in the case that nf is an integer ($B = nf - 1$). Note that $\{\hat{m} = k, U = 0\} \;\; = \;\; \left\{Y_{k-1}^{LE} \le B\right\} \cap \left\{Y_k^{LE} = B + 1\right\}$. The situation requires that Y_k^{LE} be equal to $B + 1 = nf$ exactly. Then

$$\begin{aligned} P\left(\hat{m} = k, U = 0\right) \;\; &= \;\; P\left[\; \left\{Y_{k-1}^{LE} \le B\right\} \cap \left\{Y_k^{LE} = B + 1\right\} \;\right] \\ &= \;\; (1 - p_k^{LE})^{n-B-1} \binom{n}{B+1} \left[\; \left(p_k^{LE}\right)^{B+1} - \left(p_{k-1}^{LE}\right)^{B+1} \;\right]. \quad (5) \end{aligned}$$

The computation of expression (5) is shown in the appendix. We now consider a simulation example illustrating (4) and (5).

Example 1 Consider the random variable X of Table 1 defined on the set {0, 1, 2,. . . , 20}, and suppose that the estimation of quantile $f = 0.3$ is sought based on a sample of size 100. The first three columns in Table 1 show only that part of the distribution of X most relevant to the problem. The population 0.30$-$quantile is 7. Here nf is an integer, and the sample quantile need not be unique. The fourth column was computed as $P\left(\hat{m} = k\right) - P\left(\hat{m} = k, U = 0\right)$ and fifth column as $P\left(\hat{m} = k, U = 0\right)$ using equations (4) and (5). The theoretical probability that the sample quantile will be uniquely 7 is 0.7940, and the probability that both 7 and 8 will qualify as sample quantile is 0.0269. The sixth and seventh columns show the results of 1,000,000 simulation runs. It can be seen, as expected, that the simulated probabilities are close to the theoretical ones. There were no cases in these million trials in which the sample quantile was shared among three integers.

With smaller values of n, multiple quantiles (three or more) are more likely. For the same situation with $n = 40$, there were 801 simulation runs, out of 1,000,000, in which three or more values qualified as the sample quantile. Of these 801, there were two cases in which the sample quantile was shared among four consecutive integers.

3.2 Asymptotic behavior of sample quantile

We consider limiting behavior of the sample quantile as n increases. We will first show that the sample quantile converges in probability to the population quantile and specify the rate at which this convergence takes place. This convergence rate also shows that the sample quantile converges to the population quantile almost surely.

Using the usual normal approximation to the binomial distribution we obtain, for large n, an approximate expression for the probability in (4)

$$P(\hat{m} = k) = P(Y_{k-1}^{LE} < B) - P(Y_k^{LE} < B) \approx \Phi\left(\sqrt{n}g_{k-1}\right) - \Phi\left(\sqrt{n}g_k\right) \tag{6}$$

where Φ is the standard normal cumulative, $g_k = \left(f - p_k^{LE}\right)/\sqrt{p_k^{LE}(1-p_k^{LE})}$, and we used the approximations: $B = \lfloor nf \rfloor / n \approx f$ and $B = (nf-1)/n \approx f$, which hold for large n. Assume that ξ, the population f -quantile, is unique. Then by (6)

$$P(\hat{m} = \xi) = P(Y_{\xi-1}^{LE} < B) - P(Y_\xi^{LE} < B) \approx \Phi\left(\sqrt{n}g_{\xi-1}\right) - \Phi\left(\sqrt{n}g_\xi\right)$$

By the definition of the unique population population quantile we have that $g_{\xi-1} > 0$ and $g_\xi < 0$, so $\Phi\left(\sqrt{n}g_{\xi-1}\right)$ converges to 1 and $\Phi\left(\sqrt{n}g_\xi\right)$ converges to 0. Thus $P(\hat{m} = \xi)$ converges to 1 as $n \to \infty$. If $\phi(x)$ is the standard normal density function, we can investigate the asymptotic rate of this convergence through the approximation $1 - \Phi(x) \approx \phi(x)/x$ which holds when $x \to \infty$ and through the approximation $1 - \Phi(x) = 1 - (1 - \Phi(-x)) = 1 - \left(\frac{\phi(-x)}{-x}\right) = 1 + \frac{\phi(x)}{x}$ when $x \to -\infty$, see Feller (1950, page 166) or Pollard (1984, Appendix B). Thus

$$P(\hat{m} = \xi) \approx \left[1 - \Phi\left(\sqrt{n}g_\xi\right)\right] - \left[1 - \Phi\left(\sqrt{n}g_{\xi-1}\right)\right] \approx 1 + \frac{\phi\left(\sqrt{n}g_\xi\right)}{\sqrt{n}g_\xi} - \frac{\phi(\sqrt{n}g_{\xi-1})}{\sqrt{n}g_{\xi-1}},$$

from which we see that $P(\hat{m} = \xi)$ converges to 1 on the order of $\frac{1}{\sqrt{n}}\exp(-n)$, considerably faster than $\frac{1}{\sqrt{n}}$ convergence.

