Time-series and Cross-sectional Variations of Expected Returns

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Abstract

The paper develops a general equilibrium stochastic growth model of a multi-sector economy subject to i.i.d. taste shocks. Each sector produces one good, and each firm has a linear production technology and faces a quadratic capital adjustment cost. The model contains a standard intertemporal capital asset pricing theory of consumption and portfolio demands with dynamically complete and frictionless markets and a standard q-theory of investment under uncertainty. We show that the equilibrium stochastic investment opportunity set is driven by the relative shares of firms' nominal capital stocks, and the equilibrium dynamics of the state vector is driven by firms' relative investment intensities. Key implications of the model include: (i) the expected equity returns are endogenously predictable both over time and in the cross-section; and (ii) the "value anomaly" arises in a rational expectations equilibrium due to a negative (positive) hedging demand for value (growth) stocks against the risk of cross-sectional dispersion of firms' nominal capital stocks.

1 Introduction

There is growing empirical evidence that (i) equity returns are predictable over time based on lagged variables including short term interest rates, term spreads, stock volatility dividend yields, and book-to-market ratios; and (ii) expected equity returns are cross-sectionally different not only because of cross-sectional differences in market risk (covariance or beta with respect to the market return), but also because of cross-sectional differences in firm characteristics such as earnings-to-price ratios or book-to-market ratio (the value effect), firm size (the size effect).[1] These findings are often viewed as "anomalies" because they are inconsistent with standard intuitions derived from traditional asset pricing theories. Although a growing minority of financial researchers have resorted to explanations based on irrational behavior (due to either psychological illusions and/or agency problems) and limits to arbitrage (due to some form of market frictions and/or incompleteness), it is far from clear that these empirical findings are inconsistent with a rational expectations equilibrium in an economy with frictionless and dynamically complete markets. One plausible (rational) explanation is advanced by Fama and French (1992, 1993, and 1996) who posit that, in addition to the aggregate risk factor that drives the market return, there are at least two other risk factors, proxied by the size and the book-to-market ratio of a common stock, that are priced in equilibrium. Although it appears to be a reasonable description of the factor structure of equity risk premiums (with respect to which a number of anomalies disappear), the model does not make explicit how the risk factors emerge from a fully specified general equilibrium model and why they vary over time and cross-sectionally in a predictable manner.

The purpose of this paper is to develop some intuitions on how time-series and cross-sectional variations of expected returns may arise in a general equilibrium setting, through the lens of a multi-sector continuous-time stochastic growth model with commodity price or demand uncertainty. At the partial equilibrium level, the model contains a version of the standard intertemporal model of consumption and portfolio demands a.k.a. Merton (1973) with dynamically complete and frictionless markets and a version of the standard q-theory of investment a.k.a. Tobin (1969) with convex adjustment costs. At the general equilibrium level, the model may be viewed as a multi-sector extension of Brock and Mirman (1972). The general equilibrium aspect of

[1]For a complete survey of these and other empirical findings, see Schwert (2002).

the model allows us to identify the state vector driving the stochastic evolution of the investment opportunity set as the relative shares of firms' capital stocks.

In standard one-sector growth models, cyclic variation in risk premium and other endogenous variables can be generated endogenously by production shocks impinging on a decreasing-return-to-scale production technology (see, e.g., Brock and Mirman (1972) and Prescott and Mehra (1980)). In our model, the cyclic variation is induced by cross-sectional dispersion of taste shocks or commodity price shocks impinging on a convex capital adjustment cost associated with each linear technology. To see how this works, consider the symmetric case in which (i) firms have the same profit rates and adjustment costs; and (ii) taste shocks have same the volatility and same pair-wise correlations, but the correlation is not perfect; and (iii) initially all firms have the same level of capital stocks, same level of investment intensity and therefore the same equity market shares and the same expected returns. Now suppose that commodity j produced by sector j receives a positive price shock (due to a positive taste shock, say) relative to other commodities. Because commodity j is now relatively more expensive, firms in sector j will have a higher market price or market share relative to firms in other sectors. This is possible in equilibrium only if the expected return for firms in sector j is also higher (which induces a higher portfolio demand for firms in sector j to clear the equity markets). From the perspective of those firms in sector j, higher expected returns imply higher costs of capital, which in turn imply lower investment/capital ratios. This is possible only if firms in sector j have lower Tobin's q. Thus, there is a positive association between the expected return and the book-to-market ratio (interpreted as an empirical proxy for the inverse of Tobin's q). The cross-sectional dispersion does not disappear immediately because firms can only adjust their capital stocks gradually due to the presence of a convex capital adjustment cost. This leads to time-series predictability of expected returns in horizons commensurate with the doubling times associated with the firms' capital adjustment costs. If the demand shocks are perfectly correlated, then there is no cross-sectional dispersion in relative capital stocks, relative market shares, and relative expected returns. Consequently there is also no time-series variation and predictability in expected returns (there is, however, constant endogenous growth).

This work is complimentary to a recent paper by Gomes, Kogan, and Zhang (2001) (henceforth GKZ), who show how cross-sectional variations in expected equity returns may be attributed to cross-sectional variations in firms' "assets in place" and their "growth options", which are interpreted as the size and book-to-market factors. In our model, expected equity returns are shown to co-vary directly with the size (i.e., either the market capitalization or the capital stock) and book-to-market ratio (inverse of Tobin's q) in the cross-section. Both models are general equilibrium in nature in the sense that consumption and portfolio choices are affected by the aggregate level of investment and production/investment decisions are in turn affected by aggregate consumption growth, and both goods and equity markets must clear. However, the structures of the two models are very different. First, in GKZ, the economy-wide state variable is exogenously specified and enters the model as a systematic component of the production shock. Time-variation and predictability of expected returns are driven by this systematic factor, which is assumed to be mean-reverting. In our model, there is no predictable external shocks. The dynamics of the state vector are endogenously determined in a rational expectations equilibrium. Second, in GKZ, the output function is concave and the investment cost is linear but irreversible. In my model, the output function is linear and the investment cost is quadratic but reversible.

A key prediction of the model is that the so-called "value anomaly", namely, value stocks earn an abnormal positive return (positive alpha) over and above the risk premium associated with their market risk, arises endogenously in a rational expectations equilibrium of our model due to a negative (positive) hedging demand for value (growth) stocks against the risk of cross-sectional dispersion of firms' capital stocks. In the context of our model, therefore, the "value anomaly" does not represent an arbitrage opportunity because it arises from underlying economic fundamentals. In an interesting empirical study that captures much of the same spirit, Brennan, Wang, and Xia (2002) use the bond yields and inflation data to back out the implied stochastic investment opportunity set, and construct two self-financing portfolios that track the innovations in the implied stochastic investment opportunity set. They find that the expected returns earned from these two tracking portfolios eliminate the "abnormal" returns almost entirely when they replace the size and book-to-market factors in the Fama-French regressions and does a better job in explaining industry portfolios than the Fama-French three-factor model. In contrast, Gomes, Kogan, and Zhang (2001) and Berk, Green, and Naik (1999) show that the "value anomaly" is not an intrinsic feature of their models, and argue that the observed empirical regularity associated with the "value anomaly" is due to mis-measurement of market betas.

