DYNAMIC CONDITIONAL CORRELATION – A SIMPLE CLASS OF MULTIVARIATE GARCH MODELS

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July 1999

Revised May 2000

Abstract

Time varying correlations are often estimated with Multivariate Garch models that are linear in squares and cross products of returns. A new class of multivariate models called dynamic conditional correlation (DCC) models is proposed. These have the flexibility of univariate GARCH models coupled[1] with parsimonious parametric models for the correlations. They are not linear but can often be estimated very simply with univariate or two step methods based on the likelihood function. It is shown that they perform well in a variety of situations and give sensible empirical results.

[1] This research has been supported by NSF grant SBR-9730062 and NBER.

I. INTRODUCTION

The quest for reliable estimates of correlations between financial variables has been the motivation for countless academic articles, practitioner conferences and back room Wall Street research. These correlations are needed for derivative pricing, portfolio optimization, risk management and hedging. Simple methods such as rolling historical correlations and exponential smoothing have been widely used because of the complexity and potential unreliability of methods such as multivariate GARCH or Stochastic Volatility and the unavailability of implied correlations for most markets.

In this paper Dynamic Conditional Correlation (DCC) estimators are proposed which have the flexibility of univariate GARCH but not the complexity of multivariate GARCH. These models, which parameterize the conditional correlations directly, are naturally estimated in two steps – the first is a series of univariate GARCH estimates and the second the correlation estimate.

The next section of the paper will give an overview of various models for estimating correlations. Section 3 will introduce the new method and compare it with some of the other cited approaches. Section 4 will investigate some properties of the method including correlation forecasting in Section 5. Section 6 carries out a series of Monte Carlo experiments. Section 7 presents empirical results for several pairs of daily time series and Section 8 concludes.

II. CORRELATION MODELS

The correlation between two random variables *r1* and *r2* that each have mean zero, is defined to be:

$$\rho_{12} = \frac{E(r_1 r_2)}{\sqrt{E(r_1^2)E(r_2^2)}} \tag{1}$$

Similarly, the conditional correlation is defined as:

$$\rho_{12,t} = \frac{E_{t-1}(r_{1,t} r_{2,t})}{\sqrt{E_{t-1}(r_{1,t}^2)E_{t-1}(r_{2,t}^2)}} \quad . \tag{2}$$

In this definition, the conditional correlation is based on information known the previous period, however multi-period forecasts of the correlation can be defined in the same way.

By the laws of probability, all correlations defined in this way must lie within the interval [–1,1]. The conditional correlation satisfies this constraint for all possible realizations of the past information.

To clarify the relation between conditional correlations and conditional variances, it is convenient to write the returns as the conditional standard deviation times the standardized disturbance:

$$h_{i,t} = E_{t-1}\left(r_{i,t}^2\right), \quad r_{i,t} = \sqrt{h_{i,t}}\,\varepsilon_{i,t}, \ \ i=1,2 \tag{3}$$

Epsilon is a standardized disturbance which has mean zero and variance one for each series. Substituting into (2) gives

$$\rho_{12,t} = \frac{E_{t-1}\left(\varepsilon_{1,t}\varepsilon_{2,t}\right)}{\sqrt{E_{t-1}\left(\varepsilon_{1,t}^2\right)E_{t-1}\left(\varepsilon_{2,t}^2\right)}} = E_{t-1}\left(\varepsilon_{1,t}\varepsilon_{2,t}\right). \tag{4}$$

Thus, the conditional correlation is also the conditional covariance between the standardized disturbances.

Many estimators have been proposed for conditional correlations. The ever popular rolling correlation estimator is defined for returns with a zero mean as:

$$\hat{\rho}_{12,t} = \frac{\sum_{s=t-n-1}^{t-1} r_{1,s}r_{2,s}}{\sqrt{\left(\sum_{s=t-n-1}^{t-1} r_{1,s}^2\right)\left(\sum_{s=t-n-1}^{t-1} r_{2,s}^2\right)}}. \tag{5}$$

Substituting from (3) it is clear that this is only an attractive estimator in very special circumstances. In particular, it gives equal weight to all observations less than n periods in the past and zero weight on older observations. The estimator will always lie in the [-1,1] interval, but it is unclear under what assumptions it consistently estimates the conditional correlations.

The exponential smoother used by RiskMetrics™ uses declining weights based on a parameter λ, which emphasizes current data but has no fixed termination point in the past where data becomes uninformative.

(6) $$\hat{r}_{12,t} = \frac{\sum_{s=1}^{t-1} l^{t-j-1} r_{1,s} r_{2,s}}{\sqrt{\left(\sum_{s=1}^{t-1} l^{t-s-1} r_{1,s}^2\right)\left(\sum_{s=1}^{t-1} l^{t-s-1} r_{2,s}^2\right)}}$$

It also will surely lie in [-1,1]; however there is no guidance from the data on how to choose lambda and it is necessary that the same lambda be used for all assets. Defining the conditional covariance matrix of returns as:

(7) $$E_{t-1}(r_t r_t') \equiv H_t,$$

these estimators can be expressed in matrix notation respectively as:

(8) $$H_t = \frac{1}{n}\sum_{j=1}^{n}(r_{t-j} r_{t-j}'), \quad and \quad H_t = l(r_{t-1} r_{t-1}') + (1-l)H_{t-1}$$

A simple approach to estimating multivariate models with somewhat more flexibility than these methods is the Orthogonal GARCH method or principle component GARCH method. This has recently been advocated by Alexander(1999). The procedure is simply to construct unconditionally uncorrelated linear combinations of the series *r*. Then univariate GARCH models are estimated for some or all of these and the full covariance matrix is constructed by assuming the conditional correlations are all zero. More precisely, let $y_t = Ar_t$, $E(y_t y_t') \equiv V$ is diagonal. Univariate GARCH models are estimated for the elements of y and combined into the diagonal matrix V_t. Assuming in addition that $E_{t-1}(y_t y_t') = V_t$ is diagonal (a strong assumption), then

(9) $$H_t = A'^{-1} V_t A^{-1}$$

In the bivariate case, the matrix A can be chosen to be triangular and estimated by least squares where r_1 is one component and the residuals from a regression of r_1 on r_2 are the second. In this simple situation, a slightly better approach is to run this regression as a GARCH regression, thereby obtaining residuals which are orthogonal in a GLS metric.

Multivariate GARCH models are natural generalizations of this problem. Many specifications have been considered, however most have been formulated so that the covariances and variances are linear functions of the squares and cross products of the

data. The most general expression of this type is called the vec model and is described in Engle and Kroner(1995). The vec model parameterizes the vector of all covariances and variances expressed as vec(H_t). In the first order case this is given by

$$(10) \qquad vec(H_t) = vec(\Omega) + \mathrm{A}vec(r_{t-1}r_{t-1}') + \mathrm{B}vec(H_{t-1})$$

where A and B are $n^2 x n^2$ matrices with much structure following from the symmetry of H. Without further restrictions, this model will not guarantee positive definiteness of the matrix H.

Useful restrictions are derived from the BEKK representation, also introduced by Engle and Kroner(1995), which in the first order case can be written as:

$$(11) \qquad H_t = \Omega + A(r_{t-1}r_{t-1}')A' + BH_{t-1}B'$$

Various special cases have been discussed in the literature starting from models where the *A* and *B* matrices are simply a *scalar* or *diagonal* rather than a whole matrix, and continuing to very complex highly parameterized models which still ensure positive definiteness. See for example Engle and Kroner(1995), Bollerslev, Engle and Nelson(1994) and Engle and Mezrich(1996) for examples. In this study the scalar BEKK and the diagonal BEKK will be estimated.

As discussed in Engle and Mezrich(1996), these models can be estimated subject to the constraint that the long run variance covariance matrix is the sample covariance matrix. This constraint differs from MLE only in finite samples but reduces the number of parameters and often gives improved performance. In the general vec model of equation (9), this can be expressed as

$$(12) \qquad vec(\Omega) = (I - \mathrm{A} - \mathrm{B})vec(S), \text{ where } S = \frac{1}{T}\sum_t (r_t r_t')$$

This expression simplifies in the scalar and diagonal BEKK cases. For example for the scalar BEKK the intercept is simply

$$(13) \qquad \Omega = (1 - \alpha - \beta)S$$

III. DYNAMIC CONDITIONAL CORRELATIONS

This paper introduces a new class of multivariate GARCH estimators which can best be viewed as a generalization of Bollerslev(1990)'s constant conditional correlation estimator. In Bollerslev's model,

(14) $H_t = D_t R D_t, \quad where \quad D_t = diag\left\{\sqrt{h_{i,t}}\right\}$

where R is a correlation matrix containing the conditional correlations as can directly be seen from rewriting this equation as:

(15) $E_{t-1}(\boldsymbol{e}_t \boldsymbol{e}_t') = D_t^{-1} H_t D_t^{-1} = R$, since $\boldsymbol{e}_t = D_t^{-1} r_t$

The expressions for h are typically thought of as univariate GARCH models, however, these models could certainly include functions of the other variables in the system as predetermined variables.