Suppose next that the population quantile is nonunique. Let ξ_1 and ξ_2 be the two values in the population distribution which are population quantiles with $\xi_1 < \xi_2$. This means that $p_{\xi_1}^{LE} = f, p_{\xi_2}^{GE} = 1 - f$. The first equality is equivalent to $g_{\xi_1} = 0$ and $g_{\xi_1 - 1} > 0$. If the sample quantile is nonunique, then $\hat{m}$ is the smallest value so qualifying. Then by (6)

$$P(\hat{m} = \xi_1) \approx \Phi\left(\sqrt{n}g_{\xi_1-1}\right) - \Phi\left(\sqrt{n}g_{\xi_1}\right) = \left[1 - \Phi\left(\sqrt{n}g_{\xi_1-1}\right)\right] - 0.5$$

and thus $\lim_{n\to\infty} P(\hat{m} = \xi_1) = 0.5$. By a nearly identical calculation $\lim_{n\to\infty} P(\hat{m} = \xi_2) = 0.5$. Hence in the case of nonuniqness the sample quantile converges in probability to a random variable D which takes values ξ_1 and ξ_2 each with probability 0.5. Machado and Santos Silva (2003) obtained this result in the context of Bernoulli distribution. It can also be shown that $\hat{m} \to \xi$ (D) almost surely in case of uniqness (nonuniqness).

4 Confidence Sets for the Population Quantile

We first develop a new large sample confidence set with nominal confidence level $1 - \alpha$ for the population quantile and compare it with the confidence set that is given implicitly by David's formula (1). The proposed confidence set always contains a sample quantile and, in general, also other values.

4.1 Confidence set based on test inversion (TI)

This confidence set will be derived by inverting the following hypothesis test $H_0 : p_k^{LE} \geq f$ and $p_k^{GE} \geq 1 - f$ vs H_1 : one of the statements of H_0 is violated. The null hypothesis states that the population $f-$quantile is k and the alternative states that it is not. Our $(1-\alpha)100\%$ confidence set will consist of the integers k for which H_0 would be accepted with the Type I error probability of at most α. We propose this rule: reject H_0 if $\hat{p}_k^{LE} \leq f - c$ or if $\hat{p}_k^{GE} \leq 1 - f - c$ where $\hat{p}_k^{LE}$ and $\hat{p}_k^{GE}$ are defined in section 3 and c is a positive constant. The two rejection events are mutually exclusive. We now compute c so that the test has required level α :

$$P(\text{rejecting } H_0 \mid H_0) = P(\hat{p}_k^{LE} \leq f - c \mid H_0) + P(\hat{p}_k^{GE} \leq 1 - f - c \mid H_0).$$

Note that

$$P(\hat{p}_k^{LE} \leq f - c \mid H_0) \quad \leq \quad P(\hat{p}_k^{LE} \leq f - c \mid p_k^{LE} = f) \tag{7}$$

and

$$P(\hat{p}_k^{GE} \leq 1 - f - c \mid H_0) \quad \leq \quad P(\hat{p}_k^{GE} \leq 1 - f - c \mid \; p_k^{GE} = 1 - f). \tag{8}$$

We can bound each of the two probabilities above by $\alpha/2$. Given that $p_k^{LE} = f$, the distribution of Y_k^{LE} is binomial (n, f). We evaluate the second probability in (7) through a normal approximation and set it equal to $\alpha/2$:

$$\begin{aligned} P(\; Y_k^{LE} \quad \leq \quad n(f - c)\;) \quad &= \quad P\left(\frac{Y_k^{LE} - nf}{\sqrt{nf(1-f)}} \leq \frac{n(f-c) - nf}{\sqrt{nf(1-f)}}\right) \\ &\approx \quad P\left(Z \leq \frac{-nc}{\sqrt{nf(1-f)}}\right) \quad = \quad \Phi\left(\frac{-nc}{\sqrt{nf(1-f)}}\right) \quad = \quad \frac{\alpha}{2}. \end{aligned}$$

where Z is a standard normal random variable. This leads to the condition $z_{\alpha/2} = \sqrt{n}c/\sqrt{f(1-f)}$ and thus

$$c \equiv c_\alpha = z_{\alpha/2}\sqrt{\frac{f(1-f)}{n}}. \tag{9}$$

The calculation for the probability in (8) produces the same expression. Hence the $(1-\alpha)100\%$ confidence set is

$$\{k : \hat{p}_k^{LE} > f - c_\alpha \quad \text{and} \quad \hat{p}_k^{GE} > 1 - f - c_\alpha\}$$

or equivalently

$$\{k : \hat{p}_k^{LE} > f - c_\alpha \quad \text{and} \quad \hat{p}_{k-1}^{LE} < f + c_\alpha\} \tag{10}$$

where c_α is given in (9). Note that a sample quantile will always be in the confidence set.

Example 2 Suppose that $f = 0.30, n = 500$ and the sample results were $\hat{p}_{18}^{LE} = 0.236, \hat{p}_{19}^{LE} = 0.274, \hat{p}_{20}^{LE} = 0.342$, and $\hat{p}_{21}^{LE} = 0.370$. The sample quantile is 20. For a 99% confidence, $c_{0.01} = 2.5758\sqrt{\frac{0.3\times0.7}{500}} \approx 0.0528$. The 99% confidence set is $\{19, 20, 21\}$.

4.2 Confidence set based on order statistics (OS)

Let $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq ... \leq X_{(n)}$ denote the order statistics in a sample of n independent observations. We refer to (1.) in section 2, the probability inequality for the closed interval about ξ. We will use this to construct a confidence set for ξ. The right side is a function of n and f, which are given, and also r and s, which can be regarded as decision variables. As discussed in section 2, we can select r and s so that the sum exceeds $1-\alpha$ but is as close as possible to $1-\alpha$. After selecting r and s, we can claim that $[X_{(r)}, X_{(s)}]$ is a $1-\alpha$ confidence interval for ξ, in that $P(X_{(r)} \leq \xi \leq X_{(s)}) \geq 1-\alpha$. Our ($r$, s) choice is thus determined without looking at the data.