The rest of the paper is organized as follows. In Section 2, we describe the model and the general equilibrium restrictions. The state vector that drives the equilibrium stochastic investment opportunity set is identified and its equilibrium dynamics is derived. In Section 3, we solve numerically for a rational expectations equilibrium in a two-sector economy under realistic parameter values. We use the explicit solution to illustrate key properties of the model, highlighting in particular the time-series and cross-sectional predictability of expected equity returns and the "value anomaly". Section 4 concludes.

2 The Model

The economy consists of N industrial sectors, each of which is populated by a continuum of identical, competitive, and value-maximizing firms. Each firm uses a linear quadratic production technology to produce a unique consumption good, which can be either reinvested as capital for the firm or paid out as dividend and consumed by the owners of the firm. There is also a continuum of identical, competitive, utility-maximizing consumers. Each consumer owns an equity share of each firm, and consumes her share of the dividends.

For simplicity, we will assume that the economy consists of a single competitive (representative) consumer/investor, and each industrial sector consists of a single competitive (representative) firm.

Let $\Gamma_t \equiv (r_t, \mu_t, \sigma_t)$ be the investment opportunity set at time t, where r_t is the instantaneous return on a riskless asset, μ_t is the $N \times 1$ vector of instantaneous expected returns for the equity claims to the N firms, and $\sigma_t \sigma_t'$ is the $N \times N$ instantaneous variance-covariance matrix of the equity returns. In choosing optimal policies, both the representative investor and the representative firms take Γ_t as given. Since Γ_t is not known a priori, optimal policies depend on the expectation on the future evolution of Γ_t. A rational expectations equilibrium is one in which the future evolution of Γ_t is correctly anticipated.

In the rest of the section, we proceed to specify the individual optimization problems, followed by a discussion on how to solve for individual policies and the investment opportunity set in a rational expectations equilibrium.

2.1 Consumption and Portfolio Demand

The demand side of the economy is a version of Merton (1973)'s intertemporal asset pricing model, except that the investment opportunity set Γ_t is to be endogenously determined. Taking Γ_t as given, the representative consumer/investor solves

$$J(k, \zeta) = \max_{c_t, a_t : t \geq 0} E\left[\int_0^\infty e^{-\rho t} u(c_t)\, dt \,\middle|\, k_0 = k, \zeta_0 = \zeta \right], \tag{1}$$

subject to

$$dk_t = [a_t'(\mu_t - r_t)k_t + r_t k_t - c_t]\, dt + a_t' \sigma_t dB_t, \tag{2}$$

where, B_t is a $N \times 1$ vector of standard independent Brownian motions, $\rho > 0$ is the subjective discount rate, and at time t, k_t is the total wealth, a_t is $N \times 1$ vector of portfolio holdings (shares of total wealth invested in risky equity claims), and c_t is the aggregate consumption. Throughout the paper, we assume that the period utility is $u(c) = \frac{c^\gamma}{\gamma}$, where $1 - \gamma$ is the constant coefficient of relative risk aversion.

2.2 Production and Investment

For each $j = 1, 2, \ldots, N$, firm j solves

$$k^j(b^j, \zeta) = \max_{I_t^j : t \geq 0} E\left[\int_0^\infty m_s\, z_s^j\, ds \,\middle|\, b_0^j = b^j, \zeta_0 = \zeta \right], \tag{3}$$

subject to the capital accumulation equation:

$$db_t^j = I_t^j\, dt + b_t^j \xi_t^j\, dB_t, \tag{4}$$

where ξ^j is a $1 \times N$ row vector of volatility loadings to the N Brownian shocks, z_t^j is the firm's dividend payout, m_t is the pricing kernel, given by

$$\frac{dm_t}{m_t} = -r_t\, dt - \Lambda_t' dB_t, \tag{5}$$

and $\Lambda_t \equiv \sigma_t^{-1}(\mu_t - r_t)$. Firms' dividend policies are directly determined by their investment policies, through

$$z_t^j = \pi^j b_t^j - I_t^j - \frac{\alpha^j}{2} \left(\frac{I_t^j}{b_t^j} \right)^2 b_t^j, \tag{6}$$

where π^j is the net profit rate and α^j is the doubling time for firm j. The first term on the right hand side of equation (6) represents the linear output function, and the third term the quadratic capital adjustment cost.

2.3 The

where z_t^j is given by (6). The individual maximization problems defined in Sections 2.1 and 2.2 are the decentralization of the central planner's problem under the new numeraire.

While $\tilde{b}_t^j$, $1 \le j \le N$, may be viewed as real capital stocks, the variables b_t^j, $1 \le j \le N$, may be loosely referred to as nominal capital stocks. By the same token, while $\tilde{I}_t^j$, $1 \le j \le N$, may be viewed as real capital expenditures I_t^j, $1 \le j \le N$, may be viewed as nominal capital expenditures.

2.4 The Rational Expectations Equilibrium

The rational expectations equilibrium of the model is characterized in two steps. First, in Propositions 1–3, we characterize partial equilibrium restrictions between the investment opportunity set and the optimal policies, and derive the endogenous dynamics for the economy-wide state vector. Second, in Proposition 4, we impose market clearing conditions that jointly determine the equilibrium prices and quantities.

In order to solve for the rational expectations equilibrium, we need to guess correctly the state vector and its equilibrium dynamics. To this end, let $\hat{b}_t$ be the vector of capital stocks for the N firms (sectors) in the equilibrium, and let ζ_t be a $(N-1) \times 1$ vector, defined by

$$\zeta_t^i \equiv \frac{\hat{b}_t^i}{\hat{b}_t^N}, \ 1 \le i \le N-1.$$

As a side effect of solving the rational expectations equilibrium, we will be able to verify that (i) ζ_t is the state vector that drives the equilibrium investment opportunity set, i.e., $r_t = r(\zeta_t)$, $\mu_t = \mu(\zeta_t)$, and $\sigma_t = \sigma(\zeta_t)$; and (ii) ζ_t is an autonomous $(N-1)$-dimensional Markov process, i.e.,

$$d\zeta_t = \hat{\mu}_t \, dt + \hat{\sigma}_t \, dB_t, \tag{12}$$

where $\hat{\mu}_t = \hat{\mu}(\zeta_t)$ is a $(N-1) \times 1$ vector and $\hat{\sigma}_t = \hat{\sigma}(\zeta_t)$ is a $(N-1) \times N$ matrix. Intuitively, the equilibrium dynamics of the state vector ζ_t is determined as follows: being perfectly competitive, each firm j treats $\hat{b}_t$ or ζ_t as an exogenous state vector, and treats its own capital stock b^j as an endogenous state variable controlled by its investment policy; in equilibrium, we must have $\hat{b}_t^j = b_t^j$ or $\zeta^j = \frac{b^j}{b^N}$, for $1 \le \forall j \le N$.

Proposition 1 (Optimal Policies) *Suppose that $r_t = r(\zeta_t)$, $\mu_t = \mu(\zeta_t)$, and $\sigma_t = \sigma(\zeta_t)$, where ζ is given by (12). Then the representative consumer/investor's optimal consumption and portfolio problem is solved with the indirect utility function given by $J(k,\zeta) = h(\zeta)^{\gamma-1}\frac{k^\gamma}{\gamma}$, and the representative firms' optimal investment problems are solved with firm equity values given $k^j(b^j,\zeta) = q^j(\zeta)\, b^j$, $1 \le j \le N$.*

Furthermore,

1. *The optimal consumption policy is given by*

$$c_t^* = h(\zeta_t)\, k_t, \tag{13}$$

where $h(\zeta)$ is the aggregate consumption/wealth ratio, and its functional form depends on the functional forms of $r(\zeta)$, $\mu(\zeta)$, $\sigma(\zeta)$, $\hat{\mu}(\zeta)$, and $\hat{\sigma}(\zeta)$.

