Standard Risk Aversion and the Demand for Risky Assets in the Presence of Background Risk

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Abstract

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We consider the demand for state contingent claims in the presence of a zero-mean, non-hedgeable background risk. An agent is defined to be generalized risk averse if he/she reacts to an increase in background risk by choosing a demand function for contingent claims with a smaller slope. We show that the conditions for standard risk aversion: positive, declining absolute risk aversion and prudence are necessary and sufficient for generalized risk aversion. We also derive a necessary and sufficient condition for the agent's derived risk aversion to increase with a simple increase in background risk.

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D52, D81, G11."

1 Introduction

Recent advances in the theory of risk bearing have concentrated on the effect of a non-tradeable background risk on the risk aversion of an agent to a second independent risk. For example, Gollier and Pratt (1996) define a rather general class of utility functions such that risk-averse individuals become even more risk averse towards a risk, when a second, independent, unfair background risk is added. They compare the risk aversion of an agent with no background risk to that of an agent who faces the background risk. They term the set of functions under which the agent becomes more risk averse, the class of "risk-vulnerable" utility functions. The set of risk-vulnerable functions is larger than the set of proper risk averse functions introduced earlier by Pratt and Zeckhauser (1987), who consider utility functions such that the expected utility of an undesirable risk is decreased by the presence of an independent, undesirable risk. Kimball (1993) has considered the effect of the [even larger] set of expected marginal utility increasing background risks. This led him to define the more restrictive class of standard risk averse utility functions. Standard risk aversion characterises those functions where the individual responds to an expected marginal utility increasing background risk by reducing the demand for a marketed risk. Kimball shows that standard risk averse functions are characterized by positive, decreasing absolute risk aversion and absolute prudence. The set of standard risk averse functions is a subset of the set of proper risk averse functions, which, in turn, are a subset of the risk vulnerable functions, as discussed by Gollier and Pratt (1996, pp 1118-9). In a related paper, Eeckhoudt, Gollier and Schlesinger (1996) extend this analysis by considering a rather general set of changes in background risk, which take the form of first or second order stochastic dominance changes. They establish a set of very restrictive conditions on the utility function such that agents become more risk averse when background risk increases in this sense.

The purpose of this paper is twofold. First, we consider a smaller set of increases in background risk than Eeckhoudt, Gollier and Schlesinger (1996) and derive less restrictive conditions for an increase in background risk to increase the derived risk aversion of agents. We restrict the set of increases in the risk of background income y, with $E(y) = 0$, to simple increases (see also Eeckhoudt, Gollier and Schlesinger (1995)). A *simple* increase in background risk is a change Δ to y such that $\Delta \leq [=][\geq] 0$ for $y < [=] > y_0$ for some y_0 and $E(\Delta) = 0$. We derive a necessary and sufficient condition on the utility function for a simple increase in background risk to make the agent more risk averse. We show that standard risk aversion is sufficient, but not necessary for a simple increase in background risk to increase derived risk aversion.

The second and the main purpose of the paper is to investigate restrictions on utility functions which guarantee a more risk averse behaviour in the presence of an increased, independent background risk, when the agent faces a choice between state-contingent claims. In this setting, changes in risk-averse behaviour are reflected in the slope of the demand function for contingent claims.[1] Gollier (2000) considers a model where the agent can buy state-contingent claims on consumption, given no background risk. Let ϕ be the probability deflated price of obtaining one unit of consumption if a state occurs and nothing otherwise. Then, in this model, the higher is ϕ for a given state, the lower is the agent's demand for claims on that state, w. In other words, the demand function, $w(\phi)$, that relates the consumption in a state to the price, is downward sloping. Gollier [Proposition 51] shows that, if two agents with utility functions u_1 and u_2 have the same endowment, and if u_1 is more risk averse than u_2, then the demand function of agent 1, $w_1(\phi)$, 'single-crosses from below' the demand function of agent 2, $w_2(\phi)$. This single-crossover property is illustrated in Figure 1. Gollier goes on to conclude that "risk-vulnerable investors will select a safer consumption plan", when they face background risk. Our results, showing the effect of a simple increase in background risk on risk aversion, therefore imply that an agent facing an increase in background risk will respond by choosing a demand function similar to investor 1 rather than that chosen by investor 2, in Figure 1.

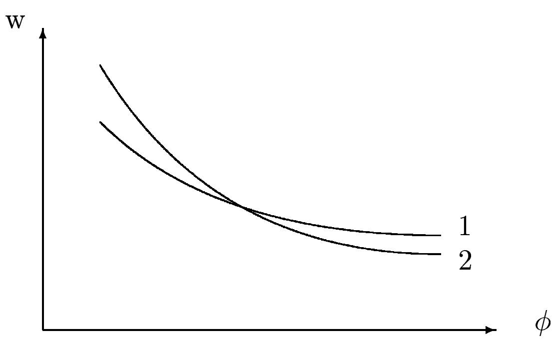

Figure 1: Demand curve 1 is less steep than demand curve 2 everywhere

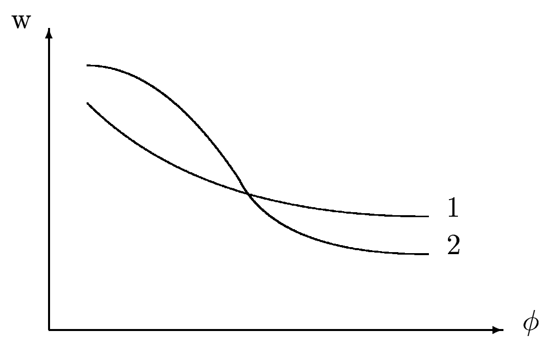

Figure 2: Demand curve 1 is less steep than demand curve 2 in some range and steeper in other ranges

[1] In a state-contingent claims model, risk-averse behaviour can be characterized by the slope of the demand curve for contingent claims. In the case of extreme risk aversion, the agent buys an equal amount of claims on each state, despite the higher prices of the claims on some of the states. A less risk-averse agent buys a schedule of claims more weighted towards claims that are relatively cheap. In the Pratt (1964) characterization of risk aversion, the more risk averse agent buys less of a single risky asset and more of a risk-free asset. This also has the effect of producing a demand curve with a lower slope. The equating of 'less risk-averse behavior' with a smaller slope of the demand function for contingent claims is therefore a natural generalization of Pratt's characterization of risk aversion.