Example 3 (Example 2 revisited) With $\hat{p}_{18}^{LE} = 0.236 = 118/500$ we know that 118 of the 500 values are 18 or less. Since $\hat{p}_{19}^{LE} = 0.274 = 137/500$, we see that 137 values are 19 or less. This means that $X_{(119)} = X_{(120)} = ...= X_{(137)} = 19$. A similar strategy leads us to $X_{(138)} = X_{(139)} = ... = X_{(171)} = 20$ and also $X_{(172)} = X_{(173)} = ... = X_{(185)} = 21$.

The calculation that dictates the set is $\sum_{i\,=\,125}^{176} \binom{500}{i}\, 0.30^i\, 0.70^{n-i} \approx 0.990185 > 0.99 = 1-\alpha$. It follows that the 99% confidence set for the 0.30 quantile is $[X_{(125)}\ , X_{(177)}]$. The 125th value is 19, and the 177th value is 21. This leads to the 99% confidence set {19, 20, 21 }, which is identical to the set found by test inversion.

We note that 0.99185 is the closest confidence to the desired confidence of 0.99 that can be attained when constructing confidence set using OS method. For future reference this attained nominal confidence for OS confidence set, will be denoted by $1-\alpha'$.

5 Ex-post confidence level

As will be seen from the development below the discreteness of the population causes the confidence sets for a population quantile to be conservative, meaning that a $1-\alpha$ confidence set fails much less than α of the time. To overcome this (sometimes extreme) conservativism, we introduce the notion of *ex-post* confidence. Once the nominal $1-\alpha$ confidence level set has been obtained our objective is to find the largest confidence level that would have produced exactly the same set. We refer to this value as the ex-post confidence level. We show through simulations that the ex-post confidence gives a more realistic appraisal of confidence than the nominal value $1-\alpha$. We define the ex-post confidence for the TI and OS confidence sets.

Suppose that the $1-\alpha$ confidence set obtained by using test inversion method is $\{k,...,l\}$. This notation should be understood as permitting the singleton case $k=l$. We now seek the largest value of c (the smallest α) given in (9) that excludes $k-1$ and $l+1$ from the confidence set. Note from (10) that value $k-1$ will be excluded when $\hat{p}_{k-1}^{LE} \leq f-c$; that is, when $c \leq f-\hat{p}_{k-1}^{LE}$. Value $l+1$ will be excluded when $\hat{p}_{l}^{LE} \geq f+c$, which is $c \leq \hat{p}_{l}^{LE}-f$. The largest value of c, $c_{\max}$, that would produce the same confidence set $\{k,...,l\}$ is

$$c_{\max} \quad = \quad \min\left(f-\hat{p}_{k-1}^{LE}, \hat{p}_{l}^{LE}-f\right), \tag{11}$$

from which, using (9), we solve $z_{\alpha/2} \quad = (\sqrt{n}c_{\max})\,/\left(\sqrt{f(1-f)}\right)$ for α and denote the solution by α_{TI}. The *ex-post* confidence in the set $\{k,...,l\}$ for the test inversion method is defined as $1-\alpha_{\text{TI}}$, where

$$\alpha_{\text{TI}} \quad = \quad 2\left\{1-\Phi\left[\frac{\sqrt{n}\min\left(f-\hat{p}_{k-1}^{LE}, \hat{p}_{l}^{LE}-f\right)}{\sqrt{f(1-f)}}\right]\right\}. \tag{12}$$

We next define the *ex-post confidence* for the confidence set obtained using order statistics. This confidence set has the form $[X_{(r)}, X_{(s)}]$ for some $r \leq s$. Let q be the smallest integer such that $q \leq r$ with $X_{(q)} = X_{(r)}$ and t a largest integer such that $t \geq r$ with $X_{(s)} = X_{(t)}$. The set of values $[X_{(q)}, X_{(t)}]$ is the same set as $[X_{(r)}, X_{(s)}]$. For the order statistics method we define the *ex-post* confidence $1-\alpha_{\text{os}}$ of the set $[X_{(r)}, X_{(s)}]$ as

$$1-\alpha_{\text{OS}} \quad = \quad \sum_{i\,=\,q}^{t-1} \binom{n}{i} f^{\,i}\,(1-f)^{n-i}. \tag{13}$$

Example 4 (Example 2 revisited) We compute the *ex-post* confidence using test inversion method for the 99% confidence set $\{19, 20, 21\}$ The largest value of c that excludes 18 and 22 is $c_{\max} \quad = \quad \min\left(0.3-\hat{p}_{18}^{LE}, \hat{p}_{21}^{LE}-0.3\right) \quad = \quad \min(0.3-0.236, 0.37-0.3) \quad = \quad 0.064$ which gives $z_{\alpha_{\text{TI}}/2} \quad = \left(\sqrt{500}\times 0.064\right)/\left(\sqrt{0.3\times 0.7}\right) \approx \quad 3.1229$ and α_{TI}=0.0018, leading to *expost confidence* of 0.9982. This is of course larger than the nominal 0.99 used to construct the confidence set $\{19, 20, 21\}$. To compute the ex-post confidence using order statistics method we recall that the 99% confidence set was identified as $[X_{(125)}, X_{(177)}]$, corresponding to $\{19, 20, 21\}$. However, the data value 19 corresponds to rank positions $119, 120, ..., 137$. Also the data value 21 corresponds to rank positions $172, 173, ..., 185$. The set $[X_{(119)}, X_{(185)}]$ is also $\{19, 20, 21\}$. The *ex-post confidence* obtained by the order statistics method is the value $\sum_{i=119}^{184} \binom{500}{i} (0.30)^i (0.70)^{500-i} \approx 0.998678$. This is slightly larger than the ex-post confidence for the test inversion confidence set.