2. *The optimal portfolio policy is given by*

$$a_t^* = a^*(\zeta_t) = a_t^M + a_t^H, \tag{14}$$

where a_t^M and a_t^H are respectively the myopic and hedging demands, given by

$$a_t^M = \frac{(\sigma_t\sigma_t')^{-1}(\mu_t - r_t)}{1-\gamma}, \quad a_t^H = -\sigma_t^{'-1}\hat{\sigma}_t'\frac{\partial \log h}{\partial \zeta}. \tag{15}$$

3. *The optimal investment policies are given by*

$$I_t^{*j} = \mathbb{I}^j(\zeta_t)\, b_t^j, \; 1 \le j \le N, \tag{16}$$

where $\mathbb{I}^j(\zeta) \equiv \frac{q^j(\zeta)-1}{\alpha^j}$, $q^j(\zeta)$ is the average q for firm j, and its functional form depends on the functional forms of $r(\zeta)$, $\mu(\zeta)$, $\sigma(\zeta)$, $\hat{\mu}(\zeta)$, and $\hat{\sigma}(\zeta)$.

Proof: See Appendix A.

Proposition 2 (State Dynamics) *If a rational expectations equilibrium exists with $r_t = r(\zeta_t)$, $\mu_t = \mu(\zeta_t)$, and $\sigma_t = \sigma(\zeta_t)$, where ζ is given by (12), then:*

$$\hat{\mu}_t^j = \left[\mathbb{I}^j(\zeta_t) - \mathbb{I}^N(\zeta_t) - (\xi^j - \xi^N)\xi^{N'}\right]\zeta_t^j, \tag{17}$$

$$\hat{\sigma}_t^j = \left(\xi^j - \xi^N\right)\zeta_t^j. \tag{18}$$

In other words,

$$\frac{d\zeta_t^j}{\zeta_t^j} = \left[\mathbb{I}^j(\zeta_t) - \mathbb{I}^N(\zeta_t) - (\xi^j - \xi^N)\,\xi^{N'}\right] dt + (\xi^j - \xi^N)\, dB_t. \tag{19}$$

Proof: At the optimal investment policies, firms' capital stocks evolve according to

$$\frac{db_t^j}{b_t^j} = \mathbb{I}^j(\zeta_t)\, dt + \xi^j\, dB_t,\ 1 \le j \le N.$$

In equilibrium, we must have $\zeta_t^j = \frac{b_t^j}{b_t^N}$, $1 \le \forall j \le N$. Equations (17)–(18) follow immediately from Ito's lemma.

Proposition 3 (Investment Opportunity Set) *If a rational expectations equilibrium exists with the aggregate consumption/wealth ratio given by $h(\zeta)$ and firms' average q given by $q^j(\zeta)$, $1 \le j \le N$, then*

$$\mu_t^i = \left[\mathbb{Z}^i(\zeta_t) + \mathbb{I}^i(\zeta_t)\right] + \left[\frac{\hat{\mathcal{A}} q^i(\zeta_t)}{q^i(\zeta_t)} + \sum_{j=1}^{N} \xi^i W_t^{ij} \xi^{j'}\right], \tag{20}$$

$$\sigma_t = \xi + W_t \xi, \tag{21}$$

where $\mathbb{Z}^i \equiv \frac{z^{*i}}{k^i} = \pi^i - \mathbb{I}^i - \frac{\alpha^i}{2}(\mathbb{I}^i)^2$ is the optimal dividend/price ratio, and $\hat{\mathcal{A}}$ and W are given by (36) and (44), respectively.

Furthermore, the equilibrium riskfree rate solves:

$$r_t = \frac{\rho}{\gamma} - \frac{\Lambda_t' \Lambda_t}{2(1-\gamma)} - \frac{1-\gamma}{\gamma}\left[h(\zeta_t) - h^H(\zeta_t)\right], \tag{22}$$

where $h^H(\zeta)$ is determined by the first and second order derivatives of $h(\zeta)$ (see equation (42)).

Proof: By definition, for each i,

$$\frac{dk^i}{k^i} = \left(\mu^i - \frac{z^{*i}}{k^i}\right) dt + \sigma^i dB_t, \tag{23}$$

where σ_t^i is the i^{th} row of σ_t. From the firm's optimization problem, we have $k^i = q^i(\zeta)b^i$. Ito's lemma implies that

$$\begin{aligned}\frac{dk^i}{k^i} &= \left(\frac{I^{*i}}{b^i} + \sum_{j=1}^{N-1} \frac{\partial \log q^i}{\partial \zeta^j}\, \hat{\sigma}^j \xi^{i'} + \frac{\hat{\mathcal{A}} q^i}{q^i}\right) dt \\ &\quad + \left(\xi^i + \sum_{j=1}^{N-1} \frac{\partial \log q^i}{\partial \zeta^j}\, \hat{\sigma}^j\right) dB.\end{aligned} \tag{24}$$

Matching terms between equations (23) and (24), we obtain

$$\begin{aligned}\sigma^i &= \xi^i + \sum_{j=1}^{N} W^{ij}\xi^j, \ \ (\text{or } \sigma = \xi + W\xi), \\ \mu^i &= \frac{z^{*i}}{k^i} + \frac{I^{*i}}{b^i} + \frac{\hat{\mathcal{A}} q^i}{q^i} + \sum_{j=1}^{N} \xi^i W^{ij} \xi^{j'}.\end{aligned}$$

Finally, equation (22) is obtained by substituting the optimal consumption and portfolio policies into the Bellman equation (34).

Equations (20)–(21) characterize the *cost of risky capital* faced by firms, and equation (22) characterizes the cost of riskless capital faced by both the representative consumer and the firms. The proof of Proposition 3 shows that how they are backed out from the optimality conditions of the representative consumer and the representative firms in terms of the indirect utility functions.

We are now ready to close the loop. For notational simplicity, let $\zeta^N \equiv 1$ (however, ζ_t still denotes the $(N-1) \times 1$ vector of capital ratios).

Proposition 4 (Rational Expectations Equilibrium) *If there exists a set of functions* $(h(\zeta); q^j(\zeta) : 1 \le j \le N)$ *such that for* $\forall \zeta \in \mathbb{R}_+^{N-1}$*,*

$$h = \frac{\sum_{j=1}^{N} \left[\pi^j - \frac{q^j - 1}{\alpha^j} - \frac{\alpha^j}{2}\left(\frac{q^j - 1}{\alpha^j}\right)^2\right] \zeta^j}{\sum_{j=1}^{N} q^j \zeta^j}, \tag{25}$$

$$a^{*i} = \frac{q^i\, \zeta^i}{\sum_{j=1}^{N} q^j\, \zeta^j}, \ \ 1 \le i \le N, \tag{26}$$

where a^* *is given by (14), then there exists a rational expectations equilibrium.*

Proof: Noting that, in equilibrium, $k = \sum_{j=1}^{N} k^j = \sum_{j=1}^{N} q^j b^j$, we can see easily that equation (25) follows from the market clearing condition in the goods market: $c^* = \sum_{j=1}^{N} z^{*j}$, and equation (26) follows from the market clearing conditions in the asset markets: $a^{*j} = \frac{k^j}{k}$, $1 \leq j \leq N$.