However, Gollier's analysis highlights a problem. Even though agent 1 is more risk averse, he could have a demand function that has a smaller slope at the crossover point, but has a greater slope over some range of ϕ, as Figure 2 illustrates. This means that the more risk-averse investor actually exhibits less risk-averse behaviour over some range . As Gollier notes, the single-crossover property only throws light on local risk-taking behaviour in the range around the crossover point. In this paper, we wish to look at local risk-taking behaviour over all ranges, hence we employ a stricter definition of more risk-averse behaviour in the contingent claims model. We define an agent 1 to behave in a more risk averse manner than an agent 2, if his demand function has a smaller (absolute) slope than that of agent 2 *everywhere.* We then consider how the slope of the demand function changes as background risk increases. If the agent responds to an increase in background risk by choosing a demand function with a smaller slope everywhere, we say that the agent is *generalized risk averse.*

This concept of generalized risk aversion relates closely to the previously discussed concepts of 'risk vulnerability' and 'standard risk aversion'. In the case of 'risk vulnerability', an agent responds to the introduction of background risk by reducing his demand for a single risky asset. In the case of standard risk aversion, an agent responds similarly to a marginal utility-increasing background risk. In the case of generalized risk aversion the idea of the response of risk-taking behaviour to an increase in background risk is extended to the case of state-contingent claims.[2]

We consider the effect of an independent background risk on the demand for state-contingent claims, using an extension of the analysis of Back and Dybvig (1993), who establish conditions for the optimality of an agent's demand. We investigate the set of [restrictions on] utility functions such that the agent responds to *monotonic* increases in zero-mean background risk by choosing a demand function that has a smaller slope at all price levels. In the context of this choice problem, we need to further restrict the set of changes in background risk that are considered to the set of monotonic increases. A monotonic increase in background risk y is defined as a change Δ in y, where $\partial\Delta/\partial y \geq 0, \forall y$, and where $E(\Delta) = 0$. Hence, a monotonic increase in background risk is a change, Δ, that itself increases with y. The simplest example of a monotonic increase is a proportionate increase where Δ is proportionate to y. Assuming monotonic increases in background risk, we find that the set of generalized risk-averse utility functions is the standard risk-averse class of Kimball (1993). Hence, risk vulnerability is not sufficient for background risk to reduce the slope of the demand function for state-contingent claims.

[2]Various papers have analysed the impact of certain types of increases in background risk on the demand for insurance, where the amount of insurance is measured by the coinsurance rate and the deductible, see, for example Eeckhoudt and Kimball (1992) and Meyer and Meyer (1998). While these papers show that standard risk aversion is sufficient to guarantee a higher demand for insurance, we derive here necessary and sufficient conditions on preferences to yield *generalized* risk averse behaviour.

The conditions for standard risk aversion - positive, declining absolute risk aversion, and positive, declining absolute prudence - are sufficient for a monotonic increase in background risk to increase derived risk aversion. They are also sufficient for the slope of the demand function for contingent claims to become smaller everywhere. What is more surprising is that these conditions are also *necessary* for generalized risk aversion. Necessity arises from the fact that the slope of the demand function for contingent claims must become less steep at all possible values of ϕ. As Kimball argues, declining absolute risk aversion and declining absolute prudence are natural attributes of the utility function. They are shared, also, by the HARA class of functions with an exponent less than one. The larger set of risk-vulnerable utility functions, used by Gollier and Pratt, is not restrictive enough, when we consider the effect of background risk on the slope of the demand function. Our result adds to the case for the standard risk-averse functions to be the natural class of functions to use when analysing the impact of background risk.

In section 2, we look again at the effect of an increase in background risk on the risk aversion of the derived utility function. Here we are concerned, as were Eeckhoudt, Gollier and Schlesinger (1996) with changes in background risk. However, in order to avoid the restrictive conditions on utility they found, we restrict the analysis to simple increases in background risk. In section 3, we then introduce the problem of analysing the slope of the demand function for contingent claims. We then present our main result: agents choosing state-contingent claims become more risk averse in their choice, if and only if they are standard risk averse, i.e. positive and declining absolute risk aversion and prudence is the necessary and sufficient condition for generalized risk aversion.

2 The Effect of an Increase in Background Risk on Derived Risk Aversion

We consider an individual agent who can buy a set of contingent claims on future consumption and faces background risk. The agent's total income at the end of the period, W, is therefore composed of an income from tradeable claims, w, plus the background risk income y, i.e. $W = w + y$. We assume that background risk, y, has a zero mean, and is bounded from below, $y \geq a$. Moreover we assume that y is distributed independently of w. A state of the world determines both the agent's income from tradeable claims and the background risk income. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the probability space on which the random variables are defined.

The agent's utility function is $u(W)$. We assume that the utility function is state-independent, strictly increasing, strictly concave, and four times differentiable on $W\varepsilon(\underline{W},\infty)$, where $\underline{W}$ is the lower bound of W. We assume that there exist integrable functions on $\omega\varepsilon\Omega$, u_0 and u_1 such that

$$u_0(\omega) \leq u(W) \leq u_1(\omega)$$

We also assume that similar conditions hold for the derivatives $u'(W)$, $u''(W)$ and $u'''(W)$. The agent's expected utility, conditional on w, is given by the derived utility function, as defined by Kihlstrom et al. (1981) and Nachman (1982):

$$\nu(w) = E_y[u(W)] \equiv E[u(w+y) \mid w] \tag{1}$$

where E_y indicates an expectation taken over different outcomes of y. Thus, the agent with background risk and a von Neumann-Morgenstern concave utility function $u(W)$ acts like an individual without background risk and a concave utility function $\nu(w)$.[3] The coefficient of absolute risk aversion is defined as $r(W) = -u''(W)/u'(W)$ and the coefficient of absolute prudence as $p(W) = -u'''(W)/u''(W)$. From Kimball (1993), the agent is standard risk averse if and only if $r(W)$ and $p(W)$ are both positive and declining. The absolute risk aversion of the agent's derived utility function is defined as the negative of the ratio of the second derivative to the first derivative of the derived utility function with respect to w, i.e.,

$$\hat{r}(w) = -\frac{\nu''(w)}{\nu'(w)} = -\frac{E_y[u''(W)]}{E_y[u'(W)]} \tag{2}$$

We first investigate the question of how an agent's derived risk aversion is affected by a "simple increase" in background risk. A simple increase in background risk, which Eeckhoudt, Gollier and Schlesinger (1995) term 'a simple spread across y_0', is defined as a change in y, Δ, such that for a given y_0,

$$\Delta \leq [=][\geq] 0, \text{ if } y < [=][>] y_0 \text{ and } E(\Delta) = 0.$$

Not surprisingly, the condition for an agent's derived risk aversion to increase, when there is a marginal increase in zero-mean background risk, is stronger than the condition of Gollier and Pratt(1996). This is because the "risk vulnerability" condition of Gollier and Pratt only considers changes in background risk from zero to a finite level, whereas we consider any changes in background risk.

[3] See, for example, Eeckhoudt, Gollier and Schlesinger (1996), p. 684.

It is worth noting that, in the absence of background risk , $\hat{r}(w)$ is equal to $r(W)$, the coefficient of absolute risk aversion of the original utility function. In the proposition that follows, we characterize the behavior of $\hat{r}(w)$ in relation to $r(W)$, and explore the properties of derived risk aversion in the presence of increasing zero-mean background risk. We will proceed by proving a proposition about the condition under which any marginal increase in background risk raises derived risk aversion. Since the condition holds for *any* marginal increase in background risk, the same condition must hold for a finite increase to raise derived risk aversion. It is convenient to define an index of background risk, $s \epsilon R^+$, where $s = 0$ if no background risk exists. A marginal increase in background risk is represented by a marginal increase in s. We assume that the background risk income y is differentiable in s.[4]

Proposition 1 *(Derived Risk Aversion and Simple Increases in Background Risk)*

If $u'(W) > 0$ and $u''(W) < 0$, then

$$\frac{\partial \hat{r}(w)}{\partial s} > [=][<] 0, \forall (w,s) \Longleftrightarrow$$
$$u'''(W_2) - u'''(W_1) < [=][>] - r(W)[u''(W_2) - u''(W_1)],$$
$$\forall (W, W_1, W_2), W_1 \leq W \leq W_2$$

Proof: See Appendix 1.