It can be shown that if the confidence sets produced by the two methods (TI and OS) are the same then $1 - \alpha_{\text{OS}} \geq 1 - \alpha_{\text{TI}}$. But simulations indicate that $1 - \alpha_{\text{OS}}$ is only slightly larger than $1 - \alpha_{\text{TI}}$.

6 Singleton confidence set

In case of a unique sample quantile, the singleton set is defined as consisting of that sample quantile. In case of nonuniqness, the "singleton" set consists of both observed sample quantiles and all values in between them. For example, if 23 and 25 are two observations in the sample that both qualify as sample quantiles then we would report the singleton confidence set as {23,24,25}. We first give give necessary and sufficient conditions for a $1 - \alpha$ TI confidence set for the population $f-$quantile, to be a singleton when the sample quantile is unique (Proposition 1) and when it is nonunique (Proposition 2).

Proposition 1 *Suppose that k is a unique sample quantile. Then the necessary and sufficient condition for the $1 - \alpha$ TI confidence set to be a singleton (contain only this sample quantile) is: $\hat{p}_{k-1}^{LE} \leq f - c_\alpha \leq f + c_\alpha \leq \hat{p}_k^{LE}$ or equivalently,*

$$c_\alpha \leq \min(f - \hat{p}_{k-1}^{LE}, \hat{p}_k^{LE} - f), \tag{14}$$

A necessary condition for the $1 - \alpha$ confidence set to be a singleton set is

$$\hat{p}_k \geq 2c_\alpha. \tag{15}$$

Proof**:** If k is a unique sample quantile then by an extension of definition (3) $\hat{p}_k^{LE} > f$, $\hat{p}_{k-1}^{LE} < f$, and thus both quantities in (14) are positive, which makes c_α well defined. Since k is a sample quantile, it is certainly in the confidence set (10), so that $\hat{p}_{k-1}^{LE} < f + c_\alpha$. But $\hat{p}_{k-2}^{LE} \leq \hat{p}_{k-1}^{LE} < f + c_\alpha$, which shows that the second inequality in (10) is automatically satisfied by integer $k - 1$. Hence for an integer $k - 1$ not to be in the confidence set we must have

$$\hat{p}_{k-1}^{LE} \leq f - c_\alpha \tag{16}$$

Similarly for an integer $k + 1$ not to be in confidence set we must have

$$\hat{p}_k^{LE} \geq f + c_\alpha. \tag{17}$$

Combining (16) and (17) gives (14). The necessary condition (15) follows by combining (16) and (17).■

Proposition 2 *Suppose that sample quantile is not unique, and k_1 and k_2, with $k_1 < k_2$, are two sample observations that are both sample quantiles. Then the confidence set does not contain any values outside $[k_1, k_2]$ if and only if*

$$\hat{p}_{k_1} \geq c_\alpha \tag{18}$$

and

$$\hat{p}_{k_2} \geq c_\alpha, \tag{19}$$

or equivalently

$$c_\alpha \leq \min(\hat{p}_{k_1}, \hat{p}_{k_2}). \tag{20}$$

Proof: Since both k_1 and k_2 are sample quantiles we must have

$$\begin{aligned} \hat{p}^{LE}_{k_1} &= f, \quad \hat{p}^{GE}_{k_1} > 1-f, \\ \hat{p}^{LE}_{k_2} &> f, \quad \hat{p}^{GE}_{k_2} = 1-f. \end{aligned}$$

Now (18) holds if and only if $\hat{p}^{LE}_{k_1-1} = \hat{p}^{LE}_{k_1} - \hat{p}_{k_1} = f - \hat{p}_{k_1} \leq f - c_\alpha$, that is, iff $k_1 - 1$ is not in the confidence set. Similarly (19) holds if and only if $\hat{p}^{GE}_{k_2+1} = \hat{p}^{GE}_{k_2} - \hat{p}_{k_2} \leq 1 - f - c_\alpha$, that is, if $k_2 + 1$ is not in the confidence set. ■

These necessary and sufficient conditions for a singleton confidence set will be rarely satisfied in practice with typical values of $1-\alpha$. This will be illustrated with the simulations below. However, if the confidence set is not a singleton we can obtain a TI singleton confidence set by selecting c according to (14), or in case of nonuniqness, according to (20). Similarly we can compute the ex-post confidence in the singleton set.

Example 5 In Example 2 the sample quantile is 20 and confidence set is $\{19, 20, 21\}$. We want to declare $\{20\}$ as the confidence set. The value of c which excludes values 19 and 21 from the original confidence set and thus produces the singleton set $\{20\}$ is

$$c_{\max} = \min\left(f - \hat{p}^{LE}_{19}, \hat{p}^{LE}_{20} - f\right) = \min(0.026, 0.042) = 0.026,$$

which leads to $z_{\alpha/2} = (\sqrt{n} c_{\max})/\sqrt{f(1-f)} \approx 1.2687$. The TI *ex-post* confidence level for the singleton set is $1 - \alpha = 0.7954$, and we would claim that $\{20\}$ is a 79.54% confidence set. The OS *ex-post* confidence in $\{20\}$ is

$$\sum_{i=138}^{170} \binom{500}{i} (0.30)^i (0.70)^{500-i} = 0.86569149 \approx 0.8657.$$

Example 5 illustrates that restricting the confidence set to a sample quantile(s) may result in a low confidence level. Thus, we propose the following multistage procedure for obtaining a singleton confidence set with a specified confidence level. There is a parallel with the classic problem of obtaining a fixed-width confidence interval for a continuous parameter. Stein (1945) does this for a nominal confidence, and he uses a two-stage procedure.