This paper proposes an estimator called dynamic conditional correlation model or DCC. The dynamic correlation model differs only in allowing *R* to be time varying giving a model:

(16) $H_t = D_t R_t D_t$

Parameterizations of R have the same requirements that H did except that the conditional variances must be unity.

Probably the simplest and one of the most successful is the exponential smoother which can be expressed as:

(17) $$\boldsymbol{r}_{i,j,t} = \frac{\sum_{s=1} \boldsymbol{l}^s \boldsymbol{e}_{i,t-s} \boldsymbol{e}_{j,t-s}}{\sqrt{\left(\sum_s \boldsymbol{l}^s \boldsymbol{e}_{i,t-s}^2\right)\left(\sum_s \boldsymbol{l}^s \boldsymbol{e}_{j,t-s}^2\right)}} = [R_t]_{i,j},$$

a geometrically weighted average of standardized residuals. Clearly these equations will produce a correlation matrix at each point in time. A simple way to construct this correlation is through exponential smoothing.

(18) $$q_{i,j,t} = (1-\boldsymbol{l})(\boldsymbol{e}_{i,t-1}\boldsymbol{e}_{j,t-1}) + \boldsymbol{l}(q_{i,j,t-1}), \quad \boldsymbol{r}_{i,j,t} = \frac{q_{i,j,t}}{\sqrt{q_{ii,t} q_{jj,t}}}$$

A natural alternative is suggested by the GARCH(1,1) model.

(19) $$q_{i,j,t} = \bar{\rho}_{i,j} + \alpha\left(\varepsilon_{i,t-1}\varepsilon_{j,t-1} - \bar{\rho}_{i,j}\right) + \beta\left(q_{i,j,t-1} - \bar{\rho}_{i,j}\right)$$

Rewriting gives,

(20) $$q_{i,j,t} = \bar{\rho}_{i,j}\left(\frac{1-\alpha-\beta}{1-\beta}\right) + \alpha\sum_{s=1,\infty}\beta^{s}\varepsilon_{i,t-s}\varepsilon_{j,t-s}$$

The unconditional expectation of the cross product is $\bar{\rho}$ while for the variances:

(21) $$\bar{\rho}_{i,i} = 1.$$

The correlation estimator

(22) $$\rho_{i,j,t} = \frac{q_{i,j,t}}{\sqrt{q_{i,i,t}q_{j,j,t}}}$$

will be positive definite as the covariance matrix, $Q_t = \left[q_{i,j,t}\right]$, is a weighted average of a positive definite and a positive semidefinite matrix. The unconditional expectation of the numerator of (22) is $\bar{\rho}$ and each term in the denominator has expected value one. This model is mean reverting as long as $\alpha + \beta < 1$ and when the sum is equal to one it is just the model in (18). Matrix versions of these estimators can be written as:

(23) $$Q_t = (1-\lambda)(\varepsilon_{t-1}\varepsilon_{t-1}') + \lambda Q_{t-1}, \quad \text{and}$$

(24) $$Q_t = S(1-\alpha-\beta) + \alpha(\varepsilon_{t-1}\varepsilon_{t-1}') + \beta Q_{t-1}$$

where S is the unconditional correlation matrix of the epsilons.

Clearly more complex positive definite multivariate GARCH models could be used for the correlation parameterization as long as the unconditional moments are set to the sample correlation matrix. The goal however is to keep this simple.

IV. ESTIMATION

The log likelihood for this estimator can be expressed as

$$
\begin{aligned}
& r_t \left| \Im_{t-1} \right. \sim N(0, H_t) \\
& L = -\frac{1}{2} \sum_t \left(n \log(2\pi) + \log \left| H_t \right| + r_t' H_t^{-1} r_t \right) \\
& L = -\frac{1}{2} \sum_t \left(n \log(2\pi) + \log \left| D_t R_t D_t \right| + r_t' D_t^{-1} R_t^{-1} D_t^{-1} r_t \right) \\
& L = -\frac{1}{2} \sum_t \left(n \log(2\pi) + 2 \log \left| D_t \right| + \log \left| R_t \right| + \varepsilon_t' R_t^{-1} \varepsilon_t \right)
\end{aligned} \tag{25}
$$

which can simply be maximized over the parameters of the model. However one of the objectives of this formulation is to allow the model to be estimated more easily even when the covariance matrix is very large. In the next few paragraphs several estimation methods will be presented which give simple consistent but inefficient estimates of the parameters of the model. There will be no attempt to develop the properties of such estimators although they will be illustrated on both real and artificial data.

Let the parameters in D be denoted θ and the additional parameters in R be denoted ϕ. Suppose for a moment that θ is known, then the relevant part of the log likelihood becomes

$$
L_C(\phi) = -\frac{1}{2} \sum_t (\log \left| R_t \right| + \varepsilon_t' R_t^{-1} \varepsilon_t), \tag{26}
$$

which can be maximized directly. If consistent estimates of θ can be found, then the two step estimation strategy will be consistent but not fully efficient. In the two dimensional case, this can be written quite simply as:

$$
L_C(\phi) = -\frac{1}{2} \sum_t \left(\log \left(1 - \rho_t^2 \right) + \frac{\left(\varepsilon_{1,t}^2 + \varepsilon_{2,t}^2 - 2 \rho_t \varepsilon_{1,t} \varepsilon_{2,t} \right)}{\left(1 - \rho_t^2 \right)} \right) \tag{27}
$$

where ρ_t is given either by (18) or (22).

An even simpler approach is available. Rewrite (19) as

$$
e_{i,j,t} = \bar{\rho}(1 - \alpha - \beta) + (\alpha + \beta) e_{i,j,t-1} - \beta \left(e_{i,j,t-1} - q_{i,j,t-1} \right) + \left(e_{i,j,t} - q_{i,j,t} \right) \tag{28}
$$

where $e_{i,j,t} = \varepsilon_{i,t} \varepsilon_{j,t}$. This equation is an ARMA(1,1) since the errors are a Martingale difference by construction. The autoregressive coefficient is slightly bigger than the negative of the moving average if α is positive. This equation can therefore be estimated with conventional time series software to recover consistent estimates of the parameters.

The drawback to this method is that ARMA with nearly equal roots are numerically unstable and tricky to estimate. These parameters would then be used to construct the correlation estimates in (22). The problem is even easier if the model is (18) since then the autoregressive root is assumed to be one. The model is simply an integrated moving average or IMA with no intercept.

$$\Delta e_{i,j,t} = -\beta\left(e_{i,j,t-1} - q_{i,j,t-1}\right) + \left(e_{i,j,t} - q_{i,j,t}\right), \tag{29}$$

which is simply an exponential smoother with parameter $\lambda = (1-\beta)$.

In a multivariate context each of these approaches remains feasible although slightly more complicated. The regressions in (24) would necessarily have to fit all the covariance equations to the same parameters. This could be done by stacking the off diagonal elements and estimating one model possibly with breaks between each series. Possibly, estimating each covariance equation separately and then averaging the coefficients could even do it. One would hope that the results would not be very sensitive to these choices.

To complete the discussion it is necessary to propose how to consistently estimate the parameters θ that appear in the individual GARCH models. The original likelihood in (25) can be viewed as a GLS estimator for $D^{-1}r$. An inefficient but consistent estimator can be found by replacing R by the identity matrix. In this case the univariate quasi-likelihood function becomes:

$$QL_U(\theta) = -\frac{1}{2}\sum_t \left(n\log(2\pi) + \sum_{i=1}^{n}\left(\log(h_{i,t}) + \frac{r_{i,t}^2}{h_{i,t}} \right)\right) \tag{30}$$

that is the sum of the QL_U for each of the individual assets. Since the parameters for each asset can be different, these can all be estimated as univariate models and the standard QMLE properties will hold. Thus consistent estimates of all the parameters can be obtained by estimating the univariate models and then using these models to define the standardized residuals and finally using one of the listed methods to estimate the parameters of the correlation process.

The sum of the likelihood in (26) plus (30) plus the total sum of squared standardized residuals, which is given almost exactly by NT/2, equals the log likelihood in (25). Thus it is possible to compare the log likelihood of this method with other

methods and similarly to determine the likelihood sacrificed by the two step estimation procedure.