In order to interpret the necessary and sufficient condition under which an increase in a zero-mean, background risk will raise the risk aversion of the derived utility function, first consider the special case in which background risk changes from zero to a small positive level. This is the case analysed previously by Gollier and Pratt (1996). In this case, we have

Corollary 1 *In the case of small risks, Proposition 1 becomes*

$$\hat{r}(w) > [=][<] r(W) \quad \textit{iff} \quad \frac{\partial \theta}{\partial W} < [=][>] 0, \forall W$$

[4] This assumption in no way restricts the type of background risk increases assumed in the analysis. Consider, for example, jumps in background risk. These can be analysed as sums of small increases. Our proof derives conditions for the derived risk aversion to change in a certain manner, given a small increment in background risk. The same conditions assure that derived risk aversion changes in a similar manner in response to jumps in background risk.

where $\theta(W) \equiv u'''(W)/u'(W)$.

Proof: Let $W_2 - W_1 \rightarrow dW$. In this case, $u'''(W_2) - u'''(W_1) \rightarrow u''''(W)dW$. Similarly $u''(W_2) - u''(W_1) \rightarrow u'''(W)dW$.

Hence, the condition in Proposition 1 yields, in this case, $u''''(W) < [=][>] - r(W)u'''(W)$. This is equivalent to $\partial\theta/\partial W < [=][>]0, \forall W$.□

In Corollary 1, we define an additional characteristic of the utility function $\theta(W) = u'''(W)/u'(W)$ as a *combined* prudence/risk aversion measure. This measure is defined by the product of the coefficient of absolute prudence and the coefficient of absolute risk aversion. The corollary says that for a small background risk derived risk aversion exceeds [is equal to] [is smaller than] risk aversion if and only if $\theta(W)$ decreases [stays constant] [increases] with W. Hence, it is significant that *neither* decreasing prudence *nor* decreasing absolute risk aversion is necessary for derived risk aversion to exceed risk aversion. However, the combination of these conditions is sufficient for the result to hold, since the requirement is that the product of the two must be decreasing. The condition is thus weaker than standard risk aversion, which requires that *both* absolute risk aversion and absolute prudence should be positive and decreasing. Note that the condition in this case is the same as the 'local risk vulnerability' condition derived by Gollier and Pratt (1996). Local risk vulnerability is $r'' > 2rr'$, which is equivalent to $\theta' < 0$. We now apply Proposition 1 to show that standard risk aversion is a sufficient, but not a necessary condition, for an increase in background risk to cause an increase in the derived risk aversion [see also Kimball (1993)]. We state this as

Corollary 2 *Standard risk aversion is a sufficient, but not necessary, condition for derived risk aversion to increase with a simple increase in background risk.*

Proof: Standard risk aversion requires both positive, decreasing absolute risk aversion and positive decreasing absolute prudence. Further, $r'(W) < 0 \Rightarrow p(W) > r(W)$. Also, standard risk aversion requires $u'''(W) > 0$. It follows that the condition in Proposition 1 for an *increase* in the derived risk aversion can be written as[5]

$$\frac{u'''(W_2) - u'''(W_1)}{u''(W_2) - u''(W_1)} < -r(W_1)$$

[5]Note that whenever $r'(W)$ has the same sign for all W, the three-state condition in Proposition 1 (i.e. the condition on W, W_1, and W_2) can be replaced by a two-state condition (a condition on W_1 and W_2).

or, alternatively,

$$p(W_1)\left[1-\frac{u'''(W_2)}{[u'''(W_1)]}\right] / \left[1-\frac{u''(W_2)}{[u''(W_1)]}\right] > r(W_1)$$

Since $p(W_1) > r(W_1)$, a sufficient condition is that the term in the square bracket exceeds 1. This, in turn, follows from decreasing absolute prudence, $p'(W) < 0$. Hence, standard risk aversion is a sufficient condition.

To establish that standard risk aversion is not necessary, consider a case that is not standard risk averse. Suppose, in particular, that $u'''(W) < 0, u''''(W) < 0$, that is, the utility function exhibits increasing risk aversion and negative prudence.[6] In this case, it follows from Proposition 1 that $\partial \hat{r}/\partial s > 0, \forall(w,s)$. □

In order to obtain more insight into the meaning of the condition in Proposition 1, consider the case where the increase in background risk raises derived risk aversion. Defining $y' = \partial y/\partial s$,

$$\hat{r}(w) = E_y\left[\frac{u'(W)}{E_y[u'(W)]}r(W)\right],$$

$$\frac{\partial \hat{r}(w)}{\partial s} = E_y\left[\frac{u'(W)}{E_y[u'(W)]}r'(W)y'\right] + E_y\left[r(W)\frac{\partial}{\partial y}\left[\frac{u'(W)}{E_y[u'(W)]}\right]y'\right] \tag{3}$$

As shown in appendix 1, it suffices to consider a three-point distribution of background risk with $y_1 < 0, y_2 > 0, y_1 < y_0 < y_2$ and $y_0' = 0, y_1' < 0, y_2' > 0$. The first term in equation (3) is positive whenever r is declining and convex. This follows since $E(y') = 0$ and $y_2' > y_1'$ implies that $E[r'(W)y'] \geq 0$. Since $u'(W)$ is declining, it follows that the first term in (3) is positive. Now consider the second term: $\partial[u'(W)/E_y[u'(W)]]/\partial y$ is positive for y_1 and negative for y_2 and has zero expectation. Therefore a declining r implies that the second term is positive. Hence a sufficient condition for $\partial\hat{r}(w)/\partial s \geq 0$ is a declining and convex r.[7]

The first term is higher, the more convex is r. Therefore, $\partial\hat{r}(w)/\partial s \geq 0$ is also possible for an increasing r, if convexity is sufficiently high. Therefore, there are utility functions with

[6] As an example, consider the utility function

$$u(W) = \frac{1-\gamma}{\gamma}\left[A + \frac{W}{1-\gamma}\right]^{\gamma}, \text{where } \gamma \in (1,2), W < A(\gamma - 1)$$

This utility function exhibits *increasing* risk aversion and *negative* prudence. Still, $\theta(W)$ decreases with wealth even in this case and the derived risk aversion increases with background risk.

[7] See also Corollary 1 of Gollier and Pratt (1996).

increasing risk aversion which still imply that simple increases in zero-mean background risk raise the derived risk aversion.

3 The Effect of Changes in Background Risk on the Optimal Demand Function for Contingent Claims

In the previous section we derived the condition under which an increase in background risk increases the agent's derived risk aversion. As will be shown, this condition is not sufficient to guarantee that the increase in background risk reduces the slope of the agent's demand curve for state-contingent claims, everywhere, i.e., it is not sufficient for generalized risk aversion. In this section we derive the necessary and sufficient condition for the utility function to exhibit generalized risk aversion.