If the obtained confidence set in (10) with confidence level $1-\alpha$ is not a singleton, we increase the sample size, but keep the confidence level fixed, until we reach a singleton set or the bail-out sample size $N_{\max}$. The multistage procedure is based on the TI method of constructing the confidence set and uses Propositions 1 and 2. The OS confidence set does not allow us in any obvious way to create a multistage procedure for achieving a singleton confidence set.

Suppose that $\hat{m}=k$ is a unique sample quantile based on the initial sample of size $n<N_{\max}$. If $\{\hat{m}\}$ is the $1-\alpha$ confidence set, then the procedure stops. If the sample quantile is unique and the confidence set consists of two or more values then according to Proposition 1 we must have

$$c_\alpha^0 \equiv z_{\alpha/2}\sqrt{\frac{f(1-f)}{n}} > \min(f-\hat{p}_{k-1}^{LE}, \hat{p}_k^{LE}-f) \equiv \min(\varepsilon_{k-1}^0, \zeta_k^0),$$

where we set $\varepsilon_{k-1}^0 = f-\hat{p}_{k-1}^{LE}$ and $\zeta_k^0=\hat{p}_k^{LE}-f$, and the superscript "0" refers to the initial sample size n. The idea, motivated by lemma 1, is now to increase the sample size from n to n_1, so that the corresponding c_α, namely $c_\alpha^1 = z_{\alpha/2}\sqrt{\frac{f(1-f)}{n_1}}$, satisfies $c_\alpha^1 \le \min(\varepsilon_{k-1}^0, \zeta_k^0)$. This requires a sample size of at least

$$n_1 = \left\lceil \frac{\left(z_{\alpha/2}\right)^2 f(1-f)}{\left[\min(\varepsilon_{k-1}^0, \zeta_k^0)\right]^2} \right\rceil, \tag{21}$$

where $\lceil x\rceil$ is the function which gives the smallest integer greater than, or equal to, x. We enlarge the sample size to n_1 by taking an additional n_1-n observations. If n_1 exceeds $N_{\max}$ or if the resulting confidence set based on a sample of size n_1 is a singleton we stop. Otherwise we repeat the procedure.

If the sample quantile is not unique then, by Proposition (2), the expression in (21) should be modified by replacing $\min(\varepsilon_{k-1}^0, \zeta_k^0)$ with $\min(\hat{p}_{k_1}^0, \hat{p}_{k_2}^0)$, where k_1, k_2 are two sample quantiles satifying $k_1<k_2$. The above procedure may converge to an incorrect singleton confidence set, but, as the simulations results show, this is a rare occurence unless the required quantile is extremely diificult to identify.

7 Simulation results

The simulation study was conducted to assess whether ex-post confidence gives a more realistic appraisal of confidence than the nominal confidence, and to evaluate the performance of multistage procedure. We also compare confidence sets generated by TI and OS methods for nominal confidence.

We used a simulation approach based on a conceptual betting game. This was necessary because, unlike the nominal confidence, ex-post confidence changes from sample to sample. The betting game is this. In the j^{th} simulation round when sample j produces ex-post confidence $1-\alpha_j$, place a conceptual bet of $1-\alpha_j$ dollars on coverage against α_j dollars on noncoverage. This bet will win α_j on coverage and win $-(1-\alpha_j)$ dollars when the confidence set does not cover the population quantile. The betting game for the nominal confidence is the same, but each bet uses the same nominal confidence α. More precisely, the average ex-post payout over N simulation runs is

$$\text{average ex-post payout} \quad = \quad \frac{1}{N}\sum_{j\,=\,1}^{N}\ [\alpha_j C_j-(1-\alpha_j)(1-C_j)] = \ \frac{N_C}{N}\ -\ (1-\frac{\sum\alpha_j}{N}),$$

where N_C is the number of times the confidence set covered population quantile in N simulation rounds and $C_j = 1$ when the j^{th} confidence set covers the population quantile and $C_j = 0$ if not. This is compared to the average payout with nominal confidence $1-\alpha$ given by $N_C/N-(1-\alpha)$ for TI method and by $N_C/N-(1-\alpha')$ for OS method.The criterion for the appraisal of the two ways of assigning confidence is the fairness of the betting game. The preferred way of assigning confidence is the one for which the corresponding betting game has the average payout closer to zero.

The results of the simulations are reported in Tables 2 - 6. We used three populations, and asked for the 0.30 quantile in each. The relevant information about these populations is given in Table 3 and repeated in subsequent tables. In each population the 0.30 quantile is 79. The populations are listed in the order of the level of difficulty in identifying the 0.30 quantile from the "easy" to a very "difficult" one. In the first population the quantile was well identified, in the second it is difficult to know whether 79 or 80 is the 0.30 quantile. In the third population it is extremely difficult to know whether 78, 79 or 80 is the quantile.

In Tables 2 and 3 we used 0.99 as the nominal confidence and sample size of 83 and 803 respectively. For each combination of the sample size and the population the simulation consisted of $N = 10,000$ repetitions. It is seen that for all simulations, except for the last one in Table 2, the actual coverage is more than the nominal of 0.99. Consequently the average ex-post confidence payout for both TI and OS methods is closer to zero than the nominal one except for the one simulation, which involves the "very difficult" population and a small sample size of 83. The ex-post confidence payouts are comparable by OS and TI methods. These simulations illustrate that nominal level is in general too low and one can do better by using ex-post confidence with either TI or OS methods.