(31) $$L(\theta,\phi) = L_C(\phi,\theta) + QL_U(\theta) + \sum_t \varepsilon_t' \varepsilon_t / 2$$

V. FORECASTING CORRELATIONS

In all of the models for dynamic conditional correlations, the correlation coefficient is expressed as a ratio with a square root in the denominator. Thus unbiased forecasts cannot easily be computed. In fact, for all multivariate GARCH models, the correlation coefficient is not itself forecast, it is the ratio of the forecast of the covariance to the square root of the product of the forecasts of the variances. To develop a forecasting expression for the DCC models, it will be necessary to approximate the correlation coefficient by its first order Taylor series expansion.

Consider the mean reverting model in (19) that specializes to the integrated model in (18) if $\alpha + \beta = 1$.

(32) $$E_t\left(q_{i,j,t+k}\right) = \bar{r}(1-\alpha-\beta) + \beta E_t\left(q_{i,j,t+k-1}\right) + \alpha E_t\left(\varepsilon_{i,t+k-1}\varepsilon_{j,t+k-1}\right)$$

The last expectation is by construction equal to 1 for $i=j$ since these are standardized residuals.

$$\text{For } i \neq j, \quad E_t\left(\varepsilon_{i,t+k-1}\varepsilon_{jk,t+k-1}\right) = E_t\left(\rho_{i,j,t+k-1}\right).$$

Finally, by expanding the correlation coefficient about the point $\{\bar{q}_{i,j}\}$, the correlation function can be expressed

(33)

$$\rho_{i,j,t+k} \cong \frac{\bar{q}_{i,j}}{\sqrt{\bar{q}_{ii}\bar{q}_{jj}}} + \frac{1}{\sqrt{\bar{q}_{ii}\bar{q}_{jj}}}\left(q_{i,j,t+k} - \bar{q}_{i,j}\right) - .5 * \frac{\bar{q}_{i,j}}{\sqrt{\bar{q}_{ii}\bar{q}_{jj}}}\left(\frac{q_{ii,t+k} - \bar{q}_{ii}}{\bar{q}_{ii}} + \frac{q_{jj,t+k} - \bar{q}_{jj}}{\bar{q}_{jj}}\right)$$

By successively solving forward equations (31)-(33), forecasts of correlations can be built up.

To determine the whether these are satisfactory approximations, data are generated following the DCC model. In the integrated case the RiskMetrics parameters (.94,.06) are chosen while in the mean reverting case, (.90,.06) are used with an

unconditional correlation of .5. The starting point for the forecast is taken to be $\{q_{12}, q_{11}, q_{22}\} = \{1,1,2\}$ so the initial correlation is .707. In the integrated case, the correlation coefficient is expanded in (33) around the starting point of the forecast. In the mean reverting case, this option is computed as well as the expansion around the unconditional values of *{.5,1,1}*. With 1000 replications of the forecast period, the average rho is plotted against the forecast calculated as above.

From these pictures, this approximation is reasonably close to giving accurate correlation forecasts. The forecasts incorporate mean reversion when the model has mean reversion. They incorporate some predictability also in the integrated model that arises from deviations of the smoothed standardized residuals from the unconditional value of one. Probably, better approximations can be found that give yet more accurate forecasts.

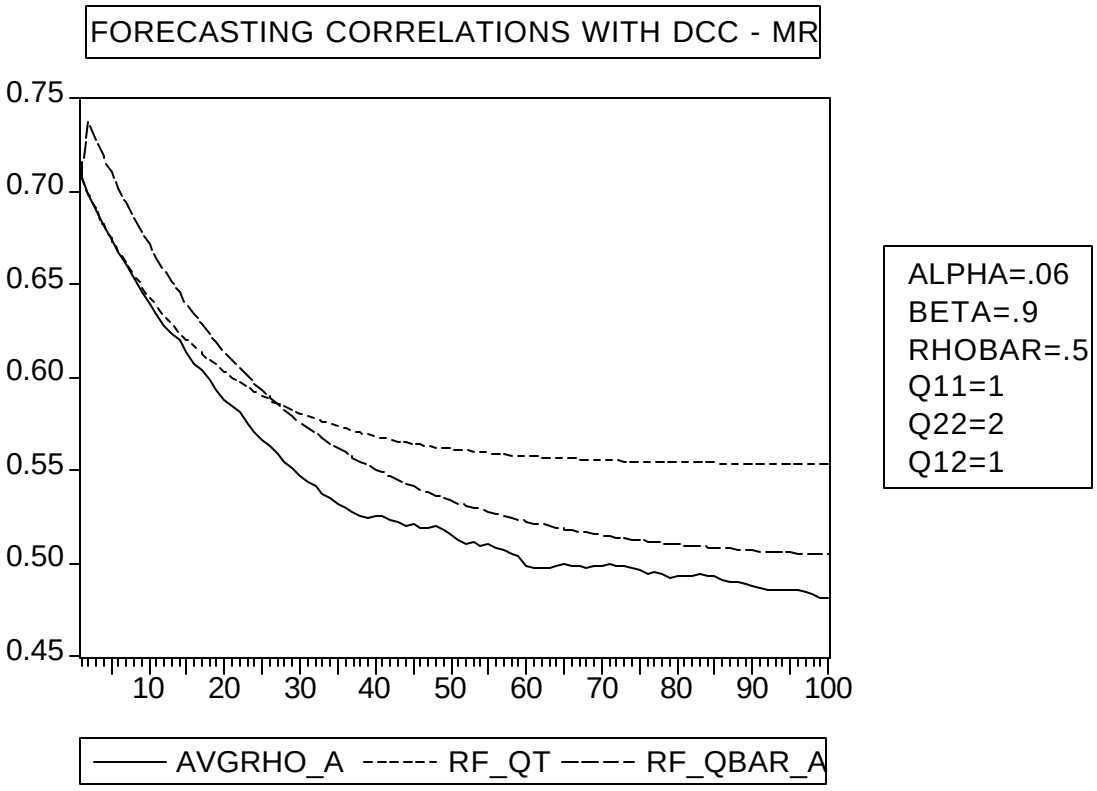

Figure 1.

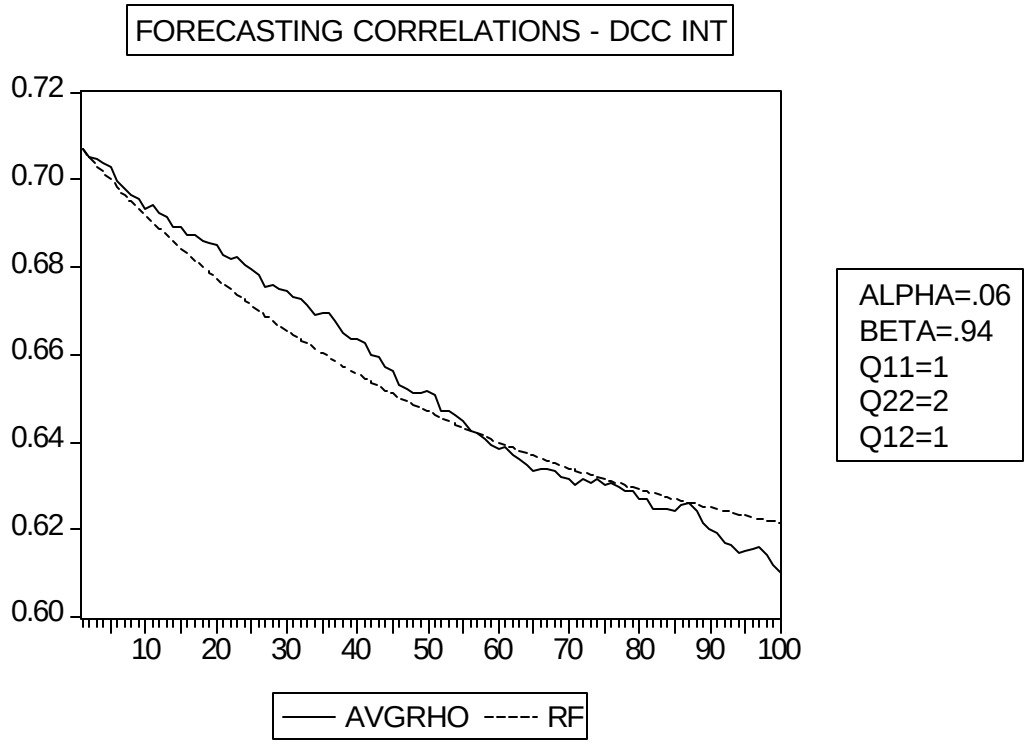

Figure 2

VI. COMPARISON OF ESTIMATORS

In this section, several correlation estimators will be compared in a setting where the true correlation structure is known. A bivariate GARCH model will be simulated 200 times for 1000 observations or approximately 8 years of daily data for each correlation process. Alternative correlation estimators will be compared in terms of simple goodness of fit statistics, multivariate GARCH diagnostic tests and Value at Risk tests.