We assume that the capital market is perfect. A state of nature determines both the agent's tradeable income w and his background income y. We partition the state space into subsets of states that differ only in the background income, y. We call these subsets "traded states" since they represent states on which state-contingent claims can be traded. We assume there is a continuum of such states and, for convenience, we label these states by a continuous variable $x \epsilon R^+$. We assume the market, in the traded states, is complete. We also assume that there exists a pricing kernel, $\phi = \phi(x)$ with the property $\phi > 0$, where $\phi(x)$ is a continuous function. [8]

Let $w = g(x)$ be the agent's income from the purchase of state-contingent claims. The agent chooses $w = g(x)$, subject to the constraint that the cost of acquiring this set of claims is equal to his/her initial endowment. The agent's consumption at the end of the single period, W, is equal to the chosen marketed claim, w, plus an independent, zero-mean background risk y, i.e. $W = w + y$. The background risk y affects his/her choice of the function $w = g(x)$. We assume that the agent has sufficient endowment to ensure that w can be chosen to obtain $W \geq \underline{W}$ in all traded states. We also assume certain properties of

[8] The market is complete in the sense of Nachman (1988). The agent can buy a digital option which pays one unit of consumption, if $x \geq k$, and 0 otherwise, $\forall k \epsilon R^+$. The price of such an option is

$$\int_k^\infty \phi(x) f(x) d(x),$$

where $\phi(x)$ is the pricing kernel and the probability density function is $f(x)$. A contingent claim is a contract (a portfolio of digital options) paying one unit of consumption if $x \epsilon [k, k+\eta)$ and nothing otherwise, for positive, infinitely small η.

the utility function. First, the marginal utility has the limits:

$$u'(W) \to \infty \text{ if } W \to \underline{W},$$

$$u'(W) \to 0 \text{ if } W \to \infty.$$

Second, the risk aversion goes to zero at high levels of income, i.e.

$$r(W) \to 0 \text{ if } W \to \infty.$$

These reasonable restrictions are satisfied, for example, by the HARA class with an exponent less than 1.

The agent solves the following maximization problem:

$$\max_{w=g(x)} E_x[\nu(w)] = E_x[\nu(g(x))] \tag{4}$$

$$\text{s.t.} \quad E_x\left[(g(x) - g^0(x))\phi(x)\right] = 0$$

In the budget constraint, $w^0 = g^0(x)$ is the agent's endowment of claims. $\phi(x)$, the pricing kernel, is given exogenously. The maximisation problem (4) is a standard state-preference maximisation problem. The expectation, $E_x(.)$, is taken only over the traded states. Note that the background income, y, has only an indirect impact on problem (4) through its effect on the derived utility function. This is defined by equation (1) as the expected value of utility over different outcomes of y, given the traded income w.

The first order condition for a maximum is

$$\nu'(g(x)) = \lambda\phi(x),$$

or simply

$$\nu'(w) = \lambda\phi, \tag{5}$$

where λ is a positive Lagrange multiplier which reflects the tightness of the budget constraint. Equation (5) holds as an equality since, by assumption, $u'(W) \to \infty$ for $W \to \underline{W}$ and $u'(W) \to 0$ for $W \to \infty$. The demand for claims in equation (5) can be shown to be optimal and unique under some further finiteness restrictions.[9] This follows from the results of Back and Dybvig (1993).

[9] $E[w\phi] < \infty$ for any $\lambda > 0$ and each w satisfying (5) is assumed.

From the first order condition (5), it follows that we can define a function $w = w(\phi) = \nu'^{(-1)}(\lambda\phi)$. Hence, given the derived utility function and the initial endowment, the demand for claims contingent on a traded state x depends only on $\phi(x)$. Thus $w(\phi)$ is a deterministic function relating the demand for state-contingent claims to the pricing kernel. It follows from our assumptions that $w(\phi)$ is a twice differentiable function of ϕ.[10]

Our aim is to find the necessary and sufficient conditions on the utility function, which guarantee that the agent's demand function becomes less steep when background risk increases. First we define

Definition 1 *An agent is generalized risk-averse if the absolute value of the slope of his/her demand function for state-contingent claims $w(\phi)$ becomes smaller for all ϕ, given an increase in background risk.*

Differentiating equation (5) with respect to ϕ, for a given level of background risk, and dividing by $\lambda\phi$, yields the slope of the demand function

$$\frac{\partial w}{\partial \phi} = \frac{-1/\phi}{\hat{r}(w)}, \forall \phi \tag{6}$$

Suppose that background risk increases the derived risk aversion of the agent, $\hat{r}(w)$. It follows from equation (6) that the background risk affects the slope of the demand function. We now consider the effect of changes in the level of background risk, assuming that the pricing function ϕ is given. From equation (6) it appears at first sight that the slope of the demand function becomes less steep whenever the increase in background risk increases the agent's derived risk aversion. In fact, it follows from Gollier (2000, Proposition 51) that:

Proposition 2 *Suppose that an increase in background risk raises the agent's derived risk*

[10] Consider the function

$$F(w, \phi, s) = \nu'(w) - \lambda\phi = 0.$$

The partial derivative F_w exists and is continuous, since the utility function $u(w + y)$ and its first three derivatives are assumed to exist and to be integrable. Also $F_w \neq 0$ for $w < \infty$. Hence, by the implicit function theorem, the function $w = w(\phi)$ is differentiable with

$$\partial w/\partial \phi = -\frac{F_\phi}{F_w}.$$

Also, since $y = y(s)$ is differentiable, and since F_ϕ and F_w are differentiable in y, then F_ϕ and F_w are also differentiable in s. It follows that $\partial^2 w/\partial\phi\partial s$ also exists.

aversion, everywhere. Then the new demand curve for contingent claims intersects the original one once from below.

Proof: At an intersection of the new demand curve, $w_1(\phi)$, and the original demand curve, $w_0(\phi)$, $w_1 = w_0$ so that, by equation (6), $\partial w_1/\partial\phi > \partial w_0/\partial\phi$ follows from $\hat{r}_1 > \hat{r}_0$. A second intersection would require $\partial w_1/\partial\phi < \partial w_0/\partial\phi$, which contradicts (6). Also, at least one intersection must exist, in order for the budget constraint to be satisfied.□

However, as noted by Gollier (2000), the one-intersection property does not imply that the new demand curve is less steep than the original one, everywhere. This is because a change in background risk, affects $\hat{r}(w)$ both directly and through the induced change in w. This is stated in the following proposition.