As an illustration Table 4 enumerates all confidence sets that were achieved by TS and OS methods in simulation involving the second population and sample size of 803 and also gives the most frequent confidence sets obtained by TS and OS methods for the same population when sample size was 83.

Tables 5 and 6 report simulation results for obtaining a singleton confidence set $\{79\}$ using the procedure described in Section 6. Three values were specified for the nominal confidence: 0.90, 0.95 and 0.999. For each combination of population and nominal confidence the initial sample size was 1000 and there were 10,000 simulaton runs. The bail-out sample size for each simulation run was 500,000. Table 6 lists all confidence sets achieved. The second column gives the number of times population quantile was covered in 10,000 simulations either when the singleton set was achieved or when the bail out sample size was reached. For the first population the correct singleton was achieved in all simulations for confidence levels of 0.95 and 0.999. For a lower confidence of 0.90, the procedure converged to a wrong singleton confidence set 7 out of 10,000 times. For the second population the number of wrong singleton sets decreased with the confidence level from 309 to 7. For the extremely "difficult" population at 0.999 confidence no simulation converged to a singleton confidence set, but all confidence set resulting from these simulations covered population quantile. We note that very large sample sizes are required to obtain a singleton confidence set for each considered population. In particular a sample size larger than 500,000 is required to obtain a singleton confidence set for the extremely difficult population. Further illustrations of obtaining singleton confidence set are given in the next section.

8 Estimation Examples

We illustrate our methods with the data set from the KDD-CUP competition in data mining available at http://kdd.ics.uci.edu/databases/kddcup98/kddcu. This is the 1998 competition set, which concerns contributions to a charity following a mailing solicitation. The participants were asked to devise an algorithm to predict the amount contributed. The data set has 95,413 data lines, each representing a regular donor. There are 481 data fields. Two of those data fields will be used to illustrate our methods: Field 24: Number of children living in household and Field 468: Time-lag in months between first and second gifts. For each data field we regard the set of 95,413 values as the population and want to obtain a *singleton* confidence set with 95% confidence.for the population 70th percentile based on a sample of size 200.

In a random sample of 200 from field 24 the data are these:

Value	Count	Cumulative Count
0	165	165
1	23	188
2	10	198
3	2	200
4	0	200
Total	200	

The confidence set procedure based on test inversion gives the 95% confidence set as {0}, with ex-post confidence of 0.99991. (We only report the test inversion confidence set because in this and next example the order statistics confidence set is the same as the test inversion one and has practically the same ex-post confidence.) With more than 80% of the data values at 0, this result was obvious. For the full set of 95,413 values, there are 83,027 zeroes, which gives $83,027/95,413 = 87.01\%$ of zeros so that indeed 0 is the population 70th percentile. Thus, for this population estimating the 70th percentile is a very easy problem. Our small starting sample size of 200 was sufficient to produce a singleton confidence set.

We now consider field 468, months between first and second gifts. The data from the sample of size 200 is this:

Value	Count	Cumulative Count
6	109	109
7	11	120
8	20	140
9	9	149
10	9	158
11	9	167
12	7	174
13	5	179
14	21	200
Total	200	

There are two observations in this data that qualify for a sample quantile: 8 and 9. A 95% confidence set is {7, 8, 9, 10} with ex-post confidence of 0.995132. Since the sample quantile is not unique and the confidence set contains other values than samples quantiles we now use a modified version of (21) to enlarge the sample size. With $k_1 = 8$ and $k_2 = 9$ we obtain

$$n_1 = \left\lceil \frac{\left(z_{\alpha/2}\right)^2 f(1-f)}{[\min(\widehat{p}_8, \widehat{p}_9)]^2} \right\rceil = \left\lceil \frac{(1.96)^2 \times 0.7 \times 0.30}{[\min(0.1, 0.045)]^2} \right\rceil = 399$$

so that we should increase the sample size to 399. With $n = 399$, the data results are:

Value	Count	Cumulative Count	Cumulative Proportion
6	209	209	0.523810
7	42	251	0.629073
8	22	273	0.684211
9	18	291	0.729323
10	15	306	0.766917
11	20	326	0.817043
12	15	341	0.854637
13	8	349	0.874687
14	50	399	1.000000
Total	399		

This time, the sample quantile is unique at 9. The confidence set is {8, 9, 10}, with ex-post confidence of 0.997211. We can apply formula (21) to suggest a sample size that would produce a singleton confidence set. This leads to

$$n_1 = \left\lceil \frac{\left(z_{\alpha/2}\right)^2 f(1-f)}{\left[\min(\varepsilon_8^0, \zeta_9^0)\right]^2} \right\rceil = \left\lceil \frac{(1.96)^2 \times 0.7 \times 0.30}{[\min(0.015789, 0.029323)]^2} \right\rceil = 3,237$$

We will enlarge the sample size to 3,237. With $n = 3,237$,

Value	Count	Cumulative Count	Cumulative Proportion
6	1,750	1,750	0.540624
7	213	1,963	0.606426
8	172	2,135	0.659561
9	159	2,294	0.708681
10	146	2,440	0.753784
11	138	2,578	0.796416
12	130	2,708	0.836578
13	81	2,789	0.861600
14	448	3,237	1.000000
Total	3,237		

The sample quantile is again unique at 9, but the confidence set is {9, 10} with the ex-post confidence of 1.0000. As this is not a singleton, we will use formula (21) again to get a new sample size:

$$n_1 = \left\lceil \frac{\left(z_{\alpha/2}\right)^2 f(1-f)}{\left[\min(\varepsilon_8^0, \zeta_9^0)\right]^2} \right\rceil = \left\lceil \frac{(1.96)^2 \times 0.7 \times 0.30}{[\min(0.040439, 0.008681)]^2} \right\rceil = 10,706.$$