Equations (25)–(26) characterize a fixed-point problem: on the one hand, the set of functions $[h(\zeta); q^j(\zeta) : 1 \leq j \leq N]$ are implicit functionals of $a^*(\zeta)$. On the other hand, elements of a^* are implicit functionals of $[r(\zeta), \mu(\zeta), \sigma(\zeta)]$ (Proposition 1), which in turn are implicit functionals of $[h(\zeta); q^j(\zeta) : 1 \leq j \leq N]$ (Proposition 3). Although Proposition 4 is not a formal proof for the existence and uniqueness of a rational expectations equilibrium, it establishes an operational procedure for computing the rational expectations equilibrium – if it exists.

2.5 ICAPM

It is well known since Merton (1973) that if the investment opportunity set is stochastic, then the one-factor conditional CAPM need not hold. This is because a stochastic investment opportunity may induce a hedging demand, which need not be perfectly correlated with the market portfolio.

In our model, the investment opportunity set is stochastic, and its equilibrium dynamics is endogenously determined. The cross-sectional restriction on individual equity returns can be derived as a corollary of Propositions 1 and 2:

$$\mu_t - r_t = (1-\gamma)\sigma_t \sigma_t' (a_t^* - a_t^H), \tag{27}$$

where $a_t^H = -(\mathbf{I} + W_t)'^{-1}\Delta$, $\mathbf{I}$ is the $N \times N$ identity matrix and Δ is a $N \times 1$ vector defined by

$$\Delta^j = \frac{\partial \log h}{\partial \log \zeta^j}, \; 1 \leq j \leq N-1; \quad \Delta^N = -\sum_{j=1}^{N-1} \Delta^j. \tag{28}$$

Let $\mu_t^* \equiv a_t^{*'} \mu_t = r_t + (1-\gamma) a_t^{*'} \sigma_t \sigma_t' (a_t^* - a_t^H)$ denote the expected market return, and $\beta_t^j \equiv \frac{\sigma_t^j \sigma_t' a_t^*}{a_t^{*'} \sigma_t \sigma_t' a_t^*}$ the conditional beta of equity j with respect to the market. Then we can write

$$\mu_t^j - r_t = \lambda_t^j \beta_t^j \, (\mu_t^* - r_t), \tag{29}$$

where

$$\lambda_t^j = \frac{1 - \frac{\sigma_t^j \sigma_t' a_t^H}{\sigma_t^j \sigma_t' a_t^*}}{1 - \frac{a_t^{*\prime} \sigma_t \sigma_t' a_t^H}{a_t^{*\prime} \sigma_t \sigma_t' a_t^*}}. \tag{30}$$

When the hedging demands are zero, i.e., $a^H = 0$ and $\Lambda_t^j = 1$, so that equation (29) recovers the one-factor conditional CAPM. In general, however, the hedging demands are non-zero, and $\lambda_t^j \neq 1$, $1 \leq j \leq N$. This generates a systematic deviation from the one-factor CAPM. The "abnormal" differential return over and above the risk premium associated with the market return is given by

$$\alpha_t^j \equiv (\lambda_t^j - 1)\beta_t^j(\mu_t^* - r_t), \tag{31}$$

which can be computed explicitly once the model has been solved.

3 Numerical Solution and Illustrations

In this section, we characterize some key properties of the rational expectations equilibrium, by numerically solving the fixed-point problem: (25)–(26), together with (13), (14), (16), (20), (21), and (22). The numerical problem is complicated by the fact that equations (14), (20), (21), and (22) involve the first and second derivatives of $h(\zeta)$ and $q^j(\zeta)$, $1 \leq j \leq N$. To get around this problem, we adopt an approximation which allows us to obtain the solution state by state through a simple iterative procedure.

To illustrate the idea, let us recall that the partial derivatives of $h(\zeta)$ and $q^j(\zeta)$, $1 \leq j \leq N$, are captured by the $N \times 1$ vector Δ and the $N \times N$ matrix W, which are defined by (28) and (44), respectively. An individual consumer/investor who holds the belief

$$\Delta = 0_{N\times 1}, \text{ and } W = 0_{N\times N}, \tag{32}$$

adopts a myopic portfolio policy ($a^H = 0$) and a myopic consumption policy ($h^H = 0$). Hence (32) may be characterized as *myopic belief*. It is easy to show that the myopic belief is inconsistent with the rational expectations equilibrium: under the myopic belief, the fixed-point problem reduces to

$$\begin{aligned}
h &= \sum_{j=1}^{N} a^{*j}\left[\frac{\pi^j}{q^j} - \frac{(q^j-1)(q^j+1)}{2\alpha^j q^j}\right], \\
a^{*j} &= \frac{q^j\zeta^j}{\sum_{j=1}^N q^j\zeta^j}, \\
\text{where}: \ a^* &= \frac{(\xi\xi')^{-1}(\mu-r)}{1-\gamma}, \\
\mu^j &= \frac{\pi^j}{q^j} + \frac{(q^j-1)^2}{2\alpha^j q^j}, \quad \sigma = \xi, \\
r &= \frac{\rho}{\gamma} - \frac{(\mu-r)'(\xi\xi')^{-1}(\mu-r)}{2(1-\gamma)} - \frac{1-\gamma}{\gamma}h
\end{aligned}$$

It is easy to verify that the solution $h(\zeta)$ and $q^j(\zeta)$, $1 \le j \le N$ must be state-dependent, thus contradicting the myopic belief.

We assume that agents and firms are rational, in that they try to forecast the state-dependence of $h(\zeta)$ and $q^j(\zeta)$, $1 \le j \le N$ in the best way they can. They believe that $\Delta \neq 0_{N\times 1}$ and $W \neq 0_{N\times N}$, and that in general even Δ and W themselves are state-dependent. To solve the model, however, we will adopt an approximation that is equivalent to assuming that agents and firms hold the following expectations:

$$\frac{\partial \log \Delta}{\partial \log \zeta} = 0_{N\times(N-1)}, \ \frac{\partial \log W}{\partial \log \zeta} = 0_{N\times N\times(N-1)}, \tag{33}$$

that is, Δ and W are locally constant, or equivalently, $\log h(\zeta)$ and $\log q^j(\zeta)$, $1 \le j \le N$, are locally linear in $\log \zeta$. The resulting solution is therefore referred to as the "linear rational expectations equilibrium" (L.R.E.E.). Operationally, equation (33) means that in each state of the world ζ and its immediate vicinity, agents and firms expect a constant Δ and a constant W in computing their optimal policies. The equilibrium values of Δ and W are determined through the principle of rational expectations: after $h(\zeta)$ and $q^j(\zeta)$, $1 \le j \le N$, have been computed from market clearing conditions at ζ and its immediate vicinity, their partial derivatives can be computed and must be consistent with prior expectations on Δ and W. Thus, on top of the fixed-point problem that determines $h(\zeta)$ and $q^j(\zeta)$, $1 \le j \le N$, there is another fixed-point problem that determines Δ and W.[2]

As an illustration, consider a two-sector economy with the following baseline parameterization: $\pi^1 = \pi^2 = 10\%$, $\alpha^1 = \alpha^2 = 7$(years), $\xi^1 = (18.4\%, 0)$, $\xi^2 = (0, 18.4\%)$, $1-\gamma = 3.3$, $\rho = 5\%$. Under this parameterization, the two sectors (representative firms) have the same net profit rates, are subject to demand shocks with the same volatility, and the demand shocks are uncorrelated. These parameter values are roughly consistent with those calibrated by Friend and Blume (1975): if W were $0_{N\times N}$, the volatilities of for individual sectors would be given by $\sqrt{\xi^{j\prime}\xi^j} = 18.4\%$, $j = 1, 2$, and if in addition the two firms had equal market shares, the volatility of the market return would be $\sigma^* = 18.4\%/\sqrt{2} = 13\%$ With a relative risk aversion of $1-\gamma = 3.3$, the aggregate equity premium would be $(1-\gamma)\times\sigma^{*2} = 5.58\%$.