The data generating process consists of two gaussian GARCH models; one is highly persistent and the other is not.

$$h_{1,t} = .01 + .05r_{1,t-1}^2 + .94h_{1,t-1}, \quad h_{2,t} = .5 + .2r_{2,t-1}^2 + .5h_{2,t-1} \tag{34}$$
$$r_{1,t} = \sqrt{h_{1,t}}\varepsilon_{1,t}, \quad r_{2,t} = \sqrt{h_{2,t}}\varepsilon_{2,t}, \quad \rho_t = E_{t-1}\varepsilon_{1,t}\varepsilon_{2,t}$$

The correlations follow several processes that are labeled as follows:

- *Constant* $\rho_t = .9$
- *Sine* $\rho_t = .5 + .4\cos(2\pi t/200)$
- *Fast Sine* $\rho_t = .5 + .4\cos(2\pi t/40)$
- *Step* $\rho_t = .9 - .5(t > 500)$
- *Ramp* $\rho_t = \mathrm{mod}(t/200)$

These processes were chosen because they exhibit rapid changes, gradual changes and periods of constancy. Various other experiments are done with different error distributions and different data generating parameters but the results are quite similar.

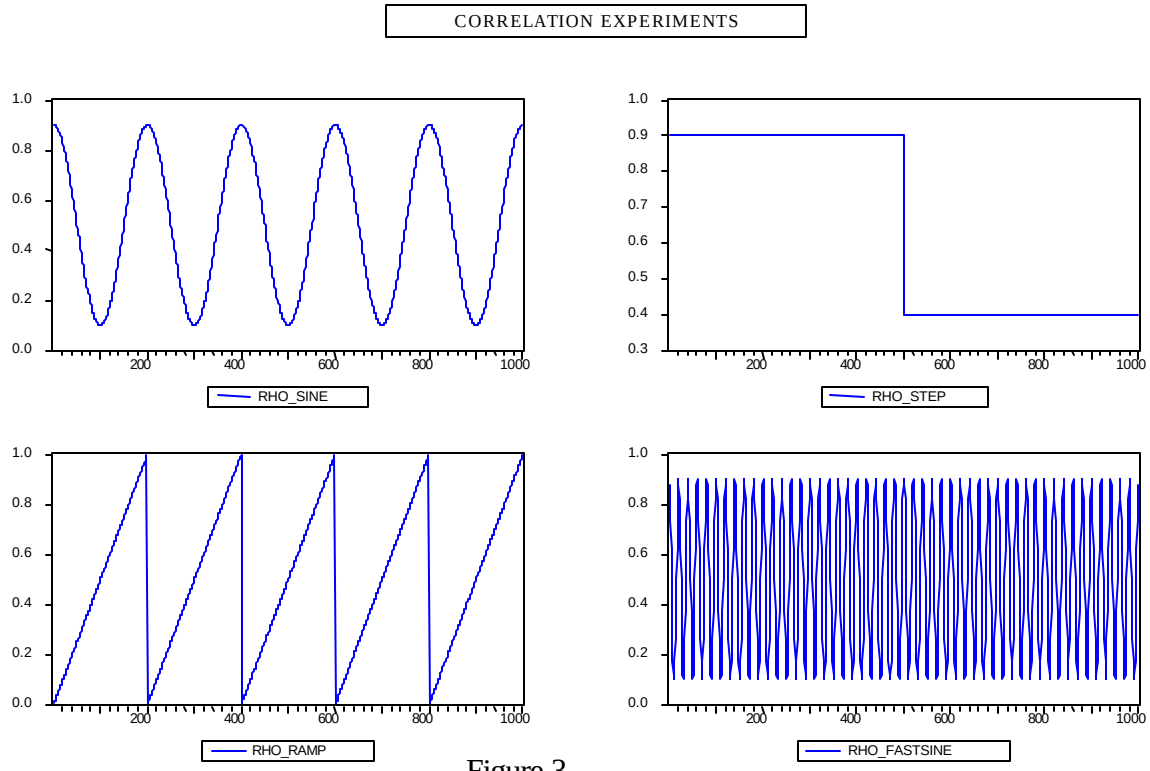

Figure 3

Eight different methods are used to estimate the correlations – two multivariate GARCH models, Orthogonal GARCH, two integrated DCC models and one mean reverting DCC plus the exponential smoother from RISKMETRICS and the familiar 100 day moving average. The methods and their descriptions are:

- SCALAR BEKK – scalar version of (10) with variance targeting as in (12)
- DIAG BEKK- diagonal version of (10) with variance targeting as in (11)
- DCC IMA – Dynamic Conditional Correlation with integrated moving average estimation as in (26)
- DCC INT –Dynamic Conditional Correlation by Log Likelihood for integrated process
- DCC LL MR – Dynamic Conditional Correlation by Log Likelihood with mean reverting model as in (24)
- MA100- Moving Average of 100 days
- EX .06 –Exponential smoothing with parameter=.06
- OGARCH- orthogonal GARCH or principle components GARCH as in (9).

Three performance measures are used. The first is simply the comparison of the estimated correlations with the true correlations by mean absolute error. This is defined as:

$$MAE = \frac{1}{T}\sum \left| \hat{\rho}_t - \rho_t \right| \tag{35}$$

and of course the smallest values are the best. A second measure is a test for autocorrelation of the squared standardized residuals. For a multivariate problem, the standardized residuals are defined as

$$\eta_t = H_t^{-1/2} r_t \tag{36}$$

which in this bivariate case is implemented with a triangular square root defined as:

$$\begin{aligned} \eta_{1,t} &= r_{1,t} / \sqrt{H_{11,t}} \\ \eta_{2,t} &= r_{2,t} \frac{1}{\sqrt{H_{22,t}\left(1-\hat{\rho}_t^2\right)}} - r_{1,t} \frac{\hat{\rho}_t}{\sqrt{H_{11,t}\left(1-\hat{\rho}_t^2\right)}} \end{aligned} \tag{37}$$

The test is computed as an F test from the regression of $\eta_{1,t}^2$ and $\eta_{2,t}^2$ on 5 lags of the squares and cross products of the standardized residuals plus an intercept. The number of rejections using a 5% critical value is a measure of the performance of the estimator since the more rejections, the more evidence that the standardized residuals have remaining time varying volatilities. This test can obviously be used for real data.

The third performance measure is an evaluation of the estimator for calculating value at risk. For a portfolio with w invested in the first asset and (1-w) in the second, the value at risk, assuming normality, is

$$VaR_t = 1.65\sqrt{\left(w^2 H_{11,t} + (1-w)^2 H_{22,t} + 2 * \hat{\rho}_t \sqrt{H_{11,t} H_{22,t}}\right)} \tag{38}$$

and a dichotomous variable called hit should be unpredictable based on the past where hit is defined as:

$$hit_t = I\left(w * r_{1,t} + (1-w) * r_{2,t} < -VaR_t\right) - .05 \tag{39}$$

The Dynamic Quantile Test introduced by Engle and Manganelli(1999) is an F test of the hypothesis that all coefficients as well as the intercept are zero in a regression of this

variable on its past, on current VaR, and any other variables. In this case 5 lags are used and the number of days since the last hit (lagged one day) are used. The number of rejections using a 5% critical value is a measure of model performance. The reported results are for w =.5, but similar results were obtained for a hedge portfolio with weights 1,-1. As these tests are both done "in sample" it is not surprising to find that often they have less than a 5% rejection rate.

V. RESULTS

Table I presents the results for the Mean Absolute Error for the eight estimators for 6 experiments with 200 replications. In four of the six cases the DCC mean reverting model has the smallest MAE. When these errors are summed over all cases, this model is the best. Very close second and third place models are DCC integrated with log likelihood estimation, and scalar BEKK.

In Table III the second standardized residual is tested for remaining autocorrelation in its square. This is the more revealing test since it depends upon the correlations; the test for the first residual does not. For five out of six cases, the DCC mean reverting model is the best. When summed over all cases it is a clear winner. The test for autocorrelation in the first squared standardized residual is less uniform across experiments as seen in Table IV. Overall the best model appears to be the diagonal BEKK.

The VaR based Dynamic Quantile Test is presented in Table V for a portfolio that is half invested in each asset. The number of rejections for many of the models is well below the 5% nominal level. The minimum is somewhat spread out over models although the worst cases are dramatic. The MA100 is so much worse than other models that it is not included in the graph of Figure 4. Overall, the best method is found to be DCC integrated by log likelihood.

From all of these performance measures, the Dynamic Conditional Correlation methods are either the best or very near the best method. Choosing among these models, the mean reverting model is the general winner although the integrated versions are close behind.