Proposition 3 *For the slope of the demand function for contingent claims to become smaller with an increase in background risk (generalized risk aversion), it is necessary, but not sufficient for the absolute risk aversion of the derived utility function to increase with background risk. That is*

$$\frac{d}{ds}\left[\frac{\partial w}{\partial\phi}\right] \geq 0 \Rightarrow \frac{\partial\hat{r}(w)}{\partial s} \geq 0, \tag{7}$$

but

$$\frac{\partial\hat{r}(w)}{\partial s} \geq 0$$

does not imply

$$\frac{d}{ds}\left[\frac{\partial w}{\partial\phi}\right] \geq 0.$$

Proof: Totally differentiating equation (6) with respect to s yields, since $1/\phi$ is given,

$$\frac{d}{ds}\left[\frac{\partial w}{\partial\phi}\right] = \frac{1/\phi}{[\hat{r}(w)]^2}\frac{d\hat{r}(w)}{ds}. \tag{8}$$

Since

$$\frac{1/\phi}{[\hat{r}(w)]^2} > 0,$$

$$\frac{d}{ds}\left[\frac{\partial w}{\partial\phi}\right] \geq 0 \Leftrightarrow \frac{d\hat{r}(w)}{ds} = \frac{\partial\hat{r}(w)}{\partial s} + \frac{\partial\hat{r}(w)}{\partial w}\frac{\partial w}{\partial s} \geq 0. \tag{9}$$

Given the budget constraint, $\partial w/\partial s$ has to be positive in some traded states and negative in others. It follows immediately that $\partial \hat{r}(w)/\partial s \geq 0$ is not sufficient to ensure that

$$\frac{d}{ds}\left[\frac{\partial w}{\partial \phi}\right] \geq 0.$$

Now to establish necessity, suppose that

$$\frac{d}{ds}\left[\frac{\partial w}{\partial \phi}\right] \geq 0$$

for all ϕ, then since the sign of

$$\frac{\partial \hat{r}(w)}{\partial w}\frac{\partial w}{\partial s}$$

depends on the sign of $\partial w/\partial s$, which can be positive or negative,

$$\frac{d}{ds}\left[\frac{\partial w}{\partial s}\right] \geq 0 \Rightarrow \frac{\partial \hat{r}(w)}{\partial s} \geq 0.$$

□

Having shown that increased derived risk aversion is a necessary, but not sufficient condition for generalized risk aversion, we can now establish our main result. In order to analyse the impact of background risk on the slope of the agent's demand function for contingent claims we need to make stronger assumptions. Regarding the background risk we now assume monotonic changes in background risk. This is a somewhat stronger than the previous assumption of simple increases in background risk. First we define monotonic increases in background risk.

Definition 2 *(Monotonic Increases in Background Risk)*

Let $y_i(s)$ denote a realisation $i = 1, ..., j$ of background risk income, given the index of background risk, s. Suppose that

$$y_1(s) \leq y_2(s) \leq ... \leq y_i(s) \leq ... \leq y_j(s)$$

with $y_i(0) = 0, \forall i$. Then, increases in background risk are monotonic, if for any $\overline{s} > s \geq 0$,

$$y_1(\overline{s}) - y_1(s) \leq y_2(\overline{s}) - y_2(s) \leq ... y_i(\overline{s}) - y_i(s) \leq ... \leq y_j(\overline{s}) - y_j(s)$$

The effect of assuming monotonic increases in background risk is that the rank order of the outcomes $y_1, y_2, ...$ is preserved under a monotonic increase in background risk. The main result of the paper is Proposition (4).

Proposition 4 *(Generalized Risk Aversion)*

Assume any monotonic increase in an independent, zero-mean background risk. Let $u'(W) > 0$ *and* $u''(W) < 0$*, where* $W \varepsilon (\underline{W}, \infty)$*. Suppose that* $u'(W) \to \infty$ *for* $W \to \underline{W}$ *and that* $u'(W) \to 0$ *and* $r(W) \to 0$*, for* $W \to \infty$*. Then*

$$\frac{d}{ds}\left[\frac{\partial w}{\partial \phi}\right] \geq 0, \ \forall \phi, \ \forall \, probability \ distributions \ of \ \phi$$

$$\Longleftrightarrow$$

$$utility \ is \ standard \ risk \ averse.$$

We first establish three lemmas which are required in the proof. We have

Lemma 1 *Suppose that* $u'(W) \to \infty$ *for* $W \to \underline{W}$*, then* $r(W) \to \infty$ *and* $p(W) \to \infty$ *for* $W \to \underline{W}$*.*

Proof: $u'(W) \to \infty$, for $W \to \underline{W}$, implies $\partial \ln u'(W)/\partial W \to -\infty$, and hence $r(W) \to \infty$. Also, since for $W \to \underline{W}$, $r' < 0, p > r$, and hence $p(W) \to \infty$. □

The second lemma establishes the equivalence of declining risk aversion and declining derived risk aversion. We have:

Lemma 2 $\hat{r}'(w) \leq 0$ *for any background risk* $\Leftrightarrow r'(W) \leq 0$

Proof: Kihlstrom et. al. (1981) and Nachman (1982) have shown that declining risk aversion implies declining derived risk aversion. Conversely, declining derived risk aversion implies declining risk aversion of $u(W)$. This follows from the case of small background risks.□

The third lemma establishes a condition for declining prudence, in the case of monotonic changes in background risk:

Lemma 3 *For monotonic increases in background risk,*

$$\frac{d}{d\phi}\left[-\frac{\partial \nu'(w)/\partial s}{\partial \nu'(w)/\partial w}\right] \geq 0 \Leftrightarrow p'(W) \leq 0$$

Proof: See Appendix 2.

We now present the proof of Proposition (4).

Proof of Proposition (4): Totally differentiating equation (5) with respect to s yields

$$\frac{\partial \nu'(w)}{\partial s} + \frac{\partial \nu'(w)}{\partial w}\frac{\partial w}{\partial s} = \frac{d\lambda}{ds}\phi. \tag{10}$$

Substituting λ from equation (5) then yields

$$\frac{\partial \nu'(w)}{\partial s} + \frac{\partial \nu'(w)}{\partial w}\frac{\partial w}{\partial s} = \frac{d\ln\lambda}{ds}\nu'(w).$$

Hence, the effect of the background risk on the demand for claims is given by

$$\frac{\partial w}{\partial s} = -\frac{d\ln\lambda}{ds}\frac{1}{\hat{r}(w)} - \frac{\partial \nu'(w)/\partial s}{\partial \nu'(w)/\partial w}. \tag{11}$$

The Proposition is concerned with the conditions under which

$$\frac{d}{ds}\left[\frac{\partial w}{\partial \phi}\right] = \frac{d}{d\phi}\left[\frac{\partial w}{\partial s}\right] \geq 0.$$

We investigate these conditions by looking at the behaviour of the two terms in equation (11).

Sufficiency of Standard Risk Aversion: First, we show that the first term in (11) is negative, while the second term is positive. In order to satisfy the budget constraint, $\partial w/\partial s$ has to be positive in some traded states and negative in others. Given positive prudence, $\partial \nu'(w)/\partial s > 0$, so that the second term in (11) is positive. It follows that the first term must be negative. We can now investigate

$$\frac{d}{d\phi}\left[\frac{\partial w}{\partial s}\right],$$

by taking the two terms in (11) one-by-one. First, the (negative) first term increases with ϕ, since

$$\frac{\partial \hat{r}}{\partial \phi} = \frac{\partial \hat{r}}{\partial w}\frac{\partial w}{\partial \phi}$$

is positive. This follows from $\partial w/\partial\phi < 0$ (see equation (6)) and $\partial\hat{r}/\partial w \leq 0$ (which in turn follows from $\partial r/\partial w \leq 0$ and Lemma 2). Second, the (positive) second term increases in ϕ, given declining prudence (see Lemma 3). Hence

$$\frac{d}{d\phi}\left[\frac{\partial w}{\partial s}\right]$$

is positive given standard risk aversion.