Figure 1 plots some key features of the L.R.E.E. solution. Reading from left to right, the horizontal axis is the share of the capital stock for firm 1, or $\chi \equiv \frac{\zeta}{1+\zeta}$, rather than $\zeta \equiv \frac{b^1}{b^2}$ so that the plots reflect the symmetric nature of the two sectors. The eight panels are:

(a) Riskfree rate $r(\zeta)$;

(b) Aggregate consumption/wealth ratio $h(\zeta) = \frac{z^{*1}+z^{*2}}{k^1+k^2}$;

(c) Expected equity returns: solid line for $\mu^1(\zeta)$ and dashed line for $\mu^2(\zeta)$;

(d) Average q: solid line for $q^1(\zeta)$ and dashed line for $q^2(\zeta)$;

(e) Sharpe ratios: solid line for $\Lambda^1(\zeta)$ and dashed line for $\Lambda^2(\zeta)$;

(f) Dividend/price ratios: solid line for $\mathbb{Z}^1(\zeta)$, and dashed line for $\mathbb{Z}^2(\zeta)$;

(g) Market shares: solid line for $a^{*1}(\zeta)$ and dashed line for $a^{*2}(\zeta)$;

(h) Investment/capital ratios: solid line for $\mathbb{I}^1(\zeta)$ and dashed line for $\mathbb{I}^2(\zeta)$.

The solution indicates that there is a long-run steady state distribution for the relative share of the capital stocks, centered around $\bar{\xi} = 0.5$, or $\bar{\zeta} = 1$. In particular, Panel (h) implies that the economy wide state variable ζ is

[2]Starting from the myopic expectations (32), which is subsumed by the linear expectations (33), the iteration over Δ and W mimics a price-discovery process in a decentralized market place in which rational agents and firms adopt more rational policies over time as they learn from past prices and of course past mistakes (initially when there is no price history, they act myopically).

mean-reverting around its long-run mean $\bar{\zeta} = 1$: the drift of $\log \zeta$ is given by $\mathbb{I}^1(\zeta) - \mathbb{I}^2(\zeta)$ (see equation (19)), which is negative when $\zeta > \bar{\zeta}$ and positive when $\zeta < \bar{\zeta}$. This endogenously mean-reverting behavior is a key feature of the model. The intuition is as follows. Suppose that initially the economy is at $\zeta = \bar{\zeta} = 1$. Now, suppose firm 1 receives a positive shock and firm 2 receives a negative shock (or a smaller positive shock) so that $\zeta > 1$. Since the shocks are permanent, firm 1 will generate more cash flows than firm 2 now and in future expectations if firms do not change their production plans. This means that firm 1 must be more valuable than firm 2, i.e., $a^1 > a^2$ [see Panel (g)]. In equilibrium, a higher portfolio demand for firm 1 is possible only if $\mu^1 > \mu^2$ [see Panel (c)]. The higher cost of capital for firm 1 forces it to scale back investment intensity, i.e., $\mathbb{I}^1 < \mathbb{I}^2$, which implies that, in expectation, the capital stock of firm 1 grows slower than that of firm 2, thereby pushing the ratio ζ back toward the long-run mean $\bar{\zeta} = 1$.

Ignoring the last term in the square bracket in equation (20), which is typically small, the expected return for each firm is roughly the sum of the dividend/price ratio ("current income") and the investment/capital ratio ("expected growth"). That is, $\mu^j \approx \frac{z^{*j}}{k^j} + \frac{I^{*j}}{b^j}$. A firm is a "value stock" if "current income" contributes more than "expected growth" to its expected return, and a "growth stock" if "expected growth" contributes more than "current income" to its expected return. According to this definition, we can identify firm 1 as the value stock when $\zeta > 1$, and firm 2 as the value stock when $\zeta < 1$. Panel (d) shows that this characterization of the value and growth stocks is consistent with the Fama-French definition that a value stock has a higher book-to-market ratio than a growth stock.

Two interesting asset pricing implications are apparent in Figure 1. First, in each state of the world, the expected return [Panel (c)] and the Tobin's q [Panel (d)] are inversely related. This implies that, cross-sectionally, value stocks earn higher expected returns than growth stocks. Second, the riskfree rate is very flat near the long-run mean [see Panel (a)], which means that it has a low volatility even though the primitive shocks of the economy are quite volatile.

As a comparison, Figure 2 plots the same set of features for the "myopic solution", i.e., the equilibrium reached when agents and firms have the myopic expectations (32) – without iterating over Δ and W. We can see that in many respect, the two solutions are qualitatively similar. Indeed, the general description of the L.R.E.E. solution in the preceding paragraphs is equally applicable to the myopic solution. In particular, the value stock earns a higher expected return. The quantitative differences of the two solutions are highlighted by plotting them side by side in Figure 3. which has the same layout as Figures 1 and 2 except that only firm 1 is shown in Panels (c)–(g) (firm 2 is just the mirror image), and that Panel (h) plots the welfare function scaled by $(b^1 + b^2)^\gamma$.

Although it may not be apparent in Figure 3, a key difference between the myopic solution and the L.R.E.E. solution pertains to the "value anomaly": there is no "value anomaly" under myopic expectations, and there is under the L.R.E.E. expectations. In the myopic solution, the one-factor conditional CAPM holds exactly, thus the value stock earns a higher expected return only because it has a higher market beta.[3]

In contrast, the "value anomaly" exists in the L.R.E.E. solution, as a risk compensation associated with the need to hedge against a stochastic investment opportunity set – just as Merton has predicted, or more fundamentally to hedge against the stochastic dispersion of the firms' capital stocks. To see this, let us compute the hedging demand for the symmetric two-sector case. Let $w_1 \equiv -\frac{\partial \log q^1}{\partial \log \zeta}$, $w_2 \equiv -\frac{\partial \log q^2}{\partial \log \zeta}$, and $\delta \equiv \frac{\partial \log h}{\partial \log \zeta}$. Then

$$W = \begin{pmatrix} 1 - w_1 & w_1 \\ w_2 & 1 - w_2 \end{pmatrix}, \quad \Delta = \begin{pmatrix} \delta \\ -\delta \end{pmatrix}.$$

It follows that

$$a^H = -(\mathbf{I} + W)'^{-1}\Delta = \frac{1}{1 - w_1 - w_2} \begin{pmatrix} -\delta \\ \delta \end{pmatrix}.$$