TABLE I

MEAN ABSOLUTE ERROR OF CORRELATION ESTIMATES

MODEL	SCAL BEKK	DIAG BEKK	DCC LL MR	DCC LL INT	DCC IMA	EX .06	MA 100	O-GARCH
FAST SINE	0.2292	0.2307	**0.2260**	0.2555	0.2581	0.2737	0.2599	0.2474
SINE	0.1422	0.1451	**0.1381**	0.1455	0.1678	0.1541	0.3038	0.2245
STEP	0.0859	0.0931	0.0709	0.0686	0.0672	0.0810	**0.0652**	0.1566
RAMP	0.1610	0.1631	**0.1546**	0.1596	0.1768	0.1601	0.2828	0.2277
CONST	0.0273	0.0276	0.0070	**0.0067**	0.0105	0.0276	0.0185	0.0449
T(4) SINE	0.1595	0.1668	**0.1478**	0.1583	0.2199	0.1599	0.3016	0.2423

SUM OF MEAN ABSOLUTE ERROR ESTIMATES OF CORRELATIONS

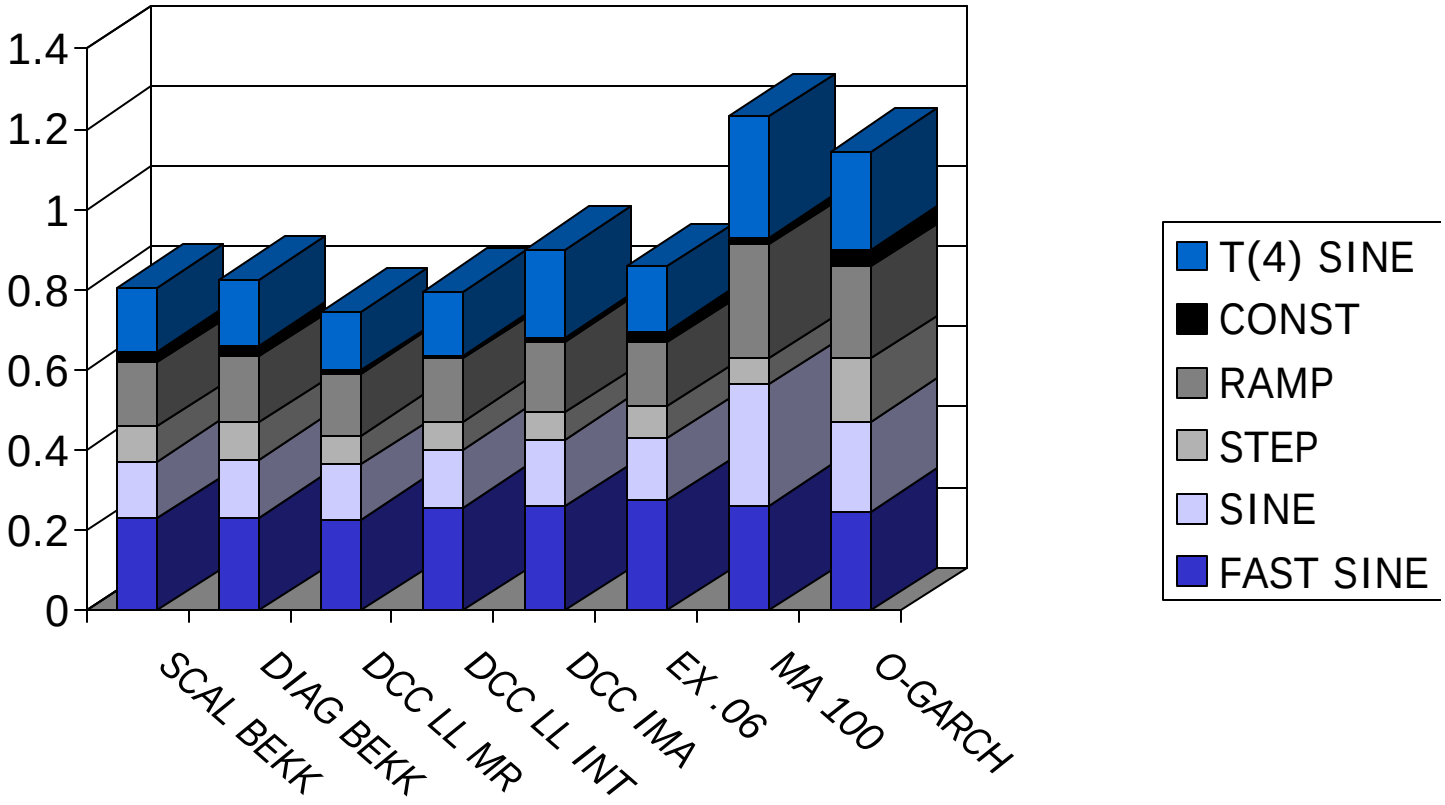

Figure 4

TABLE II

FRACTION OF 5%TESTS FINDING AUTOCORRELATION IN SQUARED STANDARDIZED SECOND RESIDUAL

MODEL	SCAL BEKK	DIAG BEKK	DCC LL MR	DCC LL INT	DCC IMA	EX .06	MA 100	O-GARCH
FAST SINE	0.1750	0.0550	**0.0450**	0.2400	0.2350	0.5750	0.9700	0.0500
SINE	0.3769	0.1313	**0.0500**	0.0800	0.1850	0.5750	1.0000	0.1200
STEP	0.7638	0.4650	**0.1616**	0.1900	0.4900	0.7500	0.9900	0.6000
RAMP	0.3550	0.1350	**0.1150**	0.4400	0.6350	0.6450	0.9950	0.1200
CONST	0.9600	0.2050	**0.0182**	0.0200	0.0250	0.9400	0.9950	0.8550
T(4) SINE	0.2000	0.1300	0.1500	0.1950	**0.1050**	0.2450	0.8450	0.1300