We will enlarge the sample size to $10,706$. With $n = 10,706$

Value	Count	Cumulative Count	Cumulative Proportion
6	5,954	5,954	0.556137
7	653	6,607	0.617131
8	560	7,167	0.669438
9	540	7,707	0.719877
10	453	8,160	0.762189
11	451	8,611	0.804315
12	416	9,027	0.843172
13	297	9,324	0.870914
≥ 14	1,382	10,706	1.000000
Total	10,706		

The sample quantile is now 9, and the 95% confidence set is the singleton $\{9\}$ with the ex-post confidence at 0.999995. For the entire set of 95,413, the summary is this:

Value	Cumulative Probability
7	0.61981071
8	0.67499187
9	0.72202949
10	0.76187731
11	0.80118013

from which we see that the population 70th percentile is indeed 9 and the method has succeded in identifying this quantile.

In both examples, the target quantile is nailed down with very high precision, although a relatively large sample size was required in the second example.

9 Appendix

We derive here the expression given in (5), which is the probability that the sample quantile is non-unique with k as the smallest value. Setting $G = B + 1$ we have

$$
\begin{aligned}
& P\left(\hat{m} = k, U = 0\right) \\
= \quad & P\left[\left\{Y_{k-1}^{LE} \le B\right\} \cap \left\{Y_k^{LE} = G\right\}\right] \quad = \quad \sum_{j\,=\,0}^{B} P(Y_{k-1}^{LE} = j)\; P(Y_k = G - j \mid Y_{k-1}^{LE} = j) \\
= \quad & \sum_{j\,=\,0}^{B} \binom{n}{j} \left(p_{k-1}^{LE}\right)^j \left(1 - p_{k-1}^{LE}\right)^{n-j} \binom{n-j}{G-j} \left(\frac{p_k}{1 - p_{k-1}^{LE}}\right)^{G-j} \left(1 - \frac{p_k}{1 - p_{k-1}^{LE}}\right)^{n-G} \\
= \quad & \sum_{j\,=\,0}^{B} \binom{n}{j} \binom{n-j}{G-j} \left(p_{k-1}^{LE}\right)^j \left(1 - p_{k-1}^{LE}\right)^{n-j} \left(\frac{p_k}{1 - p_{k-1}^{LE}}\right)^{G-j} \left(\frac{1 - p_k^{LE}}{1 - p_{k-1}^{LE}}\right)^{n-G} \\
= \quad & (1 - p_k^{LE})^{n-G} \binom{n}{G} \sum_{j\,=\,0}^{B} \binom{G}{j} \left(p_{k-1}^{LE}\right)^j \left(p_k\right)^{G-j} \\
= \quad & (1 - p_k^{LE})^{n-G} \binom{n}{G} \left[\sum_{j\,=\,0}^{G} \binom{G}{j} \left(p_{k-1}^{LE}\right)^j \left(p_k\right)^{G-j} \; - \; \left(p_{k-1}^{LE}\right)^G\right] \\
= \quad & (1 - p_k^{LE})^{n-G} \binom{n}{G} \left[\left(p_k^{LE}\right)^G - \left(p_{k-1}^{LE}\right)^G\right].
\end{aligned}
$$

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TABLE 1. The description of this table appears in Example 1

			P[$\hat{m} = x$]		# times in sample $\hat{m} = x$	
x	p_x	p_x^{LE}	x is unique sample quantile	x is minimum non-unique sample quantile	x is unique sample quantile	x is minimum non-unique sample quantile
4	0.04197170	0.07007117	0.00000000	0.00000000	0	0
5	0.07219132	0.14226250	0.00001430	0.00002486	17	37
6	0.10347423	0.24573673	0.08661756	0.04061139	86,662	40,715
7	0.12712548	0.37286221	0.79398164	0.02690271	793,861	26,898
8	0.13665989	0.50952211	0.05182996	0.00001062	51,793	8
9	0.13058612	0.64010823	0.00000693	0.00000000	9	0
10	0.11230406	0.75241229	0.00000000	0.00000000	0	0
TOT					932,342	67,658

TABLE 2: Results for 99% confidence sets, using sample size n = 83. The description of this table is given in Section 7.

Population distribution near 0.30 quantile (middle value is the quantile) $\begin{pmatrix} k-2 & k-1 & k & k+1 & k+2 \\ p_{k-2}^{LE} & p_{k-1}^{LE} & p_{k}^{LE} & p_{k+1}^{LE} & p_{k+2}^{LE} \end{pmatrix}$		Times Covered, out of 10,000	Nominal Confidence Payout	Ex-post Confidence Payout
$\begin{pmatrix} 77 & 78 & 79 & 80 & 81 \\ 0.2447 & 0.2814 & 0.3203 & 0.3613 & 0.4037 \end{pmatrix}$	Test Inv	9,958	0.005800	0.001466
	Order Stat	9,945	0.003252	-0.001259
$\begin{pmatrix} 77 & 78 & 79 & 80 & 81 \\ 0.2259 & 0.2636 & 0.3042 & 0.3471 & 0.3919 \end{pmatrix}$	Test Inv	9,957	0.005700	0.001237
	Order Stat	9,963	0.005052	0.000372
$\begin{pmatrix} 77 & 78 & 79 & 80 & 81 \\ 0.2955 & 0.2980 & 0.3005 & 0.3030 & 0.3055 \end{pmatrix}$	Test Inv	9,868	-0.003200	-0.005117
	Order Stat	9,850	-0.006248	-0.006684

TABLE 3: Results for 99% confidence sets, using sample size n = 803. The description of this table is given in Section 7.