Under the given parameters, the L.R.E.E. solution implies $w_1 > 0$, $w_2 > 0$, and $1 - w_1 - w_2 > 0$, $\delta > 0$ when $\zeta > 1$, and $\delta < 0$ when $\zeta < 1$. Thus, $a^{H,1} < 0$ when $\zeta > 1$ and $a^{H,2} < 0$ when $\zeta < 1$. In other words, the hedging demand for the value stock is positive, which implies that the value stock should earn an "abnormal" return over and above the risk premium associated with

[3]Gomes, Kogan, and Zhang (2001) find themselves in exactly the same situation. The one-factor CAPM holds in GKZ and in our model under the myopic expectations for a completely different reason. In GKZ, the one-factor CAPM holds because both the market risk and the aggregate consumption risk are driven by the same systematic production shock, which means that the hedging demand is also perfectly correlated with the market risk. In our model under the myopic expectations, the one-factor CAPM holds because there is no hedging demand.

its market risk. The expected "abnormal return", given by equation (31), is plotted in Figure 4. We see that the HML strategy (long high book-to-market namely value stock and short low book-to-market namely growth stock) earns a positive "abnormal return", which increases with the dispersion of the capital stocks. When the relative share is $\frac{b^1}{b^1+b^2} = 56.5\%$, which is a standard deviation away from the long-run mean,[4] the "abnormal return" is about 6 basis points. At $\frac{b^1}{b^1+b^2} = 80\%$, which is about five standard deviations from the long-run mean, the "abnormal return" is about 38 basis points. About one third of the "abnormal return" comes from the long position in the value stock and the rest comes from the short position in the growth stock.

4 Conclusion

In this paper, we have demonstrated that a positive value premium (high book-to-market stocks earn higher expected returns and a positive "abnormal return" for the HML strategy (long high book-to-market and short low book-to-market) can arise endogenously in a general equilibrium model. Our model has its obvious limitations. First, the model abstracts away from any production shocks. Second, the model abstracts away from labor supply and demand decisions. These simplifying assumptions are adopted so that the effect of demand shocks can be more easily analyzed and highlighted. A more realistic model that relaxes these assumptions is of obvious interest but will be left to future research.

[4] $\frac{d\chi}{\chi} = \ldots + (1-\chi)(\xi^1 - \xi^2)dB$. Thus, the standard deviation of χ at the long-run mean is about $(1-0.5) \times 13\% = 6.5\%$.

Figure 1: The L.R.E.E. Solution

$\pi^1 = \pi^2 = 10\%,\ \alpha^1 = \alpha^2 = 7,\ \xi^1 = \xi^2 = 18.4\%,\ 1-\gamma = 3.3,\ \rho = 0.05$

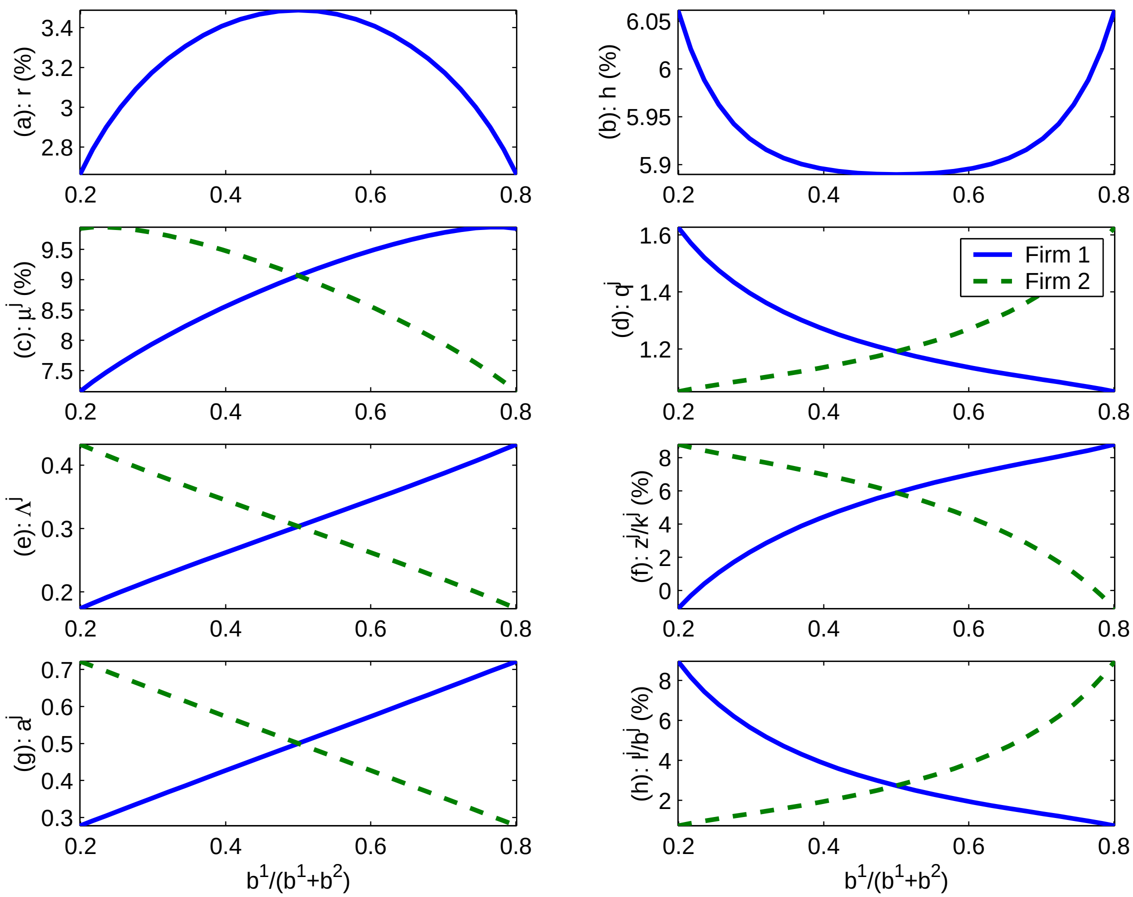


Figure 2: The Myopic Solution

$\pi^1 = \pi^2 = 10\%,\ \alpha^1 = \alpha^2 = 7,\ \xi^1 = \xi^2 = 18.4\%,\ 1-\gamma = 3.3,\ \rho = 0.05$