SUM OF REJECTIONS OF AUTOCORRELATION TEST2

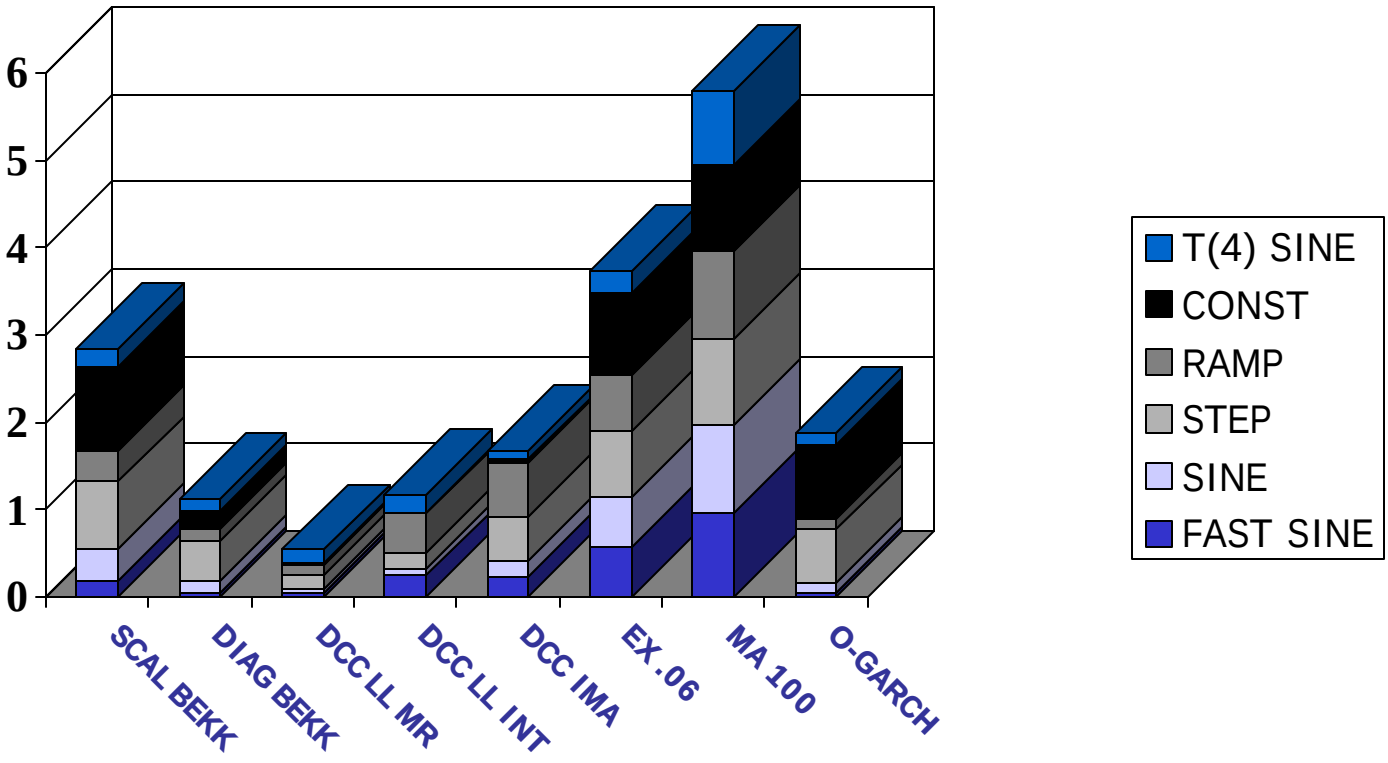

Figure 5

TABLE III

FRACTION OF 5% TESTS FINDING AUTOCORRELATION IN SQUARED STANDARDIZED FIRST RESIDUAL

MODEL	SCAL BEKK	DIAG BEKK	DCC LL MR	DCC LL INT	DCC IMA	EX .06	MA 100	O-GARCH
FAST SINE	0.0900	0.0100	**0.0050**	0.0150	0.0150	0.0300	0.5150	0.0150
SINE	0.0151	0.0051	0.0100	0.0100	**0.0050**	0.0200	0.4850	**0.0050**
STEP	0.0151	0.0100	0.0051	**0.0050**	0.0200	0.0150	0.5350	0.0100
RAMP	0.0150	**0.0050**	0.0200	0.0200	0.0150	0.0250	0.6050	0.0100
CONST	0.0150	**0.0100**	0.0121	0.0150	**0.0100**	**0.0100**	0.5050	**0.0100**
T(4) SINE	0.0500	**0.0450**	0.0550	0.0600	0.0500	0.0600	0.3950	0.0650

SUM OF REJECTIONS OF AUTOCORRELATION TEST1

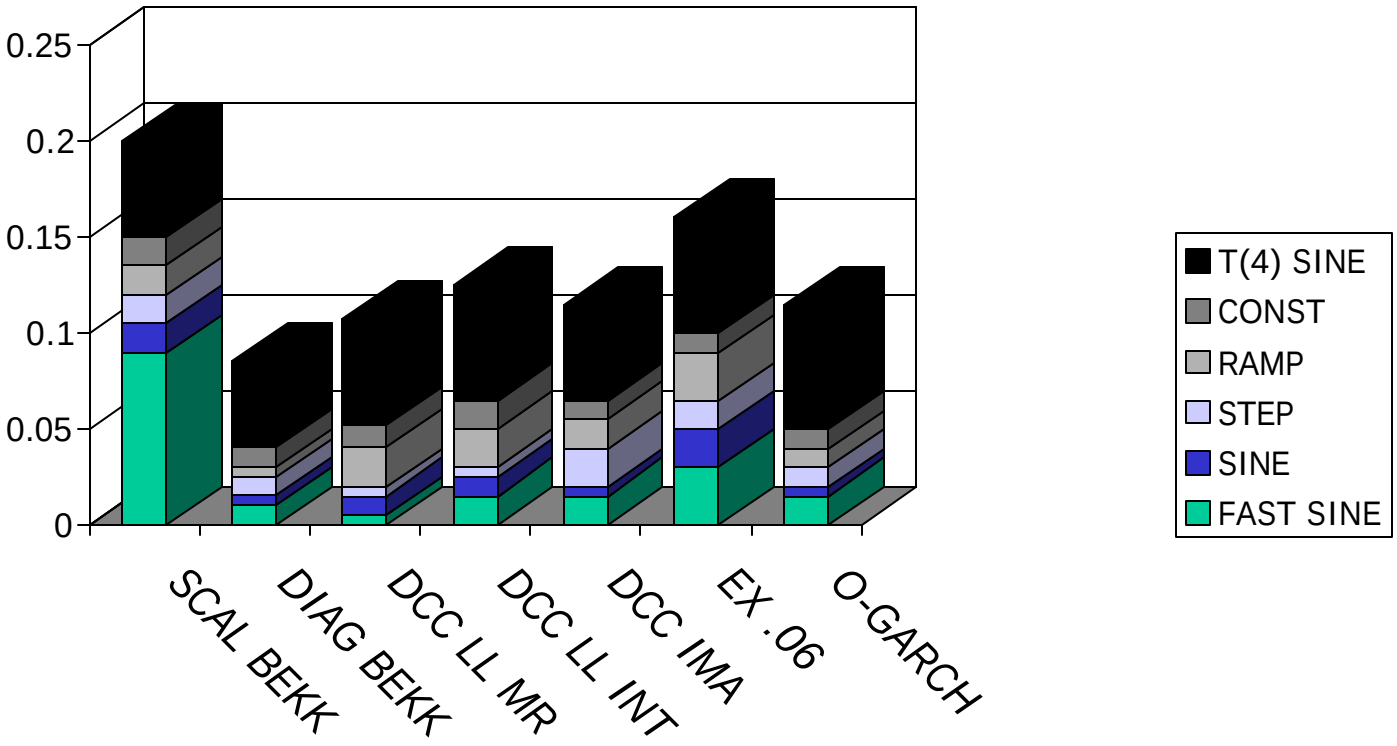

Figure 6

TABLE IV

FRACTION OF 5% DYNAMIC QUANTILE TESTS REJECTING

VALUE AT RISK

MODEL	SCAL BEKK	DIAG BEKK	DCC LL MR	DCC LL INT	DCC IMA	EX .06	MA 100	O-GARCH
FAST SINE	0.0050	0.0100	**0.0000**	0.0000	0.0050	0.0400	0.3000	0.0100
SINE	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	0.0100	0.2000	0.0650
STEP	0.0352	**0.0150**	0.0253	0.0200	0.0250	0.0600	0.2350	0.2850
RAMP	0.0100	0.0100	**0.0000**	**0.0000**	**0.0000**	0.0300	0.3300	0.0450
CONST	0.0250	0.0200	**0.0000**	**0.0000**	**0.0000**	0.0900	0.2650	0.0500
T(4) SINE	0.0300	0.0300	0.0200	**0.0100**	0.0150	0.0300	0.2100	0.0400

SUM OF REJECTIONS VALUE AT RISK

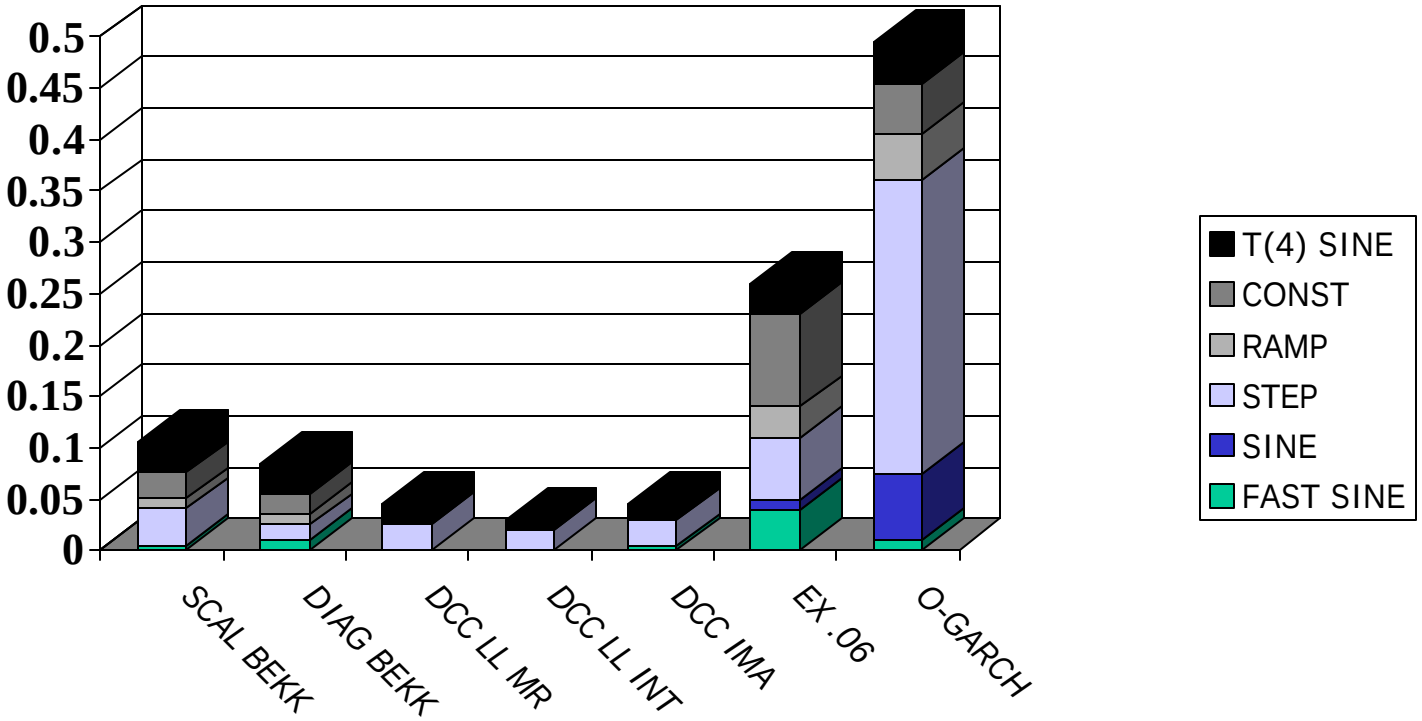

Figure 7

VII. EMPIRICAL RESULTS

Empirical examples of these correlation estimates are presented for several interesting series. First we examine the correlation between the Dow Jones Industrial Average and the NASDAQ composite for the ten years ending in March 2000. Then we look at correlations between stocks and bonds, a central feature of asset allocation models. Finally we examine the correlation between returns on several currencies around major historical events including the launch of the Euro.