Necessity of Standard Risk Aversion: We establish necessity of standard risk aversion by taking the special case of a small background risk. Also, we assume ϕ converges in probability to a degenerate distribution, ϕ_0. By assuming $w(\phi_0)$ is, in turn, large [small], we show that the first [second] term in (11) dominates. For the first term in (11) to increase in ϕ, declining risk aversion is required. For the second term in (11) to increase in ϕ, declining prudence is required. Hence, to cover both of these possibilities, standard risk aversion is required. First, we consider the term $-d\ln\lambda/ds$.

We have from equation (5),

$$E[\nu'(w)] = E[u'(w+y)] = E(\lambda\phi) = \lambda$$

and

$$\frac{d\lambda}{ds} = \frac{d}{ds}E[u'(w+y)] = \frac{d}{ds}E[u'(w-\psi)],$$

where $\psi = \psi(w)$ is the precautionary premium as defined by Kimball(1990). Hence,

$$\frac{d\lambda}{ds} = E\left\{u''(w-\psi)\left[\frac{\partial w}{\partial s} - \frac{\partial\psi}{\partial s} - \frac{\partial\psi}{\partial w}\frac{\partial w}{\partial s}\right]\right\}.$$

Assume that we start from a position of no background risk. In this case, $s = 0$, $\psi = 0$, and $\partial\psi/\partial w = 0$. Since, for small background risks with variance σ^2, the precautionary premium is[11]

$$\psi = \frac{1}{2}p(w)\sigma^2,$$

it follows that

$$\frac{d\lambda}{ds} = E\left\{u''(w)\left[\frac{\partial w}{\partial s} - \frac{\partial\psi}{\partial s}\right]\right\} = E\left\{u''(w)\left[\frac{\partial w}{\partial s} - \frac{1}{2}p(w)\sigma^2\right]\right\}.$$

[11] This follows by analogy with the Pratt-Arrow argument for the risk premium, since initially, $s = 0$.

Now we assume that ϕ converges to the degenerate distribution ϕ_0, in probability. Since we can write

$$\frac{d\lambda}{ds} = E[f(\phi)],$$

where f is a continuous, uniformly integrable function, then it follows that

$$\frac{d\lambda}{ds} \to u''(w_0)\left[-\frac{1}{2}p(w_0)\sigma^2\right],$$

where $w_0 = w(\phi_0)$, since $\partial w_0/\partial s = 0$, for the case of the degenerate distribution, ϕ_0. Dividing by $\lambda = u'(w_0)$,

$$\frac{d\ln\lambda}{ds} \to \frac{u''(w_0)}{u'(w_0)}\left[-\frac{1}{2}p(w_0)\sigma^2\right]$$

and hence

$$-\frac{d\ln\lambda}{ds} \to -r(w_0)\left[\frac{1}{2}p(w_0)\sigma^2\right].$$

Substituting in (11), we now have

$$\frac{\partial w}{\partial s} \to r(w_0)\left[\frac{1}{2}p(w_0)\sigma^2\right]\frac{-1}{\hat{r}(w)} - \frac{\partial\nu'(w)/\partial s}{\partial\nu'(w)/\partial w}.$$

Starting with no background risk, the term

$$-\frac{\partial\nu'(\cdot)/\partial s}{\partial\nu'(\cdot)/\partial w} = \frac{1}{2}p(w)\sigma^2,$$

since $\partial\psi/\partial w = 0$. Hence, we can write

$$\frac{\partial w}{\partial s} \to r(w_0)\left[\frac{1}{2}p(w_0)\sigma^2\right]\frac{-1}{\hat{r}(w)} + \frac{1}{2}p(w)\sigma^2. \tag{12}$$

Differentiating (12), we then have

$$\frac{d}{ds}\left[\frac{\partial w}{\partial\phi}\right] = \frac{d}{d\phi}\left[\frac{\partial w}{\partial s}\right] \to \left\{r(w_0)\left[\frac{1}{2}p(w_0)\sigma^2\right]\frac{\hat{r}'(w)}{\hat{r}(w)^2} + \frac{1}{2}p'(w)\sigma^2\right\}\frac{\partial w}{\partial\phi}$$

Since $\partial w/\partial\phi < 0$, the condition for a smaller slope becomes

$$r(w_0)\left[\frac{1}{2}p(w_0)\sigma^2\right]\frac{\hat{r}'(w)}{\hat{r}(w)^2} + \frac{1}{2}p'(w)\sigma^2 \leq 0. \tag{13}$$

To establish the necessity of declining absolute risk aversion, we choose ϕ_0 such that $w_0 \to \underline{W}$. By Lemma 1 hence $r(w) \to \infty$ and $p(w_0) \to \infty$, for $w \to \underline{W}$. Therefore, $\hat{r}'(w) > 0$ implies that the first term in equation (13) $\to \infty$. Then, since the second term in (13) is independent of $w_0, \hat{r}' \leq 0$ and by Lemma 2, $r' \leq 0$ is required for the condition (13) to hold. $r' \leq 0$ also establishes the necessity of positive prudence, $p > 0$.

To establish necessity of declining absolute prudence, we choose ϕ_0 such that $w_0 \to \infty$ and hence, by assumption $r(w_0) \to 0$. Then $r'(w_0) = r(w_0)[r(w_0) - p(w_0)] \to 0$ implies $r(w_0)p(w_0) \to 0$. Hence the first term in equation (13) $\to 0$. Then, since the second term in (13) is independent of w_0, $p' \leq 0$ is required for the condition (13) to hold. Hence standard risk aversion is a necessary condition for a smaller slope.□

Proposition (4) allows us to analyze the effect of a marginal increase in a zero-mean, independent background risk, given that this increase has a negligible impact on the prices of state-contingent claims. Since a finite increase in background risk is the sum of marginal increases, the condition in Proposition (4) also holds for finite increases in background risk. Proposition (4) says that an increase in background risk will reduce the steepness of the slope of this agent's demand function. As can be seen from Proposition (4), the agent reacts to a monotonic increase in background risk by purchasing more claims in traded states for which the price ϕ is high, financing the purchase by selling some claims in the traded states with low prices. Proposition (4) can also be interpreted by comparing, within an equilibrium, the demand of agents, who differ only in the size of their respective background risks. Proposition (4) suggests that agents with higher background risk will adjust their demand functions by buying state-contingent claims on high-price traded states and selling claims on low-price traded states. This is illustrated in Franke, Stapleton and Subrahmanyam (1998), for an economy in which all agents have the same type HARA-class utility function, exhibiting declining absolute risk aversion. These functions are standard risk averse and hence generalized risk averse. In this economy, agents with high background risk buy options from those with relatively low background risk. The latter agents sell portfolio insurance to the former with relatively high background risk.