Population distribution near 0.30 quantile (middle value is the quantile) $\begin{pmatrix} k-2 & k-1 & k & k+1 & k+2 \\ p_{k-2}^{LE} & p_{k-1}^{LE} & p_{k}^{LE} & p_{k+1}^{LE} & p_{k+2}^{LE} \end{pmatrix}$		Times Covered, out of 10,000	Nominal Confidence Payout	Ex-post Confidence Payout
$\begin{pmatrix} 77 & 78 & 79 & 80 & 81 \\ 0.2447 & 0.2814 & 0.3203 & 0.3613 & 0.4037 \end{pmatrix}$	Test Inv	9,999	0.009900	0.001979
	Order Stat	9,999	0.009764	0.001259
$\begin{pmatrix} 77 & 78 & 79 & 80 & 81 \\ 0.2259 & 0.2636 & 0.3042 & 0.3471 & 0.3919 \end{pmatrix}$	Test Inv	9,981	0.008100	0.000107
	Order Stat	9,981	0.008964	0.000493
$\begin{pmatrix} 77 & 78 & 79 & 80 & 81 \\ 0.2955 & 0.2980 & 0.3005 & 0.3030 & 0.3055 \end{pmatrix}$	Test Inv	9,920	0.002000	-0.000605
	Order Stat	9,916	0.001464	-0.000345

TABLE 4: 99% Confidence Sets Produced for population:

$$\begin{pmatrix} 77 & 78 & 79 & 80 & 81 \\ 0.2259 & 0.2636 & 0.3042 & 0.3471 & 0.3919 \end{pmatrix}$$

With sample size of 83:

Most likely by Test Inversion (out of 10,000):

{76, …, 82} 1,068 times
{77, …, 83} 859 times

Most likely by Order Statistics (out of 10,000):

{75, …, 82} 908 times
{76, …, 83} 1,084 times

With sample size of 803:

	Test Inversion	Order Statistics
{77, 78, 79 }	66	104
{77, 78, 79, 80 }	83	122
{ 78, 79 }	64	85
{ 78, 79, 80 }	5,105	5,615
{ 78, 79, 80, 81 }	976	967
{ 79, 80 }	852	805
{ 79, 80, 81 }	2,829	2,291
{ 79, 80, 81, 82}	6	2
{ 80, 81 }	15	8
{ 80, 81, 82}	4	1
TOTAL	10,000	10,000

TABLE 5: Simulation results for the multistage procedure. The table is described in Section 7

Population distribution near 0.30 quantile (middle value is the quantile) $\begin{pmatrix} k-2 & k-1 & k & k+1 & k+2 \\ p_{k-2}^{LE} & p_{k-1}^{LE} & p_k^{LE} & p_{k+1}^{LE} & p_{k+2}^{LE} \end{pmatrix}$	Starting confidence 1 - α	Number of times covered in 10,000 trials	Number of times correct singleton achieved	Average sample size required (bail out at $N = 500{,}000$)
$\begin{pmatrix} 77 & 78 & 79 & 80 & 81 \\ 0.2447 & 0.2814 & 0.3203 & 0.3613 & 0.4037 \end{pmatrix}$	0.90	9,993	9,993	55,257.58
	0.95	10,000	10,000	59,542.28
	0.999	10,000	10,000	96,818.69
$\begin{pmatrix} 77 & 78 & 79 & 80 & 81 \\ 0.2259 & 0.2636 & 0.3042 & 0.3471 & 0.3919 \end{pmatrix}$	0.90	9,691	9,691	162,809.53
	0.95	9,923	9,923	190,595.25
	0.999	9,993	9,993	289,378.88
$\begin{pmatrix} 77 & 78 & 79 & 80 & 81 \\ 0.2955 & 0.2980 & 0.3005 & 0.3030 & 0.3055 \end{pmatrix}$	0.90	9,936	1,195	499,919.10
	0.95	9,973	41	499,997.03
	0.999	10,000	0	500,000.00

TABLE 6: Simulation results for the multistage procedure. The table is described in Section 7.

Population distribution near 0.30 quantile (middle value is the quantile) $\begin{pmatrix} k-2 & k-1 & k & k+1 & k+2 \\ p_{k-2}^{LE} & p_{k-1}^{LE} & p_k^{LE} & p_{k+1}^{LE} & p_{k+2}^{LE} \end{pmatrix}$	Starting confidence 1 - α	Number of times producing confidence sets								
		No cover (low)	Cover population quantile						No cover (high)	
		{78}	{77,78,79}	{78,79}	{79}	{78,79,80}	{79,80}	{79,80,81}	{80}	{80,81}
$\begin{pmatrix} 77 & 78 & 79 & 80 & 81 \\ 0.2447 & 0.2814 & 0.3203 & 0.3613 & 0.4037 \end{pmatrix}$	0.90	6			9,993				1	
	0.95				10,000					
	0.999				10,000					
$\begin{pmatrix} 77 & 78 & 79 & 80 & 81 \\ 0.2259 & 0.2636 & 0.3042 & 0.3471 & 0.3919 \end{pmatrix}$	0.90				9,691				309	
	0.95				9,923				77	
	0.999				9,993				7	
$\begin{pmatrix} 77 & 78 & 79 & 80 & 81 \\ 0.2955 & 0.2980 & 0.3005 & 0.3030 & 0.3055 \end{pmatrix}$	0.90			720	1,195		8,021		53	11
	0.95			1,117	41	154	8,655	6	1	26
	0.999		3	56		5,756	3,257	928		