The dramatic rise in the NASDAQ over the last part of the 90's perplexed many portfolio managers and delighted the new internet start-ups and day traders. A plot of the GARCH volatilities of these series reveals that the NASDAQ has always been more volatile than the Dow but that this gap widens at the end of the sample.

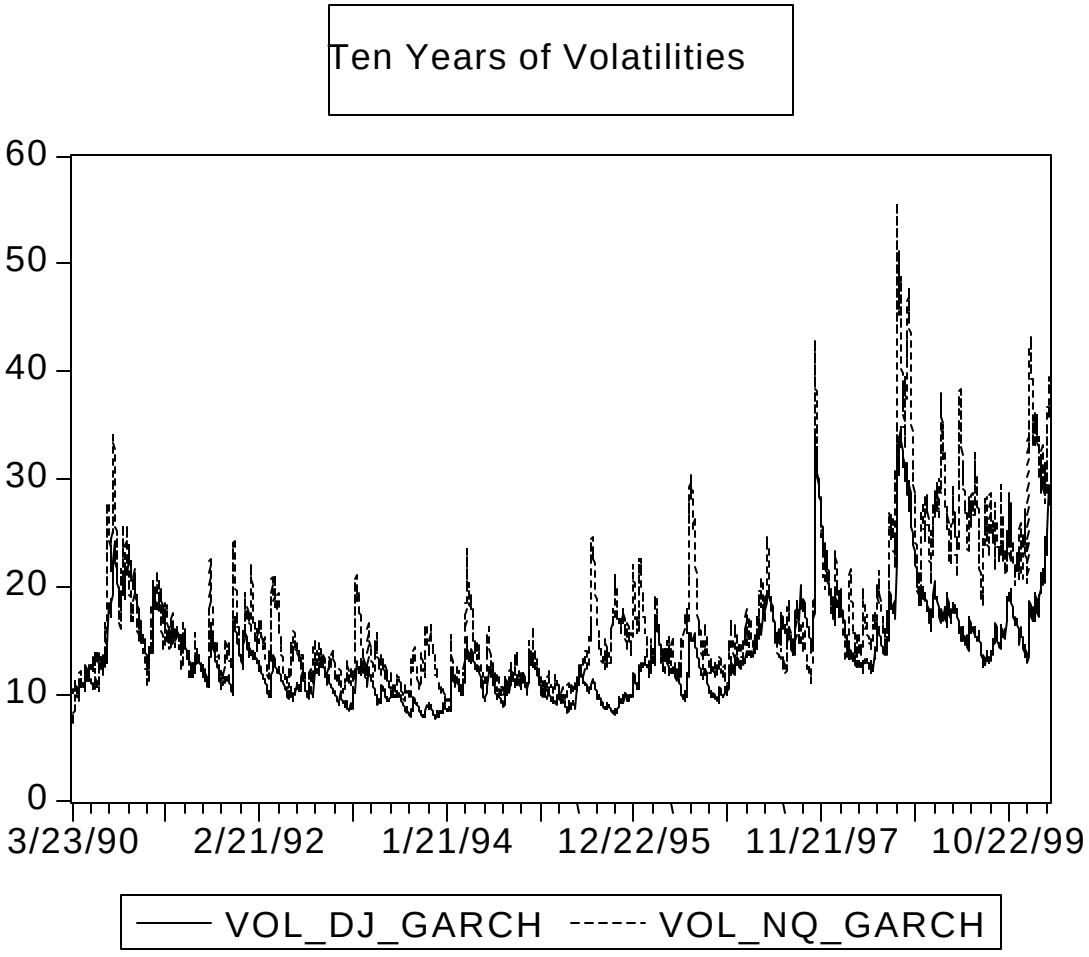

Figure 8.

The correlation between the Dow and NASDAQ was estimated with the DCC integrated method using the volatilities in the figure above. The results are quite interesting.

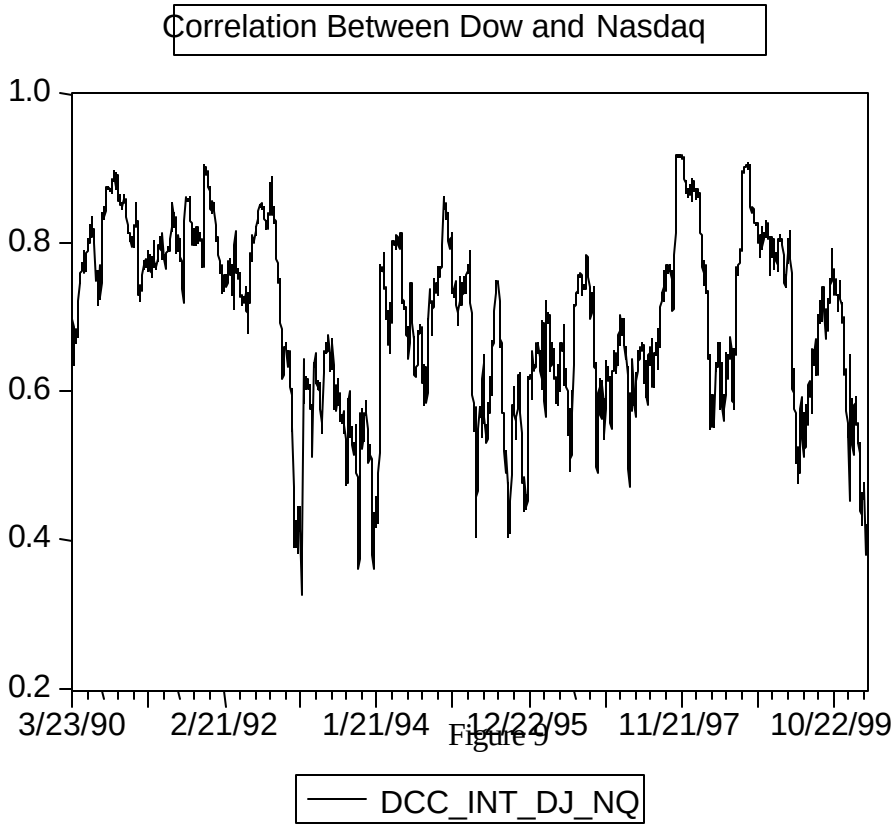

Figure 9

Figure 9

While for most of the decade the correlations were between .6 and .9, there were two notable drops. In 1993 the correlations averaged .5 and dropped below .4, and in March of 2000 they again dropped below .4. The episode in 2000 is associated with sector rotation between "new economy" stocks and "brick and mortar" stocks. The drop at the end of the sample period is more pronounced for some estimators than for others. Looking at just the last year in Figure 10, it can be seen that only the Orthogonal GARCH correlations fail to decline and that the BEKK correlations are most volatile.