4 Conclusions

The main conclusions regarding the effects of an increase in background risk, on risk aversion and on the demand for contingent claims, are summarised in the four propositions of the paper. Proposition 1 provides a necessary and sufficient condition for simple increases in background risk to increase the derived risk aversion of agents. The condition on utility is weaker than Kimball's standard risk aversion, but stronger than Gollier and Pratt's risk vulnerability. By considering only the set of simple increases in background risk, we find a larger set of utility functions which satisfy the criterion of increased derived risk aversion, than those of Eeckhoudt, Gollier and Schlesinger. We then proceed to examine the condition for 'generalized risk aversion', whereby agents react to increased background risk by reducing the slope of the demand curve for state contingent claims. We find in Proposition 3 that increased derived risk aversion is necessary, but not sufficient, for generalized risk aversion. The stronger requirement for generalized risk aversion is shown for the case of monotonic increases in background risk in Proposition 4. Standard risk aversion, i.e. positive, declining absolute risk aversion and absolute prudence, is a necessary and sufficient condition for generalized risk aversion.

Appendix 1

Proof of Proposition (1)

From the definition of $\hat{r}(w)$,

$$\hat{r}(w) = \frac{E_y[-u''(W)]}{E_y[u'(W)]} \tag{14}$$

Differentiating with respect to s, we have the following condition:

$$\frac{\partial \hat{r}(w)}{\partial s} > [=][<] 0 \Longleftrightarrow f(w,s) > [=][<] 0 \tag{15}$$

for *any* distribution of y, where $f(w,s)$ is defined as

$$f(w,s) \equiv E_y\left[y'\left\{-u'''(W) - u''(W)\hat{r}(w)\right\}\right] \tag{16}$$

with $y' \equiv \partial y/\partial s$. y' is the marginal change in y, such that $y' \leq [=][\geq] 0$ when $y < [=][>] y_0$ and $E[y'|w] = 0$ (simple increase in background risk).

<u>Necessity</u>

We now show that

$$\begin{aligned} f(w,s) &\quad > [=][<] \quad 0 \Longrightarrow \\ u'''(W_2) - u'''(W_1) &\quad < [=][>] \quad -r(W)\left[u''(W_2) - u''(W_1)\right], \forall W_1 \leq W \leq W_2 \end{aligned}$$

Consider a background risk with three possible outcomes, y_0, y_1, and y_2, such that $0 > y_1 < y_0 < y_2$ and $0 > y_1' < y_0' = 0 < y_2'$. Define

$$W_i \quad = \quad w + y_i, \quad i = 0, 1, 2.$$

and let q_i denote the probability of the outcome, y_i. For the special case of such a risk, equation (16) can be written as

$$f(w,s) = q_1|y_1'|\left\{-u'''(W_2) + u'''(W_1) - [u''(W_2) - u''(W_1)]\hat{r}(w)\right\} \tag{17}$$

since

$$E[y'] = \sum_{i=0}^{2} q_i y_i' = 0$$

so that

$$q_1|y_1'| = q_2 y_2'$$

Now $\hat{r}(w)$ can be rewritten from (14) as

$$\begin{aligned} \hat{r}(w) &= E_y\left\{\frac{u'(W)}{E_y[u'(W)]}\frac{-u''(W)}{u'(W)}\right\} \\ &= E_y\left\{\frac{u'(W)}{E_y[u'(W)]}r(W)\right\} \end{aligned} \qquad (18)$$

Hence, $\hat{r}(w)$ is the expected value of the coefficient of absolute risk aversion, using the risk-neutral probabilities given by the respective probabilities multiplied by the ratio of the marginal utility to the expected marginal utility. Thus, $\hat{r}(w)$ is a convex combination of the coefficients of absolute risk aversion at the different values of y. For the three-point distribution being considered, $\hat{r}(w)$ is a convex combination of $r(W_0)$, $r(W_1)$, and $r(W_2)$. Hence, as $q_0 \to 1$, $\hat{r}(w) \to r(W_0)$. Since W_0 can take any value in the range $[W_1, W_2]$, $f(w,s)$ must have the required sign for *every* value of $r(W_0)$, where $W_1 \leq W_0 \leq W_2$. Thus, since $q_1|y_1'| > 0$, this is true only if the condition in (17) holds. This is the same condition as stated in Proposition 1.

Sufficiency

To establish sufficiency we use a method similar to that used by Pratt and Zeckhauser (1987) and by Gollier and Pratt (1996).

a) We first show

$$\begin{aligned} u'''(W_1) - u'''(W_2) &> r(W)\left[u''(W_2) - u''(W_1)\right], \forall W_1 \leq W \leq W_2 \\ \Longrightarrow f(w,s) &> 0, \quad \forall (w,s) \end{aligned} \qquad (19)$$

We need to show that $f(w,s) > 0$, for all non-degenerate probability distributions. Hence, we need to prove that the minimum value of $f(w,s)$ over *all* possible probability distributions $\{q_i\}$, with $E(y') = 0$, must be positive. In a manner similar to Gollier and Pratt (1996), this can be formulated as a mathematical programming problem, where $f(w,s)$ is minimized, subject to the constraints that all q_i are non-negative and sum to one, and $E(y') = 0$. Equivalently, this can be reformulated as a parametric linear program where the non-linearity is eliminated by writing $\bar{r}$ as a parameter

$$\min_{\{q_i\}} f(w,s) = \sum_i q_i \left[y_i' \left\{ -u'''(W_i) - u''(W_i)\bar{r} \right\} \right] \tag{20}$$

s.t.

$$\sum_i q_i y_i' = 0 \tag{21}$$

$$\sum_i q_i = 1 \tag{22}$$

the definitional constraint for the parameter $\bar{r}$

$$\bar{r} \sum_i q_i u'(W_i) = -\sum_i q_i u''(W_i) \tag{23}$$

and the non-negativity constraints

$$q_i \geq 0, \quad \forall i \tag{24}$$

A sufficient condition for $\partial\hat{r}/\partial s > 0$ is that $f(w,s)$ as defined by (20) is positive for *any* probability distribution $\{q_i\}$ subject to $E(y') = 0$ and the definition of $\bar{r}$ given in (23).

Since we are looking for a sufficient condition for $f(w,s) > 0$, we can relax the non-negativity constraint for q_0 in the above linear program. In case even this (infeasible) resulting minimum is positive, then we know that the solution of the above linear program is always positive. We drop the non-negativity constraint on q_0, the probability of the y_0 state in the following manner. We define q_0^+ and q_0^- such that

$$q_0 = q_0^+ - q_0^- \tag{25}$$

where both q_0^+ and q_0^- are non-negative. These new variables replace q_0 in the program.