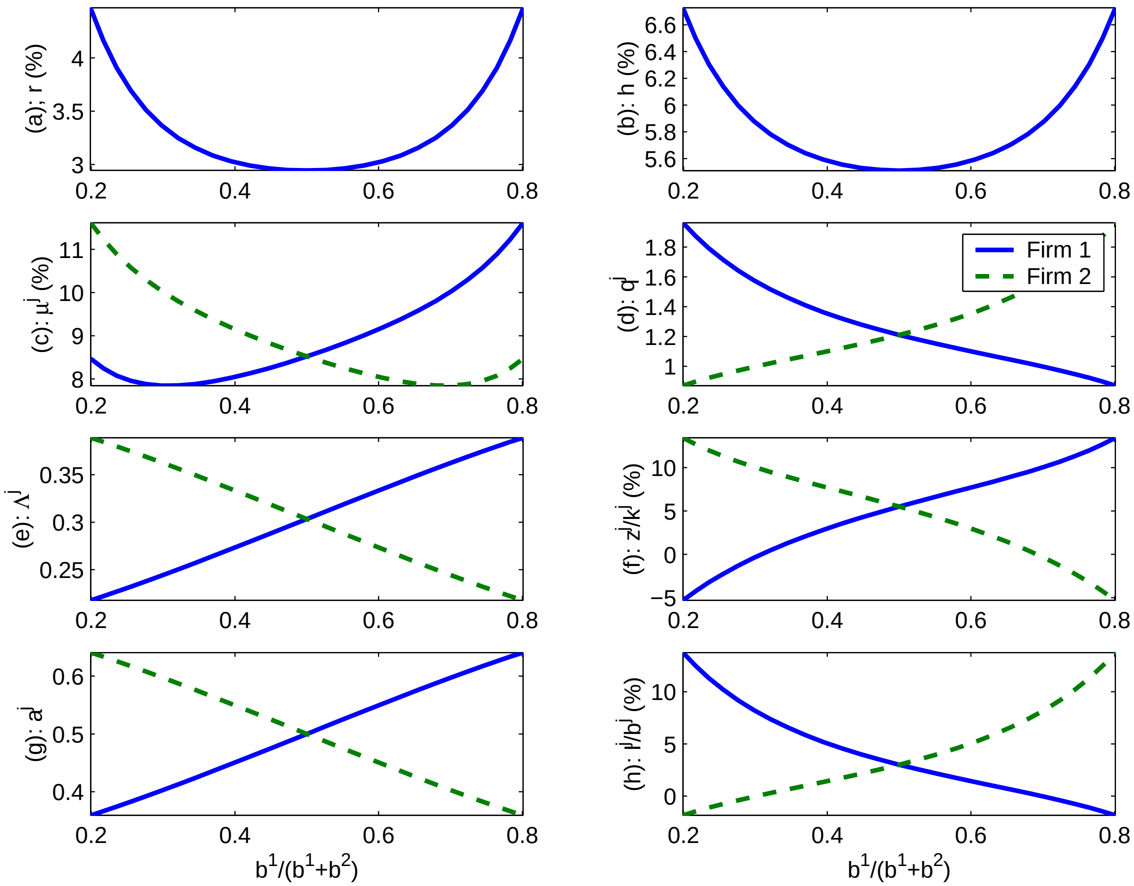


Figure 3: Comparing the Myopic and Ration Solutions

$$\pi^1 = \pi^2 = 10\%,\ \alpha^1 = \alpha^2 = 7,\ \xi^1 = \xi^2 = 18.4\%,\ 1-\gamma = 3.3,\ \rho = 0.05$$

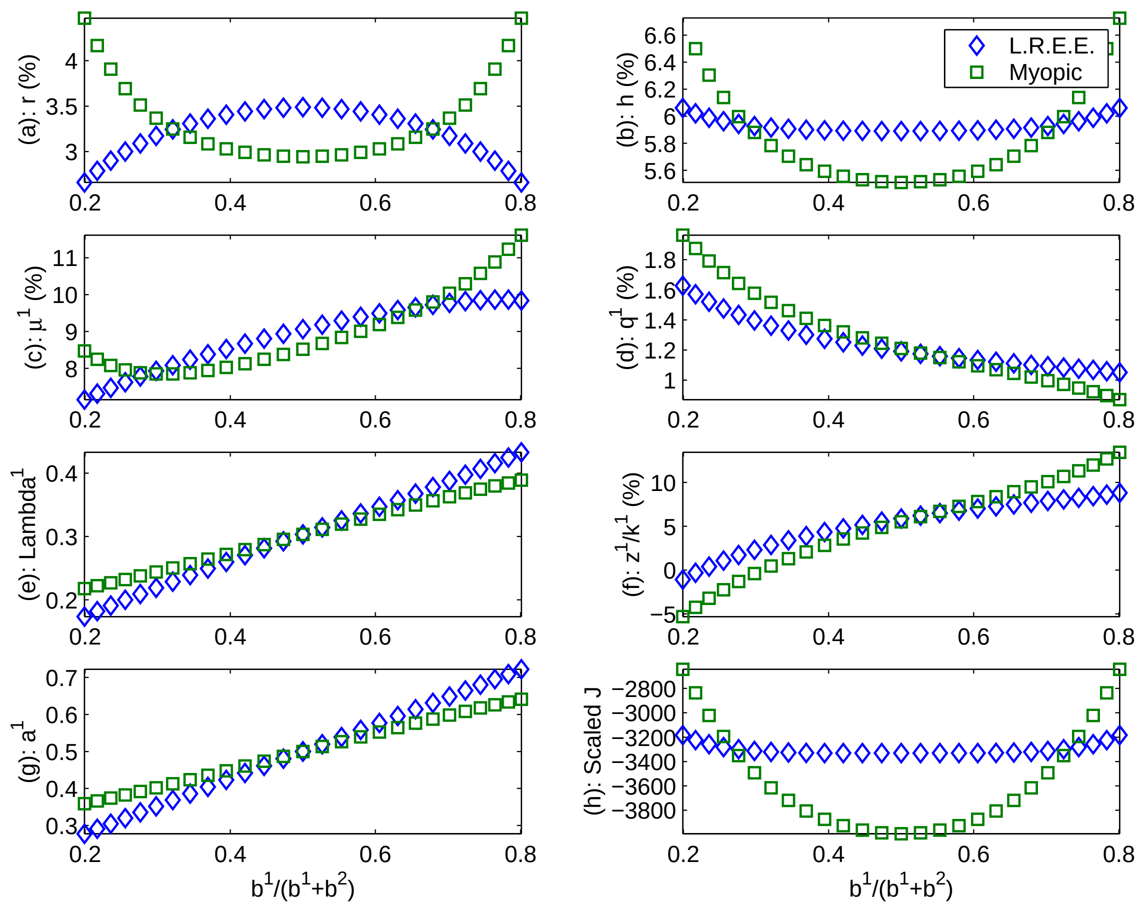
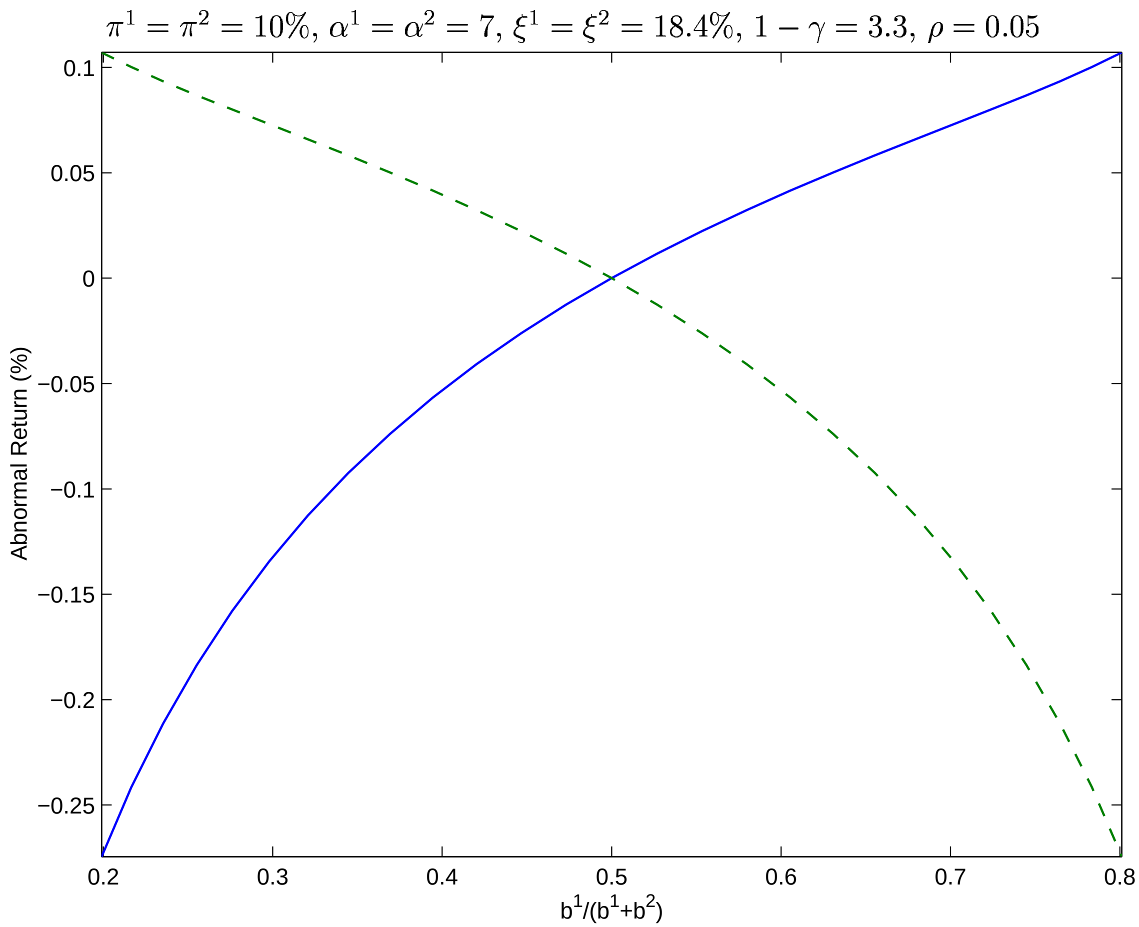