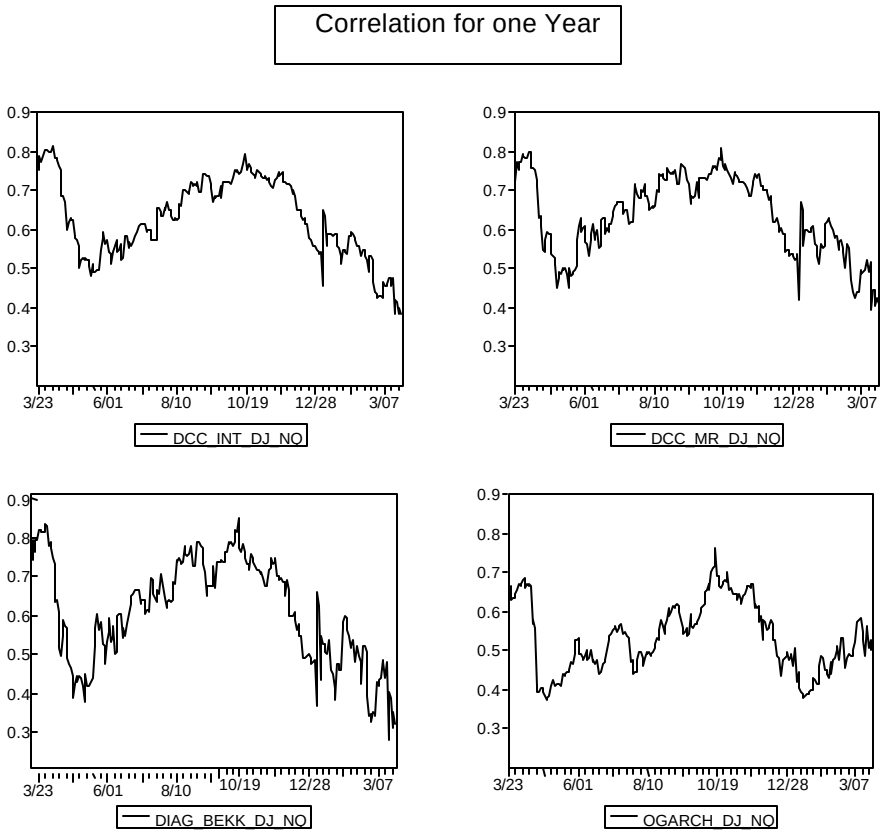

Figure 10

The second empirical example is the correlation between domestic stocks and bonds. Taking bond returns to be minus the change in the 30 year benchmark yield to maturity, the correlation between the Dow and the Nasdaq are shown in Figure 11 for the integrated DCC for the last ten years. The correlations are generally positive in the range of .4 except for the summer of 1998 when they become highly negative, and the end of the sample when they are about zero. While it is widely reported in the press that the Nasdaq does not seem to be sensitive to interest rates, the data suggests that this is also true for the Dow. Throughout the decade it appears that the Dow is more highly correlated with bond prices than is the Nasdaq.

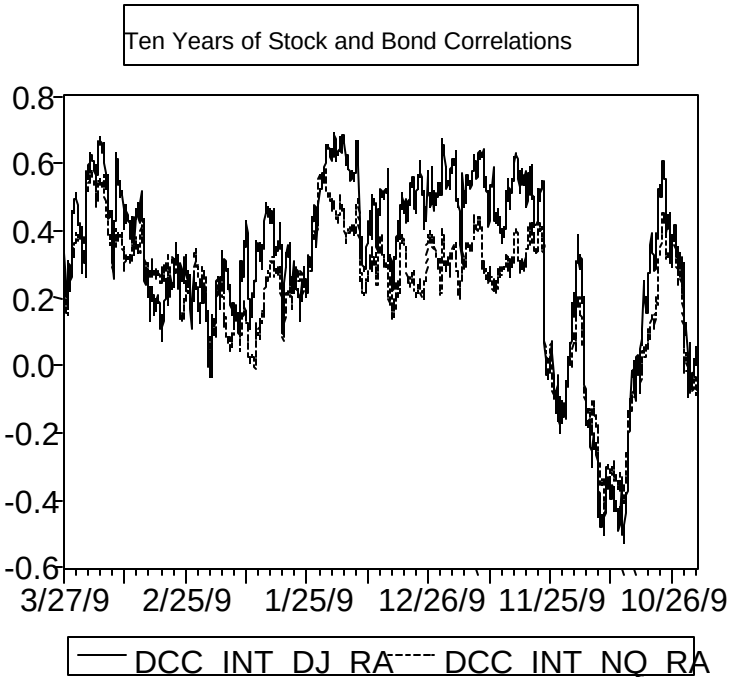

Figure 11

Currency correlations show dramatic evidence of non-stationarity. That is, there are very pronounced apparent structural changes in the correlation process. In

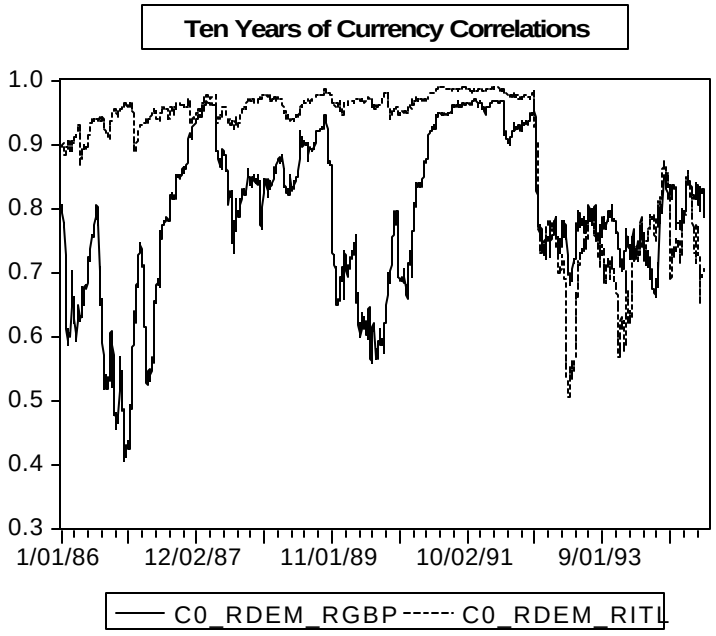

Figure 12

Figure 12, the breakdown of the correlations between the Deutschmark and the Pound and Lira in August of 1992 is very apparent. For the Pound this was a return to a more normal correlation while for the Lira it was a dramatic uncoupling.

Figure 13 shows currency correlations leading up to the launch of the Euro in January 1999. The Lira has lower correlations with the Franc and Deutschmark from 93 to 96 but then they gradually approach one. As the Euro is launched the estimated correlation has moved essentially to one. In the last year it drops below .95 only once for the Franc/Lira and not at all for the other two pairs.

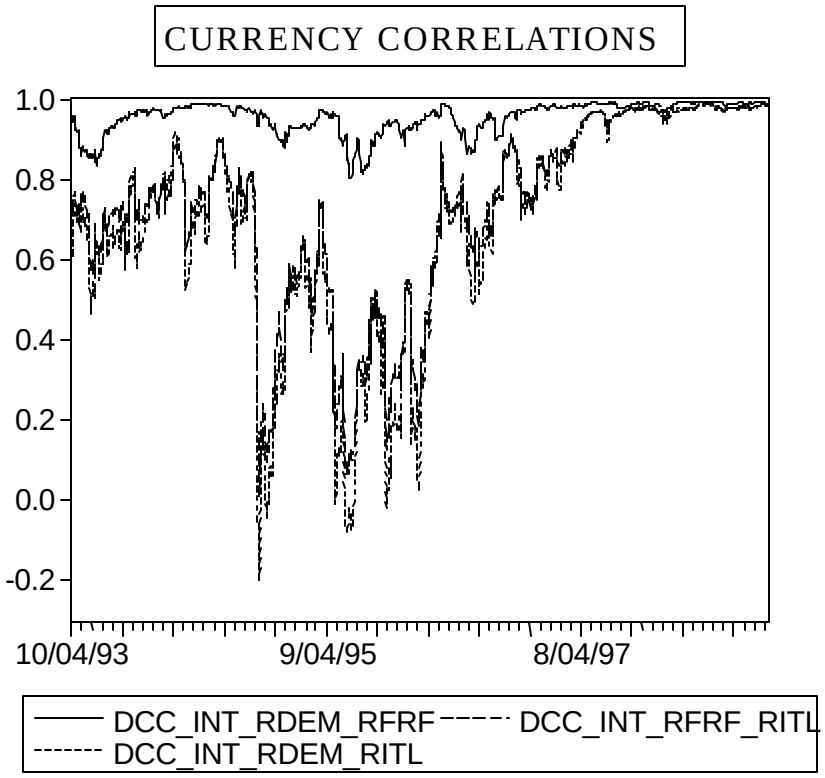

Figure 13

VII. CONCLUSIONS

In this paper a new family of multivariate GARCH models has been proposed which can be simply estimated in two steps from univariate GARCH estimates of each equation. A Maximum Likelihood estimator has been proposed and several different specifications suggested. The goal of this proposal is to find specifications that potentially can estimate large covariance matrices. In this paper, only bivariate systems have been estimated to establish the accuracy of this model for simpler structures. However, the procedure has been carefully defined and should also work for large systems. A desirable practical feature of the DCC models, is that multivariate and univariate volatility forecasts are consistent with each other. When new variables are added to the system, the volatility forecasts of the original assets will be unchanged and correlations may even remain unchanged depending upon how the model is revised.

The main finding in this paper is that the bivariate version of this model provides a very good approximation to a variety of time varying correlation processes. The comparison of DCC with simple multivariate GARCH and several other estimators shows that the DCC is often the most accurate. This is true whether the criterion is mean absolute error, diagnostic tests or tests based on value at risk calculations.

Empirical examples from typical financial applications are quite encouraging as they reveal important time varying features which might otherwise be difficult to quantify.

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