The modified linear program has three variables in the basis since there are three constraints in the program. In the optimal solution, one basis variable is either q_0^+ or q_0^-. Hence, the optimal solution of the modified linear program is (q_0, q_1, q_2) and the objective function is

$$f^*(w,s) = q_1 y_1' \left[-u'''(W_1) - u''(W_1)\bar{r} \right] + q_2 y_2' \left[-u'''(W_2) - u''(W_2)\bar{r} \right] \tag{26}$$

Since $q_1 y_1' + q_2 y_2' = 0$, it follows that (26) can be rewritten as

$$f^*(w,s) = q_1 y_1' \left[(-u'''(W_1) - u''(W_1)\bar{r}) - (-u'''(W_2) - u''(W_2)\bar{r}) \right] \tag{27}$$

Hence

$$u'''(W_1) - u'''(W_2) - \left[u''(W_2) - u''(W_1)\right]\bar{r} > 0 \tag{28}$$

is a sufficient condition for $f^* > 0$, given $\bar{r}$.

As shown in equation (18), $\bar{r}$ is a convex combination of $r(W_0)$, $r(W_1)$ and $r(W_2)$, hence $\bar{r} \in \{r(W) | W \in [W_1, W_2]\}$. Hence, a sufficient condition for (28) is that

$$u'''(W_1) - u'''(W_2) - r(W)\left[u''(W_2) - u''(W_1)\right] > 0 \tag{29}$$

for all $\{W_1 \leq W \leq W_2\}$ as given by the condition of Proposition 1.

b) By an analogous argument, it can be shown that

$$\begin{aligned} u'''(W_1) - u'''(W_2) &< r(W)\left[u''(W_2) - u''(W_1)\right], \forall W_1 \leq W \leq W_2 \\ \Longrightarrow f(w,s) &< 0 \quad \forall (w,s) \end{aligned} \tag{30}$$

c) We now show directly that

$$\begin{aligned} u'''(W_1) - u'''(W_2) &= r(W)\left[u''(W_2) - u''(W_1)\right], \forall W_1 \leq W \leq W_2 \\ \Longrightarrow f(w,s) &= 0 \quad \forall (w,s) \end{aligned} \tag{31}$$

A sufficient condition for $f(w,s) = 0, \forall (w,s)$ is that $\min_{\{q_i\}} f(w,s) = \max_{\{q_i\}} f(w,s) = 0$, subject to (21)-(23) and the nonnegativity condition for every q_i except q_0. The minimum and maximum involve three basis variables, one of which is either q_0^+ or q_0^-. Therefore, $f^*(w,s)$ is always determined by (27). Hence, the minimal and maximal value of $f^*(w,s)$ are zero if the bracketed term in (27) is zero. This is the case if

$$u'''(W_1) - u'''(W_2) = r(W)\left[u''(W_2) - u''(W_1)\right], \quad \forall W_1 \leq W \leq W_2. \tag{32}$$

□

Appendix 2

Proof of Lemma (3)

We have to show that

$$-\frac{\partial \nu'(w)/\partial s}{\partial \nu'(w)/\partial w}$$

increases in ϕ, if and only if absolute prudence is declining. This term increases in ϕ if the negative of the term increases in w since $\partial w/\partial \phi < 0$. In terms of the underlying utility function, this is the same as showing that the term

$$Z(w) = \frac{E_y[u''(W)y']}{E_y[u''(W)]}$$

is increasing in w, where $y' = \partial y/\partial s$.

Now consider a marginal increase in w. Then

$$\text{sign } \frac{\partial Z(w)}{\partial w} = \text{sign } E_y[u''(W)]E_y[u'''(W)y'] - E_y[u'''(W)]E_y[u''(W)y'],$$

or,

$$\text{sign } \frac{\partial Z(w)}{\partial w} = \text{sign } \; - E_y[\{u'''(W) - u''(W)\frac{E_y[u'''(W)]}{E_y[u''(W)]}\}y'],$$

and then it follows that

$$\text{sign } \frac{\partial Z(w)}{\partial w} = \text{sign } \; - E_y[\{u'''(W) - u''(W)\frac{E_y[u'''(W)]}{E_y[u''(W)]}\}(y' - \hat{y}')],$$

where $\hat{y}' = y'(\hat{y})$ and $\hat{y}$ is defined by

$$\frac{E_y[u'''(W)]}{E_y[-u''(W)]} - p(\hat{W} = w + \hat{y}) = 0.$$

Hence

$$\text{sign } \frac{\partial Z(w)}{\partial w} = \text{sign } E_y[-u''(W)\{\frac{E_y[u'''(W)]}{E_y[-u''(W)]} - p(W)\}(y' - \hat{y}')]. \tag{33}$$

Monotonic increases in background risk imply that

$$y' - \hat{y}' \leq [=][\geq]0, \text{for } y < [=][>]\hat{y}.$$

It follows that absolute prudence, $p(W)$, must be declining if sign $\frac{\partial Z(w)}{\partial w}$ is to be non-negative for any distribution of y, in particular for any binomial distribution. Sufficiency of declining prudence for sign $\frac{\partial Z(w)}{\partial w}$ to be non-negative follows from

$$\{\frac{E_y[u'''(W)]}{E_y[-u''(W)]} - p(W)\}(y' - \hat{y}') \geq 0, \forall y.$$

Hence, declining absolute prudence is necessary and sufficient. □

References

Back, K. and Dybvig, P.H. (1993), "On Existence of Optimal Portfolios in Complete Markets", Washington University in St Louis, Working paper.

Eeckhoudt, L., Gollier, C. and Schlesinger, H. (1995), "The Risk-Averse (and Prudent) Newsboy", *Management Science*, 41, 786-794.

Eeckhoudt, L., Gollier, C. and Schlesinger, H. (1996), "Changes in Background Risk and Risk Taking Behaviour", *Econometrica*, 64, 683-689.

Eeckhoudt, L., and Kimball, M.S. (1992), "Background Risk, Prudence, and the Demand for Insurance", in *Contributions to Insurance Economics*, G. Dionne, ed., Kluwer Academic Publishers, Boston, 239-254.

Franke, G., Stapleton, R.C. and Subrahmanyam, M.G. (1998), "Who Buys and Who Sells Options: The Role of Options in an Economy with Background Risk", *Journal of Economic Theory*, 82, 89-109.

Gollier, C. (2000), *The Economics of Risk and Time*, MIT Press, forthcoming.

Gollier, C. and Pratt, J. W. (1996), "Weak Proper Risk Aversion and the Tempering Effect of Background Risk", *Econometrica*, 64, 1109-1123.

Kihlstrom R.E., Romer D. and Williams S. (1981), "Risk Aversion with Random Initial Wealth", *Econometrica*, 49, 911-920.

Kimball, M.S. (1990), "Precautionary Saving in the Small and in the Large", *Econometrica*, 58, 53-73.

Kimball, M.S. (1993), "Standard Risk Aversion", *Econometrica*, 61, 589-64.

Meyer, P.J. and Meyer, J., (1998), "Changes in Background Risk and the Demand for Insurance", *The Geneva Papers on Risk and Insurance Theory*, 23, 29-40.

Nachman, D.C. (1982), "Preservation of 'More Risk Averse' under Expectations", *Journal of Economic Theory*, 28, 361-368.

Pratt, J.W. (1964), "Risk Aversion in the Small and in the Large", *Econometrica*, 32, 122-136.

Pratt, J.W. and Zeckhauser, R.J. (1987), "Proper Risk Aversion", *Econometrica*, 55, 143-154.