Figure 4: Abnormal Return due to Hedging Demand



A Optimal Consumption, Portfolio, and Investment Decisions

A.1 Consumption and Portfolio Decisions

To solve the investor's problem, we conjecture that the value function can be written as

$$V_t = e^{-\rho t} J(k, \zeta) = \max_{c_s, a_s : s \geq t} E\left[\int_0^\infty e^{-\rho s} \frac{c_s^\gamma}{\gamma} ds \,\middle|\, k_t = k, \zeta_t = \zeta \right].$$

The Bellman equation reads

$$\begin{aligned} 0 &= \max_{c,a} \; \frac{c^\gamma}{\gamma} - \rho J + \left[a'(\mu - r)k + rk - c\right] J_k \\ &+ \frac{1}{2} k^2 a' \sigma \sigma' a J_{kk} + \hat{\mu}' J_\zeta + \frac{1}{2} \text{Trace}\left[\hat{\sigma}\hat{\sigma}' J_{\zeta\zeta'}\right] + k a' \sigma \hat{\sigma}' J_{k\zeta}. \end{aligned} \tag{34}$$

Conjecture that the indirect utility function is given by $J(k, \zeta) = \phi(\zeta) \frac{k^\gamma}{\gamma}$. The Bellman equation reduces to

$$\begin{aligned} 0 = \max_{c,a} \; & \left[\frac{c^\gamma}{\gamma J} - \rho + \gamma \left[a'(\mu - r) + r - \frac{z}{k} \right] \right. \\ & \left. - \frac{\gamma(1-\gamma)}{2} a' \sigma \sigma' a + \frac{\hat{\mathcal{A}} \phi}{\phi} + \gamma a' \sigma \hat{\sigma}' \frac{\partial \log \phi}{\partial \zeta} \right], \end{aligned} \tag{35}$$

where $\hat{\mathcal{A}}$ is the infinitesimal generator of ζ, given by

$$\hat{\mathcal{A}} \equiv \sum_{j=1}^{N-1} \hat{\mu}' \frac{\partial}{\partial \zeta} + \frac{1}{2} \text{Trace}\left[\hat{\sigma}\hat{\sigma}' \frac{\partial^2}{\partial \zeta \partial \zeta'} \right] \tag{36}$$

First order condition with respect to c implies $c^* = h(\zeta)k$, where $h(\zeta)^{\gamma - 1} = \phi(\zeta)$. First order condition with respect to a implies

$$a^* = a^M(\zeta) + a^H(\zeta), \tag{37}$$

where a^M and a^H are, respectively, the *myopic* and *hedging* components of the portfolio demand, given by

$$a^M(\zeta) = \frac{\sigma'^{-1} \Lambda}{1 - \gamma} = \frac{(\sigma \sigma')^{-1} (\mu - r)}{1 - \gamma}, \tag{38}$$

$$a^H(\zeta) = -\sigma'^{-1} \hat{\sigma}' \frac{\partial \log h}{\partial \zeta}. \tag{39}$$

Substituting the optimal policies into the Bellman equation, we obtain

$$h(\zeta) = h^M + h^H(\zeta). \tag{40}$$

where h^M and h^H represent, respectively, the myopic and hedging components of consumption demand, given by

$$h^M = \frac{\gamma}{1-\gamma}\left[\frac{\rho}{\gamma} - r(\zeta) - \frac{\Lambda(\zeta)'\Lambda(\zeta)}{2(1-\gamma)}\right], \tag{41}$$

$$h^H(\zeta) = \frac{\gamma}{1-\gamma}\frac{\partial \log h}{\partial \zeta'}\hat{\sigma}\Lambda - \frac{\gamma}{2}\frac{\partial \log h}{\partial \zeta'}\hat{\sigma}\hat{\sigma}'\frac{\partial \log h}{\partial \zeta} - \frac{\hat{\mathcal{A}}h^{\gamma-1}}{(1-\gamma)h^{\gamma-1}}. \tag{42}$$

The hedging components of the portfolio and consumption demands arise because of the state-dependence of the aggregate consumption/wealth ratio $h(\zeta)$.

A.2 Investment Decisions

Conjecture that the value function for firm i is given by $m_t k_t^i$, and $k_t^i = q^i(\zeta_t)b^i$. Then the Bellman equation for firm i reads:

$$0 = \max_{I^i}\left[\frac{z^i}{k^i} + \frac{I^i}{b^i} - \left[r + (\xi^i + \sum_{j=1}^{N} W^{ij}\xi^j)\Lambda\right] + \frac{\hat{\mathcal{A}}q^i}{q^i} + \xi^i\sum_{j=1}^{N} W^{ij}\xi^{j'}\right], \tag{43}$$

where z^j is given by equation (6), and W_t is a $N\times N$ matrix defined by, for $1 \le i \le N$,

$$W^{ij} = \frac{\partial \log q^i}{\partial \log \zeta^j},\ 1 \le j \le N-1;\ W^{iN} = -\sum_{j=1}^{N-1} W^{ij}. \tag{44}$$

Since the last three terms on the right hand side of equation (43) depend only on the economy wide state vector ζ, the first order condition with respect to I^j implies

$$I^{*j} = \frac{q^j(\zeta) - 1}{\alpha^j}.$$

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